Soliton via Stchastic Area

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By solitons, we usually mean solitary wave solutions (behaving like a particle) to a class of non-linear wave equations including the KdV (Korteweg-de Vries) equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + \frac{3}{2}u\frac{\partial u}{\partial x} \tag{1}$$

The first giant step in the study of solitons was made by Gardner, Greene, Kruskal and Miura (1967, Phys. Rev. Lett.) where they observed that

(i) the eigenvalues of the Shorödinger operator

$$\frac{\partial}{\partial x^2} + u(t,x),$$

where u is a solution to (1), is constant in time parameter t,

(ii) one can construct a soliton solution to (1) by applying the inverse scattering method, by which we mean the (mathematical) method to construct (unknown) potentials out of given scattering data, which had already been fully developed.

The observation (i) together with the awareness of the existence of the infinite invariants in (G.G.K and M, 1967) motivated another seminal paper by P. Lax (1968, Comm. Pure Appl. Math.), where the KdV equation (1) is understood as the compatibility between the two equations:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial x^2} + u(t,x)\right) w \ (=: Pw) = k\, w & (k \text{ is an eigenvalue}) \\[2ex] \left(\dfrac{\partial^3}{\partial x^3} + \dfrac{3}{2} u \dfrac{\partial}{\partial x} + \dfrac{3}{4} \dfrac{\partial}{\partial x}\right) w \ (=: Bw) = 0 \end{cases}$$

This compatibility is rephrased as the celebrated "Lax equation":

$$\frac{\partial P}{\partial t} + [P, B] = 0$$

where the bracket is the commutator; $[P, B] = PB - BP$

By considering pseudo differential operators such as ∂^{-n} for $n \in \mathbb{N}$ and their infinite series, we have in fact $B = (P^{3/2})_+$, where $(D)_+$ is the differential operator part of the pseudo defferential operator D. In this Lax form, the existence of the infinite many invariants can be rephrased as

$$\frac{\partial P}{\partial x_k} + [P, (P^{k/2})_+] = 0 \quad , \; k = 1, 3, 5, \cdots, 2n+1, \cdots \quad ,$$

where $u \equiv u(x_1, x_3, \cdots, x_{2n+1}, \cdots)$, a function of infinitely many variables. The KdV case (2) is retrieved by setting $x_1 = t$, $x_3 = x$. Each Lax equation generates a non-linear evolution equation with respect to x_{2k+1} since $[P, (P^{k/2})_+]$'s are all multiplication operators. The totality of the generated equations is usually called KdV hierarchy.

If we instead start with the operator

$$L = \partial + \sum_{j=1}^{\infty} u_j \partial^{-j} ,$$

then we still have that $[L, (L^k)_+]$ are all multiplication operators, and hence we obtain infinitely many non-lieaner differential equations with respect to u_j's of infinitely many variables $x_1, x_2, \cdots, x_n, \cdots$ by the Lax equations :

$$\frac{\partial L}{\partial x_k} + [L, (L^k)_+] = 0 , \quad k = 1, 2, \cdots .$$

The family is called KP hierarchy since the KP (Kadomtsev - Petviashvili) equation,

$$\frac{3}{4}\frac{\partial^2 u_1}{\partial x_2^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x} - \frac{3}{2}u_1\frac{\partial u_1}{\partial x_1} - \frac{1}{4}\frac{\partial^3 u_1}{\partial x_1^3}\right),$$

which is easily seen to be a generalization of the KdV to a two dimensional model, is deduced from the equations with $k=2$ and $k=3$. The KP hierarchy as a whole is also a generalization of the KdV hierarchy since the latter hierarchy is obtained by a reduction $(L^2)_- = 0$ from the former.

The equations in KP/KdV hierarchy are all "soliton equations" in the sense that they all have exact solutions of soliton type. In fact, according to Sato's theory of infinite dimensional Grassmannian (M. Sato and Y. Sato, 1982, see also M. Sato, 1987, and T. Miwa, M. Jimbo, and E. Date, 2000), all the u_j's of the hierarchy are simultaneously generated from a single function called tau-function τ in the following way: determine w_1, w_2, etc, by

$$\frac{\tau(x_1 - \frac{1}{k}, x_2 - \frac{1}{2k^2}, \cdots)}{\tau(x_1, x_2, \cdots)} = 1 + \frac{w_1}{k} + \frac{w_2}{k^2} + \cdots \tag{3}$$

by comparing the coefficients of k^{-j}, $j = 1, 2, \cdots$, and then u_1, u_2, etc by

$$L = \left(1 + \sum_{j=1}^{\infty} w_j \partial^{-j}\right) \circ \partial \circ \left(1 + \sum_{j=1}^{\infty} w_j \partial^{-j}\right)^{-1} . \tag{4}$$

For example, we have

$$u_1 = 2\frac{\partial^2}{\partial x_1^2}\log \tau . \tag{5}$$

In particular, we see that if τ is a polynomial of $e^{\Sigma c_{ij}x_j}$'s, then u_j's are all "solitons" in that they are all rational functions of $e^{\Sigma c_{ij}x_j}$'s. The tau functions are characterized as a solution to a family of quadratic differential equations called Hirota equations, which are nothing but Plücker relations that define Sato's infinite dimensional Grassmannian. That is to say, a tau function of the KP hierarchy is a point in the Sato's Grassmannian. It should be noted that in the Sato's theory, the KP hierarchy is the most universal one, out of which many well-known soliton equations are obtained by a reduction.

A tau function τ of the n-soliton solution of the KdV equation:

$$\tau(x_1, x, \cdots) = \det\left(I + G(x_1, x_3, \cdots)\right), \tag{6}$$

with

$$G(x_1, x_3, \cdots) = \left(\frac{\sqrt{m_i m_j}}{p_i + p_j} e^{\frac{1}{2}(\xi_i + \xi_j)}\right)_{1 \le i,j \le n},$$

where

$$\xi_i = p_i x_1 + p_i^3 x_3 + \cdots, \quad i = 1, 2, \cdots, n,$$

and $m_i > 0$ and p_i are parameters.

A tau function τ of the n-soliton solution of the KP equation:

$$\tau(x_1, x_2, \cdots) = \det\left(I + G(x_1, x_2, \cdots)\right), \tag{7}$$

with

$$G(x_1, x_2, \cdots) = \left(\frac{\sqrt{m_i m_j}}{p_i - q_j} e^{\frac{1}{2}(\xi_i + \xi_j)}\right)_{1 \leq i,j \leq n},$$

where

$$\xi_i = (p_i - q_i)x_1 + (p_i^2 - q_i^2)x_2 + \cdots, \quad i = 1, 2, \cdots, n,$$

and $m_i > 0$ and p_i and q_i are parameters.

Definition of Stochastic Area

$W \equiv (W^1, W^2)$ is a 2-dimentional Brownian motion. Lévy's stochastic area of W is given by

$$S_t := \frac{1}{2}\left(\int_0^t W_s^2 \, dW_s^1 - \int_0^t W_s^1 \, dW_s^2\right).$$

Lévy's formula(s)

- The characteristic function of S_t is explicitly given as

$$E\left[e^{\sqrt{-1}\xi S_t}\right] = \left(\cosh \frac{\xi t}{2}\right)^{-1} \tag{8}$$

- The conditional one is also explicitly given as

$$E\left[e^{\sqrt{-1}\xi S_t} \,\middle|\, W_t^1 = x, W_t^2 = y\right] = \frac{\xi t}{2\sinh \frac{\xi t}{2}} \cdot e^{\frac{1}{2}(x^2+y^2)\left(1 - \frac{\xi t}{2}\coth\frac{\xi t}{2}\right)} \tag{9}$$

Setting.

- $W^\ell \equiv (W^{\ell,1}, W^{\ell,2})$, $\ell = 1, 2, \cdots, n$: mutually independent 2-dimensional Brownian motions starting at the origin

  Denote $\mathbf{W}_t^i = (W_t^{1,i}, \cdots, W_t^{n,i})$ for $i = 1, 2$
- $S^\ell := \int_0^1 (W_s^{\ell,2} dW_s^{\ell,1} - W_s^{\ell,1} dW_s^{\ell,2})$
- $\Lambda := \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, where λ_ℓ, $\ell = 1, 2, \cdots, n$ are positive numbers
- $A = (a_{i,j})$: a real $n \times n$-matrix
- $C^\pm := (A \pm A^*)/2$
- Define for $z \in \mathbb{C}$

$$\hat{S}(z) \equiv \hat{S}_{A,\Lambda}(z)$$

$$:= z \sum_{\ell=1}^{n} \lambda^\ell S^\ell + z \langle \Lambda^{\frac{1}{2}} C^- \Lambda^{\frac{1}{2}} \mathbf{W}_1^1, \mathbf{W}_1^2 \rangle_{\mathbb{R}^n} - \frac{z^2}{2} \sum_{i=1,2} \langle \Lambda^{\frac{1}{2}} C^+ \Lambda^{\frac{1}{2}} \mathbf{W}_1^i, \mathbf{W}_1^i \rangle_{\mathbb{R}^n} \qquad (10)$$

THEOREM 1

If either $\max_{\ell} |\lambda_\ell|$ or $\|C^+\|$ is surfficiently small, we have

$$E\left[e^{\hat{S}(\sqrt{-1})}\right] = \begin{vmatrix} \cosh\lambda_1 + a_{1,1}\sinh\lambda_1 & a_{1,2}\sinh\lambda_2 & \cdots & a_{1,n}\sinh\lambda_n \\ a_{2,1}\sinh\lambda_1 & \cosh\lambda_2 + a_{2,2}\sinh\lambda_2 & \cdots & a_{2,n}\sinh\lambda_n \\ \vdots & \vdots & \ddots & \\ a_{n,1}\sinh\lambda_1 & a_{n,2}\sinh\lambda_2 & \cdots & \cosh\lambda_n + a_{n,n}\sinh\lambda_n \end{vmatrix}^{-1}.$$

Proof of Theorem 1.

We first calculate the conditional expectation of $e^{\hat{S}(\sigma)}$ conditioned by $\mathbf{W}_1 = (\mathbf{W}_1^1, \mathbf{W}_1^2)$.

For sufficiently small $\sigma \in \mathbb{R}$,

$$E\left[e^{\sigma \sum_{\ell} \lambda_\ell S^\ell} \mid \mathbf{W}_1\right] = \prod_{\ell} \frac{\sigma \lambda_\ell}{\sin \sigma \lambda_\ell} \exp\left(-\frac{(W_1^{\ell,1})^2 + (W_1^{\ell,2})^2}{2} (\sigma \lambda_\ell \cot \sigma \lambda_\ell - 1)\right) . \tag{11}$$

Proof of Theorem 1.

Therefore we have

$$\mathbb{E}\left[e^{\hat{S}(\sigma)} \mid \mathbf{W}_1\right] = \prod_{\ell} \frac{\sigma\lambda_\ell}{\sin\sigma\lambda_\ell} \exp\left(-\frac{1}{2}\left\langle\left(M(\sigma) - I + C(\sigma)\right)\mathbf{W}_1, \mathbf{W}_1\right\rangle\right)$$

where

$$M(\sigma) = \begin{pmatrix} \sigma\Lambda\cot\sigma\Lambda & 0 \\ 0 & \sigma\Lambda\cot\sigma\Lambda \end{pmatrix},$$

with

$$\cot\sigma\Lambda := \operatorname{diag}\left(\cot\sigma\lambda_1, \cdots, \cot\sigma\lambda_n\right)$$

as usual, and

$$C(\sigma) := \begin{pmatrix} \sigma^2\Lambda^{\frac{1}{2}}C^+\Lambda^{\frac{1}{2}} & \sigma\Lambda^{\frac{1}{2}}C^-\Lambda^{\frac{1}{2}} \\ -\sigma\Lambda^{\frac{1}{2}}C^-\Lambda^{\frac{1}{2}} & \sigma^2\Lambda^{\frac{1}{2}}C^+\Lambda^{\frac{1}{2}} \end{pmatrix}.$$

Proof of Theorem 1.

Since $\|M(\sigma) + C(\sigma) - I\| \to 0$ as $\sigma \to 0$, we can take σ small enough to ensure that $M(\sigma) + C(\sigma)$ is positive definite. Then, applying quadratic Gaussian formula for such σ, we obtain

$$E\left[e^{\hat{S}(\sigma)}\right] = \prod_{\ell=1}^{n} \frac{\sigma \lambda_\ell}{\sin \sigma \lambda_\ell} \det\left(M(\sigma) + C(\sigma)\right)^{-\frac{1}{2}} \qquad (12)$$

Proof of Theorem 1.

We may go further as

$$\det\left(M(\sigma) + C(\sigma)\right)$$

$$= \det\begin{pmatrix} \sigma \Lambda^{\frac{1}{2}}\left(\Lambda^{\frac{1}{2}} \cot\sigma\Lambda + \sigma C^{+}\Lambda^{\frac{1}{2}}\right) & \sigma\Lambda^{\frac{1}{2}} C^{-} \Lambda^{\frac{1}{2}} \\ -\sigma\Lambda^{\frac{1}{2}} C^{-}\Lambda^{\frac{1}{2}} & \sigma\Lambda^{\frac{1}{2}}\left(\Lambda^{\frac{1}{2}}\cot\sigma\Lambda + \sigma C^{+}\Lambda^{\frac{1}{2}}\right)\end{pmatrix}$$

$$= \det\left(\sigma\Lambda^{\frac{1}{2}}\left(\cot\sigma\Lambda + \sigma C^{+} + \sqrt{-1}\,C^{-}\right)\Lambda^{\frac{1}{2}}\right)\cdot \det\left(\sigma\Lambda^{\frac{1}{2}}\left(\cot\sigma\Lambda + \sigma C^{+} - \sqrt{-1}\,C^{-}\right)\Lambda^{\frac{1}{2}}\right)$$

((Since C^{-} is skew symmetric))

$$= \left(\prod_{\ell}(\sigma\lambda_{\ell}) \det\left(\cot\sigma\Lambda + \sigma C^{+} + \sqrt{-1}\,C^{-}\right)\right)^{2} .$$

Proof of Theorem 1.

Hence (12) is turned into the following equality:

$$E\left[e^{\hat{S}(\sigma)}\right] = \det\left(\cos\sigma\Lambda + \left(\sigma C^{+} + \sqrt{-1}\,C^{-}\right)\sin\sigma\Lambda\right)^{-1} \tag{13}$$

where $\sin\sigma\Lambda := \operatorname{diag}\left(\sin\sigma\lambda_1, \cdots, \sin\sigma\lambda_n\right)$.

Proof of Theorem 1.

Now, we prove that an analytic continuation to a domain including $z=\sqrt{-1}$ is possible. We need to check the integrability of

$$E\left[\frac{d}{dz}e^{\hat{S}(z)}\right]$$
$$=E\left[e^{\hat{S}(z)}\left(\sum_{\ell=1}^{n}\lambda_\ell S^\ell+\langle\Lambda^{\frac{1}{2}}C^-\Lambda^{\frac{1}{2}}\mathbf{W}_1^1,\mathbf{W}_1^2\rangle_{\mathbb{R}^n}-z\sum_{i=1,2}\langle\Lambda^{\frac{1}{2}}C^+\Lambda^{\frac{1}{2}}\mathbf{W}_1^i,\mathbf{W}_1^i\rangle_{\mathbb{R}^n}\right)\right].$$

Since $\hat{S}$ is quadratic Gaussian, the integrability is inherited from that of $e^{\hat{S}(a)}$ itself, which is guaranteed if either $\max_\ell|\lambda_\ell|$ or $\|C^+\|$ is sufficiently small.

$\square$

THEOREM 2

Let $\mathcal{P} = \left(\frac{1}{p_i - q_j}\right)_{1 \le i,j \le n}$, and assume that $\min_{i,j} |p_i - q_j|$ is sufficiently large so that $I + \mathcal{P}$ is invertible. Then, if we put $A = (I - \mathcal{P})(I + \mathcal{P})^{-1}$ and

$\Lambda := \mathrm{diag}\left(-\frac{1}{2}(\xi_1 + \log m_1), \cdots, -\frac{1}{2}(\xi_n + \log m_n)\right)$, we have that $\left(E\left[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}\right]\right)^{-1}$, where $\hat{S}_{A,\Lambda}$ is defined by (10), defines a tau function of KP solitons.

Proof of Theorem 2.

Since

$$G = e^{-\Lambda} P e^{-\Lambda} ,$$

we have

$$\tau = \det(I + e^{-\Lambda} P e^{-\Lambda})$$
$$= \det(I + P e^{-2\Lambda}) .$$

Proof of Theorem 2.

On the other hand,

$$\det(\cosh\Lambda + A\sinh\Lambda)$$
$$= \det\left(\frac{e^{\Lambda}+e^{-\Lambda}}{2} + A\,\frac{e^{\Lambda}-e^{-\Lambda}}{2}\right)$$
$$= 2^{-n}\det\left((I+A)e^{\Lambda} + (I-A)e^{-\Lambda}\right)$$
$$= 2^{-n}\det\left((I+A)e^{\Lambda}\right)\det\left(I + (I+A)^{-1}(I-A)e^{-2\Lambda}\right)$$
$$= 2^{-n}\det(I+A)\, e^{-\frac{1}{2}\Sigma(\xi_i+\log m_i)}\det\left(I+\varrho e^{-2\Lambda}\right).$$

$2^{-n}\det(I+A)e^{-\frac{1}{2}(\xi_i+\log m_i)}$ is a trivial factor and thus by Theorem 1 we have the assertion. $\square$

Ikeda-Taniguchi's construction.

They obtained a construction of the Gaussian process X^a in

$$q^a(x) = -4\frac{\partial^2}{\partial x^2}\log E\left[\exp\left(-\frac{1}{2}\int_0^x |X^a(y)|^2 dy\right)\right].$$

They set

$$X_t^a = \sqrt{a}\langle c, \xi_t^p\rangle,$$

where $a>0$, $c \in \mathbb{R}_+^n$, $p \in \mathbb{R}^n$ and ξ^p is an Ornstein-Uhlenbeck process in $\mathbb{R}^n$ starting at 0 defined as the solution to the following SDE :

$$d\xi_t = dW_t + \mathrm{diag}(p_1, \cdots, p_n)\,\xi_t\,dt.$$

PROPOSITION 1.

Suppose that A in Theorem 1 is symmetric. Then

$$E\left[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}\right] = \left(E\left[e^{-\int_0^1 X_s^{A,\Lambda} ds} \right] \right)^2 e^{\operatorname{tr} \Lambda A},$$

where $X^{A,\Lambda} = \langle (\Lambda - A\Lambda A)\xi, \xi \rangle$ and ξ is an Ornstein-Uhlenbeck process on $\mathbb{R}^d$ starting at 0 and satisfying

$$d\xi_t = \Lambda^{\frac{1}{2}} dB_t + \Lambda A \xi_t dt, \tag{14}$$

with B being an n-dimensional standard Brownian motion.

Proof of Proposition 1.

$$E\left[e^{\sqrt{-1}\sum_\ell \lambda_\ell S^\ell} \,\middle|\, \mathbf{W}_1\right] = E\left[e^{-\sum_\ell \frac{\lambda_\ell^2}{2}\int_0^1\left((W_s^{\ell,1})^2+(W_s^{\ell,2})^2\right)ds} \,\middle|\, \mathbf{W}_1\right]$$

Then since $C^+ = A$ and $C^- = 0$, we have

$$E\left[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}\right] = \prod_{i=1,2} E\left[e^{-\sum_\ell \frac{\lambda_\ell^2}{2}\int_0^1 (W_s^{\ell,i})ds + \frac{1}{2}\langle \Lambda^{\frac12} A \Lambda^{\frac12}\mathbf{W}_1^i, \mathbf{W}_1^i\rangle}\right]$$

$$= \left(E\left[e^{-\sum_\ell \frac{\lambda_\ell^2}{2}\int_0^1 (W_s^{\ell,1})ds + \frac{1}{2}\langle \Lambda^{\frac12} A \Lambda^{\frac12}\mathbf{W}_1^1, \mathbf{W}_1^1\rangle}\right]\right)^2$$

$$= \left(E\left[e^{\frac12 \operatorname{tr}\Lambda A}\, e^{\int_0^1\langle \Lambda^{\frac12} A \Lambda^{\frac12}\mathbf{W}_s^1, d\mathbf{W}_s^1\rangle - \frac12\int_0^1 |\Lambda^{\frac12} A \Lambda^{\frac12}\mathbf{W}_s^1|^2 ds} \times e^{-\frac12\int_0^1\langle(\Lambda - A\Lambda A)\Lambda^{\frac12}\mathbf{W}_s^1, \Lambda^{\frac12}\mathbf{W}_s^1\rangle ds}\right]\right)^2.$$

PROOF of PROPOSITION 1.

Define Q by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_1} = e^{\int_0^1 \langle \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_s^1, d\mathbf{W}_s^1 \rangle - \frac{1}{2}\int_0^1 |\Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \mathbf{W}_s^1|^2 ds}.$$

Then by the Maruyama - Girsanov theorem, we see that $\mathbf{W}$ under Q has the same law as ξ of (14). This completes the proof. $\square$

Setting.

- $W \equiv (W^1, W^2)$: a 2-dimensional Brownian motion starting at the origin
- $\{\delta_i = (\delta_{i,1}, \cdots, \delta_{i,n}), i = 1, \cdots, n\}$: an orthonormal basis of $\mathbb{R}^n$
- $$f_i(t) := \sqrt{n} \sum_{\ell=1}^{n} \delta_{i,\ell} \mathbf{1}_{[\frac{\ell-1}{n}, \frac{\ell}{n})}(t), \quad i = 1, 2, \cdots, n$$
- $$S^+_{i,j} := \sum_{a=1,2} \left(\int_0^1 f_i(t) \, dW^a_s \right) \left(\int_0^1 f_j(t) \, dW^a_s \right),$$

and

$$S^-_{i,j} := \int_0^1 \left(\int_0^t f_j(s) \, dW^2_s \right) f_i(t) \, dW^1_t - \int_0^1 \left(\int_0^t f_i(s) \, dW^1_s \right) f_j(t) \, dW^2_t$$

- $\lambda_i > 0$ for all i
- $\Lambda^{\frac{1}{2}} C^+ \Lambda^{\frac{1}{2}} = (\lambda^+_{i,j})$, $\Lambda^{\frac{1}{2}} (I + C^-) \Lambda^{\frac{1}{2}} = (\lambda^-_{i,j})$, and $\lambda^-_{ii} = \lambda_i$
- $\max_{\ell} |\lambda_\ell|$ or $\|C^+\|$ is sufficiently small to ensure the integrability

PROPOSITION 2.

We have that

$$E\left[e^{\sum_{i,j}\left(\sqrt{-1}\,\lambda^{-}_{i,j}S^{-}_{i,j}+\frac{1}{2}\lambda^{+}_{i,j}S^{+}_{i,j}\right)}\right]\left(=E\left[e^{\hat{S}_{A,\Lambda}(\sqrt{-1})}\right]\right)$$

$$=\det\left(\cosh\Lambda+A\sinh\Lambda\right)^{-1}.$$

Proof of Proposition 2.

We will show the following equivalence in law :

$$\left(\sum_{i,j} \lambda^-_{i,j} S^-_{i,j} \quad , \quad \sum_{i,j} \lambda^+_{i,j} S^+_{i,j}\right)$$

$$\stackrel{d}{=} \left(\sum_{\ell=1}^{n} \lambda_\ell S^\ell + \left\langle \Lambda^{\frac{1}{2}} C^- \Lambda^{\frac{1}{2}} \mathbf{W}^1_1 \, , \mathbf{W}^2_1 \right\rangle_{\mathbb{R}^n} \quad , \quad \sum_{i=1,2} \left\langle \Lambda^{\frac{1}{2}} C^+ \Lambda^{\frac{1}{2}} \mathbf{W}^i_1 \, , \mathbf{W}^i_1 \right\rangle_{\mathbb{R}^n} \right).$$

Proof of Proposition 2.

Let us start with the following direct calculation :

$$S^{+}_{i,j} = n \sum_{a=1,2} \left(\sum_{k=\ell} \delta_{i,\ell}\, \delta_{j,k} \left(W^{a}_{\frac{k}{n}} - W^{a}_{\frac{k-1}{n}}\right)\left(W^{a}_{\frac{\ell}{n}} - W^{a}_{\frac{\ell-1}{n}}\right)\right),$$

and

$$S^{-}_{i,j} = n \sum_{\ell=1}^{n} \delta_{\ell,i}\, \delta_{\ell,j} \int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}} \left(\left(W^{2}_{t} - W^{2}_{\frac{\ell-1}{n}}\right) dW^{1}_{t} - \left(W^{1}_{t} - W^{1}_{\frac{\ell-1}{n}}\right) dW^{2}_{t}\right)$$

$$+ n \sum_{k<\ell} \left(\delta_{k,i}\, \delta_{\ell,j} \left(W^{2}_{\frac{k}{n}} - W^{2}_{\frac{k-1}{n}}\right)\left(W^{1}_{\frac{\ell}{n}} - W^{1}_{\frac{\ell-1}{n}}\right) - \delta_{i,\ell}\, \delta_{j,k} \left(W^{1}_{\frac{k}{n}} - W^{1}_{\frac{k-1}{n}}\right)\left(W^{2}_{\frac{\ell}{n}} - W^{2}_{\frac{\ell-1}{n}}\right)\right).$$

Proof of Proposition 2.

By the scaling property of the Brownian motion, the process

$$\left\{\left(W^1_{\frac{s+\ell-1}{n}} - W^1_{\frac{\ell-1}{n}},\ W^2_{\frac{s+\ell-1}{n}} - W^2_{\frac{\ell-1}{n}}\right) : 0 \le s \le 1,\ \ell = 1, \cdots, n\right\}$$

is identically distributed as

$$\left\{ n^{-\frac{1}{2}} \left(W^{\ell,1}_s,\ W^{\ell,2}_s\right) : 0 \le s \le 1,\ \ell = 1, \cdots, n\right\}$$

Here $\{W^{\ell,1}, W^{\ell,2}, \ell = 1, \cdots, n\}$ are $2n$-dimensional Brownian motions starting at the origin. In particular,

$$\left\{\int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}} \left(\left(W^2_t - W^2_{\frac{\ell-1}{n}}\right) dW^1_t - \left(W^1_t - W^1_{\frac{\ell-1}{n}}\right) dW^2_t\right),\ \ell = 1, \cdots, n\right\}$$

is identically distributed as

$$\left\{ n^{-1} \int_0^1 \left(W^{\ell,2}_t \, dW^{\ell,1}_t - W^{\ell,1}_t \, dW^{\ell,2}_t\right),\ \ell = 1, \cdots, n\right\},$$

and

$$\left\{\left(W^1_{\frac{k}{n}} - W^1_{\frac{k-1}{n}}\right)\left(W^2_{\frac{\ell}{n}} - W^2_{\frac{\ell-1}{n}}\right),\ 1 \le k, \ell \le n\right\}$$

is identically distributed as $\left\{W^{k,1}_1 W^{\ell,2}_1,\ 1 \le k, \ell \le n\right\}$.

Proof of Proposition 2.

Therefore, we have the following identity in law:

$$\sum_{i,j} \lambda^{+}_{i,j} S^{+}_{i,j} \overset{d}{=} \sum_{a=1,2} \sum_{k,l} \sum_{i,j} \lambda^{+}_{i,j} \delta_{i,l} \delta_{j,k} W^{k,a}_{1} W^{l,a}_{1}$$

$$= \sum_{a=1,2} \langle \Lambda^{\frac{1}{2}} C^{+} \Lambda^{\frac{1}{2}} \mathbf{W}^{a}_{1}, \mathbf{W}^{a}_{1} \rangle \qquad (15)$$

Proof of Proposition 2.

$$
\begin{aligned}
\sum_{\bar{i},\bar{j}} \lambda_{\bar{i},\bar{j}} S_{\bar{i},\bar{j}} \overset{d}{=} & \sum_{\ell=1}^{n} \sum_{\bar{i},\bar{j}} \lambda_{\bar{i},\bar{j}} \delta_{\bar{i},\ell} \delta_{\bar{j},\ell} \int_0^1 \left(W_t^{\ell,2} dW_t^{\ell,1} - W_t^{\ell,1} dW_t^{\ell,2} \right) \\
& + \sum_{k<\ell} \left(\sum_{\bar{i},\bar{j}} \lambda_{\bar{i},\bar{j}} \delta_{\bar{i},k} \delta_{\bar{j},\ell} W_1^{k,2} W_1^{\ell,1} - \sum_{\bar{i},\bar{j}} \lambda_{\bar{i},\bar{j}} \delta_{\bar{i},\ell} \delta_{\bar{j},k} W_1^{k,1} W_1^{\ell,2} \right) \\
= & \int_0^1 \langle \Lambda \mathbf{W}_t^2, d\mathbf{W}_t^2 \rangle - \int_0^1 \langle \Lambda \mathbf{W}_t^1, d\mathbf{W}_t^2 \rangle \\
& + \langle \Lambda^{\frac{1}{2}} C^- \Lambda^{\frac{1}{2}} \mathbf{W}_1^1, \mathbf{W}_1^2 \rangle \\
\overset{d}{=} & \sum_{\ell=1}^{n} \lambda_\ell \int_0^1 \left(W_t^{\ell,2} dW_t^{\ell,1} - W_t^{\ell,1} dW_t^{\ell,2} \right) + \langle \Lambda^{\frac{1}{2}} C^- \Lambda^{\frac{1}{2}} \mathbf{W}_1^1, \mathbf{W}_1^2 \rangle \quad .
\end{aligned}
\tag{16}
$$

Proof of Proposition 2.

Here $\mathbf{W}^a$, $a = 1, 2$ are n-dimensional Brownian motions. Note that $\Lambda^{\frac{1}{2}} C^- \Lambda^{\frac{1}{2}}$ is also skew-symmetric matrix and $(\Lambda^{\frac{1}{2}} C^- \Lambda^{\frac{1}{2}})_{ii} = 0$ for $i = 1, \cdots, n$. Using the equivalences (15) and (16) in law, we can establish

$$E\left[e^{\sum_{i,j} \left(\sqrt{-1}\, \lambda^-_{i,j} S^-_{i,j} + \frac{1}{2} \lambda^+_{i,j} S^+_{i,j} \right)} \right] = \det \Lambda \, \det\left(\cosh \Lambda + \Lambda^{\frac{1}{2}} A \Lambda^{\frac{1}{2}} \sinh \Lambda \right)^{-1}$$

in the same manner as we have done in the proof of Theorem 1. Since $\Lambda^{\frac{1}{2}}$ and $\sinh \Lambda$ commute, we have the assertion. $\square$

THANK YOU
VERY MUCH FOR YOUR
KIND ATTENTION