Poincare's lemma on domains defined by Brownian rough path

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Introduction

- G : simply connected compact Lie group
- $H^{2k+1}(L_e(G), \mathbb{R}) = \{0\}$, where

$L_e(G) = C([0,1] \to G \mid \gamma(0) = \gamma(1) = e)$.

- The exterior differential operator d and Hodge-Kodaira type Laplacian $\Box$ can be defined on Sobolev spaces of differential forms on $L_e(G)$ based on the pinned Brownian motion measure and Malliavin calculus.

Question Can one prove $\dim \ker \Box|_p = 0$? Here p is odd and $\Box|_p$ stands for the Hodge-Kodaira Laplacian acting p-forms.

We can prove the following.

Theorem 1 (1) *Let* $\boldsymbol{\alpha}$ *be a* $\mathbf{1}$*-form on* $\boldsymbol{L_e(G)}$ *satisfying* $\boldsymbol{d\alpha = 0}$. *Then there exists a function* $\boldsymbol{f}$ *such that* $\boldsymbol{df = \alpha}$.

$\mathbf{(2)}$ $\mathbf{Ker}\Box|_{\mathbf{1}} = \{\mathbf{0}\}$.

This theorem is proved by using a Poincaré type vanishing lemma on a certain domain in a Wiener space.

The related general results were studied by Kusuoka based on his capacity and Sobolev spaces.

Plan of Talk

1. Poincaré type vanishing lemma : convex case
2. Brownian rough path
3. Poincaré type vanishing lemma : A domain defined by Brownian rough path
4. Precise statement of Theorem 1
5. Proof of Theorem 1

Poincaré type vanishing lemma : convex case

Let (B, H, μ) be an abstract Wiener space. Let $F \in \mathbb{D}^\infty(B, \mathbb{R})$. Assume

(1) $|DF(w)|_H^{-1} \in \cap_{p \geq 1} L^p(B, \mu)$

(2) $D^2 F(w) \geq 0$ μ-almost sure w.

Let $U = \{w \in B \mid F(w) < 1\}$. We assume $\mu(U) > 0$. Let d be the exterior differential operator acting on smooth forms $\mathbb{D}^\infty(B, \wedge^p H^*)$ on B. Let d^* be the adjoint opertaor of d. We define

$$\Box = dd^* + d^* d.$$

We restrict this operator on $\boldsymbol{U}$ by specifying a boundary condition. Let

$$\begin{aligned} \boldsymbol{D}(\Box|_p) &= \{\boldsymbol{\alpha} \in \mathbb{D}^\infty(\boldsymbol{B}, \wedge^p \boldsymbol{H}^*) \mid \boldsymbol{\iota}(\boldsymbol{n})\boldsymbol{\alpha}|_S(\boldsymbol{w}) = \boldsymbol{0}, \\ &\qquad \boldsymbol{\iota}(\boldsymbol{n})\boldsymbol{d\alpha}|_S(\boldsymbol{w}) = \boldsymbol{0}\ \boldsymbol{\nu} - \boldsymbol{a.s.w} \in \boldsymbol{S}\}. \end{aligned}$$

Here

$$\begin{aligned} &\boldsymbol{d\nu} = |\boldsymbol{DF}(\boldsymbol{w})|\boldsymbol{\delta}_1(\boldsymbol{F})\boldsymbol{d\mu}, \quad \boldsymbol{S} = \{\boldsymbol{F} = \boldsymbol{1}\}, \\ &\qquad \boldsymbol{n}(\boldsymbol{w}) : \text{unit outer normal vector at } \boldsymbol{w} \in \boldsymbol{S} \end{aligned}$$

This is called the absolute boundary condition.

More precisely, we define $\Box$ in the following way. Let

$$\begin{aligned}
\mathcal{E}(\alpha,\beta) &= \int_U (d\alpha, d\beta)d\mu + \int_U (d^*\alpha, d^*\beta)d\mu, \\
\alpha,\beta &\in \{\theta \in \mathbb{D}^\infty(B, \wedge^p H^*) \mid \\
&\qquad \iota(n)\theta(w) = 0 \ a.s. w \in S\}.
\end{aligned}$$

We define $\Box$ as the self-adjoint operator corresponding to the closed form which is obtained by the closure of $\mathcal{E}$. Shigekawa proved the following theorem.

Theorem 2 (Shigekawa) $\inf \sigma(\Box|_p) \geq p$.

$\boldsymbol{L^2}$-Poincaré lemma:

Suppose that $\boldsymbol{d\alpha = 0}$, $\boldsymbol{\alpha \in D(\overline{\mathcal{E}|_p})}$. Let $\boldsymbol{\beta = d^* \Box|_p^{-1} \alpha}$. Then

$$\begin{aligned} \boldsymbol{d\beta} &= \boldsymbol{dd^* \Box^{-1} \alpha = (dd^* + d^* d)\, \Box^{-1} \alpha} \\ &= \boldsymbol{\alpha,} \end{aligned}$$

where we have used $\boldsymbol{d\Box^{-1} = \Box^{-1} d}$ and $\boldsymbol{d\alpha = 0}$. This implies that closed form is exact on $\boldsymbol{U}$. However, it seems that this argument require the essentially self-adjointness of $\Box$ on some suitable domain (domain issue).

Proposition 3 *Assume*

(1) $\forall w \in B$, $F(w+\cdot) : H \longrightarrow \mathbb{R}$ *is convex and* $\mathrm{esssup}|DF(w)|_H < \infty$.

(2) *There exist projection operators* P_N *on* H *such that* $P_N \longrightarrow I_H$, $P_N \in L(B, B^*)$, $\|F(P_N^{\perp}\cdot)\|_{\mathbb{D}^{1,4}(\mathbb{R})} \longrightarrow 0$.

(3) $\mu(U_r) > 0$, *where* $U_r = \{w \mid F(w) < r\}$.

We fix $p > 1$. *Assume* $\alpha \in L^2(B, H^*) \cap \mathbb{D}^{\infty,p}(B, H^*)$ *satisfies* $d\alpha = 0$ *on* U_r.

Then there exists $f_{r'} \in \mathbb{D}^{1,2}(B, \mathbb{R})$ *such that* $df_{r'} = \alpha$ *on* $U_{r'}$ *for any* $r' < r$.

Proof of Proposition 3

We write $P_N w = \xi, \;\; w - P_N w = \eta$ and define

$$\begin{aligned} R_N &= \{\eta \mid F(\eta) < r/2\}, \; U(\eta) = \{\xi \mid F(\xi + \eta) < r\} \\ U_N &= \{\xi + \eta \mid \xi \in U(\eta), \eta \in R_N\} \subset U. \end{aligned}$$

Then $U(\eta)$ is a convex set and $0 \in U(\eta)$. We can write

$$\begin{aligned} \alpha(w) &= \sum_i \alpha_i(w) d\xi^i + \sum_j \alpha_j(w) d\eta^j \\ &:= \alpha_N(\xi, \eta) + \alpha_N^{\perp}(\xi, \eta). \end{aligned}$$

$d_N \alpha_N(\cdot, \eta) = 0$ on $U(\eta)$ for $a.s.\eta$, where $d_N = P_N d$.

Define

$$
\begin{aligned}
f_N(\xi, \eta) &= \int_0^1 \sum_i \alpha_i(t\xi + \eta)\xi^i dt \\
g_N(\xi, \eta) &= f_N(\xi, \eta) - \frac{\int_{U(\eta)} f_N(\xi, \eta) d\mu_N(\xi)}{\mu_N(U(\eta))}
\end{aligned}
$$

Then

$$
\begin{aligned}
d_N g_N(\xi, \eta) &= \alpha_N(w) \\
\|g_N(\cdot, \eta)\|^2_{L^2(U(\eta), \mu_N)} &\leq C\|\alpha_N(\cdot, \eta)\|^2_{L^2(U(\eta), \mu_N)}. \quad (*)
\end{aligned}
$$

$(*)$ follows from the Poincaré inequality on the convex domain $U(\eta)$. Let $\hat{g}_N(w) = g_N(w) 1_{U_N}(w)$ $w \in B$.

$(*)$ implies $\sup_N \|\hat{g}_N\|_{L^2(B,\mu)} < \infty.$

Hence a subsequence $\hat{g}_{N(k)}$ converges weakly to some g_∞ in $L^2(B, \mu)$.

$g_\infty \varphi(F)$ is the desired function, where φ is a smooth cut-off function such that

$$\varphi(x) = 1 \ \ (x \le r_1), \qquad \varphi(x) = 0 \ \ (x \ge r_2),$$

where $r' < r_1 < r_2 < r$.

Let us consider the case:

$$\begin{aligned} \boldsymbol{B} = \boldsymbol{W}^d &= \boldsymbol{C}([\mathbf{0},\mathbf{1}] \longrightarrow \mathbb{R}^d \mid \boldsymbol{w}(\mathbf{0}) = \mathbf{0}), \\ \boldsymbol{H} &= \boldsymbol{H}^1([\mathbf{0},\mathbf{1}] \longrightarrow \mathbb{R}^d \mid \boldsymbol{h}(\mathbf{0}) = \mathbf{0}). \end{aligned}$$

There exists the Wiener measure $\boldsymbol{\mu}$ on $\boldsymbol{W}^d$. Using solutions of SDE on $\boldsymbol{G}$, we can change the problem on $\boldsymbol{L_e}(\boldsymbol{G})$ in Theorem 1 to a problem on some domains in Wiener spaces.

However, the above Poincaré lemma on convex domains cannot be applied to the problem.

We need Poincaré's lemmas on non-convex domains defined by Brownian rough path.

Brownian rough path

Let $x = x_t = (x_t^1, \ldots, x_t^d)$,
$y = y_t = (y_t^1, \ldots, y_t^d)$ $(0 \le t \le 1)$ be continuous paths. Suppose that x or y is a bounded variation. Then we can define for $0 \le s \le t \le 1$

$$\begin{aligned} C(x, y)_{s,t} &= \int_s^t (x_u - x_s) \otimes dy_u \\ &= \sum_{1 \le i,j \le d} \left(\int_s^t (x_u^i - x_s^i) dy_u^j \right) e_i \otimes e_j \\ &\in \mathbb{R}^d \otimes \mathbb{R}^d \end{aligned}$$

as a Stieltjes integral. Here $e_i = {}^t(0, \ldots, \overset{i}{1}, \ldots, 0)$. Let $\Delta = \{(s,t) \in \mathbb{R}^2 \mid 0 \le s \le t \le 1\}$. For a continuous mapping $\phi : \Delta \longrightarrow V$ with values in a normed linear space V, define

$$\|\phi\|_{m,\theta} = \left\{ \int_0^1 \int_0^t \frac{|\phi(s,t)|_V^m}{(t-s)^{2+m\theta}} ds dt \right\}^{1/m},$$

where, m is an even number such that $m(1-\theta) > 2$ and $2/3 < \theta < 1$.

$$W_{m,\theta}(\Delta \to V) := C(\Delta \to V \mid \|\phi\|_{m,\theta} < \infty).$$

Also we define for a continuous path $\boldsymbol{x}$ starting at $\mathbf{0}$ on $\mathbb{R}^d$

$$\|\boldsymbol{x}\|_{m,\theta} = \left\{ \int_0^1 \int_0^t \frac{|\boldsymbol{x}_t - \boldsymbol{x}_s|^m}{|\boldsymbol{t} - \boldsymbol{s}|^{2+m\theta}} \boldsymbol{dsdt} \right\}^{1/m}$$

and

$$\boldsymbol{W}_{m,\theta}(\mathbb{R}^d) = \Big\{ \boldsymbol{x} : [\mathbf{0}, \mathbf{1}] \longrightarrow \mathbb{R}^d \mid \boldsymbol{x}_0 = \mathbf{0}, \|\boldsymbol{x}\|_{m,\theta} < \infty, \\ \boldsymbol{x} \text{ is continuous.} \Big\}.$$

- $\boldsymbol{W}_{m,\theta/2}(\mathbb{R}^d)$ is a subset of the $\boldsymbol{\theta/2}$-Hölder continuous function space
- $\boldsymbol{\mu}(\boldsymbol{W}_{m,\theta/2}(\mathbb{R}^d)) = \mathbf{1}$.

From now on, we fix the indexes

$$m(1-\theta') > 4, \quad 2/3 < \theta < \theta' < 1.$$

For $w \in W_{m,\theta'/2}$, define

$w(N) =$ dyadic polygonal approximation of w such that

$w(N)_{k/2^N} = w_{k/2^N}$ for all $0 \leq k \leq 2^N$ and

$$\begin{aligned} w(N)^i &= (w(N), e_i) \\ w(N)^{\perp} &= w - w(N) \\ w(N)^{\perp,i} &= w^i - w(N)^i \end{aligned}$$

Theorem 4 *Let* $\boldsymbol{\Omega}$ *be the set of* $\boldsymbol{w} \in \boldsymbol{W}_{\boldsymbol{m},\boldsymbol{\theta}'/2}(\mathbb{R}^{\boldsymbol{d}})$ *satisfying the following* (i)-(iii).

(i) $\lim_{N\to\infty} \boldsymbol{w}(\boldsymbol{N}) = \boldsymbol{w}$ *in* $\boldsymbol{W}_{\boldsymbol{m},\boldsymbol{\theta}'/2}(\mathbb{R}^{\boldsymbol{d}})$.

(ii) $\lim_{N\to\infty} \boldsymbol{C}(\boldsymbol{w}(\boldsymbol{N}), \boldsymbol{w}(\boldsymbol{N}))$ *converges in* $\boldsymbol{W}_{\boldsymbol{m},\boldsymbol{\theta}}(\boldsymbol{\Delta} \longrightarrow \mathbb{R}^{\boldsymbol{d}} \otimes \mathbb{R}^{\boldsymbol{d}})$.

(iii) $\lim_{N\to\infty} \boldsymbol{C}(\boldsymbol{w}(\boldsymbol{N})^{\perp}, \boldsymbol{w}(\boldsymbol{N}))$ *and* $\lim_{N\to\infty} \boldsymbol{C}(\boldsymbol{w}(\boldsymbol{N}), \boldsymbol{w}(\boldsymbol{N})^{\perp})$ *converge to* $\mathbf{0}$ *in* $\boldsymbol{W}_{\boldsymbol{m},\boldsymbol{\theta}}(\boldsymbol{\Delta} \longrightarrow \mathbb{R}^{\boldsymbol{d}} \otimes \mathbb{R}^{\boldsymbol{d}})$.

Then $\boldsymbol{\Omega}^{c}$ *is a slim set,* $\boldsymbol{H} \subset \boldsymbol{\Omega}$ *and* $\boldsymbol{\Omega} + \boldsymbol{H} \subset \boldsymbol{\Omega}$.

For $\boldsymbol{w} = (\boldsymbol{w}^1, \ldots, \boldsymbol{w}^d) \in \boldsymbol{\Omega}$, we define

$$\begin{aligned} \boldsymbol{C}(\boldsymbol{w}, \boldsymbol{w})_{s,t} &= \lim_{N\to\infty} \boldsymbol{C}(\boldsymbol{w}(N), \boldsymbol{w}(N))_{s,t} \\ \boldsymbol{C}(\boldsymbol{w}^i, \boldsymbol{w}^j)_{s,t} &= \lim_{N\to\infty} \boldsymbol{C}(\boldsymbol{w}(N)^i, \boldsymbol{w}(N)^j)_{s,t} \end{aligned}$$

For $\boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{\Omega}$, let

$$\begin{aligned} &\boldsymbol{d}_{\boldsymbol{\Omega}}(\boldsymbol{w}, \boldsymbol{z}) \\ &\quad = \max\Big\{ \|\boldsymbol{w} - \boldsymbol{z}\|_{m,\theta'/2}, \|\boldsymbol{C}(\boldsymbol{w}, \boldsymbol{w}) - \boldsymbol{C}(\boldsymbol{z}, \boldsymbol{z})\|_{m,\theta} \Big\}. \end{aligned}$$

Let $\boldsymbol{\varphi} \in (\boldsymbol{W}^d)^*$, where $(\boldsymbol{W}^d)^*(\subset \boldsymbol{H}^* \simeq \boldsymbol{H})$ is the set of functions whose first derivatives are bounded variation.

We define a ball like set in rough path analysis.

$$
\begin{aligned}
\boldsymbol{U_{r,\varphi}} \;&=\; \Big\{ \boldsymbol{w} \in \boldsymbol{\Omega} \;\Big|\; \max_{1\le i\le d} \|\boldsymbol{w}^i\|_{m,\theta'/2} < \boldsymbol{r}, \\
&\qquad \max_{1\le j<k\le d} \|\boldsymbol{C}(\boldsymbol{w}^j, \boldsymbol{w}^k)\|_{m,\theta} < \boldsymbol{r}, \\
&\qquad \max_{1\le i\le j\le d} \|\boldsymbol{C}(\boldsymbol{\varphi}^i, \boldsymbol{w}^j)\|_{m,\theta} < \boldsymbol{r}, \\
&\qquad\qquad \sup_{1\le i\le j\le d} \|\boldsymbol{C}(\boldsymbol{w}^i, \boldsymbol{\varphi}^j)\|_{m,\theta} < \boldsymbol{r} \Big\}.
\end{aligned}
$$

Let

$$
\boldsymbol{U_r(\varphi)} = \{\boldsymbol{w} + \boldsymbol{\varphi} \mid \boldsymbol{w} \in \boldsymbol{U_{r,\varphi}}\}.
$$

Poincaré type vanishing theorem

Theorem 5 *Let* $\beta \in L^2(W^d, H^*)$ *and assume* $\beta \in \mathbb{D}^{\infty,p}(W^d, H^*)$ *for some* $p > 1$. *Let* $\varphi \in (W^d)^*$. *Suppose that* $d\beta = 0$ *on* $U_r(\varphi)$. *Then for any* $r' < r$, *there exists* $f \in \mathbb{D}^{\infty,p}(W^d, \mathbb{R}) \cap \mathbb{D}^{1,2}(W^d, \mathbb{R})$ *such that* $df = \beta$ *on* $U_{r'}(\varphi)$.

Remark 6 The sets $U_r(\varphi), U_{r,\varphi}$ are not H-convex.

Sketch of Proof in the case where $d = 2$, $\varphi = 0$

$$\begin{aligned} U_{r,0} \;&=\; \Big\{ w = (w^1, w^2) \in \Omega \mid \max_{i=1,2} \|w^i\|_{m,\theta'/2} < r, \\ &\qquad \|C(w^1, w^2)\|_{m,\theta} < r \Big\}. \end{aligned}$$

We use the notation

$$\begin{aligned} \Omega_N \;&=\; \{w(N) \mid w \in \Omega\} \\ \Omega_N^{\perp} \;&=\; \{w(N)^{\perp} \mid w \in \Omega\} \end{aligned}$$

and we write

$$w(N) = \xi, \quad w(N)^{\perp} = \eta$$

$$
\begin{aligned}
\boldsymbol{R}_N \;&=\; \left\{ \boldsymbol{\eta} \in \boldsymbol{\Omega}_N^{\perp} \;\middle|\; \max_{i=1,2} \|\boldsymbol{\eta}^i\|_{m,\theta'/2} < r/4, \right. \\
&\qquad \left. \|C(\boldsymbol{\eta}^1, \boldsymbol{\eta}^2)\|_{m,\theta} < r/4 \right\}.
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{U}_{r,0}(\boldsymbol{\eta}) \;&=\; \left\{ \boldsymbol{\xi} \in \boldsymbol{\Omega}_N \;\middle|\; w = \boldsymbol{\xi} + \boldsymbol{\eta} \in \boldsymbol{U}_{r,0}, \right. \\
&\qquad \|C(\boldsymbol{\xi}^1, \boldsymbol{\eta}^2)\|_{m,\theta} < r/4, \\
&\qquad \left. \|C(\boldsymbol{\eta}^1, \boldsymbol{\xi}^2)\|_{m,\theta} < r/4 \right\}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{U}_{r,0,N} \;&=\; \left\{ w = \boldsymbol{\xi} + \boldsymbol{\eta} \in \boldsymbol{\Omega} \mid \boldsymbol{\xi} \in \boldsymbol{U}_{r,0}(\boldsymbol{\eta}), \right. \\
&\qquad \left. \boldsymbol{\eta} \in \boldsymbol{R}_N \right\}
\end{aligned}
$$

- Poincaré's inequality on $U_{r,0}(\eta) \subset \Omega_N$:

For $f \in C_b^\infty(\Omega_N)$ with $\int_{U_{r,0}(\eta)} f d\mu_N = 0$,

$$\|f\|^2_{L^2(U_{r,0}(\eta),\mu_N)} \;\leq\; C \int_{U_{r,0}(\eta)} |Df(\xi)|_H^2 d\mu_N(\xi),$$

where C is a positive constant independent of N, η. μ_N is the Wiener measure on Ω_N.

- Poincaré lemma on $U_{r,0}(\eta)$:

Let θ be a smooth closed 1-form on $U_{r,0}(\eta)$. Then there exists a smooth function g on $U_{r,0}(\eta)$ such that $dg = \theta$ on $U_{r,0}(\eta)$.

• (General results) Let us consider a bounded open subset $\boldsymbol{U}(\subset \mathbb{R}^{n+m}) \ni \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{x} \in \mathbb{R}^{n}, \boldsymbol{y} \in \mathbb{R}^{m}$. Let $\boldsymbol{A} = \{\boldsymbol{x} \mid (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{U}\}$, $\boldsymbol{B} = \{\boldsymbol{y} \mid (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{U}\}$, $\boldsymbol{U}_{\boldsymbol{x}} = \{\boldsymbol{y} \in \mathbb{R}^{m} \mid (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{U}\}$, $\boldsymbol{U}^{\boldsymbol{y}} = \{\boldsymbol{x} \in \mathbb{R}^{n} \mid (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{U}\}$. Let $\boldsymbol{\mu}$ be the standard normal distribution on $\mathbb{R}^{n+m}, \mathbb{R}^{n}, \mathbb{R}^{m}$. Assume

(1) $\boldsymbol{U}_{\boldsymbol{x}}, \boldsymbol{U}^{\boldsymbol{y}}$ are convex sets $\forall \boldsymbol{x} \in \boldsymbol{A}, \forall \boldsymbol{y} \in \boldsymbol{B}$.
$\mathbf{0} \in \boldsymbol{U}_{\boldsymbol{x}}\ \forall \boldsymbol{x} \in \boldsymbol{A}$. $\boldsymbol{A}$ is convex.

(2) $\boldsymbol{\delta} = \inf_{x, x' \in A} \boldsymbol{\mu}(\boldsymbol{U}_{\boldsymbol{x}} \cap \boldsymbol{U}_{\boldsymbol{x}'}) > \mathbf{0}$

Poincaré's inequality and Poincaré's lemma holds on $\boldsymbol{U}$ and the Poincaré constant depends only on $\boldsymbol{\mu}(\boldsymbol{U})$ and $\boldsymbol{\delta}$.

Let $\boldsymbol{\beta_N = P_N \beta}$.

$\boldsymbol{d\beta = 0}$ implies $\boldsymbol{d_N \beta_N = 0}$ on $\boldsymbol{U_{r,0}(\eta)}$ for $\boldsymbol{a.s.\ \eta \in R_N}$.

As already explained, $\exists \boldsymbol{f_N(w) = f_N(\xi, \eta)}$ on $\boldsymbol{U_{r,0,N}}$ such that $\int_{\boldsymbol{U_{r,0}(\eta)}} \boldsymbol{f_N(\xi,\eta) d\mu_N(\xi) = 0}$ and $\boldsymbol{d_N f_N = \beta_N}$.

Let $\boldsymbol{\hat{f}_N = f_N 1_{U_{r,0,N}}}$.

By the Poincaré inequality

$$\begin{aligned}
\|\boldsymbol{\hat{f}_N}\|_{\boldsymbol{L^2(U_{r,0})}} = \|\boldsymbol{f_N}\|_{\boldsymbol{L^2(U_{r,0,N})}} &\le \boldsymbol{C}\|\boldsymbol{\beta_N}\|_{\boldsymbol{L^2(U_{r,0,N})}} \\
&\le \boldsymbol{C}\|\boldsymbol{\beta_N}\|_{\boldsymbol{L^2(U_{r,0})}} \\
&\le \boldsymbol{C}\|\boldsymbol{\beta}\|_{\boldsymbol{L^2(U_{r,0})}}.
\end{aligned}$$

Therefore there exists a subsequence $\hat{\boldsymbol{f}}_{\boldsymbol{N(k)}}$ which converges weakly to some $\hat{\boldsymbol{f}}_{\infty}$ in $\boldsymbol{L^2(W^d, d\mu)}$.

We can show that $\boldsymbol{d\hat{f}_{\infty} = \beta}$ on $\boldsymbol{U_{r'}(\varphi)}$.

Stochastic differential equations

Let G be a compact Lie group $(\dim G = d)$. Let us consider the SDE on G:

$$\begin{aligned} dX(t,a,w) &= (L_{X(t,a,w)})_* \circ dw_t, \\ X(0,a,w) &= a \in G. \end{aligned} \tag{1}$$

Here $L_g a = ga$ is the left-multiplication and w_t is the d-dimensional standard Brownian motion on $\mathbb{R}^d \simeq \mathfrak{g} \simeq T_e(G)$ whose starting point is 0.

Theorem 7 *There exists a measurable map* $X : [0,\infty) \times G \times \Omega \longrightarrow G$ *which satisfies the following.*

(1) $X(t,a,w)$ *is a version of the solution to the SDE* (1).

(2) *For any* t, a, *the map* $w \longrightarrow X(t,a,w)$ *is continuous in the sense that for any* $w, z \in \Omega$ *with* $\max\left(d_\Omega(0,w), d_\Omega(0,z)\right) \le R$, *there exists* $C(R) > 0$ *such that*

$$\sup_{0\le t\le 1} d(X(t,a,w), X(t,a,z)) \ \le \ C(R) d_\Omega(w,z).$$

Precise statements of Theorem 1

Below we assume that

G is a simply connected compact Lie group.

Let ε be a sufficiently small positive number and set

$$\mathcal{D}_\varepsilon = \{w \in \Omega \mid d(e, X(1, e, w)) < \varepsilon\}.$$

Let ν_e be the pinned Brownian motion measure on $L_e(G)$.

First theorem is a vanishing theorem on $\mathcal{D}_\varepsilon$.

Theorem A Let us fix $p > 1$. Assume

(1) $\beta \in \mathbb{D}^{\infty,p}(W^d, H^*)$ and $\beta \in L^2(U_r(\varphi), d\mu)$ for any $r > 0$ and $\varphi \in (W^d)^*$

(2) $d\beta = 0$ on $\mathcal{D}_\varepsilon$.

Then there exist $\{\varphi_i\}_{i=1}^{\infty} \subset (W^d)^*$, $r_i > 0$, $f_i \in \mathbb{D}^{\infty,p}(W^d, \mathbb{R}) \cap \mathbb{D}^{1,2}(W^d, \mathbb{R})$ $(i \geq 1)$ and a measurable function F on $\mathcal{D}_\varepsilon$ such that

$$\begin{aligned} \mathcal{D}_\varepsilon &= \cup_{i=1}^{\infty} U_{r_i}(\varphi_i) \\ df_i &= \beta \quad \text{a.s. on } U_{r_i}(\varphi_i) \\ F(w) &= f_i(w) \quad \text{a.s. on } U_{r_i}(\varphi_i) \end{aligned}$$

Theorem B Let
$\alpha \in \mathbb{D}^{1,2}(\wedge^1 T^* L_e(G), d\nu_e) \cap \mathbb{D}^{\infty,p}(\wedge^1 T^* L_e(G), d\nu_e)$
and assume that $d\alpha = 0 \;\; \nu_e - a.s.$ on $L_e(G)$. Let

$$H_0 = \{h \in H^1([0,1] \longrightarrow T_e(G)) \mid h(0) = h(1) = 0\}.$$

There exists a measurable function f on $L_e(G)$ such that for any $h \in H_0$, $\varepsilon > 0$,

$$f(e^{\varepsilon h}\cdot) - f(\cdot) \in L^q(L_e(G), d\nu_e)$$

and for any $1 \leq q < p$,

$$\lim_{\varepsilon \longrightarrow 0} \left\| \frac{f(e^{\varepsilon h}\cdot) - f(\cdot)}{\varepsilon} - (\alpha, h) \right\|_{L^q(\nu_e)} = 0.$$

Let

$$\begin{aligned} \boldsymbol{\mathcal{E}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}) &= (\boldsymbol{d\alpha}, \boldsymbol{d\alpha})_{\boldsymbol{L^2(\nu_e)}} + (\boldsymbol{d^*\alpha}, \boldsymbol{d^*\alpha})_{\boldsymbol{L^2(\nu_e)}} \\ \boldsymbol{\alpha} &\in \mathfrak{F}\boldsymbol{C_b^\infty}(\wedge^1 \boldsymbol{T^* L_e(G)}) \end{aligned}$$

The Hodge-Kodaira operator $\square$ is the generator corresponding to the closed form which is the closure of $\boldsymbol{\mathcal{E}}$.

Theorem C Let G be a simply connected compact Lie group. Then $\ker \square = \{0\}$.

Proof of Theorem A

(1) By the H-connectedness of $\mathcal{D}_\varepsilon$, there exist $\{U_{r_i}(\varphi_i)\}_{i=1}^{\infty}$ such that $\mathcal{D}_\varepsilon = \cup_{i=1}^{\infty} U_{r_i}(\varphi_i)$ and $\mu\left((\cup_{i=1}^{k} U_{r_i}(\varphi_i)) \cap U_{r_{k+1}}(\varphi_{k+1})\right) > 0$ (for any k).

(2) By Poincaré's vanishing theorem, there exist $f_i \in \mathbb{D}^{\infty,p}(W^d, \mathbb{R}) \cap \mathbb{D}^{1,2}(W^d, \mathbb{R})$ $(1 < p < 2)$ such that $df_i = \beta$ on $U_{r_i}(\varphi_i)$.

(3) Using the H-connectedness and H-simply connectedness of $\mathcal{D}_\varepsilon$ and the Stokes theorem in H-direction, we see that there exists a measurable

function $\boldsymbol{F}$ on $\boldsymbol{\mathcal{D}}_{\varepsilon}$ and $\boldsymbol{c_i} \in \mathbb{R}$ such that for any $\boldsymbol{i} \in \mathbb{N}$,

$$\boldsymbol{F(w) = f_i(w) + c_i} \quad \boldsymbol{\mu}\text{-a.s.}\boldsymbol{w} \in \boldsymbol{U_{r_i}(\varphi_i)}.$$

These $\boldsymbol{F}$ and $\boldsymbol{f_i + c_i}$ are desired functions.

We explain (3). We denote $\boldsymbol{D_k} = \cup_{\boldsymbol{i=1}}^{\boldsymbol{k}} \boldsymbol{U_{r_i}(\varphi_i)}$. We assume that there exists a measurable function $\boldsymbol{F_k}$ on $\boldsymbol{D_k}$ and constants $\boldsymbol{c_i}$ $(\boldsymbol{1 \le i \le k})$ such that $\boldsymbol{F_k = f_i + c_i}$ on $\boldsymbol{U_{r_i}(\varphi_i)}$ for all $\boldsymbol{1 \le i \le k}$.

It suffices to prove that there exists $\boldsymbol{c_{k+1}}$ such that $\boldsymbol{F_k = f_{k+1} + c_{k+1}}$ on $\boldsymbol{D_k \cap U_{r_{k+1}}(\varphi_{k+1})}$.

Take a $h \in (W^d)^*$ and $r > 0$ such that $U_r(h) \subset D_k \cap U_{r_{k+1}}(\varphi_{k+1})$. Since

$$d(F_k - f_{k+1}) = \beta - \beta = 0 \quad \text{on } U_r(h),$$

(by the irreducibility of the Dirichlet form on $U_r(h)$) there exists a constant c_{k+1} such that

$$F_k = f_{k+1} + c_{k+1} \quad \text{on } U_r(h).$$

We prove that

$$F_k = f_{k+1} + c_{k+1} \quad \text{on } D_k \cap U_{r_{k+1}}(\varphi_{k+1}).$$

Suppose that there exists $B \subset D_k \cap U_{r_{k+1}}(\varphi_{k+1})$ and $\delta > 0$ such that

$$|F_k - (f_{k+1} + c_{k+1})| > \delta \quad \text{on } B.$$

Then by the H-connectedness of D_k and $U_{r_{k+1}}(\varphi_{k+1})$, there exists $A \subset U_r(h)$ and $h_0(\cdot), h_1(\cdot) \in H^1([0,1] \to H \mid h(0) = 0)$ such that $h_0(1) = h_1(1) = v$, $A + v \subset B$ and

$$A + h_0(\tau) \subset D_k, \quad A + h_1(\tau) \subset U_{r_{k+1}}(\varphi_{k+1})$$
$$\text{for all } 0 \le \tau \le 1.$$

By the $\boldsymbol{H}$-simply connectedness of $\mathcal{D}_\varepsilon$, there exists $\boldsymbol{\mathcal{H}}(\boldsymbol{\sigma},\boldsymbol{\tau}) \in \boldsymbol{H}$ $(\mathbf{0} \le \boldsymbol{\sigma},\boldsymbol{\tau} \le \mathbf{1})$ such that for all $\boldsymbol{\tau},\boldsymbol{\sigma}$

$$\boldsymbol{\mathcal{H}}(\mathbf{0},\boldsymbol{\tau}) = \boldsymbol{h}_0(\boldsymbol{\tau}), \quad \boldsymbol{\mathcal{H}}(\mathbf{1},\boldsymbol{\tau}) = \boldsymbol{h}_1(\boldsymbol{\tau}),$$

$$\boldsymbol{\mathcal{H}}(\boldsymbol{\sigma},\mathbf{0}) = \mathbf{0}, \quad \boldsymbol{\mathcal{H}}(\boldsymbol{\sigma},\mathbf{1}) = \boldsymbol{v}(= \boldsymbol{h}_0(\mathbf{1}) = \boldsymbol{h}_1(\mathbf{1}))$$

$$\boldsymbol{A} + \boldsymbol{\mathcal{H}}(\boldsymbol{\sigma},\boldsymbol{\tau}) \subset \boldsymbol{\mathcal{D}}_\varepsilon.$$

It holds that for almost all $\boldsymbol{w} \in \boldsymbol{A}$

$$\begin{aligned} \boldsymbol{F}_k(\boldsymbol{w}+\boldsymbol{v}) - \boldsymbol{F}_k(\boldsymbol{w}) &= \int_0^1 (\boldsymbol{\beta}(\boldsymbol{w}+\boldsymbol{h}_0(\boldsymbol{\tau})), d\boldsymbol{h}_0(\boldsymbol{\tau})) \\ \boldsymbol{f}_{k+1}(\boldsymbol{w}+\boldsymbol{v}) - \boldsymbol{f}_{k+1}(\boldsymbol{w}) &= \int_0^1 (\boldsymbol{\beta}(\boldsymbol{w}+\boldsymbol{h}_1(\boldsymbol{\tau})), d\boldsymbol{h}_1(\boldsymbol{\tau})) \end{aligned}$$

Also we have (Stokes theorem),

$$\begin{aligned}\int_0^1 &(\beta(w+h_1(\tau)), d_\tau h_1(\tau)) \\ &\qquad - \int_0^1 (\beta(w+h_0(\tau)), d_\tau h_0(\tau)) \\ &= \iint_{(\sigma,\tau)\in[0,1]^2} (d\beta)(w+\mathcal{H}(\sigma,\tau))\,(d_\sigma\mathcal{H}(\sigma,\tau), d_\tau\mathcal{H}(\sigma,\tau)) \\ &= 0.\end{aligned}$$

This shows

$$F_k(w+v) - f_{k+1}(w+v) = c_{k+1} \quad \text{for almost all } w \in A.$$

This is a contradiction.

Proof of Theorem B

Let $\mathcal{D}'_{2\varepsilon} = \{\gamma \in L_e(G) \mid d(\gamma(1), e) < 2\varepsilon\}$. Define a map $\Psi : \mathcal{D}'_{2\varepsilon} \longrightarrow L_e(G)$ by

$$\Psi(\gamma)(t) = \exp(-t \log \gamma(1))\gamma(t).$$

Let $\beta = \varphi_\varepsilon(X(1, e, w))(\Psi \circ X)^* \alpha \ \in \mathbb{D}_p^\infty(W^d, H^*)$,

$\varphi_\varepsilon(x) = 1 \ (d(x, e) \leq 3\varepsilon/2), \varphi_\varepsilon(x) = 0 \ (d(x, e) \geq 2\varepsilon)$.

Then $d\beta = 0$ on $\mathcal{D}_\varepsilon$.

By Theorem A, $\exists g$ such that $dg = \beta$.

$f(\gamma) = g\left(X^{-1}(\gamma)\right) \ (\gamma \in L_e(G))$ is the desired function.

Proof of Theorem C

Suppose $\Box \boldsymbol{\alpha} = \mathbf{0}$.

(1) Weitzenböck type formula:

$$\Box = \boldsymbol{\nabla}^* \boldsymbol{\nabla} + \boldsymbol{I} + \boldsymbol{T}_{\boldsymbol{b}(1)} + \boldsymbol{T}_2 + \boldsymbol{T}_3,$$

where $\boldsymbol{b}(\boldsymbol{t}) = \int_0^t \circ d\gamma(s)\gamma(s)^{-1}$.

(2) Using the Weitzenböck formula and the hypoellipticity of $\boldsymbol{\nabla}^* \boldsymbol{\nabla}$, $\boldsymbol{\alpha} \in \cap_{1<p<2} \mathbb{D}^{\infty,p}(\wedge^1 T^* L_e(G))$. Since $\Box \boldsymbol{\alpha} = \mathbf{0}$ implies $d\boldsymbol{\alpha} = \mathbf{0}$, there exists f such that $df = \boldsymbol{\alpha}$ by Theorem B. For any C_b^1-function ψ, $\psi(f) \in \mathbb{D}^{1,2}(L_e(G))$ and $d(\psi(f)) = \psi'(f)\boldsymbol{\alpha}$.

Let ψ be a C_b^1-function such that $\psi(x) = x$ $(|x| \le 1)$. Let $\psi_K(x) = K\psi(x/K)$. Using $d^*\alpha = 0$,

$$\begin{aligned}
&\int_{L_e(G)} |\alpha(\gamma)|^2_{T_\gamma L_e(G)} d\nu_e(\gamma) \\
&\quad = \lim_{K\to\infty} \int_{L_e(G)} \big(\alpha(\gamma), \psi_K'(f)\alpha(\gamma)\big)_{T_\gamma L_e(G)} d\nu_e(\gamma) \\
&\quad = \lim_{K\to\infty} \int_{L_e(G)} \left(\alpha(\gamma), d\left(\psi_K(f)\right)\right)_{T_\gamma L_e(G)} d\nu_e(\gamma) \\
&\quad = \lim_{K\to\infty} \int_{L_e(G)} (d^*\alpha)(\gamma)\psi_K(f) d\nu_e(\gamma) = 0
\end{aligned}$$

which implies $\alpha = 0$.

Final Remarks

- $\inf\{\sigma(\Box) \setminus \{0\}\} > 0$?

- Higher dimensional case: Let β be a closed p-form on $U_r(\varphi)$. Then $\exists$ $(p-1)$-form γ such tha $d\gamma = \beta$?

- Other kind of non-convex domain? Non-smooth boundary? Local-Sobolev spaces?

- Probably similar kind of theorem holds in the case of $L_x(M)$ when $H^1(L_x(M), \mathbb{R}) = \{0\}$.

- The case $H^1(L_x(M), \mathbb{R}) \neq \{0\}$?

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Appendix: Kusuoka's result

- Submanifolds in Wiener spaces

M : compact Riemannian manifold

$M \subset \mathbb{R}^d$: isometry

Let $P(x) : \mathbb{R}^d \longrightarrow T_xM$ be the projection operator and

$$\begin{aligned} dX(t,x,w) &= P(X(t,x,w)) \circ dw_t, \\ X(0,x,w) &= x \in M. \end{aligned}$$

There exists a probability measure $d\mu_x = p(1,x,x)^{-1}\delta_x(X(1,x,w))d\mu$ on the submanifold:

$$\boldsymbol{S} = \{\boldsymbol{w} \in \boldsymbol{W}^d \mid \boldsymbol{X}(1, \boldsymbol{x}, \boldsymbol{w}) = \boldsymbol{x}\} \subset \boldsymbol{W}^d.$$

The exterior differential operator on $\boldsymbol{S}$ as the submanifold is well-defined and some Sobolev calculus were developped by many people.

Shigeo Kusuoka defined a local Sobolev spaces

$$\boldsymbol{\mathcal{D}}_{loc}^{\infty,q}(\boldsymbol{U}, d\boldsymbol{\mu})$$

where $\boldsymbol{U}$ is a subset of $\boldsymbol{W}^d$ and $\boldsymbol{q}$ is the index of the integrability. Based on this Sobolev spaces, he proved the following (ICM proceedings 1990, Kyoto):

Theorem 8

$$\boldsymbol{H^p(\mathcal{D}_{loc}^{\infty,q}(\wedge^p T^* S)) \simeq H^p(\mathcal{M}_x, \mathbb{R})}$$

where

$$\boldsymbol{\mathcal{M}_x = \Big\{ h \in H \mid \xi(1,x,h) = x, where}$$
$$\boldsymbol{\xi(t,x,h)} \text{ is the solution to}$$
$$\boldsymbol{\dot{\xi}(t,x,h) = P(\xi(t,x,h))\dot{h}(t), \xi(0,x,h) = x \; t \geq 0 \Big\}}$$

Note that $\boldsymbol{H^1}$ loop space $\boldsymbol{H^1 L_x(M)}$ and $\boldsymbol{\mathcal{M}_x}$ is $\boldsymbol{C^\infty}$-homotopy equivalent.

Let $\Box = d_S^* d_S + d_S d_S^*$ and $\Box|_p$ be the restriction on p-forms. They are defined as the Friedrichs extension of them on some cores.

Theorem 9 (S.Kusuoka) *There exists a map* $j_p : \ker \Box|_p \longrightarrow H^p(\mathcal{M}_x, \mathbb{R})$ *such that*

(1) j_p *is surjective for* $p = 0, 1, 2, \ldots$.

(2) j_p *is injective for* $p = 0, 1$.