Large deviation for heat kernel measures on loop spaces via rough paths

(joint work with Yuzuru INAHAMA)

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§1 Introduction

Main Object of This Talk

- LDP for Banach space valued Brownian rough path (=RP) under the "exactness" condition.
  - Ledoux-Qian-Zhang (2002) $\cdots$ finite-dim case
- Heat kernel measures & heat processes on loop spaces are concrete example to which RP theory can be applied.
- LDP for heat processes & heat kernel measures on loop spaces.
  - Fang-Zhang (2001) $\cdots$ loop group case

$\Rightarrow$ Thanks to the continuity of the Itô map in rough path sense, we can show LDP for any measures of this kind!

Framework

- $\mathcal{L}_0^d := \{x \in C([0,1], \mathbb{R}^d) \mid x(0)=x(1)=0\}$
- $H_0^d \subset \mathcal{L}_0^d$: Cameron-Martin space
- μ_0^d : pinned Wiener measure on $\mathcal{L}_0^d$

$\Rightarrow (\mathcal{L}_0^d, H_0^d, \mu_0^d)$ is an abstract Wiener space.

· $(w_t)_{t\geq 0}$: $\mathcal{L}_0^d$-valued BM with $w_0 = 0$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

$$\Rightarrow \ (w_t)_{t\geq 0} = \{w_t(\tau) : 0 \leq \tau \leq 1\}_{t\geq 0}$$
$$= \{w_t^j(\tau) : 0 \leq \tau \leq 1, \ j = 1, \cdots, d\}_{t\geq 0}$$

: d-dim BMs with

$$\langle w_\cdot^j(\tau), w_\cdot^k(\tau')\rangle_t = t(\tau \wedge \tau' - \tau\tau')\delta_{jk}, \ t \geq 0$$

for $0 \leq \tau, \tau' \leq 1$, $1 \leq j, k \leq d$.

Let $\varepsilon > 0$. For each $\tau \in [0,1]$, we consider the following τ-wise $\mathbb{R}^r$-valued SDEs $(\star)$:

$$\boxed{\begin{aligned} dX_t^\varepsilon(\tau) &= \sigma(X_t^\varepsilon(\tau)) \circ \varepsilon dw_t(\tau) \\ &\qquad + b(X_t^\varepsilon(\tau))\varepsilon^2 dt + \beta(X_t^\varepsilon(\tau))dt, \\ X_0^\varepsilon(\tau) &= 0, \qquad\qquad \textbf{(Heat Process)} \end{aligned}}$$

where $\sigma = (\sigma_{ij})_{1\le i\le r, 1\le j\le d}, b = (b_i)_{1\le i\le r}, \beta = (\beta_i)_{1\le i\le r}$ and $\sigma_{ij}, b_i, \beta_i \in C_b^\infty(\mathbb{R}^r, \mathbb{R})$.

- Malliavin (1990), Driver (1997), Fang-Zhang (2001)
- Brzeźniak-Elworthy (2000)

(Remark) $\cdot\ b \equiv 0 \Longrightarrow$ Wentzell-Freidlin type!

$\cdot\ \beta \equiv 0 \Longrightarrow (X_t^\varepsilon)_{t\geq 0} \sim (X_{\varepsilon^2 t}^1)_{t\geq 0}$

Definition $\varepsilon > 0$, σ, b, β: as above.

(1) We denote by $\mathcal{V}_\varepsilon$ the law of heat process $(X_t^\varepsilon)_{t\geq 0}$. (probability measure on $P(\mathcal{L}_0^r)$)

(2) Assume $\beta \equiv 0, \varepsilon = 1$. We denote by μ_t the law of X_t^1. (probability measure on $\mathcal{L}_0^r$)

Main Theorem

(1) $\mathcal{V}_\varepsilon$ satisfies LDP as $\varepsilon \searrow 0$ with the good rate function I_1.

(2) μ_t satisfies LDP as $t \searrow 0$ with the good rate function I_2.

(We give the precise representations of I_1, I_2 later!)

§2 LDP for Banach Space Valued Brownian Rough Path

Framework

- (B, H, μ) : an abstract Wiener space
- $(w_t)_{t \geq 0}$: B-valued BM with $w_0 = 0$
- $2 < p < 3$: roughness (fixed)
- $B \otimes B$: closure of $B \otimes_a B$ w.r.t

  a tensor norm $|\cdot|$, i.e.,

$$|x \otimes y| \leq |x|_B \cdot |y|_B$$

- $G\Omega_p(B)$: the set of geometric RPs on B

Assumption (Exactness of $B \otimes B$ w.r.t μ)

$\exists C > 0$, $0 \leq \exists \alpha < 1$ s.t., $\forall N \in \mathbb{N}$, $\forall \{G_l\}_{l=1}^{2N}$ of independent B-valued random variables with common distribution μ, it holds that

$$\mathbb{E}\Big[\Big\| \sum_{l=1}^{N} G_{2l-1} \otimes G_{2l} \Big\|\Big] \leq C N^{\alpha}$$

(Remark) $\dim(\mathrm{B}) < \infty \Longrightarrow \alpha = 1/2$

- $w(n)$: n-th dyadic polygonal approximation of w
- $\overline{w(n)} := (1, \overline{w(n)}_1, \overline{w(n)}_2)$ is the smooth RP associated with $w(n)$:

$$\overline{w(n)}_1(s,t) := \quad w(n)_t - w(n)_s,$$

$$\overline{w(n)}_2(s,t) := \int_s^t (w(n)_u - w(n)_s) \otimes dw(n)_u$$

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$\boxed{\text{Fact}}$ [Ledoux-Lyons-Qian (2002)]

Under "exactness", Brownian RP exists, i.e.,

$$\exists\ \overline{w} = \lim_{n\to\infty} \overline{w(n)} \quad \text{in } G\Omega_p(B),\ \mathbb{P}\text{-a.s.},$$

- $\varepsilon\overline{w} := (1, \varepsilon\overline{w}_1, \varepsilon^2\overline{w}_2),\ \ \varepsilon > 0$
- $\mathcal{H} := L_0^{2,1}(H) \subset P(B)$: Cameron-Martin space

Theorem 1 Under "exactness", the law of $\varepsilon\overline{w}$ satisfies LDP on $G\Omega_p(B)$ with the good rate function

$$I(\overline{x}) = \begin{cases} \frac{1}{2}\|h\|_{\mathcal{H}}^2, & \text{if } \exists\ h \in \mathcal{H} \text{ s.t. } \overline{x} = \overline{h}, \\ \infty, & \text{otherwise.} \end{cases}$$

§3 Sketch of the Proof for the Main Theorem

♣ We show that any tensor product $\mathcal{L}_0^d \otimes \mathcal{L}_0^d$ is exact w.r.t. μ_0^d.

$$\Downarrow$$

We can use the RP theory for our model ! i.e., SDE $(\star)$ for $X_t(\tau)$ can be treated in Banach space valued RP theory.

♣ The Itô map is continuous in the RP theory.

♣ Combine Theorem 1 & the contraction principle for LDP.

Characterization of the Itô map

Step 1. We define the Nemytski map

$$\hat{\sigma}: \mathcal{L}_0^r \to L(\mathcal{L}_0^d \oplus \mathbb{R}^2, \mathcal{L}_0^r) \text{ by}$$

$$\hat{\sigma}(y)[(\gamma, u)](\tau) := \sigma(y(\tau))\gamma(\tau) + b(y(\tau))u_1 + \beta(y(\tau))u_2$$

for $y \in \mathcal{L}_0^r, \gamma \in \mathcal{L}_0^d, u = (u_1, u_2) \in \mathbb{R}^2, \tau \in [0, 1]$.

$\Rightarrow \hat{\sigma}$: bdd, smooth with bdd derivatives in Fréchet sense !

Step 2. We consider the ODE (in RP-sense)

$$\boxed{dy_t = \hat{\sigma}(y_t)dx_t, \; y_0 = 0 \qquad (\star')}$$

$$\Downarrow$$

$\boxed{\text{Fact}}$ (By Lyons' continuity theorem...)

For any $\bar{x} \in G\Omega_p(\mathcal{L}_0^d \oplus \mathbb{R}^2)$, the ODE $(\star')$ has the unique solution $\bar{y} \in G\Omega_p(\mathcal{L}_0^r)$. Moreover the map

$$\Phi \; : \; G\Omega_p(\mathcal{L}_0^d \oplus \mathbb{R}^2) \ni \bar{x} \longmapsto \bar{y} \in G\Omega_p(\mathcal{L}_0^r)$$

is (locally Lipschitz) continuous.

Step 3. We define a map

$\iota : G\Omega_p(\mathcal{L}_0^d) \times \mathbf{BV}(\mathbb{R}^2) \to G\Omega_p(\mathcal{L}_0^d \oplus \mathbb{R}^2)$ by

$$\iota(\overline{\gamma}, \lambda)_1(s,t) := \left(\overline{\gamma}_1(s,t), \lambda_t - \lambda_s\right),$$

$$\begin{aligned}\iota(\overline{\gamma}, \lambda)_2(s,t) := \Bigg(\overline{\gamma}_2(s,t), &\int_s^t \overline{\gamma}_1(s,u) \otimes d\lambda_u, \\ &\int_s^t (\lambda_u - \lambda_s) \otimes d\overline{\gamma}_1(0,u), \\ &\int_s^t (\lambda_u - \lambda_s) \otimes d\lambda_u\Bigg).\end{aligned}$$

(Remark) We used the isometry

$$(A_1 \oplus A_2) \otimes (B_1 \oplus B_2) \simeq \oplus_{i,j=1}^2 (A_i \oplus B_j).$$

Step 4. For $\varepsilon \geq 0$, we define $\boldsymbol{\lambda}^{(\varepsilon)} \in \mathbf{BV}(\mathbb{R}^2)$ by

$$\boldsymbol{\lambda}^{(\varepsilon)}(t) := (\varepsilon^2 t, t), \quad t \geq 0.$$

$$\Downarrow$$

The Itô map $\Psi_\varepsilon : G\Omega_p(\mathcal{L}_0^d) \to G\Omega_p(\mathcal{L}_0^r)$ is given by

$$\Psi_\varepsilon(\overline{x}) := \Phi\Big(\iota(\overline{x}, \boldsymbol{\lambda}^{(\varepsilon)})\Big), \quad \overline{x} \in G\Omega_p(\mathcal{L}_0^d).$$

(Remark) $(t, \tau) \mapsto \Psi_\varepsilon(\varepsilon\overline{w})_1(0, t)(\tau)$ is a bi-continuous modification of $(t, \tau) \mapsto X_t^\varepsilon(\tau)$.

Theorem 2 Rate functions I_1 and I_2 are given by

$$I_1(\xi) = \begin{cases} \frac{1}{2} \inf \left\{ \|\gamma\|_{\mathcal{H}}^2 \mid \xi = \Psi_0(\overline{\gamma})_1(0,\cdot) \right\} \\ \qquad\qquad \text{if } \exists\, \gamma \in \mathcal{H} \text{ s.t. } \xi = \Psi_0(\overline{\gamma})_1(0,\cdot), \\ \infty, \qquad\qquad \text{otherwise.} \end{cases}$$

$$I_2(y) = \begin{cases} \frac{1}{2} \inf \left\{ \|\gamma\|_{\mathcal{H}}^2 \mid y = \Psi_0(\overline{\gamma})_1(0,1) \right\} \\ \qquad\qquad \text{if } \exists\, \gamma \in \mathcal{H} \text{ s.t. } y = \Psi_0(\overline{\gamma})_1(0,1), \\ \infty, \qquad\qquad \text{otherwise.} \end{cases}$$

$$(\xi \in P(\mathcal{L}_0^r), \;\; y \in \mathcal{L}_0^r)$$

$\boxed{\text{Example}}$ $M = S^m \subset \mathbb{R}^{m+1}$ (Nash's embedding)

♣ M is not Lie group if $m \neq 1, 3$.

$$M := \{x = (x_1, \cdots x_{m+1}) \mid$$
$$x_1^2 + \cdots + x_m^2 + (x_{m+1} - 1)^2 = 1\}$$

$\Rightarrow$ Consider the case of $d = r = m + 1$.

Set $b = \beta \equiv 0$ and

$$\sigma_{ij}(x) = \delta_{i,j} - u_i u_j \text{ for } x \in M,$$

$1 \leq i, j \leq m + 1$, where $u_i := x_i - \delta_{i,m+1}$.

$\Rightarrow$ $(X_t^\varepsilon)_{t \geq 0}$ is a continuous process on $\mathcal{L}_0(M)$.