An algorithm for computing generators of $\mathbb{G}_a$-invariant rings

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1. Introduction

Let k be an infinite field of arbitrary characteristic $p \geq 0$ and let $\mathbb{G}_a$ be its additive group. For a $\mathbb{G}_a$-action on an affine variety $\mathrm{Spec}A$, we have the corresponding k-algebra homomorphism $\varphi : A \to A \otimes_k k[t]$, where t is an indeterminate over the field k. Write

$$\varphi(f) = \sum_{n \geq 0} D_n(f) t^n,$$

where $D_n(f) \in A$ for all $n \geq 0$. Clearly, each D_n is a k-linear endomorphism of A and the set $\{D_n\}_{n \geq 0}$ satisfies the following conditions (1), (2), (3) and (4):

(1) D_0 is the identity map of A;

(2) $D_n(ab) = \sum_{i+j=n} D_i(a) D_j(b)$ for all $n \geq 0$ and for all $a, b \in A$;

(3) For all $a \in A$, there exists $n \geq 0$ such that $D_m(a) = 0$ for all $m \geq n$;

(4) $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$ for all $i, j \geq 0$.

Let $\mathbb{G}_a$ act on A by

$$t \cdot f = \sum_{n \geq 0} D_n(f) t^n$$

for all $t \in k$ and $f \in A$. Denote by $A^{\mathbb{G}_a}$ the invariant ring for the $\mathbb{G}_a$-action on A. So, we have the equality

$$A^{\mathbb{G}_a} = \{a \in A \mid D_n(a) = 0 \text{ for all } n \geq 1\}.$$

An element σ of A is said to be a *local slice* of the $\mathbb{G}_a$-action on A if σ satisfies the following conditions (1) and (2):

(1) $\sigma \notin A^{\mathbb{G}_a}$;

(2) $\deg_t(t \cdot \sigma) = \min\{\deg_t(t \cdot f) \mid f \in A \backslash A^{\mathbb{G}_a}\}$.

An element s of A is said to be a *slice* of the $\mathbb{G}_a$-action if s is a local slice of the $\mathbb{G}_a$-action and the leading coefficient of $t \cdot s \in A[t]$ is 1.

In this report, we solve the following problem:

Problem. Assume that the finitely generated k-algebra A is a domain and the $\mathbb{G}_a$-action on A has a slice s. Give an algorithm for computing generators of the $\mathbb{G}_a$-invariant ring $A^{\mathbb{G}_a}$.

2. Dixmier operator and answer

We know from Miyanishi's theorem [2, 1.5.] that

$$A = A^{\mathbb{G}_a}[s]$$

and s can be considered as an indeterminate over $A^{\mathbb{G}_a}$. So, we have a natural surjective k-algebra homomorphism $\varepsilon_s : A \to A^{\mathbb{G}_a}$ by substituting 0 for s. We call this surjection the *Dixmier operator*.

The following theorem gives another description of the Dixmier operator. Let $P_s(t) := t \cdot s \in A[t]$.

Theorem. *For any $f \in A$, express $t \cdot f$ as*

$$t \cdot f = P_s(t) \cdot Q_f(t) + R_f(t),$$

where $Q_f(t)$, $R_f(t) \in A[t]$ and $\deg_t(R_f(t)) < \deg_t(P_s(t))$. Then $R_f(0) \in A^{\mathbb{G}_a}$ and $\varepsilon_s(f) = R_f(0)$.

Now, we give an answer to the Problem. Let $A = k[\alpha_1, \ldots, \alpha_n]$. Calculate $\varepsilon_s(\alpha_i)$ for all $1 \leq i \leq n$ by using the above Theorem. Thus we have an answer.

3. Proof of the Theorem

Let $D := \{D_n\}_{n\geq 0}$. We denote $t \cdot f$ by $\varphi_{D,t}(f)$, where $t \in \mathbb{G}_a$ and $f \in A$. We define the *D-degree* of $f \in A$ by $\deg_D(f) := \deg_t(\varphi_{D,t}(f))$. By substituting $t + t'$ for t of the equality $\varphi_{D,t}(f) = \varphi_{D,t}(s) \cdot Q_f(t) + R_f(t)$, we have

$$\varphi_{D,t+t'}(f) = \varphi_{D,t+t'}(s) \cdot Q_f(t + t') + R_f(t + t').$$

The left hand side $\varphi_{D,t+t'}(f)$ can be written as

$$\begin{aligned} \varphi_{D,t+t'}(f) &= (\varphi_{D,t} \circ \varphi_{D,t'})(f) \\ &= \varphi_{D,t}(\varphi_{D,t'}(s) \cdot Q_f(t') + R_f(t')) \\ &= \varphi_{D,t+t'}(s) \cdot \varphi_{D,t}(Q_f(t')) + \varphi_{D,t}(R_f(t')). \end{aligned}$$

Thus we have

$$\varphi_{D,t+t'}(s) \cdot (Q_f(t + t') - \varphi_{D,t}(Q_f(t'))) = \varphi_{D,t}(R_f(t')) - R_f(t + t').$$

Let $d := \deg_{t'}(\varphi_{D,t'}(s))$. If $Q_f(t+t') - \varphi_{D,t}(Q_f(t')) \neq 0$, the left hand side is of degree $\geq d$ in t' (note that $\varphi_{D,t+t'}(s)$ is a monic polynomial of degree d in t' over $A[t]$). On the other hand, the right hand side is of degree $< d$ in t'. Hence, we know

$$Q_f(t + t') = \varphi_{D,t}(Q_f(t')) \qquad \text{and} \qquad \varphi_{D,t}(R_f(t')) = R_f(t + t').$$

Substituting 0 for t' in the above two equalities, we have

$$Q_f(t) = \varphi_{D,t}(Q_f(0)) \qquad \text{and} \qquad \varphi_{D,t}(R_f(0)) = R_f(t).$$

We know from the latter of the above two equalities that $\deg_D(R_f(0)) < \deg_D(s)$. Since s is a slice of the $\mathbb{G}_a$-action, we have $R_f(0) \in A^{\mathbb{G}_a}$. Evaluating the equality

$$\varphi_{D,t}(f) = \varphi_{D,t}(s) \cdot Q_f(t) + R_f(t)$$

at $t = 0$, we have the equality

$$f = s \cdot Q_f(0) + R_f(0)$$

in A. Evaluating this equality at $s = 0$, we have $\varepsilon_s(f) = R_f(0)$. This completes the proof of the Theorem.

4. Dixmier operator in characteristic 0

In the following, assume that the characteristic of k is zero. The Dixmier operator $\varepsilon_s : A \to A^{\mathbb{G}_a}$ has the following simple form. This means that ε_s coincides with φ_{-s} defined in [1, Page 26].

Corollary. *The Dixmier operator ε_s can be written as*

$$\varepsilon_s(f) = \sum_{n\geq 0} \frac{D_1^n(f)}{n!}(-s)^n \quad \textit{for all } f \in A.$$

Proof. Since k is a field of characteristic zero, we have

$$\varphi_{D,t}(f) = \sum_{n\geq 0} \frac{D_1^n(f)}{n!} t^n \quad \text{for all } f \in A$$

and $\varphi_{D,t}(s) = s + t$. Express $\varphi_{D,t}(f)$ as

$$\varphi_{D,t}(f) = (t + s) \cdot Q_f(t) + R_f,$$

where $R_f \in A$. By substituting $-s$ for t of the above equality, we have

$$\sum_{n\geq 0} \frac{D_1^n(f)}{n!}(-s)^n = R_f.$$

We know from the above Theorem that $\varepsilon_s(f) = R_f$. Hence, we have the desired expression of ε_s. Q.E.D.

References

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