LINEARITY DEFECT AND REGULARITY OVER A KOSZUL ALGEBRA

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Abstract. Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a Koszul algebra over a field $K = A_0$, and $^*\mathrm{mod}\, A$ the category of finitely generated graded left A-modules. The *linearity defect* $\mathrm{ld}_A(M)$ of $M \in {}^*\mathrm{mod}\, A$ is an invariant defined by Herzog and Iyengar. An exterior algebra E is a Koszul algebra which is the Koszul dual of a polynomial ring. Eisenbud et al. showed that $\mathrm{ld}_E(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, E$. Improving their result, we show that the Koszul dual $A^!$ of a Koszul *commutative* algebra A satisfies the following.

- Let $M \in {}^*\mathrm{mod}\, A^!$. If $\{ \dim_K M_i \mid i \in \mathbb{Z} \}$ is bounded, then $\mathrm{ld}_{A^!}(M) < \infty$.
- If A is complete intersection, then $\mathrm{reg}_{A^!}(M) < \infty$ and $\mathrm{ld}_{A^!}(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A^!$.
- If $E = \bigwedge \langle y_1, \dots, y_n \rangle$ is an exterior algebra, then $\mathrm{ld}_E(M) \leq c^{n!} 2^{(n-1)!}$ for $M \in {}^*\mathrm{mod}\, E$ with $c := \max\{ \dim_K M_i \mid i \in \mathbb{Z} \}$.

1. Introduction

Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a (not necessarily commutative) graded algebra over a field $K := A_0$ with $\dim_K A_i < \infty$ for all $i \in \mathbb{N}$, and $^*\mathrm{mod}\, A$ the category of finitely generated graded left A-modules. Throughout this paper, we assume that A is *Koszul*, that is, $K = A / \bigoplus_{i \geq 1} A_i$ has a graded free resolution of the form

$$\cdots \longrightarrow A(-i)^{\beta_i(K)} \longrightarrow \cdots \longrightarrow A(-2)^{\beta_2(K)} \longrightarrow A(-1)^{\beta_1(K)} \longrightarrow A \longrightarrow K \longrightarrow 0.$$

Koszul duality is a certain derived equivalence between A and its Koszul dual algebra $A^! := \mathrm{Ext}^{\bullet}_A(K, K)$.

For $M \in {}^*\mathrm{mod}\, A$, we have its minimal graded free resolution $\cdots \to P_1 \to P_0 \to M \to 0$, and natural numbers $\beta_{i,j}(M)$ such that $P_i \cong \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{i,j}(M)}$. We call

$$\mathrm{reg}_A(M) := \sup\{ j - i \mid i \in \mathbb{N}, j \in \mathbb{Z} \text{ with } \beta_{i,j}(M) \neq 0 \}$$

the *regularity* of M. When A is a polynomial ring, $\mathrm{reg}_A(M)$ has been deeply studied. Even for a general Koszul algebra A, $\mathrm{reg}_A(M)$ is still an interesting invariant closely related to Koszul duality (see Theorem 3.5 below).

Let $P_\bullet$ be a minimal graded free resolution of $M \in {}^*\mathrm{mod}\, A$. The *linear part* $\mathrm{lin}(P_\bullet)$ of $P_\bullet$ is the chain complex such that $\mathrm{lin}(P_\bullet)_i = P_i$ for all i and its differential maps are given by erasing all the entries of degree ≥ 2 from the matrices representing the differentials of $P_\bullet$. According to Herzog-Iyengar [8], we call

$$\mathrm{ld}_A(M) := \sup\{ i \mid H_i(\mathrm{lin}(P_\bullet)) \neq 0 \}$$

the *linearity defect* of M. This invariant is related to the regularity via Koszul duality (see Theorem 3.8 below).

In §4, we mainly treat a Koszul *commutative* algebra A or its dual $A^!$. Even in this case, it can occur that $\mathrm{ld}_A(M) = \infty$ for some $M \in {}^*\mathrm{mod}\, A$ (c.f. [8]), while Avramov-Eisenbud [1] showed that $\mathrm{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$. On the other hand, Herzog-Iyengar [8] proved that if A is complete intersection or Golod then $\mathrm{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$. Initiated by these results, we will show the following.

Theorem A. *Let A be a Koszul commutative algebra (more generally, a Koszul algebra with $\mathrm{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$). Then we have;*

(1) *Let $N \in {}^*\mathrm{mod}\, A^!$. If $\mathrm{reg}_{A^!}(N) < \infty$ (e.g. $\dim_K N < \infty$), then $\mathrm{ld}_{A^!}(N) < \infty$.*

(2) *The following conditions are equivalent.*

  (a) $\mathrm{ld}_A(M) < \infty$ *for all $M \in {}^*\mathrm{mod}\, A$.*

  (a') $\mathrm{ld}_A(M) < \infty$ *for all $M \in {}^*\mathrm{mod}\, A$ with $M = \bigoplus_{i=0,1} M_i$.*

  (b) *If $N \in {}^*\mathrm{mod}\, A^!$ has a finite presentation, then $\mathrm{reg}_{A^!}(N) < \infty$.*

In Theorem A (2), the implications $(a) \Rightarrow (a') \Leftrightarrow (b)$ hold for a general Koszul algebra.

If A is a complete intersection, then the Koszul dual $A^!$ is left (and right) noetherian and admits a *balanced dualizing complex*, hence we have $\mathrm{reg}_{A^!}(N) < \infty$ for all $N \in {}^*\mathrm{mod}\, A^!$ by [9]. So $\mathrm{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$ by Theorem A (2). This is a special case of the above mentioned result of [8], but the proof is different.

Let ${}^*\mathrm{fp}\, A^!$ be the full subcategory of ${}^*\mathrm{mod}\, A^!$ consisting of finitely presented modules.

Theorem B. *If A is a Koszul algebra such that $\mathrm{ld}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$, then $A^!$ is left coherent (in the graded context), and ${}^*\mathrm{fp}\, A^!$ is an abelian category. If further A is commutative, then Koszul duality gives $\mathcal{D}^b({}^*\mathrm{mod}\, A) \cong \mathcal{D}^b({}^*\mathrm{fp}\, A^!)^{\mathsf{op}}$. In particular, if A is a Koszul <u>complete intersection</u>, then we have*

$$\mathcal{D}^b({}^*\mathrm{mod}\, A) \cong \mathcal{D}^b({}^*\mathrm{mod}\, A^!)^{\mathsf{op}}.$$

We remark that the last statement of Theorem B also follows from the existence of a balanced dualizing complex and [10, Proposition 4.5].

Let $E := \bigwedge \langle y_1, \dots, y_n \rangle$ be an exterior algebra. Eisenbud et al. [6] showed that $\mathrm{ld}_E(N) < \infty$ for all $N \in {}^*\mathrm{mod}\, E$ (now this is a special case of Theorem A, since E is the Koszul dual of a polynomial ring). If $n \geq 2$, then $\sup\{\, \mathrm{ld}_E(N) \mid N \in {}^*\mathrm{mod}\, E \,\} = \infty$. On the other hand, we will see that

$$\mathrm{ld}_E(N) \leq c^{n!} 2^{(n-1)!} \quad (c := \max\{\, \dim_K N_i \mid i \in \mathbb{Z} \,\}) \tag{1}$$

for $N \in {}^*\mathrm{mod}\, E$. But a computer experiment suggests that the bound could be very far from sharp. R. Okazaki and the author found a graded ideal $I \subset E$ with $n = 6$ and $\mathrm{ld}_E(E/I) = 9$. This is our "best record", but still much lower than the value given in (1).

2. Koszul Algebras and Koszul Duality

Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be a graded algebra over a field $K := A_0$ with $\dim_K A_i < \infty$ for all $i \in \mathbb{N}$, $^*\mathrm{Mod}\, A$ the category of graded left A-modules, and $^*\mathrm{mod}\, A$ the full subcategory of $^*\mathrm{Mod}\, A$ consisting of finitely generated modules. We say $M = \bigoplus_{i \in \mathbb{Z}} M_i \in {}^*\mathrm{Mod}\, A$ is *quasi-finite*, if $\dim_K M_i < \infty$ for all i and $M_i = 0$ for $i \ll 0$. If $M \in {}^*\mathrm{mod}\, A$, then it is clearly quasi-finite. We denote the full subcategory of $^*\mathrm{Mod}\, A$ consisting of quasi-finite modules by $\mathrm{qf}\, A$. (In this paper, we mainly treat a Koszul *commutative* algebra A and its dual $A^! := \mathrm{Ext}^\bullet_A(K, K)$. Even in this case, $A^!$ is *not* left noetherian in general. In fact, it is known that $A^!$ is left noetherian if and only if A is complete intersection. So $^*\mathrm{mod}\, A^!$ is not necessarily abelian, and we have to treat $\mathrm{qf}\, A^!$.) Clearly, $\mathrm{qf}\, A$ is an abelian category with enough projectives. For $M \in {}^*\mathrm{Mod}\, A$ and $j \in \mathbb{Z}$, $M(j)$ denotes the shifted module of M with $M(j)_i = M_{i+j}$. For $M, N \in {}^*\mathrm{Mod}\, A$, set $\underline{\mathrm{Hom}}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{^*\mathrm{Mod}\, A}(M, N(i))$ to be a graded K-vector space with $\underline{\mathrm{Hom}}_A(M, N)_i = \mathrm{Hom}_{^*\mathrm{Mod}\, A}(M, N(i))$. Similarly, we also define $\underline{\mathrm{Ext}}^i_A(M, N)$.

Let $\mathcal{C}(\mathrm{qf}\, A)$ be the homotopy category of cochain complexes in $\mathrm{qf}\, A$, and $\mathcal{C}^-(\mathrm{qf}\, A)$ its full subcategory consisting of complexes which are bounded above (i.e., $X^\bullet \in \mathcal{C}(\mathrm{qf}\, A)$ with $X^i = 0$ for $i \gg 0$). We say $P^\bullet \in \mathcal{C}^-(\mathrm{qf}\, A)$ is a free resolution of $X^\bullet \in \mathcal{C}^-(\mathrm{qf}\, A)$, if each P^i is a free module and there is a quasi-isomorphism $P^\bullet \to X^\bullet$. We say a free resolution $P^\bullet$ is *minimal*, if $\partial(P^i) \subset \mathfrak{m} P^{i+1}$ for all i. Here ∂ denotes the differential map, and $\mathfrak{m} := \bigoplus_{i>0} A_i$ is the graded maximal ideal. Any $X^\bullet \in \mathcal{C}^-(\mathrm{qf}\, A)$ has a minimal free resolution, which is unique up to isomorphism.

Regard $K = A/\mathfrak{m}$ as a graded left A-module, and set

$$\beta^i_j(X^\bullet) := \dim_K \underline{\mathrm{Ext}}^{-i}_A(X^\bullet, K)_{-j} \quad \text{and} \quad \beta^i(X^\bullet) := \sum_{j \in \mathbb{Z}} \beta^i_j(X^\bullet)$$

for $X^\bullet \in \mathcal{C}^-(\mathrm{qf}\, A)$ and $i, j \in \mathbb{Z}$. In this situation, if $P^\bullet \in \mathcal{C}^-(\mathrm{qf}\, A)$ is a minimal free resolution of $X^\bullet$, then we have $P^i \cong \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta^i_j(X^\bullet)}$ for each $i \in \mathbb{Z}$. It is easy to see that $\beta^i_j(X^\bullet) < \infty$ for each i, j.

Following the usual convention, we often describe (the invariants of) a free resolution of a module $M \in \mathrm{qf}\, A$ in the homological manner. So we have $\beta_{i,j}(M) = \beta^{-i}_j(M)$, and a minimal free resolution of M is of the form

$$P_\bullet : \ \cdots \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{1,j}(M)} \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0.$$

We say A is *Koszul*, if $\beta_{i,j}(K) \neq 0$ implies $i = j$, in other words, K has a graded free resolution of the form

$$\cdots \longrightarrow A(-i)^{\beta_i(K)} \longrightarrow \cdots \longrightarrow A(-2)^{\beta_2(K)} \longrightarrow A(-1)^{\beta_1(K)} \longrightarrow A \longrightarrow K \longrightarrow 0.$$

Even if we regard K as a right A-module, we get the equivalent definition.

The polynomial ring $K[x_1, \ldots, x_n]$ and the exterior algebra $\bigwedge \langle y_1, \ldots, y_n \rangle$ are primary examples of Koszul algebras. Of course, there are many other important examples. In the noncommutative case, many of them are not left (or right) noetherian. In the rest of the paper, we assume that A is Koszul.

Koszul duality is a derived equivalence between a Koszul algebra A and its dual $A^!$. A standard reference of this subject is Beilinson et al. [3]. But, in the present paper, we follow the convention of Mori [10].

Recall that Yoneda product makes $A^! := \bigoplus_{i\in\mathbb{N}} \operatorname{Ext}^i_A(K,K)$ a graded K-algebra. (In the convention of [3], $A^!$ denotes the opposite algebra of our $A^!$. So the reader should be careful.) If A is Koszul, then so is $A^!$ and we have $(A^!)^! \cong A$. The Koszul dual of the polynomial ring $S := K[x_1,\dots,x_n]$ is the exterior algebra $E := \bigwedge\langle y_1,\dots,y_n\rangle$. In this case, since S is regular and noetherian, Koszul duality is very simple. It gives an equivalence $\mathcal{D}^b({}^*\mathrm{mod}\, S) \cong \mathcal{D}^b({}^*\mathrm{mod}\, E)$ of the bounded derived categories. In the general case, the description of Koszul duality is slightly technical. For example, if A is not left noetherian, then ${}^*\mathrm{mod}\, A$ is not an abelian category. So we have to treat $\mathrm{qf}\, A$.

Let $\mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$ be the full subcategory of $\mathcal{C}(\mathrm{qf}\, A)$ (and $\mathcal{C}^-(\mathrm{qf}\, A)$) consisting of complexes $X^\bullet$ satisfying

$$X^i_j = 0 \quad \text{for } i \gg 0 \text{ or } i+j \ll 0.$$

And let $\mathcal{D}^{\uparrow}(\mathrm{qf}\, A)$ be the localization of $\mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$ at quasi-isomorphisms. By the usual argument, we see that $\mathcal{D}^{\uparrow}(\mathrm{qf}\, A)$ is equivalent to the full subcategory of the derived category $\mathcal{D}(\mathrm{qf}\, A)$ (and $\mathcal{D}^-(\mathrm{qf}\, A)$) consisting of the complex $X^\bullet$ such that

$$H^i(X^\bullet)_j = 0 \quad \text{for } i \gg 0 \text{ or } i+j \ll 0.$$

It is easy to see that $\mathcal{D}^{\uparrow}(\mathrm{qf}\, A)$ is a triangulated subcategory of $\mathcal{D}(\mathrm{qf}\, A)$.

We write V^* for the dual space of a K-vector space V. Note that if $M \in {}^*\mathrm{Mod}\, A$ then $M^* := \bigoplus_{i\in\mathbb{Z}}(M_{-i})^*$ is a graded right A-module. And we fix a basis $\{x_\lambda\}$ of A_1 and its dual basis $\{y_\lambda\}$ of $(A_1)^* (= (A^!)_1)$. Let $(X^\bullet, \partial) \in \mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$. In this notation, we define the contravariant functor $F_A : \mathcal{C}^{\uparrow}(\mathrm{qf}\, A) \to \mathcal{C}^{\uparrow}(\mathrm{qf}\, A^!)$ as follows.

$$F_A(X^\bullet)^p_q = \bigoplus A^!_{q+j} \otimes_K (X^{j-p}_{-j})^*$$

with the differential $d = d' + d''$ given by

$$d' : A^!_{q+j} \otimes_K (X^{j-p}_{-j})^* \ni a \otimes m \longmapsto (-1)^p \sum a y_\lambda \otimes m x_\lambda \in A^!_{q+j+1} \otimes_K (X^{j-p}_{-j-1})^*$$

and

$$d'' : A^!_{q+j} \otimes_K (X^{j-p}_{-j})^* \ni a \otimes m \longmapsto a \otimes \partial^*(m) \in A^!_{q+j} \otimes_K (X^{j-p-1}_{-j})^*.$$

The contravariant functor $F_{A^!} : \mathcal{C}^{\uparrow}(\mathrm{qf}\, A^!) \to \mathcal{C}^{\uparrow}(\mathrm{qf}\, A)$ is given by a similar way. (More precisely, the construction is different, but the result is similar. See the remark below.) They induce the contravariant functors $\mathcal{F}_A : \mathcal{D}^{\uparrow}(\mathrm{qf}\, A) \to \mathcal{D}^{\uparrow}(\mathrm{qf}\, A^!)$ and $\mathcal{F}_{A^!} : \mathcal{D}^{\uparrow}(\mathrm{qf}\, A^!) \to \mathcal{D}^{\uparrow}(\mathrm{qf}\, A)$.

Remark 2.1. In [10], two Koszul duality functors are defined individually. The functor denoted by $\bar{E}_A$ is the same as our $\mathcal{F}_A$. The other one which is denoted by $\tilde{E}_A$ is defined using the operations $\underline{\mathrm{Hom}}_K(A^!, -)$ and $\underline{\mathrm{Hom}}_K(-, K)$. But, in our case, it coincides with F_A except the convention of the sign ± 1. So we do not give the precise definition of $\tilde{E}_A$ here.

Theorem 2.2 (Koszul duality. c.f. [3, 10]). *The contravariant functors $\mathcal{F}_A$ and $\mathcal{F}_{A^!}$ give an equivalence*

$$\mathcal{D}^{\uparrow}(\operatorname{qf} A) \cong \mathcal{D}^{\uparrow}(\operatorname{qf} A^!)^{\mathsf{op}}.$$

The next result easily follows from Theorem 2.2 and the fact that $\mathcal{F}_A(K) = A^!$.

Lemma 2.3 (cf. [10, Lemma 2.8]). *For $X^\bullet \in \mathcal{D}^{\uparrow}(\operatorname{qf} A)$, we have*

$$\beta^i_j(X^\bullet) = \dim H^{-i-j}(\mathcal{F}_A(X^\bullet))_j.$$

3. Regularity and Linearity Defect

Throughout this section, $A = \bigoplus_{i\in\mathbb{N}} A_i$ is a Koszul algebra.

Definition 3.1. For $X^\bullet \in \mathcal{D}^{\uparrow}(\operatorname{qf} A)$, we call

$$\operatorname{reg}_A(X^\bullet) := \sup\{\, i+j \mid i, j \in \mathbb{Z} \text{ with } \beta^i_j(X^\bullet) \neq 0 \,\}$$

the *regularity* of $X^\bullet$. We set the regularity of the 0 module to be $-\infty$.

We say A is *left graded coherent*, if any finitely generated graded left ideal of A has a finite presentation. Let $^*\mathrm{fp}\, A$ be the full subcategory of $^*\mathrm{mod}\, A$ consisting of finitely presented modules. As is well-known, A is left graded coherent if and only if $^*\mathrm{fp}\, A$ is an abelian subcategory of $^*\mathrm{mod}\, A$.

Lemma 3.2. *If $\operatorname{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, A$ then A is left noetherian. Similarly, if $\operatorname{reg}_A(M) < \infty$ for all $M \in {}^*\mathrm{fp}\, A$ then A is left graded coherent.*

Proof. Assume that A is not left noetherian. Then there is a graded left ideal I which is not finitely generated. Clearly, $A/I \in {}^*\mathrm{mod}\, A$, but $\beta_{1,j}(A/I) = \beta_{0,j}(I) \neq 0$ for arbitrary large j and $\operatorname{reg}_A(A/I) = \infty$.

Assume that A is not left graded coherent. Then there is a graded left ideal I which is finitely generated but not finitely presented. Clearly, $A/I \in {}^*\mathrm{fp}\, A$, but $\beta_{2,j}(A/I) = \beta_{1,j}(I) \neq 0$ for arbitrary large j and $\operatorname{reg}_A(A/I) = \infty$. □

Remark 3.3. The author does not know any example of a Koszul algebra A which admits $M \in {}^*\mathrm{mod}\, A$ with $\operatorname{reg}_A(M) = \infty$ but $\beta_i(M) = \sum_{j\in\mathbb{Z}} \beta_{i,j}(M) < \infty$ for all i. In particular, he does not know a left noetherian (resp. graded coherent) Koszul algebra A such that $\operatorname{reg}_A(M) = \infty$ for some $M \in {}^*\mathrm{mod}\, A$ (resp. $M \in {}^*\mathrm{fp}\, A$).

Lemma 3.4. (1) *For $M \in \operatorname{qf} A$, we have*

$$\operatorname{reg}_A(M) < \infty \;\Rightarrow\; \beta_i(M) < \infty \text{ for all } i \;\Rightarrow M \text{ has a finite presentation.}$$

(2) *If $X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$ is a triangle in $\mathcal{D}^{\uparrow}(\operatorname{qf} A)$, then we have*

$$\operatorname{reg}_A(Y^\bullet) \leq \max\{\, \operatorname{reg}_A(X^\bullet), \operatorname{reg}_A(Z^\bullet) \,\}.$$

If $\operatorname{reg}_A(X^\bullet) \neq \operatorname{reg}_A(Z^\bullet) + 1$, then equality holds.

(3) *If $M \in {}^*\mathrm{mod}\, A$ has finite length, then $\operatorname{reg}_A(M) \leq \max\{\, i \mid M_i \neq 0\}$.*

(4) *For $X^\bullet \in \mathcal{D}^{\uparrow}(\operatorname{qf} A)$, we have*

$$\operatorname{reg}_A(X^\bullet) \leq \sup\{\, \operatorname{reg}_A(H^i(X^\bullet)) + i \mid i \in \mathbb{Z} \,\}.$$

The next result directly follows from Lemma 2.3.

Theorem 3.5 (Eisenbud et al [6], Mori [10]). *For $X^\bullet \in D^\uparrow(\operatorname{qf} A)$, we have*

$$\operatorname{reg}_A(X^\bullet) = -\inf\{\, i \mid H^i(\mathcal{F}_A(X^\bullet)) \neq 0 \,\}.$$

We say a complex $X^\bullet \in \mathcal{D}^\uparrow(\operatorname{qf} A)$ is *strongly bounded*, if $X^\bullet$ is bounded (i.e., $H^i(X^\bullet) = 0$ for $i \gg 0$ or $i \ll 0$) and $\operatorname{reg}_A(X^\bullet) < \infty$. Let $\mathcal{D}^{sb}(\operatorname{qf} A)$ be the full subcategory of $\mathcal{D}^\uparrow(\operatorname{qf} A)$ consisting of strongly bounded complexes. By Lemma 3.4 (2), $\mathcal{D}^{sb}(\operatorname{qf} A)$ is a triangulated subcategory of $\mathcal{D}(\operatorname{qf} A)$.

The next result follows from Theorem 3.5.

Proposition 3.6. *The (restriction of) functors $\mathcal{F}_A$ and $\mathcal{F}_{A^!}$ give an equivalence*

$$\mathcal{D}^{sb}(\operatorname{qf} A) \cong \mathcal{D}^{sb}(\operatorname{qf} A^!)^{\mathsf{op}}.$$

Let $(P^\bullet, \partial) \in \mathcal{C}^\uparrow(\operatorname{qf} A)$ be a complex of *free* A-modules such that $\partial(P^i) \subset \mathfrak{m}P^{i+1}$, in other words, $P^\bullet$ is a minimal free resolution of some $X^\bullet \in \mathcal{C}^\uparrow(\operatorname{qf} A)$. According to [6], we define the *linear part* $\operatorname{lin}(P^\bullet)$ of $P^\bullet$ as follows:

(1) $\operatorname{lin}(P^\bullet)$ is a complex with $\operatorname{lin}(P^\bullet)^i = P^i$.
(2) The matrices representing the differentials of $\operatorname{lin}(P^\bullet)$ are given by "erasing" all the entries of degree ≥ 2 (i.e., replacing them by 0) from the matrices representing the differentials of $P^\bullet$.

It is easy to check that $\operatorname{lin}(P^\bullet)$ is actually a complex. But, even if $P_\bullet$ is a minimal free resolution of $M \in \operatorname{qf} A$, $\operatorname{lin}(P_\bullet)$ is not acyclic (i.e., $H_i(\operatorname{lin}(P_\bullet)) \neq 0$ for some $i > 0$) in general.

Definition 3.7 (Herzog-Iyengar [8]). Let $M \in \operatorname{qf} A$ and $P_\bullet$ its minimal graded free resolution. We call

$$\operatorname{ld}_A(M) := \sup\{\, i \mid H_i(\operatorname{lin}(P_\bullet)) \neq 0 \,\}$$

the *linearity defect* of M.

We say $M \in {}^*\operatorname{mod} A$ has a *linear (free) resolution* if there is some $l \in \mathbb{Z}$ such that $\beta_{i,j}(M) \neq 0$ implies that $j - i = l$. In this case, the minimal free resolution $P_\bullet$ of M coincides with $\operatorname{lin}(P_\bullet)$, and $\operatorname{ld}_A(M) = 0$. As shown in [10, Theorem 5.4], we have

$$\operatorname{reg}_A(M) = \inf\{\, i \mid M_{\geq i} := \bigoplus_{j \geq i} M_j \text{ has a linear resolution}\}.$$

For $i \in \mathbb{Z}$ and $M \in \operatorname{qf} A$, $M_{\langle i \rangle}$ denotes the submodule of M generated by the degree i component M_i. We say $M \in \operatorname{qf} A$ is *componentwise linear*, if $M_{\langle i \rangle}$ has a linear resolution for all $i \in \mathbb{Z}$.

As shown in [11, 12], for $M \in \operatorname{qf} A$, we have

$$\operatorname{ld}_A(M) = \inf\{\, i \mid \Omega_i(M) \text{ is componentwise linear}\,\},$$

where $\Omega_i(M)$ is the i^{th} syzygy of M.

Clearly, we have $\operatorname{ld}_A(M) \leq \operatorname{proj.dim}_A(M)$. The inequality is strict quite often. For example, we have $\operatorname{proj.dim}_A(M) = \infty$ and $\operatorname{ld}_A(M) < \infty$ for many M. On the other hand, we sometimes have $\operatorname{ld}_A(M) = \infty$.

The next result connects the linearity defect with the regularity via Koszul duality. For a complex $X^\bullet$, $\mathcal{H}(X^\bullet)$ denotes the complex such that $\mathcal{H}(X^\bullet)^i = H^i(X^\bullet)$ for all i and all differentials are 0.

Theorem 3.8 (cf. [6, Theorem 3.1])**.** *Let $X^\bullet \in \mathcal{D}^\uparrow(\operatorname{qf} A)$, and $P^\bullet$ a minimal free resolution of $\mathcal{F}_A(X^\bullet) \in \mathcal{D}^\uparrow(\operatorname{qf} A^!)$. Then we have*

$$\operatorname{lin}(P^\bullet) = F_A \circ \mathcal{H}(X^\bullet).$$

Hence, for $M \in \operatorname{qf} A$,

$$\operatorname{ld}_A(M) = \sup\{ \operatorname{reg}_{A^!}(H^i(F_A(M))) + i \mid i \in \mathbb{Z}\}.$$

4. Koszul Commutative Algebras and their Dual

If A is a Koszul commutative algebra and $S := \operatorname{Sym}_K A_1$ is the polynomial ring, then we have $A = S/I$ for a graded ideal I of S. In this situation, A is *Golod* if and only if I has a 2-linear resolution as an S-module (i.e., $\beta_{i,j}(I) \neq 0$ implies $j = i+2$), see [8, Proposition 5.8]. We say A *comes from a complete intersection by a Golod map* (see [2, 8]), if there is an intermediate graded ring R with $S \twoheadrightarrow R \twoheadrightarrow A$ satisfying the following conditions:

(1) R is a complete intersection.
(2) Let J be the graded ideal of R such that $A = R/J$. Then J has a 2-linear resolution as an R-module.

If this is the case, R is automatically Koszul. Clearly, if A itself is complete intersection or Golod, then it comes from a complete intersection by a Golod map.

Example 4.1. Set $S = K[s, t, u, v, w]$ and $A = S/(st, uv, sw)$. Then A is neither Golod nor complete intersection, but comes from a complete intersection by a Golod map (as an intermediate ring, take $S/(st, uv)$).

The next result plays a key role in this section.

Theorem 4.2 (Avramov-Eisenbud [1])**.** *Let A be a Koszul commutative algebra, and $S := \operatorname{Sym}_K A_1$ the polynomial ring. Then we have $\operatorname{reg}_A(M) \leq \operatorname{reg}_S(M) < \infty$ for all $M \in {}^*\!\operatorname{mod} A$.*

On the other hand, even if A is Koszul and commutative, $\operatorname{ld}_A(M)$ can be infinite for some $M \in {}^*\!\operatorname{mod} A$, as pointed out in [8]. But we have the following.

Theorem 4.3 (Herzog-Iyengar [8])**.** *Let A be a Koszul commutative algebra. If A comes from a complete intersection by a Golod map (e.g., A itself is complete intersection or Golod), then $\operatorname{ld}_A(M) < \infty$ for all $M \in {}^*\!\operatorname{mod} A$.*

Now we are interested in $\operatorname{reg}_{A^!}(N)$ and $\operatorname{ld}_{A^!}(N)$ for a Koszul commutative algebra A. First, we recall that a graded left $A^!$-module has a natural graded right $A^!$-module structure in this case, and vice versa (c.f. [8, §3]). In particular, $A^!$ is left noetherian (resp. graded coherent) if and only if it is right noetherian (resp. graded coherent).

Theorem 4.4. *If A is a Koszul commutative algebra, we have the following.*

(1) *Let* $N \in {}^*\mathrm{mod}\, A^!$. *If* $\mathrm{reg}_{A^!}(N) < \infty$, *then* $\mathrm{ld}_{A^!}(N) < \infty$.
(2) *The following conditions are equivalent.*
  (a) $\mathrm{ld}_A(M) < \infty$ *for all* $M \in {}^*\mathrm{mod}\, A$.
  (a') $\mathrm{ld}_A(M) < \infty$ *for all* $M \in {}^*\mathrm{mod}\, A$ *with* $M = \bigoplus_{i=0,1} M_i$.
  (b) $\mathrm{reg}_{A^!}(N) < \infty$ *for all* $N \in {}^*\mathrm{fp}\, A^!$.
(3) *Let* $N \in \mathrm{qf}\, A^!$. *If there is some* $c \in \mathbb{N}$ *such that* $\dim_K N_i \leq c$ *for all* $i \in \mathbb{Z}$, *then* $\mathrm{ld}_{A^!}(N) < \infty$.

Proof. (1) The complex $F_{A^!}(N)$ is always bounded above. Hence if $\mathrm{reg}_{A^!}(N) < \infty$ then $H^i(F_{A^!}(N)) \neq 0$ for only finitely many i by Theorem 3.5. Thus the assertion follows from Theorems 3.8 and 4.2.

(2) The implication $(a) \Rightarrow (a')$ is clear.

$(a') \Rightarrow (b)$: First assume that $N \in {}^*\mathrm{fp}\, A^!$ has a presentation of the form $A^!(-1)^{\oplus \beta_1} \to A^{!\oplus \beta_0} \to N \to 0$. Then there is $M \in {}^*\mathrm{mod}\, A$ with $M = \bigoplus_{i=0,1} M_i$ such that $F_A(M)$ gives this presentation. Since $\mathrm{ld}_A(M) < \infty$, we have $\mathrm{reg}_{A^!}(N) < \infty$ by Theorem 3.8.

Next take an arbitrary $N \in {}^*\mathrm{fp}\, A^!$. For a sufficiently large s, $N_{\geq s} := \bigoplus_{i \geq s} N_i$ has a presentation of the form $A^!(-s-1)^{\oplus \beta_1} \to A^!(-s)^{\oplus \beta_0} \to N_{\geq s} \to 0$. (To see this, consider the short exact sequence $0 \to N_{\geq s} \to N \to N/N_{\geq s} \to 0$, and use the fact that $\mathrm{reg}_{A^!}(N/N_{\geq s}) < s$.) We have shown that $\mathrm{reg}_{A^!}(N_{\geq s}) < \infty$. So $\mathrm{reg}_{A^!}(N) < \infty$ by the above short exact sequence.

$(b) \Rightarrow (a)$: By Lemma 3.2, $A^!$ is left graded coherent. So ${}^*\mathrm{fp}\, A^!$ is an abelian category. Each term of $F_A(M)$ is a finite free $A^!$-module, in particular, $F_A(M) \in \mathcal{C}^-({}^*\mathrm{fp}\, A^!)$. Hence we have $H^i(F_A(M)) \in {}^*\mathrm{fp}\, A^!$ for all i. By the assumption, $\mathrm{reg}_{A^!}(H^i(F_A(M))) < \infty$. On the other hand, $H^i(F_A(M)) \neq 0$ for finitely many i by Theorems 3.5 and 4.2. So the assertion follows from Theorem 3.8.

(3) Let $\mathcal{S}$ be the set of all graded submodules of $A^{\oplus c}$ which are generated by elements of degree 1. By Brodmann [4], there is some $C \in \mathbb{N}$ such that $\mathrm{reg}_A(M) \leq \mathrm{reg}_S(M) < C$ for all $M \in \mathcal{S}$. Here S denotes the polynomial ring $\mathrm{Sym}_K A_1$. To prove the assertion, it suffices to show that $\mathrm{reg}_A(H^i(\mathcal{F}_{A^!}(N))) + i \leq C$ for all i. We may assume that $i = 0$. Note that $H^0(\mathcal{F}_{A^!}(N))$ is the cohomology of the sequence

$$A \otimes_K (N_1)^* \xrightarrow{\partial^{-1}} A \otimes_K (N_0)^* \xrightarrow{\partial^0} A \otimes_K (N_{-1})^*.$$

Since $\mathrm{Im}(\partial^0)(-1)$ is a submodule of $A^{\oplus \dim_K N_{-1}}$ generated by elements of degree 1 and $\dim_K N_{-1} \leq c$, we have $\mathrm{reg}_A(\mathrm{Im}(\partial^0)) < C$. Consider the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(\partial^0) \longrightarrow A \otimes_K (N_0)^* \longrightarrow \mathrm{Im}(\partial^0) \longrightarrow 0.$$

Since $\mathrm{reg}_A(A \otimes_K (N_0)^*) = 0$, we have $\mathrm{reg}_A(\mathrm{Ker}(\partial^0)) \leq C$. Similarly, we have $\mathrm{reg}_A(\mathrm{Im}(\partial^{-1})) < C$. By the short exact sequence

$$0 \longrightarrow \mathrm{Im}(\partial^{-1}) \longrightarrow \mathrm{Ker}(\partial^0) \longrightarrow H^0(\mathcal{F}_{A^!}(N)) \longrightarrow 0,$$

we are done. □

Remark 4.5. In Theorem 4.4 (2), the implications $(a) \Rightarrow (a') \Leftrightarrow (b)$ hold for a general Koszul algebra.

If A is a (not necessarily commutative) Koszul algebra satisfying $\operatorname{reg}_A(M) < \infty$ for all $M \in {}^*\operatorname{mod} A$, then Theorem 4.4 (1) and (2) hold for A.

By the above remark and Lemma 3.2, we have the following.

Corollary 4.6. *Let A be a Koszul algebra. If $\operatorname{ld}_A(M) < \infty$ for all $M \in {}^*\operatorname{mod} A$, then $A^!$ is left graded coherent.*

In [2, Corollary 3], Backelin and Roos showed that if A is a Koszul commutative algebra which comes from a complete intersection by a Golod map then $A^!$ is left graded coherent. Moreover, they actually proved that $\operatorname{reg}_{A^!}(N) < \infty$ for all $N \in {}^*\operatorname{fp} A^!$ (see [2, Corollary 2] and [8, Lemma 5.1]). So we have $\operatorname{ld}_A(M) < \infty$ for all $M \in {}^*\operatorname{mod} A$ by Theorem 4.4, that is, we get a result of Herzog and Iyengar (Theorem 4.3). Their original proof is essentially based on this line too. While, in the case when A is complete intersection, we have another proof using the notion of *balanced dualizing complex* as stated in the introduction.

Lemma 4.7. *Assume that $\operatorname{reg}_{A^!}(N) < \infty$ for all $N \in {}^*\operatorname{fp} A^!$. Let $X^\bullet \in \mathcal{D}^b(\operatorname{qf} A^!)$ be a bounded complex. Then $X^\bullet$ is strongly bounded if and only if $H^i(X^\bullet) \in {}^*\operatorname{fp} A^!$ for all i.*

Proof. (Sufficiency): If $H^i(X^\bullet) \in {}^*\operatorname{fp} A^!$, then $\operatorname{reg}_{A^!}(H^i(X^\bullet)) < \infty$. Since $X^\bullet$ is bounded, we have $\operatorname{reg}_{A^!}(X^\bullet) < \infty$ by Lemma 3.4 (4).

(Necessity): Assume that $X^\bullet$ is strongly bounded (more generally, $\beta^i(X^\bullet) < \infty$ for all i). Let $P^\bullet$ be a minimal free resolution of $X^\bullet$. Clearly, $P^\bullet \in \mathcal{C}^-({}^*\operatorname{fp} A^!)$. By Corollary 4.6, ${}^*\operatorname{fp} A^!$ is an abelian category. Hence each $H^i(P^\bullet)\,(\cong H^i(X^\bullet))$ belongs to ${}^*\operatorname{fp} A^!$. □

Theorem 4.8. *Let A be a Koszul commutative algebra such that $\operatorname{ld}_A(M) < \infty$ for all $M \in {}^*\operatorname{mod} A$ (e.g. A comes from a complete intersection by a Golod map). Then Koszul duality gives an equivalence $\mathcal{D}^b({}^*\operatorname{mod} A) \cong \mathcal{D}^b({}^*\operatorname{fp} A^!)^{\mathsf{op}}$.*

Proof. By Proposition 3.6, it suffices to show that $\mathcal{D}^b({}^*\operatorname{mod} A) = \mathcal{D}^{sb}(\operatorname{qf} A)$ and $\mathcal{D}^b({}^*\operatorname{fp} A^!) = \mathcal{D}^{sb}(\operatorname{qf} A^!)$.

Let us consider the first equality (this holds for a general Koszul commutative algebra). If $X^\bullet \in \mathcal{D}^b({}^*\operatorname{mod} A)$, then $\operatorname{reg}_A(X^\bullet) < \infty$ by Lemma 3.4 (4) and Theorem 4.2. Hence we have $X^\bullet \in \mathcal{D}^{sb}(\operatorname{qf} A)$. Conversely, if $Y^\bullet \in \mathcal{D}^{sb}(\operatorname{qf} A)$, then $\beta^i(Y^\bullet) < \infty$ for all i, and the minimal free resolution of $Y^\bullet$ is a complex of finite free modules. So we have $Y^\bullet \in \mathcal{D}^b({}^*\operatorname{mod} A)$. Hence $\mathcal{D}^b({}^*\operatorname{mod} A) = \mathcal{D}^{sb}(\operatorname{qf} A)$.

Next we will show that $\mathcal{D}^b({}^*\operatorname{fp} A^!) = \mathcal{D}^{sb}(\operatorname{qf} A^!)$. By Corollary 4.6, ${}^*\operatorname{fp} A^!$ is an abelian category, and closed under extensions in $\operatorname{qf} A^!$. Since a free $A^!$-module of finite rank belongs to ${}^*\operatorname{fp} A^!$, this category has enough projectives. So we have $\mathcal{D}^b({}^*\operatorname{fp} A^!) = \mathcal{D}^b_{{}^*\operatorname{fp} A^!}(\operatorname{qf} A^!) = \mathcal{D}^{sb}(\operatorname{qf} A^!)$. Here the first equality follows from [7, Exercise III.2.2] and the second one follows from Lemma 4.7. □

Corollary 4.9. *If A is a Koszul complete intersection, then Koszul duality gives $\mathcal{D}^b({}^*\operatorname{mod} A) \cong \mathcal{D}^b({}^*\operatorname{mod} A^!)^{\mathsf{op}}$.*

In the rest of the paper, we study the linearity defect over the exterior algebra $E := \bigwedge \langle y_1, \dots, y_n \rangle$. Eisenbud et al. [6] showed that $\operatorname{ld}_E(N) < \infty$ for all $N \in$

$^*\mathrm{mod}\,E$. Now this is a special case of Theorem 4.4. But the behavior of $\mathrm{ld}_E(N)$ is still mysterious.

If $n \geq 2$, then we have $\sup\{\, \mathrm{ld}_E(N) \mid N \in {}^*\mathrm{mod}\,E\,\} = \infty$. In fact, $N := E/\operatorname{soc}(E)$ satisfies $\mathrm{ld}_E(N) \geq 1$. And the i^{th} *cosyzygy* $\Omega_{-i}(N)$ of N (since E is selfinjective, we can consider cosyzygies) satisfies $\mathrm{ld}_E(\Omega_{-i}(N)) > i$. But we have an upper bound of $\mathrm{ld}_E(N)$ depending only on $\max\{\, \dim_K N_i \mid i \in \mathbb{Z}\,\}$ and n. Before stating this, we recall a result on $\mathrm{reg}_S(M)$ for $M \in {}^*\mathrm{mod}\,S$.

Theorem 4.10 (Brodmann and Lashgari, [5, Theorem 2.6])**.** *Let $S = k[x_1, \dots, x_n]$ be the polynomial ring. Assume that a graded submodule $M \subset S^{\oplus c}$ is generated by elements whose degrees are at most d. Then we have $\mathrm{reg}_S(M) \leq c^{n!}(2d)^{(n-1)!}$.*

When $c = 1$ (i.e., when M is an ideal), the above bound is a classical result, and there is a well-known example which shows the bound is rather sharp. For our study on $\mathrm{ld}_E(N)$, the case when $d = 1$ (but c is general) is essential.

Proposition 4.11. *Let $E = \bigwedge\langle y_1, \dots, y_n\rangle$ be an exterior algebra, and $N \in {}^*\mathrm{mod}\,E$. Set $c := \max\{\, \dim_K N_i \mid i \in \mathbb{Z}\,\}$. Then $\mathrm{ld}_E(N) \leq c^{n!}2^{(n-1)!}$.*

Proof. Similar to the proof of Theorem 4.4 (3). □

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