Generic initial ideals, graded Betti numbers and k-Lefschetz properties

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Abstract

We introduce the k-strong Lefschetz property (k-SLP) and the k-weak Lefschetz property (k-WLP) for graded Artinian K-algebras, which are generalizations of the Lefschetz properties. The main results obtained in this article are as follows:

1. Let I be a graded ideal of $R = K[x_1, x_2, x_3]$ whose quotient ring R/I has the SLP. Then the generic initial ideal of I is the unique almost revlex ideal with the same Hilbert function as R/I.

2. Let I be a graded ideal of $R = K[x_1, x_2, \ldots, x_n]$ whose quotient ring R/I has the n-SLP. Suppose that all k-th differences of the Hilbert function of R/I are quasi-symmetric. Then the generic initial ideal of I is the unique almost revlex ideal with the same Hilbert function as R/I.

3. We give a sharp upper bound on the graded Betti numbers of Artinian K-algebras with the k-WLP and a fixed Hilbert function.

1 Introduction

The strong and weak Lefschetz properties for graded Artinian K-algebras (Definition 1. SLP and WLP for short) are often used in studying generic initial ideals and graded Betti numbers ([Wat87], [Iar94], [HW03], [HMNW03], [Cim06], [ACP06], [AS07]). We generalize the Lefschetz properties, and define the k-strong Lefschetz property (k-SLP) and the k-weak Lefschetz property (k-WLP) for graded Artinian K-algebras (Definition 12). This notion was first introduced by A. Iarrobino in a private conversation with J. Watanabe in 1995. The first purpose of the article (Theorem 22) is to determine the generic initial ideals of ideals whose quotient rings have the n-SLP and have Hilbert functions satisfying some condition. The second purpose is to give upper bounds of the graded Betti numbers of graded Artinian K-algebras with the k-WLP (Theorem 31).

Let K be a field of characteristic zero. Suppose that a graded Artinian K-algebra A has the SLP (resp. WLP), and $\ell \in A$ is a Lefschetz element. If $A/(\ell)$ again has the SLP

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(resp. WLP), then we say that A has the 2-SLP (resp. 2-WLP). We recursively define the k-Lefschetz properties (Definition 12): A is said to have the k-SLP (resp. k-WLP), if A has the SLP (resp. WLP), and $A/(l)$ has the $(k-1)$-SLP (resp. $(k-1)$-WLP). We have the characterization of the Hilbert functions of graded Artinian K-algebras with the k-SLP or the k-WLP (Proposition 16). In addition, for a graded Artinian ideal $I \subset R$, we show that R/I has the k-SLP (resp. k-WLP) if and only if $R/\operatorname{gin}(I)$ has the k-SLP (resp. k-WLP), where $\operatorname{gin}(I)$ denotes the generic initial ideal with respect to the graded reverse lexicographic order.

We explain our results on generic initial ideals. A monomial ideal I is called an almost revlex ideal, if the following condition holds: for each minimal generator u of I, every monomial v with $\deg v = \deg u$ and $v >_{\text{revlex}} u$ belongs to I. Almost revlex ideals play a key role in the article. The characterization of the Hilbert functions for almost revlex ideals is given in Proposition 17. We start with the uniqueness of generic initial ideals in the case of three variables.

Theorem (see Theorem 10). *Let $I \subset R = K[x_1, x_2, x_3]$ be a graded Artinian ideal whose quotient ring has the SLP. Then $\operatorname{gin}(I)$ is the unique almost revlex ideal for the Hilbert function of R/I. In particular, $\operatorname{gin}(I)$ is uniquely determined only by the Hilbert function.*

For related results of the case of three variables, see Cimpoeaş [Cim06] and Ahn-Cho-Park [ACP06], where the uniqueness of $\operatorname{gin}(I)$ is proved under slightly stronger conditions than in the theorem above. We give some examples of complete intersection of height three whose generic initial ideals are the unique almost revlex ideals (Example 11).

By using the n-SLP, we obtain the following result for the case of n variables. In the following theorem, 'quasi-symmetric' is a notion including 'symmetric' (Definition 18).

Theorem (see Theorem 22). *Let $I \subset K[x_1, x_2, \dots, x_n]$ be a graded Artinian ideal whose quotient ring has the n-SLP, and has the Hilbert function h. Suppose that the k-th difference $\Delta^k h$ is quasi-symmetric for every integer k with $0 \le k \le n-4$. Then $\operatorname{gin}(I)$ is the unique almost revlex ideal for the Hilbert function h. In particular, $\operatorname{gin}(I)$ is uniquely determined only by the Hilbert function h.*

Here the operator Δ is defined by $(\Delta h)_i = \max\{h_i - h_{i-1}, 0\}$, and $\Delta^k h$ is the sequence obtained by applying Δ k-times.

The key to proving this theorem is a uniqueness of Borel-fixed ideals whose quotient rings have the n-SLP (Theorem 19). We give some examples of complete intersection of height n whose generic initial ideals are the unique almost revlex ideals (Example 24).

We next explain our result on the maximality of graded Betti numbers. Let $R = K[x_1, x_2, \dots, x_n]$. The following result on the maximal graded Betti numbers is first proved for $k = 1$ by Harima-Migliore-Nagel-Watanabe [HMNW03].

Theorem (see Theorem 31 and Corollary 32). *Let h be the Hilbert function of some graded Artinian K-algebra with the k-WLP. Then there is a Borel fixed ideal I of R such that R/I has the k-SLP, the Hilbert function of R/I is h, and $\beta_{i,i+j}(A) \le \beta_{i,i+j}(R/I)$ for all graded Artinian K-algebra A having the k-WLP and h as Hilbert function, and for any i and j.*

In particular, when $k = n$, the ideal I for the upper bounds is the unique almost revlex ideal for the Hilbert function h.

Some of the results of this article have been obtained independently and at the same time by Constantinescu (see [Con07]) and Cho-Park (see [CP07]).

2 Generic initial ideals in $K[x_1, x_2, x_3]$ and the SLP

In this section, we first recall the Lefschetz properties (Definition 1) and related facts. The main goal of this section is Theorem 10: for a graded Artinian ideal $I \subset K[x_1, x_2, x_3]$ whose quotient ring has the SLP, the generic initial ideal of I with respect to the the graded reverse lexicographic order is the unique almost revlex ideal for the same Hilbert function as $K[x_1, x_2, x_3]/I$.

2.1 The Lefschetz properties

Definition 1. Let A be a graded Artinian algebra over a field K, and $A = \bigoplus_{i=0}^{c} A_i$ its decomposition into graded components. The graded algebra A is said to have the *strong (resp. weak) Lefschetz property*, if there exists an element $\ell \in A_1$ such that the multiplication map $\times \ell^s : A_i \to A_{i+s}$ ($f \mapsto \ell^s f$) is full-rank for every $i \geq 0$ and $s > 0$ (resp. $s = 1$). In this case, ℓ is called a *Lefschetz element*, and we also say that (A, ℓ) has the strong (resp. weak) Lefschetz property. We abbreviate these properties as the *SLP (resp. WLP)* for short.

It is clear that if (A, ℓ) has the SLP, then (A, ℓ) has the WLP. It is also clear that Hilbert functions of graded algebras with the SLP or the WLP are *unimodal*. Namely there exists a non-negative integer i such that $h_0, h_1, \ldots, h_i$ is an increasing sequence and $h_i, h_{i+1}, \ldots$ is a weakly decreasing sequence, where $h_j = \dim_K A_j$.

Suppose that the Hilbert function of the graded Artinian algebra A is *symmetric*, that is, $A = \bigoplus_{i=0}^{c} A_i$ ($A_c \neq (0)$) and $\dim_K A_i = \dim_K A_{c-i}$ for $i = 0, 1, \ldots, \lfloor c/2 \rfloor$. In this case, it is clear that A has the SLP if and only if there exists $\ell \in A_1$ and $\times \ell^{c-2i} : A_i \to A_{c-i}$ is bijective for every $i = 0, 1, \ldots, \lfloor c/2 \rfloor$.

For a graded algebra A, we denote its Hilbert function by $\mathbf{H}_A$. Namely $\mathbf{H}_A(t)$ denotes the linear dimension of the graded component A_t of degree t. We often identify $\mathbf{H}_A$ with a finite sequence $h = (h_0, h_1, \ldots, h_c)$. A sequence $h = (h_0, h_1, \ldots, h_c)$ is called an *O-sequence* if h is a Hilbert function of some graded K-algebra. There is a classification of Hilbert functions of graded Artinian algebras with the SLP or the WLP.

Proposition 2 ([HMNW03, Corollary 4.6]). *Let $h = (h_0, h_1, \ldots, h_c)$ be a sequence of positive integers. The following three conditions are equivalent.*

(i) h is a Hilbert function of some graded algebra with the SLP,

(ii) h is a Hilbert function of some graded algebra with the WLP,

(iii) h is a unimodal O-sequence, and the sequence Δh is an O-sequence.

2.2 Almost revlex ideals and the SLP

We first recall a result of Wiebe [Wie04].

Lemma 3 (Wiebe, [Wie04, Lemma 2.7]). *If I is an Artinian stable ideal of $R = K[x_1, x_2, \ldots, x_n]$, then the following two conditions are equivalent:*

(i) R/I has the SLP (resp. WLP),

(ii) x_n is a strong (resp. weak) Lefschetz element on R/I.

We define the notion of almost revlex ideals (almost revlex-segment ideals). Let $R = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring over a field of characteristic zero. Let $>_{\text{revlex}}$ denote the graded reverse lexicographic order.

Definition 4. (i) A monomial ideal I is called a *revlex ideal*, if the following condition holds:

> for each monomial $u \in I$, every monomial v with $\deg v = \deg u$ and $v >_{\text{revlex}} u$ belongs to I.

(ii) A monomial ideal I is called an *almost revlex ideal*, if the following condition holds:

> for each monomial u in the minimal generating set of I, every monomial v with $\deg v = \deg u$ and $v >_{\text{revlex}} u$ belongs to I.

Remark 5. First it is clear that

(i) revlex ideals are almost revlex ideals.

Second,

(ii) if two almost revlex ideals have the same Hilbert function, then they are equal,

since one can determine the minimal generators from low degrees using a given Hilbert function. Finally,

(iii) almost revlex ideals are Borel-fixed,

since it is easy to see that the generating set of any almost revlex ideal is Borel-fixed.

Using the definition of almost revlex ideals (Definition 4) and combinatrics on monomials, we have the following proposition.

Proposition 6. *Let $I \subset R = K[x_1, x_2, \ldots, x_n]$ be an Artinian almost revlex ideal. Then $(R/I, x_n)$ has the SLP.*

2.3 Uniqueness of Borel-fixed ideals and generic initial ideals in $K[x_1, x_2, x_3]$

For a given O-sequence, it is known that a Borel-fixed ideal of $K[x_1, x_2]$, whose quotient ring has the O-sequence as the Hilbert function, is unique. It is the unique lex-segment ideal determined by the Hilbert function. Moreover we have the following theorem, which gives the uniqueness of Borel-fixed ideals for $n = 3$, where the quotient rings have the SLP.

Theorem 7. *Let $R = K[x_1, x_2, x_3]$ be the polynomial ring over a field of characteristic zero, and I an Artinian Borel-fixed ideal of R where R/I has the SLP. Then the ideal I is the unique almost revlex ideal for the Hilbert function. In particular, the ideal I is uniquely determined only by the Hilbert function.*

Note that Theorem 7 does not hold, if the number of variables is more than three. See Example 21 for a counterexample in the case of four variables.

The following is an immediate corollary to Proposition 2, Proposition 6 , Theorem 7 and Lemma 9 below.

Corollary 8. *Let $R = K[x_1, x_2, x_3]$ and $h = (1, 3, h_2, h_3, \dots, h_c)$ an O-sequence. The following three conditions are equivalent:*

(i) h is a Hilbert function of R/I for some almost revlex ideal I of R,

(ii) h is a Hilbert function of some graded algebra with the SLP,

(iii) h is a Hilbert function of some graded algebra with the WLP,

(iv) h is a unimodal O-sequence, and Δh is an O-sequence.

In the rest of this section, we study generic initial ideals in $K[x_1, x_2, x_3]$. We recall the definition of generic initial ideals. Fix any term order σ on the polynomial ring $R = K[x_1, x_2, \dots, x_n]$ over a field of characteristic zero. For a graded ideal I of R, there exists a Zariski open subset $U \subset GL(n; K)$ such that the initial ideals of $g(I)$ are equal to each other for any $g \in U$. This initial ideal is uniquely determined, called the *generic initial ideal* of I, and denoted by $\mathrm{gin}_\sigma(I)$. It is known that generic initial ideals are Borel-fixed with respect to any term order (see [Eis95, 15.9], e.g.).

Thus we have results on generic initial ideals of ideals whose quotient rings have the SLP, as an easy consequence of Theorem 7. We first recall another result of Wiebe. We write simply by $\mathrm{gin}(I)$ the generic initial ideal of I with respect to the graded reverse lexicographic order from now on.

Lemma 9 (Wiebe, [Wie04, Proposition 2.8]). *Take the graded reverse lexicographic order on the polynomial ring $R = K[x_1, x_2, \dots, x_n]$ over a field K of characteristic zero. Let I be a graded Artinian ideal of R. Then R/I has the SLP if and only if $R/\mathrm{gin}(I)$ has the SLP.*

We thus have the following theorem by Lemma 3, Theorem 7 and Lemma 9.

Theorem 10. *Let $R = K[x_1, x_2, x_3]$ be the polynomial ring over a field of characteristic zero, and consider the graded reverse lexicographic order on R. Let I be a graded Artinian ideal of R, and suppose that R/I has the SLP. Then the generic initial ideal* $\text{gin}(I)$ *is the unique almost revlex ideal for the same Hilbert function as R/I. In particular,* $\text{gin}(I)$ *is uniquely determined by the Hilbert function.*

Cimpoeaş [Cim06] shows that the generic initial ideals of complete intersections of height three whose quotient rings have the SLP are almost revlex ideals. Theorem 10 is an improvement of this result. In addition, Theorem 10 is an improvement of a result of Ahn, Cho and Park [ACP06]. They prove that the generic initial ideals of ideals in $K[x_1, x_2, x_3]$ whose quotient rings have the SLP are determined by their graded Betti numbers.

Example 11. We give four examples of complete intersection in $R = K[x_1, x_2, x_3]$ whose quotient rings have the SLP. The generic initial ideals of these ideals are the unique almost revlex ideals with corresponding Hilbert functions by Theorem 10.

(i) Let $I = (f, g, \ell^r) \subset R$, where f and g are any homogeneous polynomials of R, and ℓ is any homogeneous polynomial of degree one. In this case, if I is a complete intersection, then R/I has the SLP ([HW07a, Example 6.2]).

(ii) Let e_1, e_2 and e_3 be the elementary symmetric functions in three variables, where $\deg(e_i) = i$. Let r and s be positive integers, where r divides s. Then, the quotient ring of the ideal $I = (e_1(x_1^r, x_2^r, x_3^r), e_2(x_1^r, x_2^r, x_3^r), e_3(x_1^s, x_2^s, x_3^s))$ of R has the SLP ([HW07a, Example 6.4]).

(iii) Let p_i be the power sum symmetric function of degree i in three variables, and a be a positive integer. Then, the quotient ring of the ideal $I = (p_a, p_{a+1}, p_{a+2})$ of R has the SLP ([HW07b, Proposition 7.1]).

(vi) Let $I = (e_2, e_3, f) \subset R$, where f is any homogeneous polynomial of R. In this case, if I is a complete intersection, then R/I has the SLP ([HW07b, Proposition 3.1]).

3 Generic initial ideals in $K[x_1, x_2, \dots, x_n]$ and the k-SLP

Suppose that a graded Artinian algebra A has the SLP (resp. WLP), and $\ell \in A$ is a Lefschetz element. If the graded algebra $A/(\ell)$ again has the SLP (resp. WLP), then we say that A has the 2-SLP (resp. 2-WLP). We define the notion of the k-SLP and the k-WLP recursively (Definition 12). A characterization of the Hilbert functions of graded Artinian algebras having the k-SLP or the k-WLP is given in Proposition 16. Moreover, the Hilbert functions of quotient rings by almost revlex ideals are determined in terms of the n-SLP in Proposition 17.

The main goal of this section is Theorem 22: Let $I \subset K[x_1, x_2, \dots, x_n]$ be a graded Artinian ideal whose quotient ring has the n-SLP, and every k-th difference of the Hilbert function is quasi-symmetric. The generic initial ideal of I with respect to the graded reverse lexicographic order is the unique almost revlex ideal for the same Hilbert function as $K[x_1, x_2, \dots, x_n]/I$.

3.1 k-SLP and k-WLP

The first author heard from J. Watanabe that the following notion, the 'k-SLP' and the 'k-WLP', has been introduced by A. Iarrobino in a private conversation with J. Watanabe in 1995.

Definition 12. Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded Artinian K-algebra, and k a positive integer. We say that A has the *k-SLP* (resp. *k-WLP*) if there exist linear elements $g_1, g_2, \ldots, g_k \in A_1$ satisfying the following two conditions.

(i) (A, g_1) has the SLP (resp. WLP),

(ii) $(A/(g_1, \ldots, g_{i-1}), g_i)$ has the SLP (resp. WLP) for all $i = 2, 3, \ldots, k$.

In this case, we say that $(A, g_1, \ldots, g_k)$ has the k-SLP (resp. k-WLP). Note that a graded algebra with the k-SLP (resp. k-WLP) has the $(k-1)$-SLP (resp. $(k-1)$-WLP).

Remark 13. From Theorem 4.4 in [HMNW03], one knows that all graded K-algebras $K[x_1]/I$ and $K[x_1, x_2]/I$ have the SLP. Hence the following conditions are equivalent for a graded Artinian algebra $A = K[x_1, x_2, \ldots, x_n]/I$, where $I \subset (x_1, \ldots, x_n)^2$.

(i) A has the n-WLP (resp. the n-SLP),

(ii) A has the $(n-1)$-WLP (resp. the $(n-1)$-SLP),

(iii) A has the $(n-2)$-WLP (resp. the $(n-2)$-SLP).

In particular, graded algebras $K[x_1, x_2, x_3]/I$ with the SLP (resp. WLP) has the 3-SLP (resp. 3-WLP) automatically.

Example 14. For every Artinian almost revlex ideal I of $R = K[x_1, x_2, \ldots, x_n]$ where $I \subset (x_1, \ldots, x_n)^2$, the quotient ring R/I has the n-SLP. In particular, for every revlex ideal I of R where $I \subset (x_1, \ldots, x_n)^2$, the quotient ring R/I has the n-SLP.

Example 14 shows that the class of Hilbert functions for almost revlex ideals is a subset of that for ideals with the n-SLP. In fact, Proposition 17 shows that these two classes coincide. We also determine the class of Hilbert functions for ideals with the k-SLP in Proposition 16.

Let $\mathrm{gin}(I)$ denote the generic initial ideal of I with respect to the graded reverse lexicographic order. The following proposition is an analogue of Wiebe's result (Lemma 9) [Wie04, Proposition 2.8].

Proposition 15. *Let I be a graded Artinian ideal of $R = K[x_1, \ldots, x_n]$, and let $1 \leq k \leq n$. The following two conditions are equivalent:*

(i) R/I has the k-WLP (resp. the k-SLP),

(ii) $(R/\mathrm{gin}(I), x_n, x_{n-1}, \ldots, x_{n-k+1})$ has the k-WLP (resp. the k-SLP).

3.2 Hilbert functions of graded algebras with the k-SLP

We give a characterization of the Hilbert functions that can occur for graded K-algebras having the k-SLP or the k-WLP. Their characterizations are equal as in the case of the SLP and the WLP (Proposition 2). For a sequence $h = (h_0, h_1, \ldots, h_c)$ of positive integers, define a sequence of the t-th difference $\Delta^t h$ by

$$\Delta^t h = \Delta(\Delta(\cdots \Delta(h) \cdots)) \qquad \text{(apply } t \text{ times)},$$

for a positive integer t.

Proposition 16. *Let $R = K[x_1, x_2, \ldots, x_n]$ and k be an integer with $1 \leq k \leq n$. Let $h = (1, n, h_2, h_3, \ldots, h_c)$ be an O-sequence. The following three conditions are equivalent:*

(i) h is a Hilbert function of some graded algebra with the k-SLP,

(ii) h is a Hilbert function of some graded algebra with the k-WLP,

(iii) h is a unimodal O-sequence, $\Delta^t h$ is a unimodal O-sequence for every integer t with $1 \leq t < k$, and $\Delta^k h$ is an O-sequence.

In addition, we have a characterization of the Hilbert functions of quotient rings R/I for Artinian almost revlex ideals I. The characterization is the same as ideals with the n-WLP. This result is an analogue of the result of Deery [Dee96] or Marinari-Ramella [MR99, Proposition 2.13], which gives the characterization of the Hilbert functions for revlex ideals.

Proposition 17. *Let $R = K[x_1, x_2, \ldots, x_n]$ and $h = (1, n, h_2, h_3, \ldots, h_c)$ an O-sequence. The following four conditions are equivalent:*

(i) h is a Hilbert function of R/I for some almost revlex ideal I of R,

(ii) h is a Hilbert function of some graded algebra with the n-SLP,

(iii) h is a Hilbert function of some graded algebra with the n-WLP,

(iv) h is a unimodal O-sequence, and $\Delta^k h$ is a unimodal O-sequence for every integer k with $1 \leq k \leq n$.

3.3 Uniqueness of Borel-fixed ideals and generic initial ideals in $K[x_1, x_2, \ldots, x_n]$

When $n \leq 3$, we already know that Borel-fixed ideals of $K[x_1, x_2, \ldots, x_n]$ whose quotient rings have the n-SLP are the unique almost revlex ideals for given Hilbert functions. Moreover, we have the following Theorem 19 for any n. For a sequence h, we use a convention that the 0-th difference $\Delta^0 h$ is h itself. We need the following definition to state Theorem 19.

Definition 18. A unimodal sequence $h = (h_0, h_1, \ldots, h_c)$ of positive integers is said to be *quasi-symmetric*, if the following condition holds:

Let h_i be the maximum of $\{h_0, h_1, \ldots, h_c\}$. Then every integer h_j $(j > i)$ is equal to one of $\{h_0, h_1, \ldots, h_i\}$.

In particular, unimodal symmetric sequences are quasi-symmetric.

Theorem 19. *Let $I \subset R = K[x_1, x_2, \ldots, x_n]$ be an Artinian Borel-fixed ideal whose quotient ring R/I has the n-SLP, and let h be the Hilbert function of R/I. Suppose that the k-th difference $\Delta^k h$ is quasi-symmetric for every integer k with $0 \leq k \leq n-4$. Then I is the unique almost revlex ideal for which the Hilbert function of R/I is equal to h. In particular, I is determined only by the Hilbert function.*

In particular, we have the following uniqueness for Borel-fixed ideals in the case of four variables.

Corollary 20. *Let $I \subset K[x_1, x_2, x_3, x_4]$ be a Borel-fixed ideal, for which $K[x_1, x_2, x_3, x_4]/I$ has a quasi-symmetric Hilbert function h , and has the 2-SLP. Then I is the unique almost revlex ideal for the Hilbert function h.*

In Theorem 19, if we drop the condition for $\Delta^k h$ to be quasi-symmetric, then the uniqueness does not necessarily hold as follows.

Example 21. (i) There exist two different Borel-fixed ideals with the 4-SLP in $R = K[x_1, x_2, x_3, x_4]$, and their quotient rings have the same non-quasi-symmetric Hilbert function. Define the following ideals:

$$I = (x_1^2, x_1x_2, x_2^3, x_2^2x_3, x_1x_3^2, x_2x_3^2, x_3^3, x_2^2x_4) + (x_1, x_2, x_3, x_4)^4,$$
$$J = (x_1^2, x_1x_2, x_2^3, x_2^2x_3, x_1x_3^2, x_2x_3^2, x_3^3, x_1x_3x_4) + (x_1, x_2, x_3, x_4)^4.$$

We can easily check that both I and J are Borel-fixed, have the 4-SLP, and R/I and R/J have the same Hilbert function $h = (1, 4, 8, 7)$.

In the rest of this section, we study generic initial ideals in the polynomial ring $R = K[x_1, x_2, \ldots, x_n]$ over a field of characteristic zero. The following theorem, which gives a uniqueness of generic initial ideals, follows from Theorem 19 and Proposition 15.

Theorem 22. *Let $I \subset R = K[x_1, x_2, \ldots, x_n]$ be a graded Artinian ideal whose quotient ring R/I has the n-SLP, and let h be the Hilbert function of R/I. Suppose that the k-th difference $\Delta^k h$ is quasi-symmetric for every integer k with $0 \leq k \leq n-4$. Then the generic initial ideal $\mathrm{gin}(I)$ with respect to the graded reverse lexicographic order is the unique almost revlex ideal for the Hilbert function h. In particular, $\mathrm{gin}(I)$ is determined only by the Hilbert function.*

In particular, we have the following corollary, which corresponds to Corollary 20.

Corollary 23. *Let $I \subset K[x_1, x_2, x_3, x_4]$ be a graded Artinian ideal whose quotient ring has the 2-SLP. Suppose that the Hilbert function h of $K[x_1, x_2, x_3, x_4]/I$ is quasi-symmetric. Then the generic initial ideal $\mathrm{gin}(I)$ with respect to the graded reverse lexicographic order is the unique almost revlex ideal for the Hilbert function h. In particular, $\mathrm{gin}(I)$ is determined only by the Hilbert function.*

Example 24. Let $R = K[x_1, x_2, \dots, x_n]$ be the polynomial ring over a field K of characteristic zero. We consider complete intersections as follows.

(a) Let f_1 and f_2 be homogeneous polynomials of degree d_i $(i = 1, 2)$, and let $g_3, \dots, g_n$ be linear forms. Set $I = (f_1, f_2, f_3 = g_3^{d_3}, \dots, f_n = g_n^{d_n})$. Suppose that $\{f_1, f_2, g_3, \dots, g_n\}$ is a regular sequence. Example 6.2 in [HW07a] shows that R/I has the SLP.

(b) For $i = 1, 2, \dots, n$, $f_i \in K[x_i, \dots, x_n]$ be a homogeneous polynomial of degree d_i which is a monic in x_i, and set $I = (f_1, f_2, \dots, f_n)$. Then R/I is always a complete intersection. Corollary 29 in [HW03] and Corollary 2.1 in [HP05] show that R/I has the SLP.

Now, let k be an integer satisfying $1 \le k \le n-2$ and suppose that

$$d_j \ge d_1 + d_2 + \cdots + d_{j-1} - (j-1) + 1$$

for all $j = n-k+1, n-k+2, \dots, n$. Then we have the following.

(i) $A = R/I$ has the k-SLP.

(ii) In particular, when $k = n-2$, A has the n-SLP.

(iii) The generic initial ideal of I coincides with the unique almost revlex ideal determined by the Hilbert function of A.

(iv) $\mathrm{gin}(x_1^{d_1}, x_2^{d_2}, \dots, x_n^{d_n}) = \mathrm{gin}(I)$.

We conclude this section by an additional relation of initial ideals with the k-WLP or the k-SLP. Although this result is not used in the rest of this article, it is an analogue of Wiebe's result [Wie04, Proposition 2.9].

Proposition 25. *Let I be a graded Artinian ideal of $R = K[x_1, \dots, x_n]$, let $\mathrm{in}(I)$ be the initial ideal of I with respect to the graded reverse lexicographic order and let $1 \le k \le n$. If $R/\mathrm{in}(I)$ has the k-WLP (resp. the k-SLP), then the same holds for R/I.*

4 An extremal property of graded Betti numbers and the k-WLP

In the rest of this article, we study graded Betti numbers for monomial ideals. The goal is Theorem 31 on the maximality of graded Betti numbers. We give a sharp upper bound on the graded Betti numbers of graded Artinian algebras with the k-WLP and a fixed Hilbert function. The upper bounds are achieved by the quotient rings by Borel-fixed ideals having the k-SLP. In particular, when $k = n$, almost revlex ideals give the upper bounds.

4.1 Graded Betti numbers of stable ideals and the k-WLP

Let $R = K[x_1, x_2, \dots, x_n]$ be the polynomial ring over a field of characteristic zero.

Proposition 26. *Let $I \subset R$ be an Artinian stable ideal, for which R/I has the Hilbert function $h = (h_0, h_1, \dots, h_c)$, and $(R/I, x_n)$ have the WLP. Let $\overline{R} = K[x_1, x_2, \dots, x_{n-1}]$ and $\overline{I} = I \cap \overline{R}$. We have the following.*

(i) The graded Betti numbers $\beta_{i,i+j}(R/I)$ of R/I is given as follows:

$$\beta_{i,i+j}(R/I) = \beta_{i,i+j}(\overline{R}/\overline{I}) + \binom{n-1}{i-1} \times c_{j+1} \qquad (i, j \geq 0),$$
$$c_j = \max\{h_{j-1} - h_j, 0\},$$

where we use the convention that $h_{-1} = 0$.

(ii) By the same c_j, the last graded Betti numbers $\beta_{n,n+j}(R/I)$ is given as follows:

$$\beta_{n,n+j}(R/I) = c_{j+1} \qquad (j \geq 0).$$

In particular, they are determined only by the Hilbert function.

We easily generalize Proposition 26 to the case of the k-WLP.

Notation 27. For a unimodal O-sequence $h = (h_0, h_1, \dots, h_c)$, we define

$$c_j^{(h)} = \max\{h_{j-1} - h_j, 0\}$$

for all $j = 0, 1, \dots, c$, where $h_{-1} = 0$.

Proposition 28. *Let I be an Artinian Borel-fixed ideal of $R = K[x_1, x_2, \dots, x_n]$ and suppose that R/I has the k-WLP.*

(i) Let $k < n$. Set $R' = K[x_1, x_2, \dots, x_{n-k}]$ and $I' = I \cap R'$. Then

$$\beta_{i,i+j}(R/I) = \beta_{i,i+j}(R'/I') + \tbinom{n-k}{i-1} \cdot c_{j+1}^{(\Delta^{k-1}h)} + \cdots + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)}.$$

(ii) Let $k = n$. Then we have

$$\beta_{i,i+j}(R/I) = \tbinom{0}{i-1} \cdot c_{j+1}^{(\Delta^{n-1}h)} + \tbinom{1}{i-1} \cdot c_{j+1}^{(\Delta^{n-2}h)} + \cdots + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)}.$$

In particular, $\beta_{i,i+j}(R/I)$ is determined only by the Hilbert function.

Corollary 29. *Let I be a graded Artinian ideal of $R = K[x_1, x_2, \dots, x_n]$ and suppose that R/I has the k-WLP.*

(i) Let $k < n$. Set $R' = K[x_1, x_2, \dots, x_{n-k}]$ and $I' = \operatorname{gin}(I) \cap R'$. Then

$$\beta_{i,i+j}(R/\operatorname{gin}(I)) = \beta_{i,i+j}(R'/I') + \tbinom{n-k}{i-1} \cdot c_{j+1}^{(\Delta^{k-1}h)} + \cdots + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)}.$$

(ii) Let $k = n$. Then we have

$$\beta_{i,i+j}(R/\operatorname{gin}(I)) = \tbinom{0}{i-1} \cdot c_{j+1}^{(\Delta^{n-1}h)} + \tbinom{1}{i-1} \cdot c_{j+1}^{(\Delta^{n-2}h)} + \cdots + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)}.$$

In particular, $\beta_{i,i+j}(R/\operatorname{gin}(I))$ is determined only by the Hilbert function.

4.2 Maximality of graded Betti numbers and the k-WLP

Notation and Remark 30. Let h be the Hilbert function of a graded Artinian K-algebra R/I. Then there is the uniquely determined lex-segment ideal $J \subset R$ such that R/J has h as its Hilbert function. We define

$$\beta_{i,i+j}(h, R) = \beta_{i,i+j}(R/J).$$

The numbers $\beta_{i,i+j}(h, R)$ can be computed numerically without considering lex-segment ideals. Explicit formulas can be found in [EK90].

We give a sharp upper bound on the Betti numbers among graded Artinian K-algebras having the k-WLP. Moreover the upper bound is achieved by a graded Artinian K-algebra with the k-SLP. For $k = 1$, this theorem was first proved by [HMNW03, Theorem 3.20].

Theorem 31. *(i) Let $A = R/I$ be a graded Artinian K-algebra with the k-WLP and put $R' = K[x_1, \ldots, x_{n-k}]$. Then the graded Betti numbers of A satisfy*

$$\begin{aligned}\beta_{i,i+j}(A) \leq \beta_{i,i+j}(\Delta^k h, R') + \tbinom{n-k}{i-1} \cdot c_{j+1}^{(\Delta^{k-1}h)} + \cdots \\ + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)} \qquad \text{if } k < n,\end{aligned} \tag{1}$$

and

$$\begin{aligned}\beta_{i,i+j}(A) \leq \tbinom{0}{i-1} \cdot c_{j+1}^{(\Delta^{n-1}h)} + \tbinom{1}{i-1} \cdot c_{j+1}^{(\Delta^{n-2}h)} + \cdots \\ + \tbinom{n-2}{i-1} \cdot c_{j+1}^{(\Delta h)} + \tbinom{n-1}{i-1} \cdot c_{j+1}^{(h)} \qquad \text{if } k = n.\end{aligned} \tag{2}$$

(ii) Let h be an O-sequence such that there is a graded Artinian K-algebra R/J having the k-WLP and h as Hilbert function. Then there is a Borel-fixed ideal I of R such that R/I has the k-SLP, the Hilbert function of R/I is h and the equality holds in (i) for all integers i, j.

The following are immediate consequences of Theorem 31.

Corollary 32. *Let h be the Hilbert function of a graded Artinian K-algebra R/J having the n-WLP (resp. n-SLP). Let I be the unique almost revlex ideal of R whose quotient ring has the same Hilbert function h. Then R/I has the maximal Betti numbers among graded Artinian K-algebras with the same Hilbert function h and the n-WLP (resp. n-SLP).*

Corollary 33. *Let I be a graded Artinian ideal of $R = K[x_1, x_2, \ldots, x_n]$ and suppose that R/I has the n-WLP (resp. n-SLP). Then $R/\operatorname{gin}(I)$ has the maximal Betti numbers among graded Artinian K-algebras with the same Hilbert function R/I and the n-WLP (resp. n-SLP).*

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