On codimension-one $\boldsymbol{A}^1$-fibrations over Noetherian normal domains

Nobuharu Onoda (University of Fukui)

1. Introduction

This is a joint work with S. M. Bhatwadekar and A. K. Dutta. Let R be a commutative ring. For a prime ideal P of R, we denote by $k(P)$ the field R_P/PR_P. A polynomial ring in n variables over R is denoted by $R^{[n]}$.

Definition 1.1. We shall call an R-algebra A to be a *codimension-one* $\boldsymbol{A}^1$*-fibration* over R if

$$k(P) \otimes_R A = k(P)^{[1]}$$

for every $P \in \operatorname{Spec} R$ with $\operatorname{ht} P \leq 1$.

Let R be a Noetherian normal domain with field of fractions K. Then the following results were proved in ([2], 3.4) and ([1], 3.10) respectively.

Theorem 1.2. Let A be a flat R-subalgebra of $R^{[m]}$ such that $K \otimes_R A = K^{[1]}$ and $k(P) \otimes_R A$ is an integral domain for every prime ideal P in R of height one. Then $A \cong R[IX]$ for an invertible ideal I of R.

Theorem 1.3. Let A be a faithfully flat finitely generated R-algebra such that $K \otimes_R A = K^{[1]}$ and $k(P) \otimes_R A$ is geometrically integral for every prime ideal P in R of height one. Then $A \cong R[IX]$ for an invertible ideal I of R.

Recently the two results were shown to emanate from the following result.

Theorem 1.4. Let A be a faithfully flat R-algebra such that A is an R-subalgebra of a finitely generated R-algebra B and such that A satisfies the fibre conditions:

(i) $K \otimes_R A = K^{[1]}$.
(ii) For every prime ideal P in R of height one, $k(P) \otimes_R A$ is an integral domain with $\operatorname{tr.deg}_{k(P)} k(P) \otimes_R A > 0$ and $k(P)$ is algebraically closed in $k(P) \otimes_R A$.

Then $A \cong R[IX]$ for an invertible ideal I of R.

In this note, we explore the structure of a faithfully flat codimension-one $\boldsymbol{A}^1$-fibration over a Krull domain; in particular, over a Noetherian normal domain. As an application we show that all previous results described above can be deduced from this structure theorem.

2. Structure theorem

We begin by noting the following result.

Lemma 2.1. Let R be a Krull domain and A a flat R-algebra. Then A is a codimension-one $\boldsymbol{A}^1$-fibration over R if and only if $A_P = {R_P}^{[1]}$ for every $P \in \operatorname{Spec} R$ with $\operatorname{ht} P = 1$.

Set-Up

Throughout this section we will assume that

R: Krull domain with field of fractions K.

$\Delta = \{P \in \operatorname{Spec} R \mid \operatorname{ht} P = 1\}$.

A: a faithfully flat R-algebra such that $A_P = {R_P}^{[1]}$ for every $P \in \Delta$.

x: A fixed element of A such that $T^{-1}A = K[x]$, where $T = R \setminus \{0\}$.

$\Sigma = \{\Gamma \mid \Gamma \text{ is a finite subset of } \Delta\}$.

$\Gamma_a = \{P \in \Delta \mid a \in P\}$, where $0 \neq a \in R$.

Definition 2.2. For $\Gamma \in \Sigma$, we set

$$R_\Gamma = \bigcap_{P \notin \Gamma} R_P$$

and

$$A_\Gamma = S^{-1}A \cap R_\Gamma[x],$$

where $S = R \setminus (\bigcup_{P \in \Gamma} P)$. Note that

$$R_{\Gamma_a} = R[1/a]$$

for $0 \neq a \in R$.

The following result holds for the ring A_Γ defined above.

Lemma 2.3. For $\Gamma \in \Sigma$, we have

$$A_\Gamma = \bigoplus_{n \geq 0} (R \cap d^n R_\Gamma) \left(\frac{x-c}{d} \right)^n$$

for some elements $c, d (\neq 0) \in R$. In particular, for $0 \neq a \in R$,

$$A_{\Gamma_a} = \bigoplus_{n \geq 0} (R \cap d^n R[1/a]) \left(\frac{x-c}{d} \right)^n$$

for some $c, d (\neq 0) \in R$. Furthermore, we have
(1) $A_\Gamma \subseteq A$.
(2) $(A_\Gamma)_P = A_P$ for $P \in \Gamma$.
(3) $(A_\Gamma)_P = R_P[x]$ for $P \in \Delta \setminus \Gamma$.

Lemma 2.4. Let I be an ideal of R and suppose that I is R-flat. Then I is an invertible ideal of the form

$$I = R \cap dR[1/a]$$

for some $d \in I, a \in R$. Moreover we have

$$I^n = R \cap d^n R[1/a]$$

for every positive integer n.

From Lemmas 2.3 and 2.4, we have the following

Corollary 2.5. Suppose that A_{Γ_a} is flat over R. Then $A_{\Gamma_a} \cong R[IX]$ for an invertible ideal I of R.

Lemma 2.6. Let Γ_1 and Γ_2 be finite subsets of Δ. If $\Gamma_1 \subseteq \Gamma_2$, then $A_{\Gamma_1} \subseteq A_{\Gamma_2}$.

Lemma 2.6 shows that the rings A_Γ, together with inclusion maps, form a direct system

$$\{A_\Gamma \mid \Gamma \in \Sigma\}$$

indexed by Σ. We now prove the structure theorem:

Theorem 2.7. $A = \varinjlim A_\Gamma \left(= \bigcup_\Gamma A_\Gamma \right)$.

Proof. Set $C = \varinjlim A_\Gamma$. Then $C = \bigcup_\Gamma A_\Gamma$, and hence $C \subset A$. For the converse inclusion $A \subset C$, let w be an arbitrary non-zero element of A. Since $A \subset K[x]$, we can write

$$w = \xi_0 x^n + \xi_1 x^{n-1} + \cdots + \xi_n$$

for some $n \geq 0$ and $\xi_0, \ldots, \xi_n \in K$. Note that $\Delta_\xi := \{P \in \Delta \mid v_P(\xi) < 0\}$ is a finite set for $0 \neq \xi \in K$, because writing $\xi = b/c$ with $b, c (\neq 0) \in R$, we have $v_P(c) > v_P(b) \geq 0$ for $P \in \Delta_\xi$, so that $\Delta_\xi \subset \mathrm{Ass}_R(R/cR)$. Set $\Gamma = \bigcup_{i=0}^n \Delta_{\xi_i}$. Then Γ is a finite subset set of Δ, and $w \in R_P[x]$ for any $P \notin \Gamma$. Therefore

$$w \in \left(\bigcap_{P \in \Gamma} A_P \right) \cap \left(\bigcap_{P \notin \Gamma} R_P[x] \right),$$

which implies $w \in A_\Gamma \subset C$. This completes the proof. $\square$

Lemma 2.8. For $P \in \Delta$, writing $PR_P = pR_P$ with $p \in R$, we have

$$A_P = R_P \left[\frac{x-c}{p^e} \right]$$

for some $c \in R$ and $e \geq 0$. Furthermore, the integer e is uniquely determined for P.

For $P \in \Delta$, we denote by e_P the integer e given in Lemma 2.8 above. Note that

$$e_P > 0 \Longleftrightarrow A_P \neq R_P[x].$$

From Theorem 2.7, we shall now deduce that finite generation of A is equivalent to the finiteness of the set

$$\Delta_0 = \{P \in \Delta \mid e_P > 0\}.$$

Lemma 2.9. Let Γ_1, Γ_2 be elements of Σ such that $\Gamma_1 \subseteq \Gamma_2$. Then $A_{\Gamma_1} \subsetneqq A_{\Gamma_2}$ if and only if there exists $P \in \Gamma_2 \setminus \Gamma_1$ such that $P \in \Delta_0$.

We say that R is a retract of A if there exists an R-algebra map $\varphi\colon A \to R$ such that $\varphi|_R = id_R$.

Theorem 2.10. The following conditions are equivalent:
(1) A is finitely generated over R.
(2) Δ_0 is a finite set.
(3) R is a retract of A and A is a Krull ring.
(4) $A \cong R[IX]$ for an invertible ideal I of R.

Proof. We shall give a proof only for (1)⇒(2) and (2)⇒(4).

(1)⇒(2): Recall that, by Theorem 2.7, we have

$$A = \varinjlim A_\Gamma = \bigcup_{\Gamma} A_\Gamma. \tag{1}$$

Let $A = R[f_1, \ldots, f_n]$. By (1), for each i there exists $\Gamma_i \in \Sigma$ such that $f_i \in A_{\Gamma_i}$. Then, setting $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$, we have $f_i \in A_\Gamma$ for each i, which implies $A = A_\Gamma$. Now suppose that Δ_0 is an infinite set. Then there exists $P \in \Delta_0 \setminus \Gamma$, because Γ is a finite set. Let $\Gamma' = \Gamma \cup \{P\}$. Then $\Gamma \subseteq \Gamma'$ and $P \in \Gamma' \setminus \Gamma$. Thus $A_\Gamma \neq A_{\Gamma'}$ by Lemma 2.9. On the other hand, by Lemma 2.6, we have

$$A = A_\Gamma \subseteq A_{\Gamma'} \subseteq A,$$

so that $A_\Gamma = A_{\Gamma'}$, a contradiction.

(2)⇒(4): Let $\Delta_0 = \{P_1, \ldots, P_m\}$ and let a be a non-zero element of $P_1 \cap \cdots \cap P_m$. Then $\Delta_0 \subset \Gamma_a$, and hence, by Lemma 2.9, we have $A_{\Gamma_a} = A_\Gamma$ for every $\Gamma \in \Sigma$ such that $\Gamma_a \subseteq \Gamma$. It thus follows from Theorem 2.7 that

$$A = A_{\Gamma_a} = \bigoplus_{n \geq 0} (R \cap d^n R[1/a]) \left(\frac{x-c}{d}\right)^n.$$

Since A is flat over R, we have $A \cong R[IX]$ by Corollary 2.5. □

3. Applications

We now give a few applications of our results.

Theorem 3.1. Let R be a Krull domain with field of fractions K and A a faithfully flat R-algebra such that A is an R-subalgebra of a finitely generated R-algebra B and such that A satisfies the fibre conditions:

(i) $K \otimes_R A = K^{[1]}$.

(ii) For every prime ideal P in R of height one, $k(P) \otimes_R A$ is an integral domain with $\operatorname{tr.deg}_{k(P)} k(P) \otimes_R A > 0$ and $k(P)$ is algebraically closed in $k(P) \otimes_R A$.

Then $A \cong R[IX]$ for an invertible ideal I of R.

Proof. Let $T = A \backslash \{0\}$ and let $T^{-1}Q$ be a maximal ideal in $T^{-1}B$, where Q is a prime ideal in B. Then $Q \cap A = 0$ and $T^{-1}B/T^{-1}Q$ is algebraic over $T^{-1}A$. Thus, replacing B by B/Q, we may assume that B is an integral domain algebraic over A. Since B is finitely generated over R, there exist elements $f, g_1, \ldots, g_m$ in A such that $B[1/f]$ is integral over $R[g_1, \ldots, g_m, 1/f]$. Let d be a non-zero element in R such that $df \in R[x]$ and $dg_i \in R[x]$ for $1 \leq i \leq m$; such d exists because $A \subset K[x]$. Then we have

$$R[1/d][g_1, \ldots, g_m, 1/f] \subset R[1/d][x, 1/f] \subset A[1/d, 1/f] \subset B[1/d, 1/f],$$

and hence $A[1/d, 1/f]$ is integral over $R[1/d][x, 1/f]$. Note that $R[1/d][x, 1/f]$ is a Krull domain because so is R. Thus $R[1/d][x, 1/f]$ is integrally closed. Note also that both $R[1/d][x, 1/f]$ and $A[1/d, 1/f]$ have the same quotient field $K(x)$. Therefore we have

$$R[1/d][x, 1/f] = A[1/d, 1/f].$$

Let ξ be the coefficient of the highest degree term of f as a polynomial in $K[x]$, and let $b = d\xi$. We will show that $e_P = 0$ for $P \in \Delta$ with $db \notin P$. Indeed, suppose on the contrary that $e := e_P > 0$ for some P with $db \notin P$. Since $d \notin P$, we have $R[1/d] \subset R_P$, so that

$$A_P[1/f] = R_P[x, 1/f].$$

Hence, writing $A_P = R_P[(x-c)/p^e]$ with $c \in R$, we have $f^n(x-c)/p^e \in R_P[x]$ for a sufficiently large integer n. From this it follows that $b^n \in p^e R_P$, and hence $b \in PR_P \cap R = P$, a contradiction. Therefore $e_P = 0$ for P with $db \notin P$, which implies $\Delta_0 \subset \Gamma_{db}$. Thus Δ_0 is a finite set, and hence, by Theorem 2.10, $A \cong R[IX]$ for an invertible ideal I of R. $\square$

Lemma 3.2. Let R be an integral domain and A an R-domain having a retract $\varphi\colon A \to R$. Set $J = \ker\varphi$. Then the following assertions hold.
(1) If A is flat over R, then A is faithfully flat over R.
(2) If f is a non-zero element of J, then f is transcendental over R. In particular R is algebraically closed in A.
(3) Suppose that $\operatorname{tr.deg}_R A > 0$ and J is finitely generated. Let P be a prime ideal in R such that PA remains prime in A. Then $\operatorname{tr.deg}_{R/P} A/PA > 0$.

Theorem 3.3. Let R be a Krull domain and A a flat R-algebra with a retract $\varphi\colon A \to R$. Suppose that A satisfies the following conditions:

(i) $K \otimes_R A = K^{[1]}$.
(ii) For every prime ideal P in R of height one, $k(P) \otimes_R A$ is an integral domain.

If $J := \ker\varphi$ is a finitely generated ideal of A, then $A \cong R[IX]$ for an invertible ideal I in R.

Proof. Let P be a prime ideal in R of height one. Then $A/PA \subset A_P/PA_P$ because of flatness of A over R, which implies that PA is a prime ideal in A. Thus, by Theorem 2.10 and Lemma 3.2, it suffices to show that Δ_0 is a finite set. Let $g_1, \ldots, g_m$ be generators of J and let d be a non-zero element of R satisfying $dg_i \in R[x]$ for each $i = 0, \ldots, m$. Let P be an element in $\Delta \setminus \Gamma_d$. We will show that $e := e_P = 0$; if this is the case, then $\Delta_0 \subset \Gamma_d$, and hence Δ_0 is a finite set. Suppose on the contrary that $e > 0$, and write $A_P = R_P[(x-c)/p^e]$ with $c \in R$. For simplicity we set $z = (x-c)/p^e$. Let $\varphi_P\colon A_P \to R_P$ be the retract induced by φ, and let $\varphi_P(z) = c_1$. Since $A_P = R_P[z]$, it then follows that $\ker\varphi_P = (z - c_1)R_P[z]$. Replacing z by $z - c_1$, we may assume that $\ker\varphi_P = zR_P[z]$. Furthermore replacing x by $x - c$, we may assume that $z = x/p^e$. Note that $\ker\varphi_P = J_P$. Note also that $g_i \in R_P[x]$ for each i, because $dg_i \in R[x]$ and $d \notin P$. Hence, for each i, we have $g_i \in R_P[x] \cap zR_P[z] = xR_P[x]$, so that $g_i = xh_i(x)$ where $h_i(x) \in R_P[x]$. Now, since $z \in J_P$ and $J_P = (g_1, \ldots, g_m)R_P[z]$, we can write

$$\frac{x}{p^e} = xh_1(x)u_1(z) + \cdots + xh_m(x)u_m(z)$$

for some $u_1(z), \ldots, u_m(z) \in R_P[z]$. Dividing both sides of the above equation by x, and substituting $x = 0$, we have $1/p^e = h_1(0)u_1(0) + \cdots + h_m(0)u_m(0) \in R_P$. This is a contradiction, as desired. $\square$

Remark 3.4. The condition "J is finitely generated" is necessary. Consider $R = \mathbb{Z}$ and $A = \mathbb{Z}[\frac{X}{p} \mid p \ \ prime \ \ ]$.

Theorem 3.5. Let R be a locally factorial Krull domain and A a flat codimension-one $\boldsymbol{A}^1$-fibration over R. Then at each prime ideal $Q \in \operatorname{Spec} R$, either $k(Q) \otimes_R A = k(Q)^{[1]}$ or $k(Q) \otimes_R A = k(Q)$. Suppose in addition that R is a local ring with maximal ideal m and residue field $k(= R/m)$. Then the following conditions are equivalent:

(1) A is finitely generated over R.
(2) $\operatorname{tr.deg}_k A/mA > 0$.
(3) $\dim A/mA > 0$.
(4) $A = R^{[1]}$.

4. Examples

We give below some examples to illustrate the hypotheses in Theorem 3.1.

Example 4.1. The hypothesis on flatness is needed even when A is a finitely generated subalgebra of $R^{[1]}$. For instance, consider $R = k[[t_1, t_2]]$ and

$$A = R[t_1X, t_2X] \cong R[U, V]/(t_2U - t_1V).$$

Example 4.2. The hypothesis on faithful flatness is also necessary. Consider $R = k[[t_1, t_2]]$ and $A = R[U, V]/(t_1U + t_2V - 1)$.

Example 4.3. The condition "$k(P)$ is algebraically closed in $k(P) \otimes_R A$" is necessary. Let $R = \mathbb{R}[[t]]$ and

$$A = R[U, V]/(tU + V^2 + 1).$$

Then A is a finitely generated flat R-algebra, $K \otimes_R A = R^{[1]}$ and $A/tA = \mathbb{C}^{[1]}$. But $A \neq R^{[1]}$.

Example 4.4. Let k be an infinite field and $R = k[[t_1, t_2]]$. Let

$$A = R[\{X/q \mid q \text{ a square-free non-unit in } R\}].$$

Then

$$A_P = R_P[X/p] = {R_P}^{[1]}$$

for every $P \in \operatorname{Spec} R$ with $\operatorname{ht} P = 1$, where $P = pR$. However $A/(t_1, t_2)A = k$. Note that since

$$A = \bigcup_q R[X/q],$$

A is flat over R and hence faithfully flat over R. A is not finitely generated.

References

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