SEVERAL RESULTS ON FINITENESS PROPERTIES OF LOCAL COHOMOLOGY MODULES OVER COHEN MACAULAY LOCAL RINGS

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We assume that all rings are commutative and noetherian with identity throughout this paper.

1. Introduction

In 1993, Huneke and Sharp (cf. [2]) and Lyubeznik (cf. [6]) showed the following results:

Theorem 1 (Huneke, Sharp and Lyubeznik)**.** *Let $(R, \mathfrak{m})$ be a regular local ring containing a field, and I an ideal of R. Then the following assertions hold for all integers $i, j \geq 0$:*

(i) $H^j_{\mathfrak{m}}(H^i_I(R))$ *is an injective module*;
(ii) $\operatorname{inj.dim}_R(H^i_I(R)) \leq \dim H^i_I(R)$;
(iii) *the set of associated prime ideals of $H^i_I(R)$ is a finite set*;
(iv) *all the Bass numbers of $H^i_I(R)$ are finite.*

Our aims in this report are to develop results of Theorem 1 to those over Cohen-Macaulay local rings. We shall introduce the following theorems:

Theorem 2. *Let $\phi : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ be a local ring homomorphism of local rings, which is module-finite and flat. Let i be a non-negative integer. Further let I be an ideal of A satisfied with the condition that if we set $I \cap R = J$ then $I = JA$.*

(a) *If the set of associated prime ideals of $H^i_J(R)$ is a finite set, then so is the set of associated prime ideals of $H^i_I(A)$;*
(b) *if all the Bass numbers of $H^i_J(R)$ are finite, then so are all the Bass numbers of $H^i_I(A)$.*

Theorem 3. *Let $\phi : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ be a local ring homomorphism of regular local rings, which is module-finite and flat, and I an ideal of A. Let i, j be non-negative integers. Set $I \cap R = J$. Suppose that $I = JA$ and R is an unramified regular local ring. Then the following assertions hold*:

(i) $\operatorname{inj.dim}_A H^j_{\mathfrak{n}} H^i_I(A) \leq 1$;
(ii) $\operatorname{inj.dim}_A H^i_I(A) \leq \dim H^i_I(A) + 1$;
(iii) *the set of associated prime ideals of $H^i_I(A)$ is a finite set*;
(iv) *all the Bass numbers of $H^i_I(A)$ are finite.*

Here it is worth while mentioning that S. Takagi and R. Takahashi recently showed finiteness properties of local cohomlogy modules over rings with finite F-representation type (cf. [11]). Mainly we shall prove part (i) and (ii) of Theorem 3 in this report.

2. Proof of Theorem 2: Outline

Definition 1. Let T be a module over a ring A and P a prime ideal of A. We define the j-th Bass number $\mu_j(P,T)$ at P to be

$$\mu_j(P,T) = \dim_{\kappa(P)} \operatorname{Ext}^j_{R_P}(\kappa(P), T_P),$$

where $\kappa(P) = R_P/PR_P$ (cf. [1]).

Remark 1. Let $\phi : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ be a local ring homomorphism of local rings, which is module-finite and flat. Then such properties for an extension are preserved by localization: let P be a prime ideal and set $\mathfrak{p} = P \cap R$. Then the local ring homomorphism of local rings $\phi_P : R_\mathfrak{p} \longrightarrow A_P$ is module-finite and flat (cf. [9, Theorem 7.1, p.46]).

Remark 2. Let $\phi : R \longrightarrow A$ be a ring homomorphism of rings, I an ideal of A and $J = I \cap R$. The condition $I = JA$ of Theorem 2 and 3 is preserved by localization, i.e., for any prime ideal $P \subset A$ we have $JR_\mathfrak{p} = IA_P \cap R_\mathfrak{p}$ where $\mathfrak{p} = P \cap R$.

Proposition 4. *Let $A \longrightarrow B$ be a local ring homomorphism of local rings, which is flat. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be prime ideals of A. Then $\mathfrak{p}_1 = \mathfrak{p}_2$ if and only if $\operatorname{Ass}_B(B/\mathfrak{p}_1 B) = \operatorname{Ass}_B(B/\mathfrak{p}_2 B)$.*

Proof of Theorem 2. The proof follows from Bourbaki's formula and Propositition 4. □

Thanks to Theorem 1, our theorem proposes the following corollary. One can find the collection of properties for the faithfully flat and flat local ring homomorphisms in [4].

Corollary 5. *Let $(A, \mathfrak{n})$ be a Cohen-Macaulay local ring containing a field k, of dimension d, and $x_1, x_2, \dots, x_d$ a system of parameters. Let I be an ideal of A generated by polynomials over k in $x_1, x_2, \dots, x_d$. Suppose that $A/\mathfrak{n}$ is separable over k (or rather, over the image of k in $A/\mathfrak{n}$ via the natural mapping $A \to A/\mathfrak{n}$). The following statements hold for all integers $i, j \geq 0$:*

(a) *the set of associated prime ideals of $H^i_I(A)$ is finite*;
(b) *all the Bass numbers of $H^i_I(A)$ are finite.*

Example 1. Singh [10] and Katzman [3] gave the examples of rings with respect to sets of infinite associated prime ideals of the top local cohomology modules. Especially Katzman's example states that even the second local cohomology module has an infinite set of distinct associated prime ideals. The local ring R and the local cohomology module are as follows:

$$R = k[s,t,x,y,u,v]_\mathfrak{m}/(sx^2v^2 - (s+t)xyuv + ty^2u^2), \quad H^2_{(u,v)}(R),$$

where $\mathfrak{m}$ is the irrelevant maximal ideal (s,t,x,y,u,v).

On the other hand, our result states that $H^j_I(R)$ satisfies finiteness properties (a) and (b) as in Corollary 5, for all $j \geq 0$ and for the ideal I generated by polynomials $f_1 = f_1(s,x,v,t-y,t-u)$, $f_2 = f_2(s,x,v,t-y,t-u)$, $\dots$, $f_r = f_r(s,x,v,t-y,t-u)$ in $s,x,v,t-y,t-u$ over k.

Remark 3. The converse statements of (a) and (b) in Theorem 2 also hold by Proposition 4 and faithfully flatness of ϕ .

3. Several results over regular local rings

In this section, we prove part (i) and (ii) of Theorem 3.

Definition 2. A regular local ring $(R, \mathfrak{m})$ is called unramified if R contains a field or if $p \notin \mathfrak{m}^2$ in the unequal characteristic case, where p is the characteristic of the residue field $R/\mathfrak{m}$. We note that if R contains a field then the characteristic of R and its residue field are equal, and the converse also holds.

We shall introduce several lemmas.

Lemma 6. *Let $(A, \mathfrak{n})$ be a local ring with the maximal ideal $\mathfrak{n}$, M be an (not necessarily finite) A-module with support $V(\mathfrak{n})$. Let l be a non-negative integer. Suppose that M is an A-module of finite injective dimension.*

(i) *If there is an A-module N with finite length such that $\operatorname{Ext}^n_A(N, M) = 0$ for all $n \geq 1$, then M is an injective A-module*;
(ii) *if there is an A-module N with finite length such that $\operatorname{Ext}^n_A(N, M) = 0$ for all $n \geq l+1$, then* $\operatorname{inj.dim}_A M \leq l$.

Lemma 7. *Let A be a ring, and l a positive integer. Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{$\sharp$}$$

be an exact sequence of A-modules.

(1) *If M' and M'' are injective A-modules, then M is an injective A-module;*
(2) *if* $\operatorname{inj.dim}_A M' \leq l-1$ *and* $\operatorname{inj.dim}_A M'' \leq l$, *then* $\operatorname{inj.dim}_A M \leq l$.

In addition suppose that M is an injective A-module.

(3) *If M' is an injective A-module, then M'' is an injective A-module;*
(4) *if* $\operatorname{inj.dim}_A M' \leq l$, *then* $\operatorname{inj.dim}_A M'' \leq l-1$.

Lemma 8. *Let A be a ring, $\mathfrak{a}$ an ideal of A, and M an (not necessarily finite) A-module. Let l be a positive integer. Denote by I^* a minimal injective resolution of T:*

$$\begin{aligned} 0 \longrightarrow M \longrightarrow I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} I^2 \xrightarrow{\partial^2} \cdots \\ \longrightarrow I^{j-1} \xrightarrow{\partial^{j-1}} I^j \xrightarrow{\partial^j} I^{j+1} \xrightarrow{\partial^{j+1}} \cdots \end{aligned}$$

Further we denote by d^j the restriction of ∂^j to $\Gamma_{\mathfrak{a}}(I^j)$, after applying the functor $\Gamma_{\mathfrak{a}}(\,-\,)$ to I^*.

(1) *If $H^j_{\mathfrak{a}}(M)$ is an injective A-module for all $j \geq 0$, then $\ker d^j$ is an injective A-module for all $j \geq 0$, and $\operatorname{Im} d^j$ is an injective A-module for all $j \geq 0$*;
(2) *if* $\operatorname{inj.dim}_A H^j_{\mathfrak{a}}(M) \leq l$ *for all $j \geq 0$, then* $\operatorname{inj.dim}_A \ker d^j \leq l$ *for all $j \geq 0$, and* $\operatorname{inj.dim}_A \operatorname{Im} d^j \leq l-1$ *for all $j \geq 0$.*

Lemma 9. *Let $(A, \mathfrak{n})$ be a local ring and T an (not necessarily finite) A-module. Let l be a positive integer.*

(1) *If* $\operatorname{inj.dim}_{A_P} T_P \leq \dim T_P$ *for each prime ideal $P \in \operatorname{Spec}(T)$ with $P \neq \mathfrak{n}$ and $H^j_{\mathfrak{n}}(T)$ is injective for all $j \geq 0$, then T has finite injective dimension and* $\operatorname{inj.dim}_A T \leq \dim T$;
(2) *if* $\operatorname{inj.dim}_{A_P} T_P \leq \dim T_P + l$ *for each prime ideal $P \in \operatorname{Spec}(T)$ with $P \neq \mathfrak{n}$ and* $\operatorname{inj.dim}_A H^j_{\mathfrak{n}}(T) \leq l$ *for all $j \geq 0$, then T has finite injective dimension and* $\operatorname{inj.dim}_A T \leq \dim T + 2l - 1$.

Remark 4. More generally, we can slightly improve part (2) of Lemma 9 as follows: if $\operatorname{inj.dim}_{A_P} T_P \leq \dim T_P + k$ for each prime ideal $P \in \operatorname{Spec}(T)$ with $P \neq \mathfrak{n}$ and $\operatorname{inj.dim}_A H^j_{\mathfrak{n}}(T) \leq l$ for all $j \geq 0$, then T has finite injective dimension and $\operatorname{inj.dim}_A T \leq \dim T + k + l - 1$.

Proposition 10. *Let $\phi : (R, \mathfrak{m}) \longrightarrow (A, \mathfrak{n})$ be a local ring homomorphism of local rings, which is module-finite and flat, and T an R-module. Then we have $\dim_R T = \dim_A T \otimes_R A$.*

Proof of Theorem 3. We only prove (i) and (ii). By the results in [7], the assertions (iii) and (iv) in Theorem 3 follow from those of (a) and (b) in Theorem 2.

(i) First we note that $H^j_{\mathfrak{n}} H^i_I(A)$ has a finite injective dimension as an A-module by the regularity condition of A. Since the support of $H^j_{\mathfrak{n}} H^i_I(A)$ is in $V(\mathfrak{n})$, we only prove that

$$\operatorname{Ext}^p_A(A/\mathfrak{n}, H^j_{\mathfrak{n}} H^i_I(A)) = 0$$

for all $p > 1$.

By the results of Zhou [12], $H^j_{\mathfrak{m}} H^i_J(R)$ has injective dimension ≤ 1. Thus we have

$$\operatorname{Ext}^p_R(R/\mathfrak{m}, H^j_{\mathfrak{m}} H^i_J(R)) = 0$$

for all $p > 1$. Since the map $\phi : R \to A$ is a module-finite ring homomorphism (hence an integral extension), the radical of $\mathfrak{m}A$ is equal to $\mathfrak{n}$. Then it follows from flatness of ϕ that $H^j_{\mathfrak{n}} H^i_I(A) = H^j_{\mathfrak{m}A} H^i_{JA}(A) = H^j_{\mathfrak{m}} H^i_J(R) \otimes_R A$. Further, we have

$$\operatorname{Ext}^p_A(A/\mathfrak{m}A, H^j_{\mathfrak{m}A} H^i_{JA}(A)) = \operatorname{Ext}^p_R(R/\mathfrak{m}, H^j_{\mathfrak{m}} H^i_J(R)) \otimes A = 0$$

for all $p > 1$ since ϕ is flat. Since $A/\mathfrak{m}A$ is an A-module of finite length and $H^j_{\mathfrak{n}} H^i_I(A)$ has a finite injective dimension as an A-module, it follows from part (ii) of Lemma 6 that $H^j_{\mathfrak{m}A} H^i_{JA}(A)$ has injective dimension ≤ 1. Therefore the injective dimension of $H^j_{\mathfrak{n}} H^i_I(A)$ is not greater than one.

(ii) We shall show the assertion (ii) by induction on $d = \dim H^i_I(A) \geq 0$. Note that $d = \dim H^i_J(R)$ by Proposition 10.

Suppose that $d = 0$. Then the support of $H^i_I(A)$ is contained in $V(\mathfrak{n})$, so the injective dimension of $H^i_I(A) = H^0_{\mathfrak{n}}(H^i_I(A))$ is one by part (i) of the theorem.

Suppose that $d > 0$. Let $P \in \operatorname{Supp}_A(H^i_I(A))$ be a prime ideal such that P is not the maximal ideal. Set $\mathfrak{p} = P \cap R$; $\mathfrak{p}$ is not the maximal ideal of R, since the extension $R \to A$ is integral. Then the ring homomorphism $R_{\mathfrak{p}} \to A_P$ is a module finite extension and flat between regular local rings by Remark 1. The condition $I = JA$ is preserved by localization (cf. Remark 2), that is $IA_P = (JR_{\mathfrak{p}})A_P$ for a prime ideal P of A. Also, the property of a ring being unramified is preserved by localization. The dimensions of $H^i_I(A)_P$ and $H^i_J(R)_{\mathfrak{p}}$ are less than d and Proposition 10 implies that we can apply the inductive hypothesis for the local cohomology module $H^i_I(A)_P$ over A_P, that is,

$$\operatorname{inj.dim}_{A_P} H^i_I(A)_P \leq \dim H^i_I(A)_P + 1 \leq d - 1 + 1 = d.$$

Further $H^j_{\mathfrak{n}} H^i_I(A)$ has injective dimension ≤ 1 for all $j \geq 0$ by part (i) of the theorem.

So the assertion follows from Remark 4, that is $\operatorname{inj.dim}_A H^i_I(A) \leq d + 1$. The proof is completed. □

Corollary 11. *Let $(A, \mathfrak{n})$ be a complete ramified regular local ring, of dimension d, and $x_1, x_2, \ldots, x_d$ a system of parameters, where $x_1 = p$ is the characteristic of the residue field $A/\mathfrak{m}$. Suppose that I is an ideal of A generated by polynomials over $\mathbb{Z}$ in $x_2, \ldots, x_d$. Then we have the following assertions for integers $i, j \geq 0$:*

(i) $\operatorname{inj.dim}_A H^j_{\mathfrak{n}}(H^i_I(A)) \leq 1$;

(ii) $\operatorname{inj.dim}_A H^i_I(A) \le \dim H^i_I(A) + 1$;
(iii) *the set of associated prime ideals of* $H^i_I(A)$ *is finite*;
(iv) *all the Bass numbers of* $H^i_I(A)$ *are finite.*

Now we propose the following questions:

Question 1. Let i, j be non-negative integers. Let $(A, \mathfrak{n})$ be a regular local ring, I an ideal of A. Is $H^j_{\mathfrak{n}} H^i_I(A)$ injective ?

Question 2. Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be an unramified regular local ring, J an ideal of R. Is $H^j_{\mathfrak{m}} H^i_J(R)$ injective ?

If the above questions were answered affirmatively, we could prove that the upper bound of the injective dimension of local cohomology modules is its dimension over an unramified (and also any) regular local ring. Question 2 is suggested in Lyubeznik's paper [7]. We can prove this, modifying that of (iv) of Theorem 3.

Proposition 12. *Let i be a non-negative integer.*

(1) *If Question 1 has an affirmative answer for all $j \ge 0$, then* $\operatorname{inj.dim}_A H^i_I(A) \le \dim H^i_I(A)$ *holds over a regular local ring* $(A, \mathfrak{n})$ *for all ideals I of A*;
(2) *if Question 2 has an affirmative answer for all $j \ge 0$, then* $\operatorname{inj.dim}_R H^i_J(R) \le \dim H^i_J(R)$ *holds over an unramified regular local ring* $(R, \mathfrak{m})$ *for all ideal J of R.*

Although the following assertions hold not only over a regular local ring but also over other rings, we concentrate rings on regular local rings.

Example 2. Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If the dimension of I is zero, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.

Example 3. Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If the dimension of I is one, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.

Example 4. Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If I is a principal ideal, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.

Example 5. Let $(R, \mathfrak{m})$ be a regular local ring, Let i, j be non-negative integers, I an ideal of R. If I is generated by a regular sequence, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.

Proposition 13. *Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R and f is a non-zero and non-unit element of R. If I is a subideal of a principal ideal (f) up to radicals, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.*

Corollary 14. *Let i, j be non-negative integers. Let $(R, \mathfrak{m})$ be a regular local ring, I an ideal of R. If the height of I is one, then $H^j_{\mathfrak{m}} H^i_I(R)$ is injective.*

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