F-thresholds, tight closure, integral closure, and multiplicity bounds

Shunsuke Takagi (Kyushu University)

This is a joint work with Craig Huneke, Mircea Mustaţă and Kei-ichi Watanabe.

Let R be a Noetherian ring of prime characteristic p and denote by R° the set of elements of R that are not contained in any minimal prime ideal. The *tight closure* I^* of an ideal $I \subseteq R$ is defined to be the ideal of R consisting of all elements $x \in R$ for which there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all large $q = p^e$, where $I^{[q]}$ is the ideal generated by the q^{th} powers of all elements of I. The ring R is called *F-rational* if $J^* = J$ for every ideal $J \subseteq R$ generated by parameters.

Let $\mathfrak{a}$ be a fixed proper ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. To each ideal J of R such that $\mathfrak{a} \subseteq \sqrt{J}$, we associate an F-threshold as follows. For every $q = p^e$, let

$$\nu_{\mathfrak{a}}^J(q) := \max\{r \in \mathbb{N} | \mathfrak{a}^r \not\subseteq J^{[q]}\},$$

where $J^{[q]}$ is the ideal generated by the q^{th} powers of all elements of J. Since $\mathfrak{a} \subseteq \sqrt{J}$, this is a nonnegative integer (if $\mathfrak{a} \subseteq J^{[q]}$, then we put $\nu_{\mathfrak{a}}^J(q) = 0$). We put

$$\mathrm{c}_+^J(\mathfrak{a}) = \limsup_{q \to \infty} \frac{\nu_{\mathfrak{a}}^J(q)}{q}, \quad \mathrm{c}_-^J(\mathfrak{a}) = \liminf_{q \to \infty} \frac{\nu_{\mathfrak{a}}^J(q)}{q}.$$

When $\mathrm{c}_+^J(\mathfrak{a}) = \mathrm{c}_-^J(\mathfrak{a})$, we call this limit the *F-threshold* of the pair $(R, \mathfrak{a})$ (or simply of $\mathfrak{a}$) with respect to J, and we denote it by $\mathrm{c}^J(\mathfrak{a})$. The reader is referred to [3] and [4] for basic properties of F-thresholds.

Example 1. Let R be a Noetherian local ring of characteristic $p > 0$, and let $J = (x_1, \ldots, x_d)$, where $x_1, \ldots, x_d$ form a full system of parameters in R. It follows from the Monomial Conjecture that $(x_1 \cdots x_d)^{q-1} \notin J^{[q]}$ for every q. Hence $\nu_J^J(q) \geq d(q-1)$ for every q, and therefore $\mathrm{c}_-^J(J) \geq d$. On the other hand, it is easy to see that $\mathrm{c}_+^J(J) \leq d$, and we conclude that $\mathrm{c}^J(J) = d$.

We can describe the tight closure and the integral closure of parameter ideals in terms of F-thresholds.

Theorem 2. *Let $(R, \mathfrak{m})$ be a d-dimensional excellent analytically irreducible Noetherian local domain of characteristic $p > 0$, and let $J = (x_1, \ldots, x_d)$ be an ideal generated by a full system of parameters in R. Given an ideal $I \supseteq J$, we have $I \subseteq J^*$ if* *and only if* $c_+^I(J) = d$ *(and in this case* $c^I(J)$ *exists). In particular,* R *is F-rational if and only if* $c_+^I(J) < d$ *for every ideal* $I \supsetneq J$.

In order to prove Theorem 2, we start with the following lemma.

Lemma 3. *Let* $(R, \mathfrak{m})$ *be an excellent analytically irreducible Noetherian local domain of positive characteristic* p. *Set* $d = \dim(R)$, *and let* $J = (x_1, \ldots, x_d)$ *be an ideal generated by a full system of parameters in* R, *and let* $I \supseteq J$ *be another ideal. Then* I *is not contained in the tight closure* J^* *of* J *if and only if there exists* $q_0 = p^{e_0}$ *such that* $x^{q_0-1} \in I^{[q_0]}$, *where* $x = x_1 x_2 \cdots x_d$.

Proof. After passing to completion, we may assume that R is a complete local domain. Suppose first that $x^{q_0-1} \in I^{[q_0]}$, and by way of contradiction suppose also that $I \subseteq J^*$. Let $c \in R^\circ$ be a test element. Then for all $q = p^e$, one has $cx^{q(q_0-1)} \in cI^{[qq_0]} \subset J^{[qq_0]}$, so that $c \in J^{[qq_0]} : x^{q(q_0-1)} \subseteq (J^{[q]})^*$, by colon-capturing [2, Theorem 7.15a]. Therefore c^2 lies in $\bigcap_{q=p^e} J^{[q]} = (0)$, a contradiction.

Conversely, suppose that $I \not\subseteq J^*$, and choose an element $f \in I \setminus J^*$. We choose a coefficient field k, and let $B = k[[x_1, \ldots, x_d, f]]$ be the complete subring of R generated by $x_1, \ldots, x_d, f$. Note that B is a hypersurface singularity, hence Gorenstein. Furthermore, by persistence of tight closure [2, Lemma 4.11a], $f \notin ((x_1, \ldots, x_d)B)^*$. If we prove that there exists $q_0 = p^{e_0}$ such that $x^{q_0-1} \in ((x_1, \ldots, x_d, f)B)^{[q_0]}$, then clearly x^{q_0-1} is also in $I^{[q_0]}$. Hence we can reduce to the case in which R is Gorenstein. Since $I \not\subseteq J^*$, it follows from a result of Aberbach [1] that $J^{[q]} : I^{[q]} \subseteq \mathfrak{m}^{n(q)}$, where $n(q)$ is a positive integer with $\lim_{q\to\infty} n(q) = \infty$. In particular, we can find $q_0 = p^{e_0}$ such that $J^{[q_0]} : I^{[q_0]} \subseteq J$. Therefore $x^{q_0-1} \in J^{[q_0]} : J \subseteq J^{[q_0]} : (J^{[q_0]} : I^{[q_0]}) = I^{[q_0]}$, where the last equality follows from the fact that R is Gorenstein. □

Proof of Theorem 2. Note first that for every $I \supseteq J$ we have $c_+^J(I) \leq d$. Suppose now that $I \subseteq J^*$. It follows from Lemma 3 that $J^{d(q-1)} \not\subseteq I^{[q]}$ for every $q = p^e$. This gives $\nu_J^I(q) \geq d(q-1)$ for all q, and therefore $c_-^I(J) \geq d$. We conclude that in this case $c_+^I(J) = c_-^I(J) = d$.

Conversely, suppose that $I \not\subseteq J^*$. By Lemma 3, we can find $q_0 = p^{e_0}$ such that

$$\mathfrak{b} := (x_1^{q_0}, \ldots, x_d^{q_0}, (x_1 \cdots x_d)^{q_0-1}) \subseteq I^{[q_0]}.$$

If $(x_1, \ldots, x_d)^r \not\subseteq \mathfrak{b}^{[q]}$, then

$$r \leq (qq_0 - 1)(d-1) + q(q_0 - 1) - 1 = qq_0 d - q - d.$$

Therefore $\nu_J^{\mathfrak{b}}(q) \leq qq_0 d - q - d$ for every q, which implies $c^{\mathfrak{b}}(J) \leq q_0 d - 1$. Since q_0 is a fixed power of p, we deduce

$$c_+^I(J) = \frac{1}{q_0} c_+^{I^{[q_0]}}(J) \leq \frac{1}{q_0} c^{\mathfrak{b}}(J) \leq d - \frac{1}{q_0} < d.$$

□

Theorem 4. *Let $(R,\mathfrak{m})$ be a d-dimensional formally equidimensional Noetherian local ring of characteristic $p>0$. If I and J are ideals in R, with J generated by a full system of parameters, then*

(1) $\mathrm{c}_+^J(I)\le d$ *if and only if* $I\subseteq \overline{J}$.

(2) *If, in addition,* $J\subseteq I$, *then* $I\subseteq\overline{J}$ *if and only if* $\mathrm{c}_+^J(I)=d$.

Proof. Note that if $J\subseteq I$, then $\mathrm{c}_-^J(I)\ge \mathrm{c}_-^J(J)=\mathrm{c}^J(J)=d$, by Example 1. Hence both assertions in (2) follow from the assertion in (1).

One implication in (1) is easy: if $I\subseteq\overline{J}$, then we have $\mathrm{c}_+^J(I)\le \mathrm{c}_+^J(\overline{J})=\mathrm{c}^J(J)=d$. Conversely, suppose that $\mathrm{c}_+^J(I)\le d$. In order to show that $I\subseteq\overline{J}$, we may assume that R is complete and reduced. Indeed, first note that the inverse image of $\overline{J\widehat{R}_{\mathrm{red}}}$ in R is contained in $\overline{J}$, hence it is enough to show that $I\widehat{R}_{\mathrm{red}}\subseteq\overline{J\widehat{R}_{\mathrm{red}}}$. Since $J\widehat{R}_{\mathrm{red}}$ is again generated by a full system of parameters, and since we trivially have

$$\mathrm{c}^{J\widehat{R}_{\mathrm{red}}}(I\widehat{R}_{\mathrm{red}})\le \mathrm{c}^J(I)\le d,$$

we may replace R by $\widehat{R}_{\mathrm{red}}$.

Since R is complete and reduced, we can find a test element c for R.

Claim. Let $\mathfrak{a},\mathfrak{b}$ be ideals of R such that $\mathfrak{a}\subseteq\sqrt{\mathfrak{b}}$. Then $\mathrm{c}_+^{\mathfrak{b}}(\mathfrak{a})\le\alpha$ if and only if for every power q_0 of p, we have $\mathfrak{a}^{\lceil\alpha q\rceil+q/q_0}\subseteq\mathfrak{b}^{[q]}$ for all $q=p^e\gg q_0$.

Proof of Claim. First, assume that $\mathrm{c}_+^{\mathfrak{b}}(\mathfrak{a})\le\alpha$. By the definition of $\mathrm{c}_+^{\mathfrak{b}}(\mathfrak{a})$, for any power q_0 of p, there exists q_1 such that $\nu_{\mathfrak{a}}^{\mathfrak{b}}(q)/q<\alpha+1/q_0$ for all $q=p^e\ge q_1$. Thus, $\nu_{\mathfrak{a}}^{\mathfrak{b}}(q)\lneq\lceil\alpha q\rceil+q/q_0$, that is, $\mathfrak{a}^{\lceil\alpha q\rceil+q/q_0}\subseteq\mathfrak{b}^{[q]}$ for all $q=p^e\ge q_1$. For the converse implication, note that by assumption, $\nu_{\mathfrak{a}}^{\mathfrak{b}}(q)\le\lceil\alpha q\rceil+q/q_0-1$ for all large $q=p^e\gg q_0$. Dividing by q and taking the limit gives $\mathrm{c}_+^{\mathfrak{b}}(\mathfrak{a})\le\alpha+1/q_0$. Since q_0 is any power of p, we can conclude that $\mathrm{c}_+^{\mathfrak{b}}(\mathfrak{a})\le\alpha$. □

By the above claim, the assumption $\mathrm{c}_+^J(I)\le d$ implies that for all $q_0=p^{e_0}$ and for all large $q=p^e$, we have

$$I^{q(d+(1/q_0))}\subseteq J^{[q]}.$$

Hence $I^qJ^{q(d-1+(1/q_0))}\subseteq J^{[q]}$, and thus

$$I^q\subseteq J^{[q]}\colon J^{q(d-1+(1/q_0))}\subseteq\left(J^{q-d+1-(q/q_0)}\right)^*,$$

where the last containment follows from the colon-capturing property of tight closure [2, Theorem 7.15a]. We get $cI^q\subseteq cR\cap J^{q-d+1-(q/q_0)}\subseteq cJ^{q-d+1-(q/q_0)-l}$ for some fixed integer l that is independent of q, by the Artin-Rees lemma. Since c is a non-zero divisor in R, it follows that

$$I^q\subseteq J^{q-d+1-(q/q_0)-l}. \qquad (1)$$

If ν is a discrete valuation with center in $\mathfrak{m}$, we may apply ν to (1) to deduce $q\nu(I) \geq \left(q - d + 1 - \frac{q}{q_0} - l\right)\nu(J)$. Dividing by q and letting q go to infinity gives $\nu(I) \geq \left(1 - \frac{1}{q_0}\right)\nu(J)$. We now let q_0 go to infinity to obtain $\nu(I) \geq \nu(J)$. Since this holds for every ν, we have $I \subseteq \overline{J}$. $\square$

Two years ago (at the 27$^{\text{th}}$ Symposium on Commutative Algebra in Japan), we proposed the following conjecture, generalizing a result in [5].

Conjecture 5 (cf. [6, Conjecture 3.2]). *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring of characteristic $p > 0$. If $J \subseteq \mathfrak{m}$ is an ideal generated by a full system of parameters, and if $\mathfrak{a} \subseteq \mathfrak{m}$ is an $\mathfrak{m}$-primary ideal, then*

$$e(\mathfrak{a}) \geq \left(\frac{d}{\mathrm{c}_{-}^{J}(\mathfrak{a})}\right)^{d} e(J).$$

Example 6. Let $R = k[\![X, Y, Z]\!]/(X^2 + Y^3 + Z^5)$ be a rational double point of type E_8, with k a field of characteristic $p > 0$. Let $\mathfrak{a} = (x, z)$ and $J = (y, z)$. Then $e(\mathfrak{a}) = 3$ and $e(J) = 2$. It is easy to check that $\mathrm{c}^J(\mathfrak{a}) = 5/3$ and $\mathrm{c}^{\mathfrak{a}}(J) = 5/2$. Thus,

$$e(\mathfrak{a}) = 3 > \frac{72}{25} = \left(\frac{2}{\mathrm{c}^{J}(\mathfrak{a})}\right)^{2} e(J),$$
$$e(J) = 2 > \frac{48}{25} = \left(\frac{2}{\mathrm{c}^{\mathfrak{a}}(J)}\right)^{2} e(\mathfrak{a}).$$

Two years ago, we reported the following result as an evidence of Conjecture 5.

Theorem 7 ([6, Proposition 3.3]). *If $(R, \mathfrak{m})$ is a regular local ring of characteristic $p > 0$ and $J = (x_1^{a_1}, \dots, x_d^{a_d})$, with $x_1, \dots, x_d$ a full regular system of parameters for R, and with $a_1, \dots, a_d$ positive integers, then the inequality given by Conjecture 5 holds.*

We will conclude this article with a result related to the graded version of Conjecture 5.

Theorem 8. *Let $R = \bigoplus_{d \geq 0} R_d$ be an n-dimensional graded Cohen-Macaulay ring with R_0 a field of characteristic $p > 0$. If $\mathfrak{a}$ and J are ideals generated by full homogeneous systems of parameters for R, then*

$$e(\mathfrak{a}) \geq \left(\frac{n}{\mathrm{c}_{-}^{J}(\mathfrak{a})}\right)^{n} e(J).$$

Proof. Suppose that $\mathfrak{a}$ is generated by a full homogeneous system of parameters $x_1, \dots, x_n$ of degrees $a_1 \leq \cdots \leq a_n$ and J is generated by another homogeneous system of parameters $f_1, \dots, f_n$ of degrees $d_1 \leq \cdots \leq d_n$. Fix a power $q = p^e$ of p, and define the nonnegative integers $t_1^{(e)}, \dots, t_{n-1}^{(e)}$ inductively as follows: $t_1^{(e)}$ is the least integer t such that $x_1^t \in J^{[q]}$. If $i \geq 2$, then $t_i^{(e)}$ is the least integer t such that $x_1^{t_1^{(e)}-1} \cdots x_{i-1}^{t_{i-1}^{(e)}-1} x_i^t \in J^{[q]}$. We also define the integer $N^{(e)}$ to be the least integer N such that $I^N \subseteq J^{[q]}$. Note that $N^{(e)}$ is greater than $t_1^{(e)} + \cdots + t_{n-1}^{(e)} - n + 1$. Since the lim sup of the ratios $(N^{(e)} + n - 1)/p^e$ is $\mathrm{c}_+^J(\mathfrak{a})$, it suffices to prove that

$$(N^{(e)} + n - 1)^n a_1 \cdots a_n \geq n^n q^n d_1 \cdots d_n.$$

First, we will show the following inequality for every $i = 1, \dots, n-1$:

$$t_1^{(e)} a_1 + \cdots + t_i^{(e)} a_i \geq q(d_1 + \cdots + d_i). \tag{2}$$

Let $I_i^{(e)}$ be the ideal of R generated by $x_1^{t_1^{(e)}}, x_1^{t_1^{(e)}-1} x_2^{t_2^{(e)}}, \dots, x_1^{t_1^{(e)}-1} \cdots x_{i-1}^{t_{i-1}^{(e)}-1} x_i^{t_i^{(e)}}$. By the definition of $t_1^{(e)}, \dots, t_i^{(e)}$, we have that $I_i^{(e)} \subseteq J^{[q]}$. The natural surjection of $R/I_i^{(e)}$ onto $R/J^{[q]}$ induces a comparison map between the minimal free resolutions. Looking at the i^{th} free modules, we have the map

$$R(-t_1^{(e)} a_1 - \cdots - t_i^{(e)} a_i) \to \bigoplus_{1 \leq v_1 \lneq \cdots \lneq v_i \leq n} R(-q d_{v_1} - \cdots - q d_{v_i}).$$

In particular, unless this map is zero, $t_1^{(e)} a_1 + \cdots + t_i^{(e)} a_i$ must be at least as large as the minimum of the twists, which is $q(d_1 + \cdots + d_i)$. So it remains to see the reason why this map cannot be zero. Assume it is zero: then the map

$$\mathrm{Tor}_i^R(R/I_i^{(e)}, R/\mathfrak{b}_i) \to \mathrm{Tor}_i^R(R/J^{[q]}, R/\mathfrak{b}_i)$$

will be zero, where $\mathfrak{b}_i$ is the ideal generated by $x_1, \dots, x_i$. On the other hand, using the Koszul complex on $x_1, \dots, x_i$, we see that this map can be identified with the natural map

$$(I_i^{(e)} : \mathfrak{b}_i)/I_i^{(e)} \to (J^{[q]} : \mathfrak{b}_i)/J^{[q]}.$$

Since the ideal $I_i^{(e)} : \mathfrak{b}_i$ is generated by $x_1^{t_1^{(e)}-1} \cdots x_i^{t_i^{(e)}-1}$ modulo $I_i^{(e)}$, the map is zero if and only if $x_1^{t_1^{(e)}-1} \cdots x_i^{t_i^{(e)}-1}$ is in $J^{[q]}$. However, this contradicts the definition of $t_i^{(e)}$.

Next, we will prove the following estimate:

$$t_1^{(e)} a_1 + \cdots + t_{n-1}^{(e)} a_{n-1} + (N^{(e)} - t_1^{(e)} - \cdots - t_{n-1}^{(e)} + n - 1) a_n \geq q(d_1 + \cdots + d_n). \tag{3}$$

Since $\mathfrak{a}^{N^{(e)}} \subseteq J^{[q]}$, we have that

$$(x_1^{N^{(e)}}, \dots, x_n^{N^{(e)}}) : J^{[q]} \subseteq (x_1^{N^{(e)}}, \dots, x_n^{N^{(e)}}) : \mathfrak{a}^{N^{(e)}} = (x_1^{N^{(e)}}, \dots, x_n^{N^{(e)}}) + \mathfrak{a}^{(n-1)(N^{(e)}-1)}.$$

The ideal $(x_1^{N^{(e)}}, \dots, x_n^{N^{(e)}}) : J^{[q]}$ is of the form $(x_1^{N^{(e)}}, \dots, x_n^{N^{(e)}}, y^{(e)})$, where the extra generator $y^{(e)}$ has degree $N^{(e)}(a_1 + \cdots + a_n) - q(d_1 + \cdots + d_n)$. We write

$$y^{(e)} = \sum_{m_1 + \cdots + m_n = (n-1)(N^{(e)}-1)} r_{m_1 \dots m_n} x_1^{m_1} \dots x_n^{m_n}$$

modulo $(x_1^{N^{(e)}}, \ldots, x_n^{N^{(e)}})$. Since $x_1^{t_1^{(e)}-1} \cdots x_{n-1}^{t_{n-1}^{(e)}-1}$ is not in $J^{[q]}$, we see that $y^{(e)}$ is not in $(x_1^{N^{(e)}-t_1^{(e)}+1}, \ldots, x_{n-1}^{N^{(e)}-t_{n-1}^{(e)}+1}, x_n^{N^{(e)}})$. To check this, suppose that $y^{(e)}$ is in $(x_1^{N^{(e)}-t_1^{(e)}+1}, \ldots, x_{n-1}^{N^{(e)}-t_{n-1}^{(e)}+1}, x_n^{N^{(e)}})$. Then $J^{[q]} = (x_1^{N^{(e)}}, \ldots, x_n^{N^{(e)}}) : y^{(e)}$ will contain $(x_1^{N^{(e)}}, \ldots, x_n^{N^{(e)}}) : (x_1^{N^{(e)}-t_1^{(e)}+1}, \ldots, x_{n-1}^{N^{(e)}-t_{n-1}^{(e)}+1}, x_n^{N^{(e)}}) \ni x_1^{t_1^{(e)}-1} \cdots x_{n-1}^{t_{n-1}^{(e)}-1}$. Thus, some $r_{m_1 \ldots m_n}$ must be nonzero, where $m_i \le N^{(e)} - t_i^{(e)}$ for $1 \le i \le n-1$ and $m_n \le N^{(e)} - 1$. Since the degree of D is greater than or equal to the minimal degree of monomials $x_1^{m_1} \ldots x_n^{m_n}$ with $r_{m_1 \ldots m_n}$ nonzero, we can conclude that

$$\begin{aligned}\deg D &= N^{(e)}(a_1 + \cdots + a_n) - q(d_1 + \cdots + d_n) \\ &\ge (N^{(e)} - t_1^{(e)})a_1 + \cdots + (N^{(e)} - t_{n-1}^{(e)})a_{n-1} + (t_1^{(e)} + \cdots + t_{n-1}^{(e)} - n + 1)a_n,\end{aligned}$$

which implies the desired estimate.

To finish the proof, we will use the following claim.

Claim. Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be two n-tuple of real numbers, and let $1 = \gamma_1 \le \gamma_2 \le \cdots \le \gamma_n$ be another one. Assume that $\gamma_1\alpha_1 + \cdots + \gamma_i\alpha_i \ge \gamma_1\beta_1 + \cdots + \gamma_i\beta_i$ for all $i = 1, \ldots, n$. Then $\alpha_1 + \cdots + \alpha_n \ge \beta_1 + \cdots + \beta_n$.

Proof of Claim. Let $\lambda_i = \alpha_i - \beta_i$ for $1 \le i \le n$. Then $\gamma_1\lambda_1 + \cdots + \gamma_i\lambda_i \ge 0$ for all $i = 1, \ldots, n$. We will prove that $\lambda_1 + \cdots + \lambda_n \ge 0$ by induction on n. We may assume that n is greater than one. The assertion is obvious if every $\lambda_i \ge 0$. Suppose that $\lambda_i < 0$ for some i. Clearly $i \ge 2$. Since $\gamma_i \ge \gamma_{i-1}$, it follows from that $\gamma_i\lambda_i \le \gamma_{i-1}\lambda_i$. We then define $\gamma'_j = \gamma_j$ for $1 \le j \le i-1$ and $\gamma'_j = \gamma_{j+1}$ for $i \le j \le n-1$. Define also $\lambda'_j = \lambda_j$ for $1 \le j \le i-2$, $\lambda'_{i-1} = \lambda_{i-1} + \lambda_i$ and $\lambda'_j = \lambda_{j+1}$ for $i \le j \le n-1$. Since $\gamma'_1\lambda'_1 + \cdots + \gamma'_j\lambda'_j \ge 0$ for all $j = 1, \ldots, n-1$, the induction hypothesis implies that $\lambda_1 + \cdots + \lambda_n = \lambda'_1 + \cdots + \lambda'_{n-1} \ge 0$. □

Set $\alpha_i = t_i^{(e)}$ for $1 \le i \le n-1$ and $\alpha_n = N^{(e)} - t_1^{(e)} - \cdots - t_{n-1}^{(e)} + n - 1$. Set $\beta_i = qd_i/a_i$ and $\gamma_i = a_i/a_1$ for $1 \le i \le n$. Then $\gamma_1 \le \cdots \le \gamma_n$, because $a_1 \le \cdots \le a_n$. The inequalities $\gamma_1\alpha_1 + \cdots + \gamma_i\alpha_i \ge \gamma_1\beta_1 + \cdots + \gamma_i\beta_i$ for $1 \le i \le n$ follow from the estimates (1) and (2). Using the above claim, we can conclude that

$$N^{(e)} + n - 1 = \alpha_1 + \cdots + \alpha_n \ge \beta_1 + \ldots \beta_n = q\left(\frac{d_1}{a_1} + \cdots + \frac{d_n}{a_n}\right).$$

Comparing the arithmetic and geometric means of $\{qd_i/a_i\}_i$, we see that

$$(N^{(e)} + n - 1)^n a_1 \ldots a_n \ge n^n q^n d_1 \ldots d_n.$$

□

Remark 9. Theorem 8 does not imply the graded (Cohen-Macaulay) version of Conjecture 5, because a minimal reduction of an R_+-primary homogeneous ideal is not necessarily homogeneous.

References

[1] I. M. Aberbach, Extensions of weakly and strongly F-regular rings by flat maps, J. Algebra **241** (2001), 799–807.

[2] M. Hochster and C. Huneke, Tight closure, invariant theory and the Briançon-Skoda theorem, J. Amer. Math. Soc. **3** (1990), 31–116.

[3] C. Huneke, M. Mustaţă, S. Takagi and K.-i. Watanabe, F-thresholds, tight closure, integral closure, and multiplicity bounds, arXiv:0708.2394, to appear in Michigan Math. J.

[4] M. Mustaţă, S. Takagi and K.-i. Watanabe, F-thresholds and Bernstein-Sato polynomials. European congress of mathematics, 341–364, Eur. Math. Soc., Zürich, 2005.

[5] S. Takagi and K.-i. Watanabe, On F-pure thresholds, J. Algebra **282** (2004), 278–297.

[6] S. Takagi and K.-i. Watanabe, F-thresholds and multiplicities, Proceedings of "the 27th Symposium on Commutative Algebra in Japan," Toyama, 2005, 54–60 (Japanese).

Department of Mathematics, Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-8581, Japan
E-mail address: `stakagi@math.kyushu-u.ac.jp`