Local cohomology on diagrams of schemes

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Dedicated to Professor Melvin Hochster on the occasion of his sixty-fifth birthday

Abstract

We define local cohomology on a diagram of schemes, and discuss basic properties of it. In particular, we prove the independence theorem and the flat base change for local cohomology on diagrams of schemes. We also introduce the notion of G-localness of a G-scheme. It is an equivariant analogue of localness of a scheme. As an application, we prove that Cohen–Macaulay property is inherited by the affine geometric quotient under the action of a linearly reductive group scheme over a field. This generalizes the special case (that the ring in question contains a field) of the theorem of Hochster and Eagon on the Cohen–Macaulay property of an invariant subring under the action of a finite group.

1. Introduction

Let S be a scheme, G a flat S-group scheme, and X a G-scheme (i.e., an S-scheme on which G acts). In [18], a G-linearization of an invertible sheaf on X is defined. As quasi-coherent sheaves are important in studying a scheme, G-linearized quasi-coherent sheaves are important in studying a scheme with a group action. If S, G, and $X = \operatorname{Spec} A$ are all affine, then the category $\operatorname{Lin}(G, X)$ of G-linearized quasi-coherent sheaves on X is equivalent to the category of (G, A)-modules, see [8]. In particular, if $S = \operatorname{Spec} k = X$ with k a field, then $\operatorname{Lin}(G, X)$ is equivalent to the category of G-modules. However, the definition of a G-linearization in [18] is complicated, and probably it is difficult to study the homological algebra of $\mathrm{Lin}(G, X)$ only from the definition. In [9], the diagram $B_G^M(X)$ of schemes is defined, and the category of quasi-coherent sheaves $\mathrm{Qch}(G, X) = \mathrm{Qch}(B_G^M(X))$ is studied. Note that $\mathrm{Lin}(G, X)$ and $\mathrm{Qch}(G, X)$ are equivalent. The category $\mathrm{Qch}(X)$ of quasi-coherent sheaves on X is embedded in the category of $\mathcal{O}_X$-modules $\mathrm{Mod}(X)$, and this embedding gives some flexibility to the homological algebra of $\mathrm{Qch}(X)$. Similarly, $\mathrm{Qch}(G, X)$ is embedded in $\mathrm{Mod}(G, X) := \mathrm{Mod}(B_G^M(X))$, and the homological algebra of $\mathrm{Qch}(G, X)$ is considered in $\mathrm{Mod}(G, X)$. Note that $B_G^M(X)$ is a diagram of schemes of the form

$$G \times_S G \times_S X \begin{array}{c} \xrightarrow{1_G \times a} \\ \xrightarrow{\mu \times 1_X} \\ \xrightarrow{p_{23}} \end{array} G \times_S X \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{p_2} \end{array} X,$$

where $a : G \times_S X \to X$ is the action, $\mu : G \times_S G \to G$ is the product, and p_2 and p_{23} are appropriate projections. Thus the study of sheaves on diagrams of schemes is important to study $\mathrm{Lin}(G, X)$.

Local cohomology is a powerful tool in commutative ring theory. Especially, the local cohomology $H^i_{\mathfrak{m}}$ on a local ring $(A, \mathfrak{m})$ is very important. However, when we consider a group action, "local phenomena" sometimes occur on non-affine schemes, see Example 8.19. Thus, to construct a theory of equivariant local cohomology, it seems that we need to discuss local cohomology on diagrams of not necessarily affine schemes.

The objective of this paper is to give foundations of local cohomology on diagrams of schemes, and give an application to invariant theory. We also introduce the notion of G-localness of a G-scheme.

The local cohomology is a derived functor of the local section functor $\underline{\Gamma}_{U,V}$ for a pair of open subdiagrams of schemes U and V of a diagram of schemes X, such that $U \supset V$. As in the usual single schemes case, $\underline{\Gamma}_{U,V}$ depends only on $U \setminus V$. However, it is interesting to point out that $U \setminus V$ may not be a subdiagram of schemes. Moreover, not all families of locally closed subsets (Z_i) of X_i can be expressed as $Z_i = U_i \setminus V_i$ for a pair of open subdiagrams U and V such that $U \supset V$.

As unbounded homological algebra is getting more and more important, we discuss unbounded derived functor of $\underline{\Gamma}_{U,V}$. We introduce the notion of K-flabby property over a diagram of schemes.

Section 2 is preliminaries. Problems of commutativity of diagrams is inevitable in studying sheaves over diagrams of schemes. In section 3, we prove some basic commutativity of diagrams. We also prove that for a locally quasi-coherent sheaf over a locally noetherian diagram of schemes, the local section functor can be expressed in terms of some inductive limit of hom functors. In section 4, we discuss local cohomology. We discuss K-flabby property. In section 5, we slightly modify and discuss Kempf's quasi-flabby property. In section 6, we state and prove the flat base change. In the theory of local cohomology for the usual single schemes, the flat base change and the independence theorem (see Corollary 4.17) are important. We generalize and prove these theorems. In section 7, we consider the group action. We consider the local cohomology with a group action, the *equivariant local cohomology.* This is realized as a cohomology on the diagram of schemes $B_G^M(X)$. We prove that the local section functor $\underline{\Gamma}_{U,V}$ is compatible with the G-invariance functor. In section 8, we define an equivariant version of a local scheme, a *G-local G-scheme* (Definition 8.13), give some examples, and prove some basic properties. It seems that this notion has some importance in invariant theory, since if G is a (strongly) geometrically reductive k-group scheme (see for the definition, (8.22)), A a G-algebra, and $\mathfrak{p} \in \operatorname{Spec} A^G$, then $A_{\mathfrak{p}} := A \otimes_{A^G} A^G_{\mathfrak{p}}$ is G-local (Proposition 8.27).

In section 9, applying equivariant local cohomology on a G-local G-scheme, we prove

Theorem 9.5 *Let k be a field, G a linearly reductive k-group scheme, and X a Cohen–Macaulay noetherian G-scheme. Let $\pi : X \to Y$ be a geometric quotient under the action of G in the sense of* [18]. *Assume that π is an affine morphism. Then Y is noetherian and Cohen–Macaulay.*

As we will show, this is a generalization of the special case (that the ring in question contains a field) of the theorem of Hochster and Eagon on the Cohen–Macaulay property of the invariant subrings under the action of finite groups [11, Proposition 13].

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2. Preliminaries

(2.1) We use the notation, terminology, and results from [9] freely (however, see (2.11)).

(2.2) Let $f : \mathbb{Y} \to \mathbb{X}$ be a ringed continuous functor between ringed sites as defined in [9, (2.3), (2.4), (2.19)]. As in [9], let $\mathrm{PM}(\mathbb{X})$ and $\mathrm{Mod}(\mathbb{X})$ respectively denote the category of presheaves and sheaves of $\mathcal{O}_{\mathbb{X}}$-modules. Let $f^{\#}_{\heartsuit} : \heartsuit(\mathbb{X}) \to \heartsuit(\mathbb{Y})$ denote the canonical pull-back, and $f_{\#}^{\heartsuit} : \heartsuit(\mathbb{Y}) \to \heartsuit(\mathbb{X})$ denote its left adjoint, where $\heartsuit$ denotes either PM or Mod.

For $b, c \in \mathrm{Mod}(\mathbb{Y})$, $\Delta_{\mathrm{Mod}} : f_{\#}^{\mathrm{Mod}}(b \otimes c) \to f_{\#}^{\mathrm{Mod}} b \otimes f_{\#}^{\mathrm{Mod}} c$ is the composite of the following, see [9, (1.40), (2.19), (2.20), (2.52)]:

$$
\begin{aligned}
f_{\#}(b \otimes c) = af_{\#}qa(qb \otimes^p qc) &\xrightarrow{\varepsilon^{-1} \otimes^p \varepsilon^{-1}} af_{\#}qa(qaqb \otimes^p qaqc) \\
\xrightarrow{u \otimes^p u} af_{\#}qa(qaf^{\#}f_{\#}qb \otimes^p qaf^{\#}f_{\#}qc) &\xrightarrow{\theta \otimes^p \theta} af_{\#}qa(qf^{\#}af_{\#}qb \otimes qf^{\#}af_{\#}qc) \\
\xrightarrow{c \otimes^p c} af^{\#}qa(f^{\#}qaf_{\#}qb \otimes^p f^{\#}qaf_{\#}qc) &\xrightarrow{m_{\mathrm{PM}}} af^{\#}qaf^{\#}(qaf_{\#}qb \otimes^p qaf_{\#}qc) \\
\xrightarrow{\theta} af_{\#}qf^{\#}a(qaf_{\#}qb \otimes^p qaf_{\#}qc) &\xrightarrow{c} af_{\#}f^{\#}qa(qaf_{\#}qb \otimes^p qaf_{\#}qc) \\
\xrightarrow{\varepsilon} aqa(qaf_{\#}qb \otimes^p qaf_{\#}qc) &\xrightarrow{\varepsilon} a(qaf_{\#}qb \otimes^p qaf_{\#}qc) = f_{\#}b \otimes f_{\#}c.
\end{aligned}
$$

See for the notation, [9, section 2]. It is easy to see that this composite map agrees with

$$
\begin{aligned}
f_{\#}(b \otimes c) = af_{\#}qa(qb \otimes^p qc) &\xrightarrow{u^{-1}} af_{\#}(qb \otimes^p qc) \xrightarrow{\Delta_{\mathrm{PM}}} \\
a(f_{\#}qb \otimes^p f_{\#}qc) &\xrightarrow{u \otimes^p u} a(qaf_{\#}qb \otimes^p qaf_{\#}qc) = f_{\#}b \otimes f_{\#}c,
\end{aligned}
$$

see [9, Lemma 2.34].

2.3 Lemma. *Let the notation be as above. If $\Delta_{\mathrm{PM}} : f_{\#}(qb \otimes^p qc) \to f_{\#}qb \otimes^p f_{\#}qc$ is an isomorphism, then so is $\Delta_{\mathrm{Mod}} : f_{\#}(b \otimes c) \to f_{\#}b \otimes f_{\#}c$.*

Proof. Follows immediately by the discussion above and [9, Lemma 2.18]. $\square$

(2.4) The map $\Gamma(x, \Delta_{\mathrm{PM}})$, the map Δ_{PM} at the section at $x \in \mathbb{X}$, is given as follows. It is the map

$$
\begin{aligned}
\Gamma(x, f_{\#}(b \otimes^p c)) = \varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y, \mathcal{O}_{\mathbb{Y}})} (\Gamma(y, b) \otimes_{\Gamma(y, \mathcal{O}_{\mathbb{Y}})} \Gamma(y, c)) \\
\to \Gamma(x, f_{\#}b \otimes^p f_{\#}c) = \\
(\varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y', \mathcal{O}_{\mathbb{Y}})} \Gamma(y', b)) \otimes_{\Gamma(x, \mathcal{O}_{\mathbb{X}})} (\varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y'', \mathcal{O}_{\mathbb{Y}})} \Gamma(y'', c))
\end{aligned}
$$

given by $\alpha \otimes (\beta \otimes \gamma) \mapsto (\alpha \otimes \beta) \otimes (1 \otimes \gamma)$, where the colimits are taken over $y, y', y'' \in (I^f_x)^{\mathrm{op}}$, respectively. This description is obtained from the definition of Δ [9, (1.40)] and the explicit descriptions of u, m, and ε in [9, (2.20), (2.50)]. If the category $(I_x^f)^{\mathrm{op}}$ (cf. [9, (2.6)]) is filtered, then mapping $(\alpha\otimes\beta)\otimes(\alpha'\otimes\gamma)$ to $\alpha\alpha'\otimes(\beta\otimes\gamma)$, the inverse of $\Gamma(x,\Delta_{\mathrm{PM}})$ is given explicitly. Thus we have:

2.5 Lemma. *Assume that $(I_x^f)^{\mathrm{op}}$ is filtered for every $x\in\mathbb{X}$. Then $\Delta_{\mathrm{PM}}: f_\#(b\otimes^p c)\to f_\# b\otimes^p f_\# c$ is an isomorphism for $b,c\in\mathrm{PM}(\mathbb{Y})$. Hence $\Delta_{\mathrm{Mod}}: f_\#(b'\otimes c')\to f_\# b'\otimes f_\# c'$ is an isomorphism for $b',c'\in\mathrm{Mod}(\mathbb{Y})$.*

Note also that $C: f^\#\mathcal{O}_{\mathbb{Y}}\to\mathcal{O}_{\mathbb{X}}$ is also an isomorphism, if $(I_x^f)^{\mathrm{op}}$ is filtered for every $x\in\mathbb{X}$.

(2.6) For $b,c\in\mathrm{Mod}(\mathbb{X})$, the evaluation map $\mathrm{ev}:[b,c]\otimes b\to c$ is the composite

$$[b,c]\otimes b = a(q[b,c]\otimes^p qc)\xrightarrow{\bar H} a([qb,qc]\otimes^p qc)\xrightarrow{\mathrm{ev_{PM}}} aqc\xrightarrow{\varepsilon} c,$$

where $[b,c]$ denotes $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{X}}}(b,c)$ and so on.

(2.7) Let the notation be as in (2.2). Assume that for any $x\in\mathbb{X}$, the category $(I_x^f)^{\mathrm{op}}$ is filtered. Then by Lemma 2.5, Δ_{PM} and Δ_{Mod} are isomorphisms. Thus $P: f_\#[b,c]\to[f_\# b, f_\# c]$ is defined for $b,c\in\mathrm{Mod}(\mathbb{Y})$, see [9, (1.50)]. It is the composite

$$\begin{aligned}
f_\#[b,c] = af_\# q[b,c] &\xrightarrow{\varepsilon^{-1}} aqaf_\# q[b,c]\xrightarrow{\mathrm{tr}} a[qaf_\# qb, qaf_\# q[b,c]\otimes^p qaf_\# qb]\\
&\xrightarrow{\bar P}[aqaf_\# qb, a(qaf_\# q[b,c]\otimes^p qaf_\# qb)]\xrightarrow{\varepsilon^{-1}}[af_\# qb, a(qaf_\# q[b,c]\otimes^p qaf_\# qb)]\\
&\xrightarrow{(u\otimes^p u)^{-1}}[af_\# qb, a(f_\# q[b,c]\otimes^p f_\# qb)]\xrightarrow{\Delta^{-1}}[af_\# qb, af_\#(q[b,c]\otimes qb)]\\
&\xrightarrow{u}[af_\# qb, af_\# qa(q[b,c]\otimes qb)]\xrightarrow{H}[af_\# qb, af_\# qa([qb,qc]\otimes qb)]\\
&\xrightarrow{\mathrm{ev}}[af_\# qb, af_\# qaqc]\xrightarrow{\varepsilon}[af_\# qb, af_\# qc]=[f_\# b, f_\# c]
\end{aligned}$$

by [9, (2.48)] and (2.2). It is straightforward to check that this composite map agrees with

$$\text{(1)}\quad f_\#[b,c] = af_\# q[b,c]\xrightarrow{\bar H} af_\#[qb,qc]\xrightarrow{P} a[f_\# qb, f_\# qc]\xrightarrow{\bar P}[af_\# qb, af_\# qc]=[f_\# b, f_\# c].$$

(2.8) Let $\mathbb{X}$ be as in (2.7). By the definition of P ([9, (1.50)]) and the explicit descriptions of tr, Δ, and ev in [9, (2.42)], (2.4), and [9, (2.41)], P_{PM} is described as follows. It is the map

$$\begin{aligned}\Gamma(x, f_\#[b,c]) = \varinjlim \Gamma(x, \mathcal{O}_{\mathbb{X}}) \otimes_{\Gamma(y,\mathcal{O}_{\mathbb{Y}})} \operatorname{Hom}_{\mathcal{O}_{\mathbb{Y}/y}}(b|_y, c|_y) \\ \to \operatorname{Hom}_{\mathcal{O}_{\mathbb{X}/x}}((f_\# b)|_x, (f_\# c)|_x) = \Gamma(x, [f_\# b, f_\# c])\end{aligned}$$

which sends $\beta \otimes \varphi$ to the map which sends $\beta' \otimes \alpha$ to $\beta\beta' \otimes \varphi(\alpha)$ for $\beta \in \Gamma(x, \mathcal{O}_{\mathbb{X}})$, $\varphi : b|_y \to c|_y$, $\beta' \in \Gamma(x', \mathcal{O}_{\mathbb{X}})$, and $\alpha \in \Gamma(y', b)$ for some commutative diagram

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & fy' \arrow[d, "f\rho"] & y' \arrow[d, "\rho"] \\ x \arrow[r] & fy & y, \end{tikzcd}$$

where the colimit is taken over $y \in (I_x^f)^{\mathrm{op}}$.

Thus we get

2.9 Lemma. *Let $j : U \to X$ be an open immersion of ringed spaces. Then $P : j^*[b,c] \to [j^*b, j^*c]$ is an isomorphism for $b, c \in \heartsuit(X)$ for $\heartsuit = \mathrm{PM}, \mathrm{Mod}$.*

Proof. First consider the case that $\heartsuit = \mathrm{PM}$. Then for $V \subset U$, $\Gamma(V, P) : \Gamma(V, j^*[b,c]) \to \Gamma(V, [j^*b, j^*c])$ is the identity map of $\operatorname{Hom}_{\mathcal{O}_V}(b|_V, c|_V)$. Thus it is an isomorphism.

Now consider the case that $\heartsuit = \mathrm{Mod}$. Then P_{Mod} is the composite

$$j^*[b,c] = aj^*q[b,c] \xrightarrow{\bar{H}} aj^*[qb, qc] \xrightarrow{P_{\mathrm{PM}}} a[j^*qb, j^*qc] \xrightarrow{\bar{P}} [aj^*qb, aj^*qc]$$

as described in (2.7). Note that $\bar{H}$ is an isomorphism by definition [9, Lemma 2.38]. P_{PM} is an isomorphism, as we have just seen. $\bar{P}$ is also an isomorphism, since j^*qc is a sheaf, see [9, (2.39)]. Thus P_{Mod} is also an isomorphism. □

2.10 Proposition. *Let $f : X \to Y$ be a morphism of schemes and $b, c \in \mathrm{Mod}(Y)$. If one of the following conditions hold, then $P : f^*[b,c] \to [f^*b, f^*c]$ is an isomorphism:*

1 *f is locally an open immersion. That is, there is an open covering (U_λ) of X such that $f|_{U_\lambda}$ is an open immersion for every λ.*

2 *f is flat and b is coherent.*

Proof. We prove **1**. By [9, Lemma 1.54], we may assume that f is an open immersion, and this case is Lemma 2.9.

We prove **2**. By [9, Lemma 1.54] and Lemma 2.9, we may assume that both X and Y are affine. So there is a presentation of the form $\mathcal{O}_Y^n \to \mathcal{O}_Y^m \to b \to 0$. By the five lemma, we may assume that $b = \mathcal{O}_Y$, and this case is easy. $\square$

(2.11) Let I be a small category. For a category $\mathcal{C}$, the functor category $\mathrm{Func}(I^{\mathrm{op}}, \mathcal{C})$ is denoted by $\mathcal{P}(I, \mathcal{C})$. We denote the category of schemes by $\underline{\mathrm{Sch}}$. An object of $\mathcal{P}(I, \underline{\mathrm{Sch}})$ is called an I^{op}-diagram of schemes. Although in [9], whose notation we mainly use, diagrams of schemes are denoted by $X_\bullet$, $Y_\bullet$, $Z_\bullet$ and so on, we write X, Y, Z and so on for simplicity of notation. Similarly, morphisms in $\mathcal{P}(I, \underline{\mathrm{Sch}})$ are denoted by $f_\bullet$, $g_\bullet$, $h_\bullet$ and so on in [9], but we use f, g, h and so on for simplicity.

Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. For $i \in I$, $X(i)$ is denoted by X_i. For $\phi \in \mathrm{Mor}(I)$, $X(\phi)$ is denoted by X_ϕ. For a property of schemes $\mathbb{P}$, we say that X satisfies $\mathbb{P}$ if X_i satisfies $\mathbb{P}$ for every $i \in I$. Let $\mathbb{Q}$ be a property of morphisms. We say that X has $\mathbb{Q}$ arrows if X_ϕ satisfies $\mathbb{Q}$ for each $\phi \in \mathrm{Mor}(I)$. Let S be a scheme and consider the case $X \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. We say that X is $\mathbb{Q}$ over S if the structure morphism $X_i \to S$ satisfies $\mathbb{Q}$ for each i. Let $f : X \to Y$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. We say that f is $\mathbb{Q}$ if f_i is $\mathbb{Q}$ for each i. We say that f is *cartesian* if the commutative diagram $Y_\phi f_j = f_i X_\phi$ is a cartesian square for each $(\phi : i \to j) \in \mathrm{Mor}(I)$.

For a subcategory J of I, the restriction of $X_\bullet$ to J was written $X_\bullet|_J$ in [9]. In this paper, $X_\bullet$ is written X as mentioned before, and $X|_J$ is written X_J for simplicity of notation. Similarly, for a morphism f of $\mathcal{P}(I, \underline{\mathrm{Sch}})$, the restriction of f to J is denoted by f_J rather than $f|_J$.

(2.12) Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. As in [9], we denote the category of $\mathcal{O}_X$-modules by $\mathrm{Mod}(X)$. Let $\mathcal{M} \in \mathrm{Mod}(X)$. The restriction of $\mathcal{M}$ to X_i is denoted by $\mathcal{M}_i$ for $i \in I$. We say that $\mathcal{M}$ is locally quasi-coherent (resp. locally coherent) if $\mathcal{M}_i$ is quasi-coherent (resp. coherent) for each $i \in I$. We say that $\mathcal{M}$ is quasi-coherent (resp. coherent) if $\mathcal{M}$ is locally quasi-coherent (resp. locally coherent) and equivariant [9, (4.14)]. We denote the full subcategory of $\mathrm{Mod}(X)$ consisting of equivariant (resp. locally quasi-coherent, locally coherent, quasi-coherent, coherent) sheaves by $\mathrm{EM}(X)$ (resp. $\mathrm{Lqc}(X)$, $\mathrm{Lch}(X)$, $\mathrm{Qch}(X)$, $\mathrm{Coh}(X)$). The derived categories such as $D(\mathrm{Mod}(X))$, $D^+_{\mathrm{Lqc}(X)}(\mathrm{Mod}(X))$, and $D^b_{\mathrm{Qch}(X)}(\mathrm{Lqc}(X))$ are denoted by $D(X)$, $D^+_{\mathrm{Lqc}}(X)$, and $D^b_{\mathrm{Qch}}(\mathrm{Lqc}(X))$, respectively for short.

2.13 Lemma. *Let $f : X \to Y$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$, and $b, c \in \mathrm{Mod}(Y)$. If one of the following holds, then*

$$P : f^* \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(b, c) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^*b, f^*c)$$

is an isomorphism.

1 *f is locally an open immersion and b is equivariant.*

2 *f is flat and b is coherent.*

Proof. By [9, Lemma 1.59], the diagram

$$\begin{tikzcd}
(?)_i f^*[b,c] \arrow[r, "\theta^{-1}"] \arrow[d, "P"] & f_i^*(?)_i[b,c] \arrow[rr, "H"] & & f_i^*[b_i, c_i] \arrow[d, "P"] \\
(?)_i[f^*b, f^*c] \arrow[r, "H"] & {[(?)_i f^*b, (?)_i f^*c]} \arrow[rr, "{[\theta,\theta^{-1}]}"] & & {[f_i^*(?)_i b, f_i^*(?)_i c]}
\end{tikzcd}$$

is commutative for every $i \in I$, where $[\ ,\ ]$ denotes the Hom sheaf. As b is assumed to be equivariant in both cases, the horizontal arrows are isomorphisms by [9, Lemma 7.22] and [9, Lemma 6.33]. By Proposition 2.10, the right-most vertical P is an isomorphism. Thus, the left most P is also an isomorphism. As i is arbitrary, we are done. □

A morphism of schemes is said to be *concentrated* if it is quasi-compact and quasi-separated. A scheme X is said to be concentrated if the unique morphism $X \to \operatorname{Spec} \mathbb{Z}$ is concentrated.

2.14 Lemma. *Let*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

be a cartesian square in $\mathcal{P}(I, \underline{\mathrm{Sch}})$, and $\mathcal{M} \in \mathrm{Mod}(Y')$. If one of the following hold, then $\theta : f^* g_* \mathcal{M} \to h_*(f')^* \mathcal{M}$ is an isomorphism.

1 *f is locally an open immersion.*

2 *g is concentrated, f is flat, and $\mathcal{M} \in \mathrm{Lqc}(Y')$.*

Proof. Using [9, Lemma 1.22] twice, it is easy to see that the diagram

$$\begin{tikzcd} f_i^*(g_i)_*(?)_i \arrow[r, "c"] \arrow[d, "\theta"] & f_i^*(?)_i g_* \arrow[r, "\theta"] & (?)_i f^* g_* \arrow[d, "\theta"] \\ (h_i)_*(f_i')^*(?)_i \arrow[r, "\theta"] & (h_i)_*(?)_i(f')^* \arrow[r, "c"] & (?)_i h_*(f')^* \end{tikzcd}$$

is commutative. As the horizontal arrows are isomorphisms, we may assume that the all diagrams of schemes are single schemes.

We prove **1**. The case that f is an open immersion is proved in the first paragraph of the proof of [9, Lemma 7.12]. The general case follows from [9, Lemma 1.23]. **2** is nothing but [9, Lemma 7.12]. $\square$

3. Local section functor for diagrams

(3.1) Let X be an I^{op}-diagram of schemes. As in [9], the category of presheaves and sheaves of abelian groups are denoted by $\mathrm{PA}(X)$ and $\mathrm{AB}(X)$, respectively. Let U be an open subdiagram of schemes of X, and V be an open subdiagram of schemes of U. Let $f : U \to X$ be the inclusion, and $g : V \to U$ the inclusion.

For $\mathcal{M} \in \mathrm{Mod}(X)$ or $\mathcal{M} \in \mathrm{AB}(X)$, we denote the kernel of the unit of adjunction

$$u : f_*f^*\mathcal{M} \to f_*g_*g^*f^*\mathcal{M}$$

by $\underline{\Gamma}_{U,V}\,\mathcal{M}$. We denote the canonical inclusion $\underline{\Gamma}_{U,V}\,\mathcal{M} \hookrightarrow f_*f^*\mathcal{M}$ by ι. Note that the formation of $\underline{\Gamma}_{U,V}$ is compatible with the forgetful functor $\mathrm{Mod}(X) \to \mathrm{AB}(X)$.

If $U = X$, there is an exact sequence

$$0 \to \underline{\Gamma}_{X,V}\,\mathcal{M} \xrightarrow{\iota'} \mathcal{M} \xrightarrow{u} g_*g^*\mathcal{M},$$

where ι' is the composite

$$\underline{\Gamma}_{X,V}\,\mathcal{M} \xrightarrow{\iota} (\mathrm{id}_X)_*(\mathrm{id}_X)^*\mathcal{M} \xrightarrow{u^{-1}} \mathcal{M}.$$

3.2 Lemma. $\underline{\Gamma}_{U,V} : \mathrm{Mod}(X) \to \mathrm{Mod}(X)$ *is a left exact functor.*

Proof. Let $0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N}$ be an exact sequence in $\mathrm{Mod}(X)$. Since f^* and g^* are exact and f_* and g_* are left exact, the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \underline{\Gamma}_{U,V}\,\mathcal{L} \arrow[r, "\iota"] \arrow[d] & f_*f^*\mathcal{L} \arrow[r, "u"] \arrow[d] & f_*g_*g^*f^*\mathcal{L} \arrow[d] \\
0 \arrow[r] & \underline{\Gamma}_{U,V}\,\mathcal{M} \arrow[r, "\iota"] \arrow[d] & f_*f^*\mathcal{M} \arrow[r, "u"] \arrow[d] & f_*g_*g^*f^*\mathcal{M} \arrow[d] \\
0 \arrow[r] & \underline{\Gamma}_{U,V}\,\mathcal{N} \arrow[r, "\iota"] & f_*f^*\mathcal{N} \arrow[r, "u"] & f_*g_*g^*f^*\mathcal{N}
\end{tikzcd}$$

has exact rows, and the second and the third columns are exact. Hence the first column is exact, and the assertion follows. □

(3.3) Let X, U, V, f, and g be as in (3.1). Let J be a subcategory of I. Let $\mathcal{M} \in \mathrm{Mod}(X)$. Then we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \to \underline{\Gamma}_{U_J,V_J}\,\mathcal{M}_J \arrow[r, "\iota"] & (f_J)_*f_J^*\mathcal{M}_J \arrow[r, "u"] \arrow[d, "c^{-1}\theta"] & (f_J)_*(g_J)_*g_J^*f_J^*\mathcal{M}_J \arrow[d, "c^{-1}c^{-1}\theta\theta"] \\
0 \to (\underline{\Gamma}_{U,V}\,\mathcal{M})_J \arrow[r, "\iota"] & (f_*f^*\mathcal{M})_J \arrow[r, "u"] & (f_*g_*g^*f^*\mathcal{M})_J,
\end{tikzcd}$$

where $c^{-1}\theta$ is the composite isomorphism

$$(f_J)_*f_J^*(?)_J \xrightarrow{\theta} (f_J)_*(?)_Jf^* \xrightarrow{c^{-1}} (?)_Jf_*f^*,$$

see [9, Example 5.6, **2**], [9, Lemma 6.25], and [9, Lemma 1.24]. Similarly, $c^{-1}c^{-1}\theta\theta$ is the composite

$$(f_J)_*(g_J)_*g_J^*f_J^*(?)_J \xrightarrow{\theta} (f_J)_*(g_J)_*g_J^*(?)_Jf^* \xrightarrow{\theta}$$
$$(f_J)_*(g_J)_*(?)_Jg^*f^* \xrightarrow{c^{-1}} (f_J)_*(?)_Jg_*g^*f^* \xrightarrow{c^{-1}} (?)_Jf_*g_*g^*f^*.$$

Thus we get a unique natural map

$$\hat{\gamma} = \hat{\gamma}_{U,V,J} : \underline{\Gamma}_{U_J,V_J}(?)_J \to (?)_J\,\underline{\Gamma}_{U,V}$$

such that $\iota\hat{\gamma} = c^{-1}\theta\iota$. Note that $\hat{\gamma}$ is an isomorphism by the five lemma.

3.4 Lemma. *Let the notation be as above, and K a subcategory of J. Then the composite*

$$\underline{\Gamma}_{U_K,V_K}(?)_K \xrightarrow{c} \underline{\Gamma}_{U_K,V_K}(?)_{K,J}(?)_J \xrightarrow{\hat{\gamma}} (?)_{K,J}\,\underline{\Gamma}_{U_J,V_J}(?)_J \xrightarrow{\hat{\gamma}} (?)_{K,J}(?)_J\,\underline{\Gamma}_{U,V} \xrightarrow{c^{-1}} (?)_K\,\underline{\Gamma}_{U,V}$$

is $\hat{\gamma}_{U,V,K}$.

Proof. Consider the diagram

$$\begin{tikzcd}
\underline{\Gamma}_{U_K,V_K}(?)_K \arrow[r, "\iota"] \arrow[d, "c"] \arrow[dr, phantom, "\text{(a)}"] & (f_K)_* f_K^*(?)_K \arrow[d, "c"] \arrow[dddd, bend left=60, "c^{-1}\theta"] \\
\underline{\Gamma}_{U_K,V_K}(?)_{K,J}(?)_J \arrow[r, "\iota"] \arrow[d, "\hat{\gamma}"] \arrow[dr, phantom, "\text{(b)}"] & (f_K)_* f_K^*(?)_{K,J}(?)_J \arrow[d, "c^{-1}\theta"] \arrow[d, phantom, shift left=12ex, "\text{(e)}"] \\
(?)_{K,J}\,\underline{\Gamma}_{U_J,V_J}(?)_J \arrow[r, "\iota"] \arrow[d, "\hat{\gamma}"] \arrow[dr, phantom, "\text{(c)}"] & (?)_{K,J}(f_J)_* f_J^*(?)_J \arrow[d, "c^{-1}\theta"] \\
(?)_{K,J}(?)_J\,\underline{\Gamma}_{U,V} \arrow[r, "\iota"] \arrow[d, "c^{-1}"] \arrow[dr, phantom, "\text{(d)}"] & (?)_{K,J}(?)_J f_* f^* \arrow[d, "c^{-1}"] \\
(?)_K\,\underline{\Gamma}_{U,V} \arrow[r, "\iota"] & (?)_K f_* f^*
\end{tikzcd}.$$

The commutativity of (a) and (d) is trivial. The commutativity of (b) and (c) is by the definition of $\hat{\gamma}$. The commutativity of (e) follows from [9, Lemma 1.4] and [9, Lemma 1.22]. So the whole diagram is commutative, and the assertion follows from the definition of $\hat{\gamma}$. $\square$

(3.5) Let

$$\begin{tikzcd}
V' \arrow[r, "g'"] \arrow[d, "h_V"] \arrow[dr, phantom, "\text{(a)}"] & U' \arrow[r, "f'"] \arrow[d, "h_U"] \arrow[dr, phantom, "\text{(b)}"] & X' \arrow[d, "h"] \\
V \arrow[r, "g"] & U \arrow[r, "f"] & X
\end{tikzcd} \tag{2}$$

be a commutative diagram in $\mathcal{P}(I, \underline{\mathrm{Sch}})$ such that the horizontal arrows are inclusion maps of open subdiagrams.

By [9, Lemma 1.24], we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \underline{\Gamma}_{U,V}\, h_* \arrow[r, "\iota"] & f_* f^* h_* \arrow[r, "u"] \arrow[d, "c\theta"] & f_* g_* g^* f^* h_* \arrow[d, "cc\theta\theta"] \\
0 \arrow[r] & h_*\, \underline{\Gamma}_{U',V'} \arrow[r, "\iota"] & h_* f'_* (f')^* \arrow[r, "u"] & h_* f'_* g'_* (g')^* (f')^*,
\end{tikzcd} \tag{3}$$

where $c\theta$ is the composite

$$f_* f^* h_* \xrightarrow{\theta} f_*(h_U)_*(f')^* \xrightarrow{c} h_* f'_*(f')^*,$$

and $cc\theta\theta$ is the composite

$$\begin{aligned} f_* g_* g^* f^* h_* \xrightarrow{\theta} f_* g_* g^* (h_U)_* (f')^* &\xrightarrow{\theta} f_* g_* (h_V)_* (g')^* (f')^* \\ &\xrightarrow{c} f_*(h_U)_* g'_* (g')^* (f')^* \xrightarrow{c} h_* f'_* g'_* (g')^* (f')^*. \end{aligned}$$

So a unique natural map

$$\bar{\gamma} = \bar{\gamma}_{U,V,U',V',h} : \underline{\Gamma}_{U,V}\, h_* \to h_*\, \underline{\Gamma}_{U',V'}$$

such that $\iota\bar{\gamma} = c\theta\iota$ is induced. In particular, considering the case that $X' = X$ and h is the identity,

$$\bar{\gamma} = \bar{\gamma}_{U,V,U',V'} : \underline{\Gamma}_{U,V} \to \underline{\Gamma}_{U',V'}$$

is defined.

3.6 Lemma. *Assume that* (a) *and* (b) *in the diagram* (2) *are cartesian. Then $\bar{\gamma}_{U,V,U',V',h}$ is an isomorphism.*

Proof. By the five lemma, it suffices to show that $c\theta$ and $cc\theta\theta$ in (3) are isomorphisms. This is an immediate consequence of Lemma 2.14. □

3.7 Lemma. *Let*

$$\begin{tikzcd}
V'' \arrow[r, "g''"] \arrow[d, "k_V"] & U'' \arrow[r, "f''"] \arrow[d, "k_U"] & X'' \arrow[d, "k"] \\
V' \arrow[r, "g'"] \arrow[d, "h_V"] & U' \arrow[r, "f'"] \arrow[d, "h_U"] & X' \arrow[d, "h"] \\
V \arrow[r, "g"] & U \arrow[r, "f"] & X
\end{tikzcd}$$

be a commutative diagram in $\mathcal{P}(I, \underline{\mathrm{Sch}})$ such that the horizontal maps are inclusions of open subdiagrams. Then the composite

$$\underline{\Gamma}_{U,V}(hk)_* \xrightarrow{c} \underline{\Gamma}_{U,V}\, h_* k_* \xrightarrow{\bar{\gamma}} h_* \,\underline{\Gamma}_{U',V'}\, k_* \xrightarrow{\bar{\gamma}} h_* k_* \,\underline{\Gamma}_{U'',V''} \xrightarrow{c^{-1}} (hk)_* \,\underline{\Gamma}_{U'',V''}$$

equals $\bar{\gamma}$.

Proof. Consider the diagram

$$\begin{tikzcd}
\underline{\Gamma}_{U,V}(hk)_* \arrow[r, "\iota"] \arrow[d, "c"] \arrow[dr, phantom, "\text{(a)}"] & f_* f^*(hk)_* \arrow[d, "c"] \arrow[dddd, bend left=60, "c\theta"] \\
\underline{\Gamma}_{U,V}\, h_* k_* \arrow[r, "\iota"] \arrow[d, "\bar{\gamma}"] \arrow[dr, phantom, "\text{(b)}"] & f_* f^* h_* k_* \arrow[d, "c\theta"] \arrow[d, phantom, shift left=8ex, "\text{(e)}"] \\
h_* \,\underline{\Gamma}_{U',V'}\, k_* \arrow[r, "\iota"] \arrow[d, "\bar{\gamma}"] \arrow[dr, phantom, "\text{(c)}"] & h_* f'_*(f')^* k_* \arrow[d, "c\theta"] \\
h_* k_* \,\underline{\Gamma}_{U'',V''} \arrow[r, "\iota"] \arrow[d, "c^{-1}"] \arrow[dr, phantom, "\text{(d)}"] & h_* k_* f''_*(f'')^* \arrow[d, "c^{-1}"] \\
(hk)_* \,\underline{\Gamma}_{U'',V''} \arrow[r, "\iota"] & (hk)_* f''_*(f'')^*
\end{tikzcd}.$$

The commutativity of (a) and (d) is trivial. The commutativity of (b) and (c) is by the definition of $\bar{\gamma}$. The commutativity of (e) follows from [9, Lemma 1.4] and [9, Lemma 1.22]. So the whole diagram is commutative, and the assertion follows from the definition of $\bar{\gamma}$. $\square$

3.8 Lemma. *Let* (2) *be as in* (3.5) *and J a subcategory of I. Then the diagram*

$$\begin{tikzcd}
\underline{\Gamma}_{U_J,V_J}(?)_J h_* \arrow[r, "\hat{\gamma}"] \arrow[d, "c"] & (?)_J \,\underline{\Gamma}_{U,V}\, h_* \arrow[r, "\bar{\gamma}"] & (?)_J h_* \,\underline{\Gamma}_{U',V'} \arrow[d, "c"] \\
\underline{\Gamma}_{U_J,V_J}(h_J)_*(?)_J \arrow[r, "\bar{\gamma}"] & (h_J)_* \,\underline{\Gamma}_{U'_J,V'_J}(?)_J \arrow[r, "\hat{\gamma}"] & (h_J)_*(?)_J \,\underline{\Gamma}_{U',V'}
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$
\begin{tikzcd}
\underline{\Gamma}_{U_J,V_J}(?)_J h_* \arrow[r, "\hat{\gamma}"] \arrow[d, "\iota"] \arrow[ddd, bend right=60, "c"'] \arrow[dr, phantom, "\text{(a)}"] & (?)_J \underline{\Gamma}_{U,V}\, h_* \arrow[r, "\bar{\gamma}"] \arrow[d, "\iota"] \arrow[dr, phantom, "\text{(b)}"] & (?)_J h_* \underline{\Gamma}_{U',V'} \arrow[d, "\iota"] \arrow[ddd, bend left=60, "c"] \\
(f_J)_* f_J^*(?)_J h_* \arrow[r, "c^{-1}\theta"] \arrow[d, "c"] \arrow[l, phantom, "\text{(c)}"] \arrow[drr, phantom, "\text{(d)}"] & (?)_J f_* f^* h_* \arrow[r, "c\theta"] & (?)_J h_* f'_*(f')^* \arrow[d, "c"] \arrow[r, phantom, "\text{(e)}"] & {} \\
(f_J)_* f_J^*(h_J)_*(?)_J \arrow[r, "c\theta"] & (h_J)_*(f'_J)_*(f'_J)^*(?)_J \arrow[r, "c^{-1}\theta"] & (h_J)_*(?)_J f'_*(f')^* \\
\underline{\Gamma}_{U_J,V_J}(h_J)_*(?)_J \arrow[r, "\bar{\gamma}"] \arrow[u, "\iota"] \arrow[ur, phantom, "\text{(f)}"] & (h_J)_* \underline{\Gamma}_{U'_J,V'_J}(?)_J \arrow[r, "\hat{\gamma}"] \arrow[u, "\iota"] \arrow[ur, phantom, "\text{(g)}"] & (h_J)_*(?)_J \underline{\Gamma}_{U',V'} \arrow[u, "\iota"]
\end{tikzcd}
.
$$

By the definition of $\hat{\gamma}$, (a) and (g) are commutative. By the definition of $\bar{\gamma}$, (b) and (f) are commutative. The commutativity of (c) and (e) is trivial. The commutativity of (d) follows from [9, Lemma 1.4] and [9, Lemma 1.22]. Since ι is a monomorphism, the assertion follows. $\square$

3.9 Lemma. *Let* (2) *be as in* (3.5)*. Assume that* $X = X'$ *and* $h = \mathrm{id}$*. If* $U_i \setminus V_i = U'_i \setminus V'_i$ *for any* $i \in I$*, then* $\bar{\gamma}_{U,V,U',V'} : \underline{\Gamma}_{U,V} \to \underline{\Gamma}_{U',V'}$ *is an isomorphism.*

Proof. In view of Lemma 3.8, we may assume that X is a single scheme. Let $\mathcal{M} \in \mathrm{AB}(X)$ or $\mathcal{M} \in \mathrm{Mod}(X)$. For any open set $W \subset X$, we have a commutative diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & \Gamma(W, \underline{\Gamma}_{U,V}\, \mathcal{M}) \arrow[r, "\iota"] \arrow[d, "\bar{\gamma}"] & \Gamma(W \cap U, \mathcal{M}) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{res}"] & \Gamma(W \cap V, \mathcal{M}) \arrow[d, "\mathrm{res}"] \\
0 \arrow[r] & \Gamma(W, \underline{\Gamma}_{U',V'}\, \mathcal{M}) \arrow[r, "\iota"] & \Gamma(W \cap U', \mathcal{M}) \arrow[r, "\mathrm{res}"] & \Gamma(W \cap V', \mathcal{M})
\end{tikzcd}
$$

By assumption, $U = V \cup U'$ and $V' = V \cap U'$. So

$$
0 \to \Gamma(W \cap U, \mathcal{M}) \to \Gamma(W \cap V, \mathcal{M}) \oplus \Gamma(W \cap U' \mathcal{M}) \to \Gamma(W \cap V', \mathcal{M})
$$

is exact. So $\bar{\gamma}$ is bijective, as can be seen easily. $\square$

(3.10) Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. Let Y be a cartesian closed subdiagram of schemes of X. That is, Y is a subdiagram of schemes such that the inclusion $Y \hookrightarrow X$ is a cartesian closed immersion. Let Z be a cartesian closed subdiagram of schemes of Y. Then letting $U_i = X_i \setminus Z_i$, $U = U(Z)$ is a cartesian open subdiagram of schemes of X, and letting $V_i = X_i \setminus Y_i$, $V = U(Y)$ is a cartesian open subdiagram of schemes of U. Thus $\underline{\Gamma}_{Y;Z} := \underline{\Gamma}_{U(Z),U(Y)}$ is defined.

If Z is empty, $\underline{\Gamma}_{Y;\emptyset}$ is denoted by $\underline{\Gamma}_Y$. There is an exact sequence

$$0 \to \underline{\Gamma}_Y \xrightarrow{\iota'} \mathrm{Id} \xrightarrow{u} g_* g^*,$$

where $g : U(Y) \to X$ is the inclusion.

For a subcategory J of I, $U(Y)_J = U(Y_J)$ and $U(Z)_J = U(Z_J)$. Thus the isomorphism

$$\hat{\gamma}_{Y;Z;J} := \hat{\gamma}_{U(Z),U(Y),J} : \underline{\Gamma}_{Y_J;Z_J}(?)_J \to (?)_J \, \underline{\Gamma}_{Y;Z}$$

is defined, see (3.3). We denote $\hat{\gamma}_{Y;\emptyset;J}$ by $\hat{\gamma}_{Y;J}$.

(3.11) Let the notation be as above, and $h : X' \to X$ a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. Then $Y' := h^{-1}(Y)$ is a cartesian closed subdiagram of X', and $Z' := h^{-1}(Z)$ is a cartesian closed subdiagram of Y'. Thus $\bar{\gamma}_{Y;Z;h} := \bar{\gamma}_{U(Z),U(Y),U(Z'),U(Y'),h}$ is defined, see (3.5). We denote $\bar{\gamma}_{Y;\emptyset;h}$ by $\bar{\gamma}_{Y;h}$. By Lemma 3.6, we immediately have

3.12 Lemma. *Let the notation be as above. Then $\bar{\gamma}_{Y;Z;h}$ is an isomorphism.*

(3.13) Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. A collection $Z = (Z_i)_{i \in I}$ is called a *locally closed system* of X if there exist some open subdiagram of schemes U of X and an open subdiagram of schemes V of U such that $Z_i = U_i \setminus V_i$. Such a pair (U, V) is called a UV-pair of Z. If Z is a locally closed system of X, then Z_i is a locally closed subset of X_i for any i. If $((U_\lambda, V_\lambda))$ is a family of UV-pairs of Z, then $(\bigcup U_\lambda, \bigcup V_\lambda)$ is also a UV-pair of Z. So if Z is a locally closed system of X, there is a largest UV-pair $(U(Z), V(Z))$ of Z. We define $\underline{\Gamma}_Z := \underline{\Gamma}_{U(Z),V(Z)}$ for a locally closed system Z of X. If (U, V) is a UV-pair of Z, then $\bar{\gamma} : \underline{\Gamma}_Z \to \underline{\Gamma}_{U,V}$ is an isomorphism by Lemma 3.9. If Z is a cartesian closed subdiagram of schemes of X, then Z can be viewed as a locally closed system of X, and $\underline{\Gamma}_Z$ is defined. This definition of $\underline{\Gamma}_Z$ agrees with that defined in (3.10), and there is no confliction.

(3.14) Let the commutative diagram (2) be as in (3.5). Assume that h is flat. Then there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & h^* \underline{\Gamma}_{U,V} \arrow[r, "\iota"] & h^* f_* f^* \arrow[r, "u"] \arrow[d, "d\theta"] & h^* f_* g_* g^* f^* \arrow[d, "dd\theta\theta"] \\ 0 \arrow[r] & \underline{\Gamma}_{U',V'} h^* \arrow[r, "\iota"] & f'_*(f')^* h^* \arrow[r, "u"] & f'_* g'_*(g')^*(f')^* h^*, \end{tikzcd}$$

where $d\theta$ is the composite

$$h^* f_* f^* \xrightarrow{\theta} f'_* h_U^* f^* \xrightarrow{d} f'_*(f')^* h^*,$$

and $dd\theta\theta$ is the composite

$$h^* f_* g_* g^* f^* \xrightarrow{\theta} f'_* h_U^* g_* g^* f^* \xrightarrow{\theta} f'_* g'_* h_V^* g^* f^*$$
$$\xrightarrow{d} f'_* g'_*(g')^* h_U^* f^* \xrightarrow{d} f'_* g'_*(g')^*(f')^* h^*.$$

So there is a unique natural map $\bar{\delta} = \bar{\delta}_{U,V,U',V',h} : h^* \underline{\Gamma}_{U,V} \to \underline{\Gamma}_{U',V'} h^*$ such that $\iota\bar{\delta} = d\theta\iota$.

3.15 Lemma. *Assume that the squares* (a) *and* (b) *in* (2) *are cartesian, and h is flat. Let $\mathcal{M} \in \mathrm{Mod}(X)$. If one of the following conditions holds, then $\bar{\delta} : h^* \underline{\Gamma}_{U,V} \mathcal{M} \to \underline{\Gamma}_{U',V'} h^* \mathcal{M}$ is an isomorphism.*

1 *h is locally an open immersion.*

2 *f and g are quasi-compact and $\mathcal{M}$ is locally quasi-coherent.*

Proof. In both cases, $\theta : h^* f_* \to f'_* h_U^*$ and $\theta : h_U^* g_* \to g'_* h_V^*$ are isomorphisms by Lemma 2.14. The assertion follows from the five lemma. □

(3.16) Let the notation be as in (3.11). Assume that h is flat. Then we define $\bar{\delta}_{Y;Z;h} := \bar{\delta}_{U(Z),U(Y),h^{-1}(U(Z)),h^{-1}(U(Y)),h}$. By Lemma 3.15, if h is locally an open immersion, or X is locally noetherian and $\mathcal{M}$ is locally quasi-coherent, then $\bar{\delta}_{Y;Z;h}$ is an isomorphism.

3.17 Lemma. *Let the notation be as in* Lemma 3.7. *Assume that h and k are flat. Then the composite*

$$(hk)^* \underline{\Gamma}_{U,V} \xrightarrow{d^{-1}} k^* h^* \underline{\Gamma}_{U,V} \xrightarrow{\bar{\delta}} k^* \underline{\Gamma}_{U',V'} h^* \xrightarrow{\bar{\delta}} \underline{\Gamma}_{U'',V''} k^* h^* \xrightarrow{d} \underline{\Gamma}_{U'',V''}(hk)^*$$

is $\bar{\delta}$.

Proof. Consider the diagram

$$\begin{tikzcd}
(hk)^* \underline{\Gamma}_{U,V} \arrow[r, "\iota"] \arrow[d, "d^{-1}"] \arrow[dr, phantom, "\text{(a)}"] & (hk)^* f_* f^* \arrow[d, "d^{-1}"] \arrow[dddd, bend left, "d\theta"] \\
k^* h^* \underline{\Gamma}_{U,V} \arrow[r, "\iota"] \arrow[d, "\bar{\delta}"] \arrow[dr, phantom, "\text{(b)}"] & k^* h^* f_* f^* \arrow[d, "d\theta"] \arrow[r, phantom, "\text{(e)}"] & {} \\
k^* \underline{\Gamma}_{U',V'} h^* \arrow[r, "\iota"] \arrow[d, "\bar{\delta}"] \arrow[dr, phantom, "\text{(c)}"] & k^* f'_* (f')^* h^* \arrow[d, "d\theta"] \\
\underline{\Gamma}_{U'',V''} k^* h^* \arrow[r, "\iota"] \arrow[d, "d"] \arrow[dr, phantom, "\text{(d)}"] & f''_* (f'')^* k^* h^* \arrow[d, "d"] \\
\underline{\Gamma}_{U'',V''} (hk)^* \arrow[r, "\iota"] & f''_* (f'')^* (hk)^*
\end{tikzcd}.$$

The commutativity of (a) and (d) is trivial. The commutativity of (b) and (c) is by the definition of $\bar{\delta}$. The commutativity of (e) follows from the opposite assertion of [9, Lemma 1.4] and [9, Lemma 1.23]. So the whole diagram is commutative, and the assertion follows from the definition of $\bar{\delta}$. $\square$

3.18 Lemma. *Let the notation be as in* Lemma 3.8. *Assume that h is flat. Then the diagram*

$$\begin{tikzcd}
h_J^*(?)_J \underline{\Gamma}_{U,V} \arrow[r, "\hat{\gamma}^{-1}"] \arrow[d, "\theta"] & h_J^* \underline{\Gamma}_{U_J,V_J}(?)_J \arrow[r, "\bar{\delta}"] & \underline{\Gamma}_{U'_J,V'_J} h_J^*(?)_J \arrow[d, "\theta"] \\
(?)_J h^* \underline{\Gamma}_{U,V} \arrow[r, "\bar{\delta}"] & (?)_J \underline{\Gamma}_{U',V'} h^* \arrow[r, "\hat{\gamma}^{-1}"] & \underline{\Gamma}_{U'_J,V'_J}(?)_J h^*
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$\begin{tikzcd}
h_J^*(?)_J \, \underline{\Gamma}_{U,V} \arrow[d, "\iota"] \arrow[ddd, bend right=70, "\theta"'] \arrow[r, phantom, "\text{(a)}", yshift=-2.5em] & h_J^* \, \underline{\Gamma}_{U_J,V_J}(?)_J \arrow[l, "\hat{\gamma}"'] \arrow[r, "\bar{\delta}"] \arrow[d, "\iota"] \arrow[r, phantom, "\text{(b)}", yshift=-2.5em] & \underline{\Gamma}_{U'_J,V'_J} \, h_J^*(?)_J \arrow[d, "\iota"] \arrow[ddd, bend left=70, "\theta"] \\
h_J^*(?)_J f_* f^* \arrow[d, "\theta"] \arrow[d, phantom, "\text{(c)}", xshift=-3em] & h_J^*(f_J)_* f_J^*(?)_J \arrow[l, "c^{-1}\theta"'] \arrow[r, "d\theta"] \arrow[d, phantom, "\text{(d)}"] & (f'_J)_*(f'_J)^* h_J^*(?)_J \arrow[d, "\theta"] \arrow[d, phantom, "\text{(e)}", xshift=3em] \\
(?)_J h^* f_* f^* \arrow[r, "d\theta"] \arrow[r, phantom, "\text{(f)}", yshift=-2.5em] & (?)_J f'_*(f')^* h^* \arrow[r, phantom, "\text{(g)}", yshift=-2.5em] & (f'_J)_*(f'_J)^*(?)_J h^* \arrow[l, "c^{-1}\theta"'] \\
(?)_J h^* \, \underline{\Gamma}_{U,V} \arrow[u, "\iota"'] \arrow[r, "\bar{\delta}"] & (?)_J \, \underline{\Gamma}_{U',V'} \, h^* \arrow[u, "\iota"'] & \underline{\Gamma}_{U'_J,V'_J}(?)_J h^* \arrow[l, "\hat{\gamma}"'] \arrow[u, "\iota"']
\end{tikzcd}.$$

By the definition of $\hat{\gamma}$, (a) and (g) are commutative. By the definition of $\bar{\delta}$, (b) and (f) are commutative. The commutativity of (c) and (e) is trivial. The commutativity of (d) follows from [9, Lemma 1.22] and [9, Lemma 1.23]. Since ι is a monomorphism, the assertion follows. $\square$

3.19 Lemma. *Let*

$$\begin{tikzcd}
& V_{X'} \arrow[rr, "g_{X'}"] \arrow[dd, "h_V" near end] & & U_{X'} \arrow[rr, "f_{X'}"] \arrow[dd, "h_U" near end] & & X' \arrow[dd, "h"] \\
V_{Z'} \arrow[ur, "k'_V"] \arrow[rr, "g_{Z'}" near start] \arrow[dd, "\bar{h}_V"'] & & U_{Z'} \arrow[ur, "k'_U"] \arrow[rr, "f_{Z'}" near start] \arrow[dd, "\bar{h}_U" near start] & & Z' \arrow[ur, "k'"] \arrow[dd, "\bar{h}" near start] & \\
& V \arrow[rr, "g" near start] & & U \arrow[rr, "f" near start] & & X \\
V_Z \arrow[ur, "k_V"] \arrow[rr, "g_Z"] & & U_Z \arrow[ur, "k_U"] \arrow[rr, "f_Z"] & & Z \arrow[ur, "k"] &
\end{tikzcd}$$

be a commutative diagram in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. Assume that f, f_Z, $f_{X'}$, $f_{Z'}$, g, g_Z, $g_{X'}$, and $g_{Z'}$ are inclusions of open subdiagrams. Assume that h and $\bar{h}$ are flat. Then the diagram

$$\begin{tikzcd}
h^* k_* \, \underline{\Gamma}_{U_Z,V_Z} \arrow[d, "\theta"] & h^* \, \underline{\Gamma}_{U,V} \, k_* \arrow[l, "\bar{\gamma}"'] \arrow[r, "\bar{\delta}"] & \underline{\Gamma}_{U_{X'},V_{X'}} \, h^* k_* \arrow[d, "\theta"] \\
k'_* \bar{h}^* \, \underline{\Gamma}_{U_Z,V_Z} \arrow[r, "\bar{\delta}"] & k'_* \, \underline{\Gamma}_{U_{Z'},V_{Z'}} \, \bar{h}^* & \underline{\Gamma}_{U_{X'},V_{X'}} \, k'_* \bar{h}^* \arrow[l, "\bar{\gamma}"']
\end{tikzcd}$$

is commutative.

Proof. Consider the diagram

$$
\begin{tikzcd}
h^*k_*\,\underline{\Gamma}_{U_Z,V_Z} \arrow[d, "\iota"] \arrow[ddd, bend right=60, "\theta"'] \arrow[dr, phantom, "\text{(a)}"] & h^*\,\underline{\Gamma}_{U,V}\,k_* \arrow[l, "\bar\gamma"'] \arrow[r, "\bar\delta"] \arrow[d, "\iota"] \arrow[dr, phantom, "\text{(b)}"] & \underline{\Gamma}_{U_{X'},V_{X'}}\,h^*k_* \arrow[d, "\iota"] \arrow[ddd, bend left=60, "\theta"] \\
h^*k_*(f_Z)_*f_Z^* \arrow[d, "\theta"] \arrow[dr, phantom, "\text{(d)}"] & h^*f_*f^*k_* \arrow[l, "c\theta"'] \arrow[r, "d\theta"] & (f_{X'})_*f_{X'}^*h^*k_* \arrow[d, "\theta"] \\
k'_*\bar{h}^*(f_Z)_*f_Z^* \arrow[r, "d\theta"] & k'_*(f_{Z'})_*f_{Z'}^*\bar{h}^* & (f_{X'})_*f_{X'}^*k'_*\bar{h}^* \arrow[l, "c\theta"'] \\
k'_*\bar{h}^*\,\underline{\Gamma}_{U_Z,V_Z} \arrow[u, "\iota"] \arrow[r, "\bar\delta"] \arrow[ur, phantom, "\text{(f)}"] & k'_*\,\underline{\Gamma}_{U_{Z'},V_{Z'}}\,\bar{h}^* \arrow[u, "\iota"] \arrow[ur, phantom, "\text{(g)}"] & \underline{\Gamma}_{U_{X'},V_{X'}}\,k'_*\bar{h}^* \arrow[l, "\bar\gamma"'] \arrow[u, "\iota"]
\end{tikzcd}
$$

By the definition of $\bar\gamma$, (a) and (g) are commutative. By the definition of $\bar\delta$, (b) and (f) are commutative. The commutativity of (c) and (e) is trivial. The commutativity of (d) follows from [9, Lemma 1.22] and [9, Lemma 1.23]. Since ι is a monomorphism, the assertion follows. $\square$

(3.20) Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$. Assume that X has flat arrows. Let Y be a cartesian closed subdiagram of schemes of X (that is, a closed subdiagram such that the inclusion $j : Y \hookrightarrow X$ is cartesian) so that the defining ideal $\mathcal{I}$ of Y is quasi-coherent. Set $U := X \setminus Y$. Then U is an open subdiagram of schemes of X. Note that $f : U \to X$ is also cartesian.

As the sequence

$$\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{I} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}^n \to 0$$

is exact, $\mathcal{O}_X/\mathcal{I}^n$ is coherent for $n \geq 1$, since coherent sheaves are closed under tensor products and cokernels. Applying $(?)_i$ to the exact sequence, we have that $(\mathcal{O}_X/\mathcal{I}^n)_i \cong \mathcal{O}_{X_i}/\mathcal{I}_i^n$.

For $\mathcal{M} \in \mathrm{Mod}(X)$, there is a canonical monomorphism

$$\Phi_Y : \varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \to \mathcal{M}$$

induced by the obvious maps

$$\Phi_n : \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{M}) \cong \mathcal{M}.$$

The composite

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \xrightarrow{\Phi_Y} \mathcal{M} \xrightarrow{u} f_*f^*\mathcal{M} \tag{4}$$

factors through

$$f_* f^* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \cong f_* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^*(\mathcal{O}_X/\mathcal{I}^n), f^*\mathcal{M}).$$

As $f^*(\mathcal{O}_X/\mathcal{I}^n) = 0$, (4) is zero, and the monomorphism

$$\rho_Y : \varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \to \underline{\Gamma}_Y \mathcal{M}$$

such that $\iota' \rho_Y = \Phi_Y$ is induced.

By [9, Lemma 1.47], the diagram

$$\begin{array}{ccc}
(?)_i \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, ?) & \xrightarrow{\Phi_n} & (?)_i \\
\downarrow{\scriptstyle H} & & \downarrow{\scriptstyle \mathrm{id}} \\
\underline{\mathrm{Hom}}_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i}/\mathcal{I}_i^n, ?)(?)_i & \xrightarrow{\Phi_n} & (?)_i
\end{array}$$

is commutative. Hence

$$\begin{array}{ccc}
(?)_i \varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, ?) & \xrightarrow{\rho_Y} & (?)_i \underline{\Gamma}_Y \\
\downarrow{\scriptstyle \cong} & & \\
\varinjlim (?)_i \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, ?) & & \downarrow{\scriptstyle \hat{\gamma}_{Y;i}^{-1}} \\
\downarrow{\scriptstyle H} & & \\
\varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_{X_i}}(\mathcal{O}_{X_i}/\mathcal{I}_i^n, ?)(?)_i & \xrightarrow{\rho_{Y_i}} & \underline{\Gamma}_{Y_i}(?)_i
\end{array} \tag{5}$$

is also commutative.

3.21 Lemma. *Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$ be locally noetherian with flat arrows, and Y its cartesian closed subdiagram. If $\mathcal{M} \in \mathrm{Lqc}(X)$, then*

$$\rho_Y : \varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M}) \to \underline{\Gamma}_Y \mathcal{M}$$

is an isomorphism.

Proof. Since $\mathcal{O}_X/\mathcal{I}^n$ is coherent, H in (5) is an isomorphism by [9, Lemma 6.33]. Thus we may assume that X is a single scheme. This case is a special case of [6, Theorem 2.8]. □

4. Local cohomology for diagrams

(4.1) Let the notation be as in (3.1). For a complex $\mathbb{M}$ of $\mathrm{Mod}(X)$, the right derived functor $R^i\,\underline{\Gamma}_{U,V}\,\mathbb{M}$ is denoted by $\underline{H}^i_{U,V}(\mathbb{M})$, and we call it the ith *local cohomology sheaf* of $\mathbb{M}$.

For a cartesian closed subdiagram Y of X and a cartesian closed subdiagram Z of Y, $R^i\,\underline{\Gamma}_{Y;Z}\,\mathbb{M}$ is denoted by $\underline{H}^i_{Y;Z}(\mathbb{M})$. $\underline{H}^i_{Y;\emptyset}(\mathbb{M})$ is denoted by $\underline{H}^i_Y(\mathbb{M})$.

(4.2) Let $\mathbb{F} \in K(\mathrm{Mod}(X))$. We say that $\mathbb{F}$ is *K-flabby* if for any $i \in I$, $\mathbb{F}_i$ is K-flabby in the sense of [21]. By [9, Lemma 8.17], a weakly K-injective complex is K-flabby. By [9, Proposition 8.2], a K-flabby complex is K-limp. A single sheaf $\mathcal{M} \in \mathrm{Mod}(X)$ is said to be flabby, if it is K-flabby as a complex. By [21, Proposition 5.13], $\mathcal{M}$ is flabby if and only if $\mathcal{M}_i$ is a flabby sheaf on the topological space X_i in the usual sense.

4.3 Proposition. *Let the notation be as in* (3.1)*. Let* $\mathbb{I}$ *be a K-flabby complex in* $\mathrm{Mod}(X)$*. Then* $\mathbb{I}$ *is* $\underline{\Gamma}_{U,V}$*-acyclic.*

Proof. Let $\varphi : \mathbb{I} \to \mathbb{J}$ be a K-injective resolution, which exists, since $\mathrm{Mod}(X)$ is Grothendieck, see [3]. Note that $\mathbb{J}$ is K-flabby. So replacing $\mathbb{I}$ by the mapping cone of φ, we may assume that $\mathbb{I}$ is exact, and we need to prove that $\underline{\Gamma}_{U,V}(\mathbb{I})$ is exact. It suffices to prove that for any $i \in I$, $(?)_i\,\underline{\Gamma}_{U,V}(\mathbb{I}) \cong \underline{\Gamma}_{U_i,V_i}(\mathbb{I}_i)$ is exact. So we may assume that X is a single scheme. To verify that $\underline{\Gamma}_{U,V}(\mathbb{I})$ is exact, it suffices to show that $\Gamma(W, \underline{\Gamma}_{U,V}(\mathbb{I}))$ is exact for any open subset W of X. Applying the functor $\Gamma(W, ?)$ to the exact sequence

$$0 \to \underline{\Gamma}_{U,V} \xrightarrow{\iota} f_*f^* \xrightarrow{u} f_*g_*g^*f^*,$$

we get an exact sequence

$$0 \to \Gamma(W, ?) \circ \underline{\Gamma}_{U,V} \to \Gamma(U \cap W, ?) \xrightarrow{\mathrm{res}} \Gamma(V \cap W, ?).$$

Letting $Z := (U \setminus V) \cap W$, $\Gamma(W, ?)\,\underline{\Gamma}_{U,V}$ is isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_{Z\subset X}, ?)$. By [21, Proposition 5.21], $\Gamma(W, \underline{\Gamma}_{U,V}(\mathbb{I})) \cong \mathrm{Hom}^\bullet_{\mathcal{O}_X}(\mathcal{O}_{Z\subset X}, \mathbb{I})$ is exact. This is what we wanted to prove. $\square$

(4.4) Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathbb{F} \in K(\mathrm{Mod}(X))$. Then $\mathbb{F}$ is K-flabby if and only if $\mathbb{F}$ is K-flabby as a complex of sheaves of abelian groups. To verify this, it suffices to show that if $\mathbb{F}$ is a K-injective complex in $\mathrm{Mod}(X)$, then $\mathbb{F}$ is K-flabby as a complex of sheaves of abelian groups.

Let $\mathbb{G}$ be a bounded above exact complex of sheaves of abelian groups, and assume that each term of $\mathbb{G}$ is a direct sum of sheaves of the form $\mathbb{Z}_{Z\subset X}$ for some locally closed subset Z of X. Since $\mathbb{G}$ is $\mathbb{Z}$-flat, $\mathbb{G}' = \mathcal{O}_X \otimes_{\mathbb{Z}} \mathbb{G}$ is again exact. Thus

$$\operatorname{Hom}^\bullet_{\mathbb{Z}}(\mathbb{G}, \mathbb{F}) \cong \operatorname{Hom}^\bullet_{\mathcal{O}_X}(\mathbb{G}', \mathbb{F})$$

is exact by the K-injectivity of $\mathbb{F}$. So $\mathbb{F}$ is K-flabby as a complex of sheaves of abelian groups.

Similarly, a complex of $\mathcal{O}_{\mathbb{X}}$-modules on a ringed site $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ is K-limp if and only if it is K-limp as a complex of sheaves of abelian groups.

4.5 Lemma ([21, Proposition 5.15]). *Let $f : X \to Y$ be a continuous map between topological spaces. If $\mathbb{F}$ is a K-flabby complex of sheaves of abelian groups, then so is $f_*\mathbb{F}$.*

Similarly, If $f : \mathbb{Y} \to \mathbb{X}$ is an admissible continuous functor (see [9, (2.8)]) between sites, and $\mathbb{F}$ is a K-limp complex of sheaves of abelian groups on $\mathbb{X}$, then $f^\#\mathbb{F}$ is also K-limp, see [9, Lemma 3.31].

4.6 Lemma. *Let X be a topological space, U an open subset of X, and $\mathbb{F}$ a K-flabby* (*resp. K-limp*) *complex of abelian groups. Then $\mathbb{F}|_U$ is again K-flabby* (*resp. K-limp*).

Proof. Let $\varphi : \mathbb{F} \to \mathbb{I}$ be a K-injective resolution. Let $i : U \hookrightarrow X$ be the inclusion. As i^* has an exact left adjoint $i^!$, $i^*\mathbb{I}$ is K-injective. Since i^* is exact, $i^*\varphi : i^*\mathbb{F} \to i^*\mathbb{I}$ is a K-injective resolution. Let $\mathbb{J}$ be the mapping cone of φ. It suffices to show that for any locally closed subset Z (resp. open subset V) of U, $\Gamma_Z(U, i^*\mathbb{J})$ (resp. $\Gamma(V, i^*\mathbb{J})$) is exact. This is trivial, since $\Gamma_Z(U, i^*\mathbb{J}) \cong \Gamma_Z(X, \mathbb{J})$ (resp. $\Gamma(V, i^*\mathbb{J}) \cong \Gamma(V, \mathbb{J})$). □

4.7 Lemma. *Let X be a topological space, U, V, W, W' open subsets of X such that $V \subset U$ and $W' \subset W$. Set $Z := W \setminus W'$. Let F be a flabby sheaf of abelian groups on X. Then the canonical map*

$$\Gamma_{Z\cap U}(X, F) \to \Gamma_{Z\cap V}(X, F)$$

is surjective.

Proof. Let $\alpha \in \Gamma_{Z\cap V}(X, F) = \operatorname{Ker}(\Gamma(W \cap V, F) \to \Gamma(W' \cap V, F))$. Then there is a unique section $\tilde{\alpha} \in \Gamma((W' \cap U) \cup (W \cap V), F)$ such that the restriction of $\tilde{\alpha}$ to $W \cap V$ is α, and the restriction of $\tilde{\alpha}$ to $W' \cap U$ is zero. Since F is flabby, $\tilde{\alpha}$ is extended to an element β of $\Gamma(W \cap U, F)$. Then

$\beta \in \operatorname{Ker}(\Gamma(W \cap U, F) \to \Gamma(W' \cap U, F)) = \Gamma_{Z \cap U}(X, F)$, and the restriction of β to $W \cap V$ is α. This shows that the canonical map

$$\Gamma_{Z \cap U}(X, F) \to \Gamma_{Z \cap V}(X, F)$$

is surjective. $\square$

4.8 Lemma (cf. [6, Lemma 1.6]). *Let the notation be as in* (3.1). *Let* $\mathbb{I}$ *be a* K*-flabby complex in* $\operatorname{Mod}(X)$. *Then* $\underline{\Gamma}_{U,V}\,\mathbb{I}$ *is again* K*-flabby.*

Proof. We may assume that X is a single scheme.

Let $W' \subset W \subset X$ be open subsets, and $Z := W \setminus W'$. As in the proof of Proposition 4.3, it is easy to check that $\Gamma_Z(X, ?) \circ \underline{\Gamma}_{U,V}$ is isomorphic to the kernel of the map

$$\Gamma(U \cap W, ?) \to \Gamma(V \cap W, ?) \oplus \Gamma(U \cap W', ?).$$

Since this map factors through the injective map

$$\Gamma((V \cap W) \cup (U \cap W'), ?) \to \Gamma(V \cap W, ?) \oplus \Gamma(U \cap W', ?),$$

$\Gamma_Z(X, ?) \circ \underline{\Gamma}_{U,V}$ is isomorphic to $\Gamma_E(X, ?)$, where E is the locally closed subset $(U \cap W) \setminus ((V \cap W) \cup (U \cap W')) = (U \setminus V) \cap (W \setminus W')$.

First we consider the case that $\mathbb{I}$ is strictly injective (i.e., K-injective with each term injective). Then

$$0 \to \underline{\Gamma}_{U,V}\,\mathbb{I} \xrightarrow{\iota} f_* f^* \mathbb{I} \xrightarrow{u} f_* g_* g^* f^* \mathbb{I} \to 0 \tag{6}$$

is exact, since each term of $\mathbb{I}$ is flabby. By Lemma 4.5 and Lemma 4.6, $f_* f^* \mathbb{I}$ and $f_* g_* g^* f^* \mathbb{I}$ are K-flabby. So the (-1)-shift of the mapping cone of $u : f_* f^* \mathbb{I} \to f_* g_* g^* f^* \mathbb{I}$ is a K-flabby resolution of $\underline{\Gamma}_{U,V}\,\mathbb{I}$. So to verify that $\underline{\Gamma}_{U,V}\,\mathbb{I}$ is K-flabby, it suffices to show that (6) remains exact after applying $\Gamma_Z(X, ?)$ for any locally closed subset Z of X. Applying $\Gamma_Z(X, ?)$ to (6), we get a sequence

$$0 \to \Gamma_E(X, \mathbb{I}) \to \Gamma_{U \cap Z}(X, \mathbb{I}) \to \Gamma_{V \cap Z}(X, \mathbb{I}) \to 0, \tag{7}$$

which is exact by Lemma 4.7, as can be seen easily, where $E = (U \setminus V) \cap Z$. Thus the case that $\mathbb{I}$ is strictly injective is done.

Next consider the general case. Let $\varphi : \mathbb{I} \to \mathbb{J}$ be a strictly injective resolution, which exists, as $\operatorname{Mod}(X)$ is Grothendieck, see [3]. Since $\underline{\Gamma}_{U,V}\,\mathbb{J}$

is K-flabby, it suffices to show that for any locally closed subset Z of X, $\Gamma_Z(X, \underline{\Gamma}_{U,V}\,\mathbb{I}) \to \Gamma_Z(X, \underline{\Gamma}_{U,V}\,\mathbb{J})$ is a quasi-isomorphism. So letting $\mathbb{K}$ the mapping cone of φ, it suffices to show that $\Gamma_Z(X, \underline{\Gamma}_{U,V}\,\mathbb{K})$ is exact. But this is trivial, since $\Gamma_Z(X, \underline{\Gamma}_{U,V}\,\mathbb{K}) \cong \Gamma_E(X, \mathbb{K})$, and $\mathbb{K}$ is K-flabby exact, where $E = (U \setminus V) \cap Z$. $\square$

4.9 Lemma. *Let X be a topological space, and $\mathbb{F}$ a complex of sheaves of abelian groups. If $\mathbb{F}$ is K-limp and each term of $\mathbb{F}$ is flabby, then $\mathbb{F}$ is K-flabby.*

Proof. Let $\varphi : \mathbb{F} \to \mathbb{I}$ be a strictly injective resolution. Note that $\mathbb{I}$ is K-limp, and each term of $\mathbb{I}$ is flabby. So replacing $\mathbb{F}$ by the mapping cone of φ, we may assume that $\mathbb{F}$ is exact, and we are to prove that $\Gamma_Z(X, \mathbb{F})$ is exact for any locally closed subset Z of X. Let $V \subset U \subset X$ be open subsets of X such that $U \setminus V = Z$. Since each term of $\mathbb{F}$ is flabby,

$$0 \to \Gamma_Z(X, \mathbb{F}) \to \Gamma(U, \mathbb{F}) \to \Gamma(V, \mathbb{F}) \to 0$$

is a short exact sequence of complexes. Since $\mathbb{F}$ is K-limp exact, $\Gamma(U, \mathbb{F})$ and $\Gamma(V, \mathbb{F})$ are exact. Hence $\Gamma_Z(X, \mathbb{F})$ is also exact. $\square$

4.10 Lemma. *Let the notation be as in* (3.1)*. Then there is a triangle of the form*

$$R\,\underline{\Gamma}_{U,V} \xrightarrow{\iota} Rf_*f^* \xrightarrow{u} Rf_*Rg_*g^*f^* \to R\,\underline{\Gamma}_{U,V}[1].$$

Proof. Let $\mathbb{I}$ be a K-limp complex with each term of $\mathbb{I}$ flabby. Then there is a short exact sequence of complexes

$$0 \to \underline{\Gamma}_{U,V}\,\mathbb{I} \xrightarrow{\iota} f_*f^*\mathbb{I} \xrightarrow{u} f_*g_*g^*f^*\mathbb{I} \to 0.$$

The lemma follows immediately. $\square$

4.11 Corollary. *Let the notation be as in* (3.1)*. If f and g are quasi-compact, then $R\,\underline{\Gamma}_{U,V}(D_{\mathrm{Lqc}}(X)) \subset D_{\mathrm{Lqc}}(X)$. If f and g are quasi-compact cartesian and X has flat arrows, then $R\,\underline{\Gamma}_{U,V}(D_{\mathrm{Qch}}(X)) \subset D_{\mathrm{Qch}}(X)$.*

Proof. This follows from Lemma 4.10, [9, Lemma 8.5], [9, Lemma 8.7], and [9, Lemma 8.20]. $\square$

4.12 Lemma. *Let the notation be as in* (3.1)*. Assume that $f : U \hookrightarrow X$ and $g : V \hookrightarrow U$ are quasi-compact. If X is quasi-compact and I is finite, then $R\,\underline{\Gamma}_{U,V} : D_{\mathrm{Lqc}}(X) \to D_{\mathrm{Lqc}}(X)$ is way-out in both directions.*

Proof. Obvious by [9, Lemma 8.5] and Lemma 4.10. □

4.13 Lemma. *Let J be a subcategory of I. Then the canonical functor $\zeta : R(\underline{\Gamma}_{U_J,V_J}(?)_J) \to R\,\underline{\Gamma}_{U_J,V_J}(?)_J$ is an isomorphism.*

Proof. This is because if $\mathbb{I}$ is a strictly injective complex of $\mathrm{Mod}(X)$, then $\mathbb{I}_J$ is K-flabby. □

By the lemma, we have an isomorphism

$$R\,\underline{\Gamma}_{U_J,V_J}(?)_J \xrightarrow{\zeta^{-1}} R(\underline{\Gamma}_{U_J,V_J}(?)_J) \xrightarrow{R\hat{\gamma}} R((?)_J\,\underline{\Gamma}_{U,V}) \xrightarrow{\zeta} (?)_J R\,\underline{\Gamma}_{U,V},$$

which we denote simply by $\hat{\gamma}$.

4.14 Lemma. *Let the notation be as in* (3.5)*. Then the canonical map $\zeta\colon R(\underline{\Gamma}_{U,V}\,h_*) \to R\,\underline{\Gamma}_{U,V}\,Rh_*$ is an isomorphism.*

Proof. Let $\mathbb{I}$ be a K-injective complex of $\mathcal{O}_{X'}$-modules. Then $h_*\mathbb{I}$ is K-flabby by Lemma 4.5. So $h_*\mathbb{I}$ is $\underline{\Gamma}_{U,V}$-acyclic by Proposition 4.3, and the assertion follows. □

(4.15) By the lemma, the canonical map

$$R\,\underline{\Gamma}_{U,V}\,Rh_* \xrightarrow{\zeta^{-1}} R(\underline{\Gamma}_{U,V}\,h_*) \xrightarrow{\bar{\gamma}} R(h_*\,\underline{\Gamma}_{U',V'}) \xrightarrow{\zeta} Rh_*R\,\underline{\Gamma}_{U,V'},$$

which we denote by $\bar{\gamma}$, is defined.

4.16 Lemma. *Let the notation be as in* (3.5)*. Then the canonical map $\zeta : R(h_*\,\underline{\Gamma}_{U',V'}) \to Rh_*R\,\underline{\Gamma}_{U',V'}$ is an isomorphism.*

Proof. If $\mathbb{I}$ is a strictly injective complex of $\mathcal{O}_{X'}$-modules, then $\underline{\Gamma}_{U',V'}\,\mathbb{I}$ is K-flabby by Lemma 4.8. The lemma follows immediately. □

4.17 Corollary (Independence Theorem, cf. [2, (4.2.1)]). *Let the notation be as in* (3.5)*, and assume that* (a) *and* (b) *in the diagram* (2) *are cartesian. Then $\bar{\gamma} : R\,\underline{\Gamma}_{U,V}\,Rh_* \to Rh_*R\,\underline{\Gamma}_{U',V'}$ is an isomorphism.*

Proof. Follows immediately by the lemma and Lemma 3.6. □

(4.18) Let the notation be as in (3.1). Let $W \subset V$ be an open subdiagram of schemes, and $h : W \hookrightarrow V$ the inclusion. Let $\mathbb{I}$ be a complex in $\mathrm{Mod}(X)$. Assume that each term of $\mathbb{I}_i$ is flabby for any $i \in I$. Then the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & \underline{\Gamma}_{V,W}\,\mathbb{I} \arrow[d, "d^{-1}c\iota"] & \\
0 \arrow[r] & \underline{\Gamma}_{U,V}\,\mathbb{I} \arrow[r, "\iota"] \arrow[d, "\bar{\gamma}"] & f_*f^*\mathbb{I} \arrow[r, "u"] \arrow[d, "\mathrm{id}"] & f_*g_*g^*f^*\mathbb{I} \arrow[r] \arrow[d, "u"] & 0 \\
0 \arrow[r] & \underline{\Gamma}_{U,W}\,\mathbb{I} \arrow[r, "\iota"] & f_*f^*\mathbb{I} \arrow[r, "uu"] \arrow[d] & f_*g_*h_*h^*g^*f^*\mathbb{I} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

is commutative with exact rows and columns, where $d^{-1}c\iota$ is the composite

$$\underline{\Gamma}_{V,W} \xrightarrow{\iota} (fg)_*(fg)^* \xrightarrow{c} f_*g_*(fg)^* \xrightarrow{d^{-1}} f_*g_*g^*f^*.$$

Utilizing the snake lemma, it is easy to see that the sequence

$$0 \to \underline{\Gamma}_{U,V}\,\mathbb{I} \xrightarrow{\bar{\gamma}} \underline{\Gamma}_{U,W}\,\mathbb{I} \xrightarrow{\bar{\gamma}} \underline{\Gamma}_{V,W}\,\mathbb{I} \to 0$$

is exact. Thus we have a triangle

$$R\,\underline{\Gamma}_{U,V} \xrightarrow{\bar{\gamma}} R\,\underline{\Gamma}_{U,W} \xrightarrow{\bar{\gamma}} R\,\underline{\Gamma}_{V,W} \xrightarrow{\hat{\delta}} R\,\underline{\Gamma}_{U,V}[1],$$

where $\hat{\delta}$ is induced by

$$\underline{\Gamma}_{V,W} \hookrightarrow \mathrm{Cone}(\underline{\Gamma}_{U,W} \xrightarrow{\bar{\gamma}} \underline{\Gamma}_{V,W}) \xleftarrow[\simeq]{\bar{\gamma}} \underline{\Gamma}_{U,V}[1],$$

where $\simeq$ means a quasi-isomorphism.

5. Quasi-flabby sheaves

(5.1) The following definition is due to Kempf [16], although we make a slight modification here.

5.2 Definition. Let X be a topological space. A presheaf $\mathcal{M}$ of abelian groups on X is said to be *quasi-flabby* if the restriction map $\Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$ is surjective for any quasi-compact open subsets U and V such that $U \supset V$.

Note that a flabby sheaf is quasi-flabby. For the sake of completeness, we list Kempf's results for this modified definition.

5.3 Lemma ([16]). *Let X be a topological space such that the intersection of two quasi-compact open subsets is again quasi-compact. Let*

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

be a short exact sequence of sheaves of abelian groups. If $\mathcal{L}$ is quasi-flabby and U is a quasi-compact open subset of X, then the sequence

$$0 \to \Gamma(U, \mathcal{L}) \to \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N}) \to 0$$

is exact.

5.4 Corollary. *Let X be as in the lemma. Let*

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

be a short exact sequence of sheaves of abelian groups. If $\mathcal{L}$ and $\mathcal{M}$ are quasi-flabby, then so is $\mathcal{N}$.

5.5 Corollary. *Let X be as in the lemma. If $\mathcal{L}$ is quasi-flabby and U is a quasi-compact open subset, then $H^i(U, \mathcal{L}) = 0$ for $i > 0$.*

5.6 Lemma. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that Y has an open basis consisting of quasi-compact open subsets, and $f^{-1}(U)$ is quasi-compact if U is a quasi-compact open subset of Y. Assume moreover that Y has an open covering (U_λ) such that for any λ and quasi-compact open subsets V, V' of $f^{-1}(U_\lambda)$, $V \cap V'$ is again quasi-compact. Then for a short exact sequence*

$$0 \to \mathcal{L} \to \mathcal{M} \to \mathcal{N} \to 0$$

of sheaves of abelian groups on X with $\mathcal{L}$ quasi-flabby, the sequence

$$0 \to f_*\mathcal{L} \to f_*\mathcal{M} \to f_*\mathcal{N} \to 0$$

is exact.

Proof. It suffices to show that $(f|_{f^{-1}(U_\lambda)})_*\mathcal{M}|_{f^{-1}(U_\lambda)} \to (f|_{f^{-1}(U_\lambda)})_*\mathcal{N}|_{f^{-1}(U_\lambda)}$ is surjective for each λ. Since $\mathcal{L}|_{f^{-1}(U_\lambda)}$ is quasi-flabby for each λ, we may assume that for any two quasi-compact open subsets V, V' of X, $V \cap V'$ is quasi-compact, replacing $f : X \to Y$ by $f|_{f^{-1}(U_\lambda)} : f^{-1}(U_\lambda) \to U_\lambda$.

Since there is an open basis of Y consisting of quasi-compact open subsets, it suffices to show that $\Gamma(U, f_*\mathcal{M}) \to \Gamma(U, f_*\mathcal{N})$ is surjective for any quasi-compact open subset U of Y. Since $f^{-1}(U)$ is quasi-compact, this is Lemma 5.3. $\square$

5.7 Corollary. *Let $f : X \to Y$ be as in the lemma. Then if $\mathcal{L}$ is a quasi-flabby sheaf of abelian groups on X, then $R^i f_*\mathcal{L} = 0$ for $i > 0$.*

Proof. The question is local on Y, and we may assume that for any two quasi-compact open subsets V, V' of X, $V \cap V'$ is again quasi-compact. Take a short exact sequence of the form

$$0 \to \mathcal{L} \to \mathcal{I} \xrightarrow{p} \mathcal{L}' \to 0$$

with $\mathcal{I}$ injective. Since an injective sheaf is quasi-flabby, $\mathcal{L}'$ is quasi-flabby by Corollary 5.4.

We use the induction on i. Note that

$$f_*\mathcal{I} \xrightarrow{f_*p} f_*\mathcal{L}' \to R^1 f_*\mathcal{L} \to R^1 f_*\mathcal{I}$$

is exact. Since $\mathcal{I}$ is injective, $R^1 f_*\mathcal{I} = 0$. On the other hand, f_*p is surjective by the lemma. So $R^1 f_*\mathcal{L} = 0$.

Consider the case that $i \geq 2$. Then $R^i\mathcal{L} \cong R^{i-1}\mathcal{L}' = 0$ by induction. $\square$

5.8 Lemma. *Let X be a topological space. Assume that X has an open basis consisting of quasi-compact open subsets. Let U be a quasi-compact open subset of X, and $(\mathcal{M}_\lambda)$ a pseudo-filtered inductive system of sheaves of abelian groups on X. Then the canonical map*

$$\varinjlim \Gamma(U, \mathcal{M}_\lambda) \to \Gamma(U, \varinjlim \mathcal{M}_\lambda)$$

is an isomorphism.

5.9 Corollary. *Let X be as in the lemma. Then a filtered inductive limit of quasi-flabby sheaves is quasi-flabby.*

5.10 Corollary. *Let $f : X \to Y$ be a quasi-compact morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. If $(\mathcal{M}_\lambda)$ is a pseudo-filtered inductive system of sheaves of abelian groups on X, then the canonical map*

$$\varinjlim f_* \mathcal{M}_\lambda \to f_* \varinjlim \mathcal{M}_\lambda$$

is an isomorphism.

Proof. By restriction, we may assume that the problem is on single schemes. Since Y has an open basis consisting of quasi-compact open subsets, it suffices to show that for a quasi-compact open subset U of Y,

$$\Gamma(U, \varinjlim f_* \mathcal{M}_\lambda) \to \Gamma(U, f_* \varinjlim \mathcal{M}_\lambda) \tag{8}$$

is an isomorphism. Since U and $f^{-1}(U)$ are quasi-compact, the canonical maps

$$\varinjlim \Gamma(U, f_* \mathcal{M}_\lambda) \to \Gamma(U, \varinjlim f_* \mathcal{M}_\lambda)$$

and

$$\varinjlim \Gamma(f^{-1}(U), \mathcal{M}_\lambda) \to \Gamma(f^{-1}(U), \varinjlim \mathcal{M}_\lambda)$$

are isomorphisms by Lemma 5.8. Hence the map (8) is also an isomorphism, as required. □

5.11 Lemma. *Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$, U an open subdiagram of X, and V an open subdiagram of U. Assume that the inclusions $f : U \hookrightarrow X$ and $g : V \to U$ are quasi-compact. Then for a pseudo-filtered inductive system $(\mathcal{M}_\lambda)$ of $\mathcal{O}_X$-modules, the canonical map*

$$\varinjlim \underline{\Gamma}_{U,V} \mathcal{M}_\lambda \to \underline{\Gamma}_{U,V} \varinjlim \mathcal{M}_\lambda$$

is an isomorphism.

Proof. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim \underline{\Gamma}_{U,V} \mathcal{M}_\lambda \arrow[r, "\iota"] \arrow[d] & \varinjlim f_* f^* \mathcal{M}_\lambda \arrow[r, "u"] \arrow[d] & \varinjlim f_* g_* g^* f^* \mathcal{M}_\lambda \arrow[d] \\
0 \arrow[r] & \underline{\Gamma}_{U,V} \varinjlim \mathcal{M}_\lambda \arrow[r, "\iota"] & f_* f^* \varinjlim \mathcal{M}_\lambda \arrow[r, "u"] & f_* g_* g^* f^* \varinjlim \mathcal{M}_\lambda.
\end{tikzcd}$$

The middle and the right vertical arrows are isomorphisms by Corollary 5.10. By the five lemma, we are done. □

5.12 Lemma. *Let $f : X \to Y$ be a concentrated morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. Let $\mathcal{I}$ be an $\mathcal{O}_X$-module such that $\mathcal{I}_i$ is quasi-flabby for each $i \in I$. Then $\mathcal{I}$ is f_*-acyclic.*

Proof. We may assume that the problem is on a single scheme. This case is Corollary 5.7. □

5.13 Lemma. *Let $X \in \mathcal{P}(I, \underline{\mathrm{Sch}})$, and U an open subdiagram of schemes of X, and V an open subdiagram of schemes of U. Let $f : U \hookrightarrow X$ and $g : V \hookrightarrow U$ be inclusions. Assume that f and g are concentrated. If $\mathcal{M}$ is a quasi-flabby sheaf of abelian groups on X, then*

$$0 \to \underline{\Gamma}_{U,V}\, \mathcal{M} \xrightarrow{\iota} f_* f^* \mathcal{M} \xrightarrow{u} f_* g_* g^* f^* \mathcal{M} \to 0$$

is exact.

Proof. It suffices to show that for a quasi-compact open subset W of X, the restriction $\Gamma(U \cap W, \mathcal{M}) \to \Gamma(V \cap W, \mathcal{M})$ is surjective. This is trivial. □

5.14 Corollary. *Let the notation be as in the lemma. Then $\mathcal{M}$ is $\underline{\Gamma}_{U,V}$-acyclic.*

Proof. Note that $f^*\mathcal{M}$ is quasi-flabby, and hence is f_*-acyclic. Similarly, $g^* f^* \mathcal{M}$ is $(fg)_*$-acyclic. The lemma follows from the long exact sequence

$$\begin{aligned} 0 \to \underline{\Gamma}_{U,V}\, \mathcal{M} \xrightarrow{\iota} f_* f^* \mathcal{M} \xrightarrow{u} f_* g_* g^* f^* \mathcal{M} \to \\ \underline{H}^1_{U,V}\, \mathcal{M} \to R^1 f_* f^* \mathcal{M} \to R^1 (fg)_* (fg)^* \mathcal{M} \to \cdots, \end{aligned}$$

in which $u : f_* f^* \mathcal{M} \to f_* g_* g^* f^* \mathcal{M}$ is surjective. □

(5.15) Let $\mathcal{A}$ be an abelian category, and C a complex in $\mathcal{A}$. For $n \in \mathbb{Z}$, we define $\tau^{\leq n} C$ to be the truncated complex

$$\cdots \to C^{n-2} \to C^{n-1} \to \operatorname{Ker} d^n \to 0.$$

$\tau^{\geq n} C$ is defined to be the complex

$$0 \to \operatorname{Coker} d^{n-1} \to C^{n+1} \to C^{n+2} \to \cdots,$$

which is quasi-isomorphic to $C/\tau^{\leq n-1} C$.

5.16 Lemma (cf. [17, (3.9.3.1)]). *Let X, $f : U \to X$, and $g : V \to U$ be as in* Lemma 5.13. *Let (C_α) be a pseudo-filtered inductive system of complexes of $\mathcal{O}_X$-modules such that for each $j \in I$, there exists some $n_j \in \mathbb{Z}$ such that for any α, $\tau^{\leq n_j - 1}(C_\alpha)_j$ is exact. Set $C = \varinjlim C_\alpha$. Then the canonical map*

$$\varinjlim \underline{H}^i_{U,V} C_\alpha \to \underline{H}^i_{U,V} C \tag{9}$$

is an isomorphism for $i \in \mathbb{Z}$.

Proof. We may assume that the problem is on single schemes. Let n be an integer such that for any α, $\tau^{\leq n-1} C_\alpha$ is exact.

As in the proof of [17, (3.9.3.1)], let $\tau^{\geq n} C_\alpha \to F_\alpha$ be the Godement resolution so that we have a composite quasi-isomorphism $C_\alpha \to \tau^{\geq n} C_\alpha \to F_\alpha$. Note that each term of F_α is flabby. In particular, this is a $\underline{\Gamma}_{U,V}$-acyclic resolution. Then taking the inductive limit, we have a quasi-isomorphism $C \to F := \varinjlim F_\alpha$. Note that each term of F is quasi-flabby by Corollary 5.9. So this is also a $\underline{\Gamma}_{U,V}$-acyclic resolution.

So the map (9) is nothing but the composite

$$\varinjlim H^i(\underline{\Gamma}_{U,V} F_\alpha) \xrightarrow{\cong} H^i(\varinjlim \underline{\Gamma}_{U,V} F_\alpha) \xrightarrow{\cong} H^i(\underline{\Gamma}_{U,V} \varinjlim F_\alpha) = H^i(\underline{\Gamma}_{U,V} F),$$

where the second $\cong$ is an isomorphism by Lemma 5.11. This is what we wanted to prove. $\square$

6. Flat base change of local cohomology of diagrams

(6.1) Let the commutative diagram (2) be as in (3.5). Assume that h is flat. Then there is a canonical composite map

$$h^* R\,\underline{\Gamma}_{U,V} \xrightarrow{\zeta^{-1}} R(h^*\,\underline{\Gamma}_{U,V}) \xrightarrow{R\bar{\delta}} R(\underline{\Gamma}_{U',V'}\,h^*) \xrightarrow{\zeta} R\,\underline{\Gamma}_{U',V'}\,h^*,$$

which we denote by $\bar{\bar{\delta}}$.

6.2 Lemma. *Let the notation be as above. If h is an open immersion, then $\zeta : R(\underline{\Gamma}_{U',V'}\,h^*) \to R\,\underline{\Gamma}_{U',V'}\,h^*$ is an isomorphism.*

Proof. Let $\mathbb{I}$ be a K-injective complex of $\mathcal{O}_X$-modules. Then, by Lemma 4.6, $h^*\mathbb{I}$ is K-flabby, and hence is $\underline{\Gamma}_{U',V'}$-acyclic. The assertion follows. $\square$

6.3 Corollary. *Let the commutative diagram* (2) *be as in* (3.5). *If h is locally an open immersion and* (a) *and* (b) *are cartesian in* (2), *then* $\bar{\delta} : h^* R\,\underline{\Gamma}_{U,V} \to R\,\underline{\Gamma}_{U',V'}\,h^*$ *is an isomorphism.*

Proof. For $i \in I$, the diagram

$$\begin{tikzcd} h_i^* R\,\underline{\Gamma}_{U_i,V_i}(?)_i \arrow[r, "\hat{\gamma}"] \arrow[d, "\bar{\delta}"] & h_i^*(?)_i R\,\underline{\Gamma}_{U,V} \arrow[r, "\theta"] & (?)_i h^* R\,\underline{\Gamma}_{U,V} \arrow[d, "(?)_i\bar{\delta}"] \\ R\,\underline{\Gamma}_{U_i',V_i'}\,h_i^*(?)_i \arrow[r, "\theta"] & R\,\underline{\Gamma}_{U_i',V_i'}(?)_i h^* \arrow[r, "\hat{\gamma}"] & (?)_i R\,\underline{\Gamma}_{U',V'}\,h^* \end{tikzcd}$$

is commutative by Lemma 3.8. It suffices to show that the right vertical arrow $(?)_i\bar{\delta}$ is an isomorphism. So it suffices to show that the left vertical arrow $\bar{\delta} : h_i^* R\,\underline{\Gamma}_{U_i,V_i}(?)_i \to R\,\underline{\Gamma}_{U_i',V_i'}\,h_i^*(?)_i$ is an isomorphism. Hence we may assume that the problem is on single schemes.

First assume that h is an open immersion. Then the assertion follows immediately from the lemma, and Lemma 3.15.

Now consider the general case. Take an open covering $\bigcup_\lambda W_\lambda$ of X' such that $h|_{W_\lambda}$ is an open immersion for each λ. It suffices to show that $j^*\bar{\delta} : j^*h^*R\,\underline{\Gamma}_{U,V} \to j^*R\,\underline{\Gamma}_{U',V'}\,h^*$ is an isomorphism for each λ, where $j : W = W_\lambda \to X'$ is the inclusion. However, the diagram

$$\begin{tikzcd} j^*h^*R\,\underline{\Gamma}_{U,V} \arrow[rr, "d^{-1}"] \arrow[d, "j^*\bar{\delta}"] & & (hj)^*R\,\underline{\Gamma}_{U,V} \arrow[d, "\bar{\delta}"] \\ j^*R\,\underline{\Gamma}_{U',V'}\,h^* \arrow[r, "\bar{\delta}"] & R\,\underline{\Gamma}_{W\cap U',W\cap V'}\,j^*h^* \arrow[r, "d^{-1}"] & R\,\underline{\Gamma}_{W\cap U',W\cap V'}(hj)^* \end{tikzcd}$$

is commutative by Lemma 3.17, and the all arrows except for $j^*\bar{\delta}$ are isomorphisms by what we have already proved. Hence $j^*\bar{\delta}$ is also an isomorphism, as desired. $\square$

6.4 Lemma (cf. [17, (3.9.3.2)]). *Let X, $f : U \to X$, and $g : V \to U$ be as in* Lemma 5.13. *Let (C_α) be a pseudo-filtered inductive system of complexes in* $\mathrm{Mod}(X)$. *Assume one of the following.*

a) *U is locally noetherian, and for each $i \in I$, U_i admits an open covering (U_α) such that each U_α is of finite Krull dimension.*

b) *C_α has locally quasi-coherent cohomology groups for each α.*

c) *For each $j \in I$, there exists some $n_j \in \mathbb{Z}$ such that for any α, $\tau^{\leq n_j - 1}(C_\alpha)_j$ is exact.*

Then the canonical map

$$\varinjlim \underline{H}^i_{U,V} C_\alpha \to \underline{H}^i_{U,V} C$$

is an isomorphism for $i \in \mathbb{Z}$, where $C = \varinjlim C_\alpha$.

Proof. The case that **c)** is satisfied is Lemma 5.16. We consider the case that **a)** or **b)** is satisfied.

By restriction, we may assume that the problem is on a single scheme. Also, by Corollary 6.3, we may assume that X is an affine scheme.

As we assume **a)** (resp. **b)**), there exists some $d_0 \in \mathbb{Z}$ such that for any $i \in \mathbb{Z}$, any $d \geq d_0$, and any complex D in $\mathrm{Mod}(X)$ (resp. any complex D in $\mathrm{Mod}(X)$ with quasi-coherent cohomology groups), $R^i f_* f^*(\tau^{\leq i-d} D) = 0$ and $R^i (gf)_*(gf)^*(\tau^{\leq i-d} D) = 0$, see [17, Remarks in (3.9.3.2)]. This implies that $\underline{H}^i_{U,V}(\tau^{\leq i-d} D) = 0$ for any $i \in \mathbb{Z}$, $d \geq d_0 + 1$, and any complex D in $\mathrm{Mod}(X)$ (resp. any complex D in $\mathrm{Mod}(X)$ with quasi-coherent cohomology). So $\underline{H}^i_{U,V}(D) \to \underline{H}^i_{U,V}(\tau^{\geq i-d} D)$ is an isomorphism for $d \geq d_0$. As the square

$$\begin{tikzcd}
\varinjlim \underline{H}^i_{U,V}(C_\alpha) \arrow[r, "\cong"] \arrow[d] & \varinjlim \underline{H}^i_{U,V}(\tau^{\geq i-d_0} C_\alpha) \arrow[d] \\
\underline{H}^i_{U,V}(C) \arrow[r, "\cong"] & \underline{H}^i_{U,V}(\tau^{\geq i-d_0} C)
\end{tikzcd}$$

is commutative, replacing C_α by $\tau^{\geq i-d_0} C_\alpha$, we may assume that there exists some $n \in \mathbb{Z}$ such that $\tau^{\leq n-1} C_\alpha$ is exact for each α. This is the case where **c)** is assumed, and we are done. $\square$

6.5 Corollary (cf. [17, (3.9.3.3)]). *Let X, $f : U \to X$, and $g : V \to U$ be as in the lemma. Let (C_α) be a small family of complexes in $\mathrm{Mod}(X)$. If one of* **a)**, **b)**, *and* **c)** *in the lemma is satisfied, then the canonical map*

$$\bigoplus_\alpha R\,\underline{\Gamma}_{U,V}\, C_\alpha \to R\,\underline{\Gamma}_{U,V}(\bigoplus_\alpha C_\alpha)$$

is an isomorphism.

6.6 Corollary. *Let X, $f : U \to X$, and $g : V \to U$ be as in the lemma. If X is concentrated, then $R\,\underline{\Gamma}_{U,V} : D_{\mathrm{Lqc}}(X) \to D_{\mathrm{Lqc}}(X)$ has a right adjoint.*

Proof. Note that $D_{\mathrm{Lqc}}(X)$ is compactly generated by [9, Lemma 17.1]. The corollary follows from Corollary 6.5 and Neeman's theorem [19, Theorem 4.1]. $\square$

6.7 Corollary (cf. [17, (3.9.3.4)]). *Under the assumptions of* Lemma 6.4, *if each C_α is $\underline{\Gamma}_{U,V}$-acyclic, then C is $\underline{\Gamma}_{U,V}$-acyclic.*

Proof. By assumption, $H^i(\underline{\Gamma}_{U,V}\, C_\alpha) \to \underline{H}^i_{U,V}\, C_\alpha$ is an isomorphism for each $i \in \mathbb{Z}$ and α. Taking the inductive limit, the composite

$$H^i(\underline{\Gamma}_{U,V}\, C) \cong \varinjlim H^i(\underline{\Gamma}_{U,V}\, C_\alpha) \cong \varinjlim \underline{H}^i_{U,V}\, C_\alpha \cong \underline{H}^i_{U,V}\, C$$

is an isomorphism, where the first $\cong$ is an isomorphism is by Lemma 5.11, and the last $\cong$ is an isomorphism by Lemma 6.4. So C is $\underline{\Gamma}_{U,V}$-acyclic. $\square$

6.8 Corollary (cf. [17, (3.9.3.5)]). *Let X, $f : U \to X$, and $g : V \to U$ be as in* Lemma 5.13. *Let C be a complex in* $\mathrm{Mod}(X)$, *and assume one of the following.*

a) *U is locally noetherian, and for each $i \in I$, U_i admits an open covering (U_α) such that each U_α is of finite Krull dimension, and for each $i \in I$, each term of C_i is quasi-flabby.*

b) *X is locally noetherian, and for each $i \in I$, each term of C_i is an injective object of* $\mathrm{Qch}(X_i)$.

Then C is $\underline{\Gamma}_{U,V}$-acyclic.

Proof. Let $C \to \mathbb{I}$ be a K-injective resolution. Then $\mathbb{I}_i$ is $\underline{\Gamma}_{U_i,V_i}$-acyclic for each $i \in I$, since $\mathbb{I}_i$ is K-flabby. So it suffices to show that each C_i is $\underline{\Gamma}_{U_i,V_i}$-acyclic, and we may assume that the problem is on single schemes.

In each case, each term C^n of C is $\underline{\Gamma}_{U,V}$-acyclic. Indeed, in case **a)**, this is Corollary 5.14. In case **b)**, this is obvious, since an injective object of $\mathrm{Qch}(X)$ is an injective object of $\mathrm{Mod}(X)$ [7, (II.7)]. Thus the truncated subcomplex

$$\sigma^{\geq n} C : \cdots \to 0 \to 0 \to C^n \to C^{n+1} \to \cdots$$

of C is $\underline{\Gamma}_{U,V}$-acyclic for any $n \in \mathbb{Z}$. Since $C \cong \varinjlim \sigma^{\geq n} C$, the assertion follows from Corollary 6.7. $\square$

6.9 Lemma. *Let $h : X' \to X$ be a flat morphism between locally noetherian schemes. Let Y be a closed subscheme of X. Let $\mathbb{I}$ be an injective object of* $\mathrm{Qch}(X)$. *Then $h^*\mathbb{I}$ is $\underline{\Gamma}_{Y'}$-acyclic, where $Y' := h^{-1}(Y)$.*

Proof. By [7, Theorem II.7.18], $h^*\mathbb{I}$ has an injective resolution $\mathbb{J}$ in $\mathrm{Qch}(X')$. It is an injective resolution in $\mathrm{Mod}(X')$ as well, see [7, (II.7)]. Let $\mathcal{I}$ be the defining ideal sheaf of Y. Then Y' is defined by $\mathcal{I}\mathcal{O}_{Y'}$. So by Lemma 3.21 and [7, Proposition II.5.8],

$$\begin{aligned} R^i\,\underline{\Gamma}_{Y'}(\mathbb{I}) = H^i(\underline{\Gamma}_{Y'}(\mathbb{J})) &\cong H^i(\varinjlim \underline{\mathrm{Hom}}_{\mathcal{O}_{X'}}(h^*(\mathcal{O}_X/\mathcal{I}^n), \mathbb{J})) \\ &\cong \varinjlim \underline{\mathrm{Ext}}^i_{\mathcal{O}_{X'}}(h^*(\mathcal{O}_X/\mathcal{I}^n), h^*\mathbb{I}) \cong \varinjlim h^*(\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathbb{I})) = 0 \end{aligned}$$

for $i > 0$. □

6.10 Theorem (Flat base change, cf. [2, Theorem 4.3.2]). *Let $h : X' \to X$ be a flat morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}})$. Assume that X and X' are locally noetherian. Let Y be a cartesian closed subdiagram of schemes of X, and Z a cartesian closed subdiagram of schemes of Y. Then the canonical map $\bar{\delta} : h^*R\,\underline{\Gamma}_{Y;Z} \to R\,\underline{\Gamma}_{Y';Z'}\,h^*$ is an isomorphism of functors from $D_{\mathrm{Lqc}}(X)$ to $D_{\mathrm{Lqc}}(X')$, where $Y' = h^{-1}(Y)$ and $Z' = h^{-1}(Z)$.*

Proof. By an argument similar to the proof of Corollary 6.3, we may assume that the problem is on single schemes. Moreover, the question is local both on X and X' by Corollary 6.3, we may assume that both $X = \operatorname{Spec} A$ and $X' = \operatorname{Spec} B$ are affine.

Now by Lemma 4.12, $\underline{\Gamma}_{Y;Z} : D_{\mathrm{Lqc}}(X) \to D_{\mathrm{Lqc}}(X)$ and $\underline{\Gamma}_{Y';Z'} : D_{\mathrm{Lqc}}(X') \to D_{\mathrm{Lqc}}(X')$ are way-out in both directions.

By the way-out lemma [7, Proposition I.7.1], it suffices to show that $\bar{\delta} : h^*R\,\underline{\Gamma}_{Y;Z}\,\mathcal{I} \to R\,\underline{\Gamma}_{Y';Z'}\,h^*\mathcal{I}$ is an isomorphism for an injective object $\mathcal{I}$ of $\mathrm{Qch}(X)$.

Note that $\bar{\delta}$ is the composite

$$h^*R\,\underline{\Gamma}_{Y;Z}\,\mathcal{I} \xrightarrow{\zeta^{-1}} R(h^*\,\underline{\Gamma}_{Y;Z})\mathcal{I} \xrightarrow{R\bar{\delta}} R(\underline{\Gamma}_{Y';Z'}\,h^*)\mathcal{I} \xrightarrow{\zeta} R\,\underline{\Gamma}_{Y',Z'}\,h^*\mathcal{I}.$$

So it suffices to show that $R\bar{\delta}$ and ζ are isomorphisms.

By Lemma 3.15, $\bar{\delta} : h^*\,\underline{\Gamma}_{Y;Z}\,\mathcal{I} \to \underline{\Gamma}_{Y';Z'}\,h^*\mathcal{I}$ is an isomorphism. Since $\mathcal{I}$ is injective in $\mathrm{Mod}(X)$ [7, (II.7)], $R\bar{\delta}$ is an isomorphism.

To prove that ζ is an isomorphism, it suffices to prove that $h^*\mathcal{I}$ is $\underline{\Gamma}_{Y';Z'}$-acyclic. By (4.18), there is an exact sequence

$$\cdots \to \underline{H}^i_Z(h^*\mathcal{I}) \to \underline{H}^i_Y(h^*\mathcal{I}) \to \underline{H}^i_{Y;Z}(h^*\mathcal{I}) \to \underline{H}^{i+1}_Z(h^*\mathcal{I}) \to \cdots.$$

By Lemma 6.9, $\underline{H}^i_Y(h^*\mathcal{I}) = 0$ $(i > 0)$ and $\underline{H}^i_Z(h^*\mathcal{I}) = 0$ $(i > 0)$. So $\underline{H}^i_{Y;Z}(h^*\mathcal{I}) = 0$ for $i > 0$, as desired. □

7. Compatibility with G-invariance

(7.1) Let S be a scheme, G a flat S-group scheme, and X an S-scheme with a trivial G-action. As in [9, (30.1)], we denote the G-invariance functor $\mathrm{Mod}(G,X) \to \mathrm{Mod}(X)$ by $(?)^G$. By [9, Lemma 30.3], $(?)^G$ agrees with $(?)_{-1}R_{\Delta_M}$, where $R_{\Delta_M} : \mathrm{Mod}(G,X) = \mathrm{Mod}(B_G^M(X)) \to \mathrm{Mod}(\tilde{B}_G^M(X))$ is the right induction, where $\tilde{B}_G^M(X)$ is the augmented diagram described in [9, (30.2)]. If G is concentrated over S, then $(?)^G(\mathrm{Lqc}(G,X)) \subset \mathrm{Qch}(X)$. Note that $\tilde{B}_G^M(X)_{\Delta_M} = B_G^M(X)$. As in [9, section 29], for a G-morphism f, $B_G^M(f)_*$ is simply denoted by f_*, $B_G^M(f)^*$ is denoted by f^*, and so on.

$(?)^G = (?)_{-1}R_{\Delta_M}$ has an exact left adjoint $(?)_{\Delta_M}L_{-1}$. So

$$(?)^G : \mathrm{Mod}(G,X) \to \mathrm{Mod}(X)$$

preserves injectives, and $R(?)^G : D(G,X) \to D(X)$ preserves K-injectives.

It seems that the following question is fundamental.

7.2 Question. Let $\mathcal{I}$ be an injective object of $\mathrm{Qch}(G,X)$. Then $R^i(?)^G\mathcal{I} = 0$ for $i > 0$?

This is not obvious a priori, since the derived functor is computed in $D(G,X)$.

(7.3) Let $f\colon X \to Y$ be a morphism of S-schemes with trivial G-actions. Then $\tilde{B}_G^M(f)\colon \tilde{B}_G^M(X) \to \tilde{B}_G^M(Y)$ is induced. Note that $\tilde{B}_G^M(f)$ is cartesian. The composite isomorphism

$$\begin{aligned} e = e_f : f_*(?)^G = f_*(?)_{-1}R_{\Delta_M} &\xrightarrow{c^{-1}} (?)_{-1}\tilde{B}_G^M(f)_*R_{\Delta_M} \\ &\xrightarrow{\xi} (?)_{-1}R_{\Delta_M}B_G^M(f)_* = (?)^G f_* \end{aligned}$$

is induced, see [9, Corollary 6.26].

(7.4) Moreover, the natural map

$$\begin{aligned} \epsilon = \epsilon^f : f^*(?)^G = f^*(?)_{-1}R_{\Delta_M} &\xrightarrow{\theta} (?)_{-1}\tilde{B}_G^M(f)^*R_{\Delta_M} \\ &\xrightarrow{\mu} (?)_{-1}R_{\Delta_M}B_G^M(f)^* = (?)^G f^* \end{aligned}$$

is induced, see [9, (6.27)]. Note that θ is an isomorphism [9, (6.25)]. Exactly the same proof as in [9, (10.7)] shows that μ is an isomorphism of functors from $\mathrm{Lqc}(G, Y)$ to $\mathrm{Qch}(X)$, provided f is flat and G is concentrated over S. Similarly, μ is an isomorphism of functors from $\mathrm{Mod}(G, Y)$ to $\mathrm{Mod}(X)$ if f is locally an open immersion. Thus we have

7.5 Lemma. *Let $f : X \to Y$ be an S-morphism between S-schemes with trivial G-actions. If f is flat and G is concentrated over S, then*

$$\epsilon : f^*(?)^G \to (?)^G f^*$$

is an isomorphism between functors from $\mathrm{Lqc}(G, Y)$ *to* $\mathrm{Qch}(X)$*. If f is locally an open immersion, then ϵ is an isomorphism between functors from* $\mathrm{Mod}(G, Y)$ *to* $\mathrm{Mod}(X)$*.*

7.6 Lemma. *Let $f : X \to Y$ be as in* (7.3)*. Then the diagram*

$$\begin{tikzcd}
(?)^G \arrow[r, "u"] \arrow[d, "\mathrm{id}"] & f_* f^* (?)^G \arrow[d, "e\epsilon"] \\
(?)^G \arrow[r, "u"] & (?)^G f_* f^*
\end{tikzcd}$$

is commutative.

Proof. We need to prove that the composite

$$(?)_{-1} R_{\Delta_M} \xrightarrow{u} f_* f^* (?)_{-1} R_{\Delta_M} \xrightarrow{\theta} f_* (?)_{-1} \tilde{B}_G^M (f)^* R_{\Delta_M} \xrightarrow{\mu} f_* (?)_{-1} R_{\Delta_M} f^*$$
$$\xrightarrow{c^{-1}} (?)_{-1} \tilde{B}_G^M (f)_* R_{\Delta_M} f^* \xrightarrow{\xi} (?)_{-1} R_{\Delta_M} f_* f^*$$

agrees with u. Since $c^{-1}\mu$ in the composition above agrees with μc^{-1} by the naturality of c^{-1}, it suffices to show that the composite

$$(10) \qquad (?)_{-1} \xrightarrow{u} f_* f^* (?)_{-1} \xrightarrow{\theta} f_* (?)_{-1} \tilde{B}_G^M (f)^* \xrightarrow{c^{-1}} (?)_{-1} \tilde{B}_G^M (f)_* \tilde{B}_G^M (f)^*$$

agrees with u, and that the composite

$$(11) \qquad R_{\Delta_M} \xrightarrow{u} \tilde{B}_G^M (f)_* \tilde{B}_G^M (f)^* R_{\Delta_M} \xrightarrow{\mu} \tilde{B}_G^M (f)_* R_{\Delta_M} f^* \xrightarrow{\xi} R_{\Delta_M} f_* f^*$$

agrees with u.

(10) agrees with u by [9, Lemma 1.24]. (11) agrees with u by the commutativity of the diagram

$$\begin{tikzcd}
 & R_{\Delta_M} \arrow[r, "u"] \arrow[dl, "u"'] & R_{\Delta_M} f_* f^* \arrow[dl, "u"'] \arrow[dd, "\xi^{-1}"] \\
g_* g^* R_{\Delta_M} \arrow[r, "u"] \arrow[ddr, bend right, "\mu"'] & g_* g^* R_{\Delta_M} f_* f^* \arrow[d, "\xi^{-1}"] & \\
 & g_* g^* g_* R_{\Delta_M} f^* \arrow[d, "\varepsilon"] & g_* R_{\Delta_M} f^* \arrow[l, "u"'] \arrow[dl, "\mathrm{id}"'] \arrow[d, "\xi"] \\
 & g_* R_{\Delta_M} f^* \arrow[r, "\xi"] & R_{\Delta_M} f_* f^*,
\end{tikzcd}$$

where $g = \tilde{B}_G^M(f)$. $\square$

(7.7) Let X be a G-scheme, U a G-stable open subscheme of X, and V a G-stable open subscheme of U. The local section functor $\underline{\Gamma}_{B_G^M(U), B_G^M(V)} : \mathrm{Mod}(G, X) \to \mathrm{Mod}(G, X)$ is simply denoted by $\underline{\Gamma}_{U,V}$, and called the *equivariant local section functor*. The right derived functor $R^i \underline{\Gamma}_{U,V}$ is denoted by $\underline{H}^i_{U,V}$, and called the *equivariant local cohomology*. For a G-stable closed subscheme Y of X and a G-stable closed subscheme Z of Y, the local section functor $\underline{\Gamma}_{B_G^M(Y); B_G^M(Z)}$ is simply denoted by $\underline{\Gamma}_{Y;Z}$. As usual, $\underline{\Gamma}_{Y;\emptyset}$ is denoted by $\underline{\Gamma}_Y$. The derived functor $R^i \underline{\Gamma}_{Y;Z}$ is denoted by $\underline{H}^i_{Y;Z}$. $R^i \underline{\Gamma}_Y$ is denoted by $\underline{H}^i_Y$.

(7.8) Let X be an S-scheme with a trivial G-action. Let U be an open subscheme of X, and V an open subscheme of U. Let $f : U \hookrightarrow X$ be the inclusion, and $g : V \hookrightarrow U$ the inclusion.

By Lemma 7.6, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \underline{\Gamma}_{U,V}(?)^G \arrow[r, "\iota"] & f_* f^* (?)^G \arrow[r, "u"] \arrow[d, "e\epsilon"] & f_* g_* g^* f^* (?)^G \arrow[d, "e\epsilon\epsilon\epsilon"] \\
0 \arrow[r] & (?)^G \, \underline{\Gamma}_{U,V} \arrow[r, "\iota"] & (?)^G f_* f^* \arrow[r, "u"] & (?)^G f_* g_* g^* f^*.
\end{tikzcd} \tag{12}$$

So there is a unique natural map

$$E : \underline{\Gamma}_{U,V}(?)^G \to (?)^G \, \underline{\Gamma}_{U,V}$$

such that $\iota E = e\epsilon\iota$.

By Lemma 7.5, the vertical maps in (12) are isomorphisms. So E is an isomorphism.

8. G-local G-schemes

Let S be a scheme, G a flat S-group scheme concentrated over S, and X a G-scheme (i.e., an S-scheme with a left G-action).

(8.1) Let $\iota : Y \hookrightarrow X$ be a subscheme. We denote the composite

$$G \times Y \xrightarrow{1_G \times \iota} G \times X \xrightarrow{a} X$$

by a_Y, where a is the action. If a_Y factors through Y, then we say that Y is *G-stable*. In this case, Y has a unique G-scheme structure such that ι is a G-morphism.

The scheme-theoretic image of a_Y is denoted by Y^*. If ι is quasi-compact, then Y^* is the smallest closed G-stable subscheme of X containing Y, see [8, Lemma 2.1.5].

(8.2) A closed subscheme Y of X is G-stable if and only if $Y = Y^*$. Let $(Y_\lambda)_{\lambda \in \Lambda}$ be a family of closed subschemes of X. If Y_λ is defined by a quasi-coherent ideal sheaf $\mathcal{I}_\lambda$, then the sum $\sum_\lambda \mathcal{I}_\lambda$ is a quasi-coherent ideal sheaf again, and it defines the intersection $\bigcap_\lambda Y_\lambda$ (that is, the direct product of Y_λ in the category of X-schemes. It is also the usual intersection, set theoretically). If each Y_λ is G-stable, then $\bigcap Y_\lambda$ is also G-stable. The complement of a G-stable closed subscheme is a G-stable open subset.

(8.3) The intersection of finitely many G-stable open subsets is G-stable. Moreover, the union of G-stable open subsets is G-stable. Letting a G-stable open subset open, we can define a topology on X. We call this topology the *G-Zariski topology.*

If X is quasi-compact with respect to the G-Zariski topology, we say that X is *G-quasi-compact.* Since the G-Zariski-topology is coarser than the Zariski topology, a quasi-compact G-scheme is G-quasi-compact.

Let U be a G-stable open subset of X, and let Y be $X \setminus U$ with the reduced structure. It is easy to verify that Y^* does not intersect U (so $Y^* = Y$, set theoretically). Note that Y^* is G-stable. So U has a G-stable complement Y^*. Thus a closed subset in the G-Zariski topology is nothing but an underlying subset of some G-stable closed subscheme. If Y is a G-stable subscheme of X, then the G-Zariski topology of Y agrees with the induced topology of Y, induced from the G-Zariski topology of X. If $f : X \to X'$ is a G-morphism of G-schemes, then f is continuous with respect to the G-Zariski topologies.

8.4 Lemma. *If X is G-quasi-compact and Y is a G-stable closed subscheme of X, then there is a minimal non-empty closed G-subscheme of Y.*

Proof. Y is G-quasi-compact, since it is a closed subset of quasi-compact X, with respect to the G-Zariski topology. Let Ω be the set of non-empty G-stable closed subschemes of Y. For $Z, Z' \in \Omega$, we say that $Z \leq Z'$ if $Z \supset Z'$. Then by Zorn's lemma, Ω has a maximal element, and the proof is complete. □

8.5 Lemma. *Assume that $G \to S$ is universally open. Then any $x \in X$ has a quasi-compact G-stable open neighborhood.*

Proof. Let U be an affine open neighborhood of x. Since the action $a : G \times X \to X$ is an open map, $U^\star := a(G \times U)$ is open, and it is G-stable, as can be seen easily. Since U is quasi-compact and G is quasi-compact over S, $G \times U$ is quasi-compact. So $U^\star$ is quasi-compact. Since $U^\star \supset U$ is obvious, $U^\star$ is a desired open neighborhood of x. □

Since we assume that G is flat, if G is locally of finite presentation over S, then $G \to S$ is universally open ([5, (I.10.4)]).

8.6 Corollary. *Let $G \to S$ be universally open. If X is G-quasi-compact, then X is quasi-compact.*

Proof. X has an open covering U_λ consisting of quasi-compact G-stable open subschemes by the lemma. By assumption, there exists $\lambda_1, \ldots, \lambda_n$ such that $X = \bigcup_{i=1}^n U_{\lambda_i}$. Since each U_{λ_i} is quasi-compact, X is quasi-compact. □

(8.7) A topological space Γ is said to be *local* if it has a unique minimal non-empty closed subset, say Θ. In this case, we say that (Γ, Θ) is local.

8.8 Lemma. *Let Γ be a topological space. Then the following are equivalent.*

1 Γ *is local.*

2 Γ *is non-empty, and if (F_λ) is a non-empty family of non-empty closed subsets of Γ, then $\bigcap F_\lambda$ is non-empty.*

3 Γ *is non-empty, and for any open covering* (U_λ) *of* Γ*, there exists some* λ *such that* $X = U_\lambda$*.*

In particular, a local topological space is non-empty and quasi-compact.

Proof. **1⇒2** Let (Γ, Θ) be local. Then $\Gamma \supset \Theta \neq \emptyset$. Moreover, $\bigcap_\lambda F_\lambda \supset \Theta \neq \emptyset$.

2⇒1 Let Ω be the set of non-empty closed subsets of Γ. Then $\bigcap_{F \in \Omega} F$ is the desired unique minimal non-empty closed subset of Γ.

2⇔3 is trivial. □

8.9 Corollary. *If* $f : \Gamma \to \Gamma'$ *is a surjective continuous map of topological spaces and* Γ *is local, then* Γ' *is local. If* (Γ, Θ) *is local, then* (Γ', Θ') *is local, where* Θ' *is the closure of* $f(\Theta)$*.*

Proof. As f is a map and Γ is non-empty, Γ' is non-empty. Let Ω' be a non-empty set of non-empty closed subsets of Γ'. Then $f^{-1}(F') \neq \emptyset$ for $F' \in \Omega'$ by the surjectivity of f. So $f^{-1}(\bigcap_{F' \in \Omega'} F') = \bigcap f^{-1}F' \neq \emptyset$ by the localness of Γ. So Γ' is local.

We prove the last assertion. As f is surjective, $f^{-1}(\Theta')$ is a non-empty closed subset of Γ, and hence $f^{-1}(\Theta') \supset \Theta$. So the closure of $f(\Theta)$ is a non-empty closed subset of Θ'. By minimality, they agree. □

8.10 Lemma. *A* T_0*-space* Γ *is local if and only if* Γ *is quasi-compact and has exactly one closed point* γ*. In this case,* (Γ, γ) *is local.*

Proof. We prove the 'only if' part. Γ is non-empty and quasi-compact by Lemma 8.8. A non-empty quasi-compact T_0-space has a closed point. So Γ has at least one closed point γ. However, a closed point is minimal non-empty closed. Such a point must be unique, and the last assertion is also obvious.

We prove the 'if' part. Let F be a non-empty closed subset of Γ. Then F is non-empty quasi-compact T_0, and has a closed point. This closed point must be γ. So γ is the unique minimal non-empty closed subset of Γ. □

For $x, y \in \Gamma$, we define $x \equiv y$ if $\bar{x} = \bar{y}$, where the bar denotes the closure. The quotient space $\Gamma/\equiv$ is called the T_0*-ification* of Γ.

8.11 Lemma. *Let* $\pi : \Gamma \to \Gamma_0$ *be the* T_0*-ification. Then* Γ *is local if and only if* Γ_0 *is local. If* (Γ, Θ) *and* (Γ_0, Θ_0) *are local, then* $\pi(\Theta) = \Theta_0$*, and* $\Theta = \pi^{-1}(\Theta_0)$*.*

Proof. Since π is surjective and continuous, if Γ is local, then Γ_0 is local by Corollary 8.9.

We prove the converse. As Γ_0 is non-empty and π is surjective, Γ is non-empty. Let Ω be a non-empty set of non-empty closed subsets of Γ. Then for each $F \in \Omega$, $F = \pi^{-1}(\pi(F))$. Since π is submersive (i.e., for any subset F' of Γ_0, F' is closed if and only if $\pi^{-1}(F')$ is closed), $\pi(F)$ is closed, and is non-empty. So

$$\pi(\bigcap_{F \in \Omega} F) = \pi(\pi^{-1}(\bigcap \pi(F))) = \bigcap \pi(F) \neq \emptyset.$$

Hence $\bigcap F$ is non-empty, and Γ is local.

$\pi(\Theta) = \Theta_0$ follows from Corollary 8.9, since Θ_0 is a point by Lemma 8.10. Since Θ is closed, $\Theta = \pi^{-1}(\pi(\Theta)) = \pi^{-1}(\Theta_0)$. □

8.12 Lemma. *For a scheme Z, the following are equivalent.*

1 *The underlying topological space of Z is local.*

2 *Z is local, that is, $Z \cong \operatorname{Spec} A$ for some local ring $(A, \mathfrak{m})$.*

3 *Z is quasi-compact, and has a unique closed point z.*

In this case, $(Z, z) \cong (\operatorname{Spec} A, \mathfrak{m})$ are local topological spaces.

Proof. **1⇒2** Let (U_λ) be an affine open covering of Z. Then $Z = U_\lambda$ for some λ by Lemma 8.8. So $Z \cong \operatorname{Spec} A$ is affine. Since Z is non-empty, A is non-zero, and has a maximal ideal. If A has two or more maximal ideals, then Z has two or more closed points, and Z cannot be local. So A is a local ring.

2⇒3 is obvious.

3⇒1 is a consequence of Lemma 8.10, since a scheme is T_0.

The last assertion is obvious. □

8.13 Definition. We say that a G-scheme X is *G-local* if there is a unique minimal non-empty G-stable closed subscheme of X. If X is G-local and Y is the unique minimal non-empty G-stable closed subscheme, then we say that (X, Y) is G-local.

8.14 Lemma. *Let X be a G-scheme. Then the following are equivalent.*

1 *X is G-local.*

2 *X is local in the G-Zariski topology.*

In particular, a G-local G-scheme is G-quasi-compact. Moreover, if (X, Y) is G-local, then (X, Y) is local in the G-Zariski topology.

Proof. **1⇒2** Let (X, Y) be G-local. If F is a non-empty closed subset of X in the G-Zariski topology, then F is the underlying set of some non-empty G-stable closed subscheme of X. So $F \supset Y$, and (X, Y) is local in the G-Zariski topology.

2⇒1 Let $Y = \bigcap_{F \in \Omega} F$, where Ω is the set of all non-empty G-stable closed subschemes of X. Then Y is non-empty by assumption, and (X, Y) is G-local. □

8.15 Corollary. *If $G \to S$ is universally open, then a G-local G-scheme is quasi-compact.*

Proof. Follows immediately from the lemma and Corollary 8.6. □

8.16 Corollary. *Let $f : X \to X'$ be a surjective G-morphism of G-schemes. If X is G-local, then X' is G-local. If, moreover, f is concentrated, (X, Y) is G-local, and (X', Y') is G-local, then the scheme-theoretic image of $f|_Y$ is Y'.*

Proof. The first assertion is an immediate consequence of the theorem and Corollary 8.9. We prove the last assertion. As f is surjective, $Y \subset f^{-1}(Y')$. So the scheme-theoretic image of $f|_Y$ is contained in Y'. Since f is concentrated, $f|_Y$ is also concentrated, and hence the scheme-theoretic image of $f|_Y$ is G-stable closed, since $(f|_Y)_*\mathcal{O}_Y \in \mathrm{Qch}(G, X')$. By the minimality of Y', it agrees with Y'. □

Here are some examples of G-local G-schemes.

8.17 Example. Assume that G is trivial. Then the G-Zariski topology agrees with the usual Zariski topology, and X is G-local if and only if X is a local scheme by Lemma 8.12.

8.18 Example. If $S = \operatorname{Spec} k$, where k is a field, then (G, G) is G-local, where G acts on G left regularly.

Proof. It suffices to show that if Y is a non-empty G-stable closed subscheme of G, then $Y = G$. As Y is non-empty, Y has a geometric point $\eta : \operatorname{Spec} K \to Y$. Taking the base change and replacing k by K, we may assume that Y has a k-rational point y. Then $G \to G$ ($g \mapsto gy$) is an isomorphism, and hence $Y = Y^* \supset \{y\}^* = G$. $\square$

8.19 Example. If $S = \operatorname{Spec} k$, where k is a field, G affine and of finite type, and X is a homogeneous space G/H for some closed subgroup scheme H of G, then (X, X) is G-local. This example shows that even if S and G are affine, a G-local G-scheme X need not be affine in general.

Proof. Let $p : G \to X = G/H$ be the canonical projection. Then p is faithfully flat, and is surjective. As (G, G) is G-local by Example 8.18, (X, X) is G-local by Corollary 8.16. $\square$

8.20 Example. Let $S = \operatorname{Spec} \mathbb{Z}$ and $G = \mathbb{G}_m^n$, the split torus over S. Let $X = \operatorname{Spec} A$ be affine. Then A is a $\mathbb{Z}^n$-graded ring in a natural way [8, (II.1.2)]. By definition, X is G-local if and only if A is H-local in the sense of Goto–Watanabe [4].

8.21 Example. Let $S = \operatorname{Spec} k$ with k an algebraically closed field, G a reductive group, B a Borel subgroup of G, and P a parabolic subgroup of G containing B. A Schubert subvariety of G/P is a B-stable closed subvariety by definition. The point P/P is the unique minimal Schubert subvariety (see [15, Chapter 13]), and we have $(G/P, P/P)$ is B-local.

(8.22) Let $S = \operatorname{Spec} k$, with k a field. We say that G is *geometrically reductive*, if G is affine of finite type, and for any finite dimensional G-module V and any $v \in V^G \setminus 0$, there exists some $r > 0$ and $f \in (\operatorname{Sym}_r V^*)^G$ such that $f(v) \neq 0$. If, moreover, r can be taken to be 1 (for any V and v), then we say that G is linearly reductive. If r can be taken to be 1 if the characteristic of k is zero, and a power of p if the characteristic p of k is positive, then we say that G is *strongly geometrically reductive* (*SGR* for short). By definition, a linearly reductive group scheme is SGR. We can prove that G is geometrically reductive if and only if G is SGR if and only if the radical of the linear algebraic group $(\bar{k} \otimes_k G)_{\mathrm{red}}$ is a torus if and only if for any finitely generated k algebra A with a G-action, A^G is finitely generated [10]. This fact is probably well-known for linear algebraic groups. We will not use this fact later, and mainly consider SGR property.

(8.23) Assume that $S = \operatorname{Spec} R$ and G are affine. We say that A is a *G-algebra* if A is an R-algebra, and a G-scheme structure of $\operatorname{Spec} A$ is given. This is equivalent to say that A is both an R-algebra and a G-module (whose underlying R-module structure agree), and the product $A \otimes_R A \to A$ is G-linear. An ideal I of A is called a *G-ideal* if $\operatorname{Spec} A/I$ is a G-stable closed subscheme of $\operatorname{Spec} A$. Or equivalently, I is a (G, A)-submodule of A.

8.24 Lemma. *Let $S = \operatorname{Spec} k$ with k a field, and G an SGR k-group scheme. Let A be a G-algebra. If I_λ is a family of G-ideals. Let $f \in (\sum_\lambda I_\lambda)^G$. Then there exists some q such that $f^q \in \sum_\lambda I_\lambda^G$, where q is required to be a power of p if the characteristic of k is $p > 0$, and $q = 1$ if k is of characteristic zero.*

Proof. See [18, Appendix to Chapter 1, C]. $\square$

(8.25) Let S and G be affine, and A a G-algebra. A maximal element of

$$\{I \mid I \text{ is a } G\text{-ideal and } I \neq A\}$$

is said to be *G-maximal.* We say that A is *G-local* if A has a unique G-maximal G-ideal. A is G-local if and only if $\operatorname{Spec} A$ is G-local.

8.26 Lemma. *Let S and G be affine. If A is a G-algebra and $I \neq A$ a G-ideal, then there is a G-maximal ideal of A containing I.*

Proof. Since $X = \operatorname{Spec} A$ is quasi-compact, it is G-quasi-compact. Now apply Lemma 8.4. $\square$

8.27 Proposition. *Let $S = \operatorname{Spec} k$ with k a field and G SGR. Let A be a G-algebra. If $\mathfrak{p} \in \operatorname{Spec} A^G$, then $A_\mathfrak{p} := A \otimes_{A^G} A^G_\mathfrak{p}$ is G-local.*

Proof. Note that $(A_\mathfrak{p})^G = A^G_\mathfrak{p}$. Replacing A by $A_\mathfrak{p}$, we may assume that $(A^G, \mathfrak{m})$ is a local ring, and we are to prove that A is G-local.

Since A^G is nonzero, A is nonzero. By Lemma 8.26, A has a G-maximal ideal. Assume that A has two different G-maximal ideals I and J. Since $1 \notin I$ and $1 \notin J$, $I^G \subset \mathfrak{m}$ and $J^G \subset \mathfrak{m}$. On the other hand, $I + J = A$ by maximality. By Lemma 8.24, $1 \in I^G + J^G \subset \mathfrak{m}$. This is a contradiction. So A is G-local. $\square$

Note that in the proposition above, $\mathfrak{p}A_\mathfrak{p}$ may not be the G-maximal ideal. Indeed, if $G = \mathbb{G}_m$ and $A = k[x]$ with $\deg x = 1$ and $\mathfrak{p} = 0 \subset A^G = k$, then $0 = \mathfrak{p}A_\mathfrak{p}$ is not G-maximal, since $(x) \subset A$ is a G-ideal.

9. A generalization of a theorem of Hochster–Eagon

Let S, G, and X be as in the last section. In this section, we give an application of equivariant local cohomology on a G-local G-scheme to invariant theory.

9.1 Lemma. *Let $S = \operatorname{Spec} k$ with k a field, and G SGR. Let A be a G-algebra. Assume that the canonical map $\pi : \operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient in the sense of* [18]. *Then for any prime ideal $\mathfrak{p}$ of A^G, $\mathfrak{p}A_{\mathfrak{p}}$ and the G-maximal ideal P of the G-local ring $A_{\mathfrak{p}}$ have the same radical.*

Proof. Since $\mathfrak{p}A_{\mathfrak{p}}$ is a G-ideal of $A_{\mathfrak{p}}$, we have $P \supset \mathfrak{p}A_{\mathfrak{p}}$. Assume that $\sqrt{P} \neq \sqrt{\mathfrak{p}A_{\mathfrak{p}}}$. Then there is an algebraically closed extension field K of $\kappa(\mathfrak{p})$ such that, there are K-valued points ξ of $V(P)$ and η of $V(\mathfrak{p}A_{\mathfrak{p}}) \setminus V(P)$, and the set of K-valued points of $V(\mathfrak{p}A_{\mathfrak{p}})$ is one orbit with respect to the action of $G(K)$. But since $V(P)$ is G-stable, $\xi \in V(P)(K)$, and $\eta \notin V(P)(K)$, ξ and η cannot be on the same orbit. This is a contradiction. Hence $\sqrt{P} = \sqrt{\mathfrak{p}A_{\mathfrak{p}}}$. $\square$

(9.2) Let X be a locally noetherian G-scheme, and $\mathcal{M}$ a coherent $(G, \mathcal{O}_X)$-module. Then $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M})$ is also a coherent $(G, \mathcal{O}_X)$-module, as can be seen easily from [9, Lemma 6.33] and [9, Lemma 7.11]. The canonical map

$$\mathcal{O}_X \to \underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M})$$

is $(G, \mathcal{O}_X)$-linear. Hence the kernel $\underline{\operatorname{ann}}\, \mathcal{M}$ is a coherent G-ideal. Hence $\operatorname{Supp} \mathcal{M} = V(\underline{\operatorname{ann}}\, \mathcal{M})$ is a G-stable closed subscheme of X.

(9.3) Let (X, Y) be a G-local G-scheme. Assume that X is noetherian. Let Z be any irreducible component of Y, and ζ the generic point of Z.

9.4 Lemma. *The functor $(?)_\zeta : \operatorname{Qch}(G, X) \to \operatorname{Mod}(\mathcal{O}_{X,\zeta})$ is faithful exact.*

Proof. The restriction $\operatorname{Qch}(G, X) \to \operatorname{Qch}(X)$ is obviously exact. On the other hand, the stalk functor $(?)_\zeta : \operatorname{Qch}(X) \to \operatorname{Mod}(\mathcal{O}_{X,\zeta})$ is exact. Hence the composite is exact.

We prove that the functor in question is faithful. Assume the contrary, and let $\mathcal{M} \in \operatorname{Qch}(G, X)$, $\mathcal{M} \neq 0$, and $\mathcal{M}_\zeta = 0$. Then since $\operatorname{Qch}(G, X)$ is locally noetherian and a noetherian object of $\operatorname{Qch}(G, X)$ is nothing but a coherent $(G, \mathcal{O}_X)$-module by [9, Corollary 11.8], there exists some nonzero coherent $(G, \mathcal{O}_X)$-submodule $\mathcal{N}$ of $\mathcal{M}$. Let $V := \operatorname{Supp} \mathcal{N}$. Then V is nonempty, closed, and G-stable. Hence $V \supset Y \supset Z \ni \zeta$. Hence $0 = \mathcal{M}_\zeta \supset \mathcal{N}_\zeta \neq 0$, and this is a contradiction. $\square$

9.5 Theorem. *Let k be a field, G a linearly reductive k-group scheme, and X a Cohen–Macaulay noetherian G-scheme. Let $\pi : X \to Y$ be a geometric quotient under the action of G in the sense of* [18]. *Assume that π is an affine morphism. Then Y is noetherian and Cohen–Macaulay.*

Proof. Since π is surjective, Y is quasi-compact. So it suffices to show that Y is locally noetherian and Cohen–Macaulay. The question is local on Y, and we may assume that $Y = \operatorname{Spec} A$ is affine.

Since π is affine, $X = \operatorname{Spec} B$ is also affine, and $A = B^G$ by assumption. Note that A is a direct summand subring of B, since G is linearly reductive. In particular, A is noetherian, since B is (see [14, Proposition 6.15]).

It remains to show that A is Cohen–Macaulay. In order to prove this, localizing A at a maximal ideal of it, we may further assume that $(A, \mathfrak{m})$ is local. Note that π is still submersive after localization, since G is linearly reductive and $A = B^G$, see the proof of [18, Theorem 1.1]. By Proposition 8.27, X is G-local. Let Z be the unique maximal non-empty closed G-subscheme of X.

Let y be the closed point of Y. Then

$$\underline{H}^i_y(\mathcal{O}_Y) \cong H^i(R\,\underline{\Gamma}_y((\pi_*\mathcal{O}_X)^G)).$$

Let $\mathbb{J}$ be the injective resolution of $\pi_*\mathcal{O}_X$ in $\mathrm{Qch}(G, Y)$. Then $\mathbb{J}^G$ is an injective resolution of $(\pi_*\mathcal{O}_X)^G$ in $\mathrm{Qch}(Y)$, since $(?)^G : \mathrm{Qch}(G, Y) \to \mathrm{Qch}(Y)$ is exact, and preserves injectives (since it has an exact left adjoint $(?)_{\Delta_M} L_{-1}$). Any injective object of $\mathrm{Qch}(Y)$ is injective in $\mathrm{Mod}(Y)$ by [7, (II.7)]. Hence we have isomorphisms

$$\underline{H}^i_y(\mathcal{O}_Y) \cong H^i(\underline{\Gamma}_y\,\mathbb{J}^G) \cong H^i((\underline{\Gamma}_y\,\mathbb{J})^G) \cong (H^i(\underline{\Gamma}_y\,\mathbb{J}))^G$$

where the second isomorphism is by (7.8), and the third isomorphism is by the exactness of $(?)^G$ on $\mathrm{Qch}(G, Y)$ (note that $\underline{\Gamma}_y\,\mathbb{J}$ is a complex in $\mathrm{Qch}(G, Y)$ by Corollary 4.11).

So to show that Y is Cohen–Macaulay, it suffices to show that the cohomology of the complex $\underline{\Gamma}_y\,\mathbb{J}$ is concentrated in one place.

By Lemma 9.1, $\pi^{-1}(y)$ and Z agree, set theoretically. So by Corollary 4.17,

$$\begin{aligned} H^i(\underline{\Gamma}_y\,\mathbb{J}) &\cong H^i(R\,\underline{\Gamma}_y(\pi_*\mathcal{O}_X)) \cong H^i(R\,\underline{\Gamma}_y\,R\pi_*\mathcal{O}_X) \\ &\cong H^i(R\pi_* R\,\underline{\Gamma}_{\pi^{-1}(y)}\,\mathcal{O}_X) = H^i(R\pi_* R\,\underline{\Gamma}_Z\,\mathcal{O}_X). \end{aligned}$$

Note that

$$(?)_\zeta\, \underline{H}^i_Z(\mathcal{O}_X) \cong H^i((?)_\zeta R\,\underline{\Gamma}_Z\,\mathcal{O}_X) \cong H^i_\zeta(\mathcal{O}_{X,\zeta})$$

by Theorem 6.10. $H^i_\zeta(\mathcal{O}_{X,\zeta}) = 0$ for $i \neq d$ ($d := \dim \mathcal{O}_{X,\zeta}$), since $\mathcal{O}_{X,\zeta}$ is a Cohen–Macaulay local ring. Since $(?)_\zeta$ is faithful exact by Lemma 9.4, $\underline{H}^i_Z(\mathcal{O}_X) = 0$ for $i \neq d$. Let $\mathcal{M} := \underline{H}^d_Z(\mathcal{O}_X)$. Note that $\mathcal{M}$ is quasi-coherent. So

$$R\pi_* R\,\underline{\Gamma}_Z\,\mathcal{O}_X \cong R\pi_*\mathcal{M}[-d] \cong \pi_*\mathcal{M}[-d].$$

Hence

$$H^i(\underline{\Gamma}_y\,\mathbb{J}) \cong H^i(R\pi_* R\,\underline{\Gamma}_Z\,\mathcal{O}_X) = H^{d-i}(\pi_*\mathcal{M}) = 0$$

for $i \neq d$. This is what we wanted to prove. □

9.6 Corollary. *Let k be an algebraically closed field, G a linearly reductive k-group scheme, and $X = \operatorname{Spec} B$ a Cohen–Macaulay affine G-scheme of finite type. Let $\pi : X \to Y = \operatorname{Spec} B^G$ be the canonical morphism, and set*

$$U := \{x \in X \mid \dim O(x) \text{ is maximal, and } O(x) \text{ is closed}\},$$

where $O(x)$ is the G-orbit of x. Then U is a G-stable open subset of X, and $\pi(U)$ is Cohen–Macaulay.

Proof. Obvious by the theorem and [20, Proposition 3.8]. □

9.7 Corollary. *Let k be a field, G a linearly reductive finite k-group scheme, and B a noetherian and Cohen–Macaulay G-algebra. Then B^G is noetherian and Cohen–Macaulay.*

The corollary is an immediate consequence of a theorem of Hochster and Eagon [11, Proposition 12] (note that B is integral over B^G, see the proof of Lemma 9.8 below). Indeed, the case that G is a finite group is stated in their paper [11, Proposition 13] (however, note that they do not assume that B contains a field there, and Corollary 9.7 is not a complete generalization of [11, Proposition 13]). Corollary 9.7 is also obvious by Theorem 9.5 and the following lemma.

9.8 Lemma. *Let k be a field, and G a finite k-group scheme. Let B be a G-algebra. Then the canonical map $\pi : \operatorname{Spec} B \to \operatorname{Spec} B^G$ is a geometric quotient.*

Proof. As G° (the identity component of G) is normal in G, it suffices to prove that $\operatorname{Spec} B \to \operatorname{Spec} B^{G^\circ}$ and $\operatorname{Spec} B^{G^\circ} \to \operatorname{Spec}(B^{G^\circ})^{G/G^\circ}$ are geometric quotients. Thus we may assume that G is either infinitesimal or étale.

Consider the case that G is infinitesimal. We may assume that the characteristic p of k is positive, since any group scheme over a field of characteristic zero is reduced [22, Theorem 11.4]. Let H be the coordinate ring of G°. Since G° is a point set-theoretically, H is an artinian local ring. Let $\mathfrak{m}$ be the maximal ideal of H, and take $e \geq 1$ sufficiently large so that $\mathfrak{m}^{p^e} = 0$. Then it is easy to see that $b^{p^e} \in B^G$ for any $b \in B$. This shows that any base change of π is a homeomorphism (note also that B is integral over B^G). So π is a geometric quotient, as can be checked easily.

Next consider the case that G is étale. We show that B is integral over B^G. To verify this, by the base change, we may assume that k is algebraically closed. In this case, G is a finite group. Then $b \in B$ is integral over B^G, since b is a root of the monic polynomial $\prod_{g\in G}(t - gb) \in B^G[t]$.

It remains to show that π is an orbit space. To verify this, we may assume that k is algebraically closed again. Thus G is a finite group.

Then G is SGR. Indeed, let V be a finite dimensional G-module and $v \in V^G \setminus 0$, then there is a linear form $\varphi \in V^*$ such that $\varphi(v) \neq 0$. Let H be the trivial subgroup of G if the characteristic of k is zero, and a p-Sylow subgroup of G if the characteristic p of k is positive. Let r be the order of H. Let $\{g_1, \ldots, g_l\}$ be a complete set of representatives of G/H. Note that l is nonzero in k. Then $f := \sum_{i=1}^{l} g_i(\prod_{h\in H} h\varphi)$ is in $(\operatorname{Sym}_r V^*)^G$, and $f(v) = l\varphi(v)^r \neq 0$.

Now assume that π is not a geometric quotient. Then there is an algebraically closed field K and $\operatorname{Spec} K \to \operatorname{Spec} B^G$ such that the geometric fiber $\operatorname{Spec} C$ has two K-rational points x and y on different two $G(K)$-orbits, where $C := K \otimes_{B^G} B$.

For any $c \in C^G$, there exists some q such that $c^q \in K$, where $q = 1$ when the characteristic of k is zero, and q is a power of p when the characteristic p of k is positive, as can be seen easily from Lemma A.1.2 of [18, Appendix to Chapter 1, C]. Thus C^G is a ring with only one prime ideal.

On the other hand, Gx and Gy are closed orbits in $\operatorname{Spec} C$, since x and y are closed points, and G is finite. By the choice of x and y, $Gx \cap Gy = \emptyset$. By Lemma 8.24 and the proof of [18, Theorem 1.1], x and y are mapped to different points in $\operatorname{Spec} C^G$. This contradicts the fact that C^G has only one prime ideal. $\square$

(9.9) Assume that the characteristic of k is zero. In addition to the assumption of Theorem 9.5, assume that X is of finite type over k and has rational singularities. Then Y is of finite type and has rational singularities by Boutot's theorem [1], and Theorem 9.5 is unnecessary. Similarly, if the characteristic is positive and X is F-regular, then Y is F-regular by Corollary 9.11 below.

However, if D is a non-reduced artinian local G-algebra with the residue field k which is finite over k, then $\operatorname{Spec} D \times X$ is still of finite type and Cohen–Macaulay, but does not have rational singularities, since it is not even integral. By [18, Proposition 1.9], $\operatorname{Spec} D \times X$ admits an affine geometric quotient, which is Cohen–Macaulay by Theorem 9.5.

The following theorem and its corollary are due to Hochster. We include proofs, because there is no appropriate reference.

9.10 Theorem. *Let B be a ring, and A its pure subring. If A is noetherian, then for any maximal ideal $\mathfrak{m}$ of A, there exists some maximal ideal $\mathfrak{M}$ of B such that $A_{\mathfrak{m}} \to B_{\mathfrak{M}}$ is pure.*

Proof. Note that $A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is pure. By [13, (2.2)], there exists some maximal ideal $\mathfrak{M}' = \mathfrak{M}B_{\mathfrak{m}}$ of $B_{\mathfrak{m}}$ ($\mathfrak{M} = \mathfrak{M}' \cap B$) such that $A_{\mathfrak{m}} \to (B_{\mathfrak{m}})_{\mathfrak{M}'} = B_{\mathfrak{M}}$ is pure. Let M be a maximal ideal of B containing $\mathfrak{M}$. Since $\mathfrak{M}$ lies on $\mathfrak{m}$ by the purity and $\mathfrak{m}$ is maximal, M also lies on $\mathfrak{m}$. So $\mathfrak{M}' = \mathfrak{M}B_{\mathfrak{m}} \subset MB_{\mathfrak{m}} \neq B_{\mathfrak{m}}$. Since $\mathfrak{M}'$ is maximal, $\mathfrak{M}B_{\mathfrak{m}} = MB_{\mathfrak{m}}$ and hence $\mathfrak{M} = M$ is maximal. $\square$

9.11 Corollary. *Let B be a noetherian ring, and A its pure subring of B. If B is normal* (*resp. of prime characteristic and weakly F-regular, of prime characteristic and F-regular*)*, then so is A.*

Proof. Note that A is noetherian [14, Proposition 6.15]. The assertion for F-regularity follows from that for weak F-regularity by localization, and we consider normality and weak F-regularity. Note that each property in problem is local on maximal ideals, see [12, (4.15)]. So by the theorem, we may assume that both A and B are local. Since weakly F-regular implies normal by [12, (5.11)], B is a normal domain. Now the assertion for normality follows from [14, Proposition 6.15]. The assertion for weak F-regularity follows from [12, (4.12)]. $\square$

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