n-canonical modules over non-commutative algebras

Mitsuyasu Hashimoto

Okayama University

November 18, 2016

The purpose of this talk (1)

In this talk, a non-commutative algebra means a module-finite algebra over a Noetherian commutative ring. The purpose of this talk is:

- Defining the dualizing complex and the canonical module over a non-commutative algebra;
- Proving non-commutative versions of theorems on canonical modules;
- Defining n-canonical modules;
- Improving and generalizing Araya–Iima theorem on (n, C)-syzygies using n-canonical modules;
- More. . .

The purpose of this talk (2)

- Proving 'codimension-two argument' is valid over a Noetherian scheme with a full 2-canonical module;
- Defining higher-dimensional versions of symmetric, Frobenius, and quasi-Frobenius algebras and proving basic properties of them.

Notation

Throughout the talk, let R be a Noetherian commutative ring, and Λ a module-finite R-algebra. Let $d = \dim \Lambda$.
By a Λ-bimodule, we mean a left $\Lambda \otimes_R \Lambda^{\mathrm{op}}$-module.

Dualizing complex — complete case

Let (R, J) be semilocal. A dualizing complex $\mathbb{I}$ of R is said to be normalized if $\mathrm{Ext}^0_R(R/\mathfrak{m}, \mathbb{I}) \neq 0$ for each $\mathfrak{m} \in \mathrm{Max}(R)$.
If R is complete, then R has a normalized dualizing complex $\mathbb{I}_R$, which is unique up to isomorphisms in $D(R)$.

Let R be complete semilocal, and Λ a module-finite R-algebra. We define $\mathbb{I}_\Lambda := \mathbf{R}\mathrm{Hom}_R(\Lambda, \mathbb{I}_R)$, and call it the normalized dualizing complex of Λ.
$\mathbb{I}_\Lambda$ depends only on Λ, and is independent of the choice of R.

The canonical module

If (R, J) is complete semilocal and $\Lambda \neq 0$, then the canonical module of Λ, denoted by K_Λ, is defined to be the lowest nonzero cohomology group of $\mathbb{I}_\Lambda$. If $\Lambda = 0$, then K_Λ is defined to be 0.
Let (R, J) be not necessarily complete. A finite Λ-bimodule (resp. right Λ-module) M is called the canonical module (resp. right canonical module) of Λ if $\hat{M} \cong K_{\hat{\Lambda}}$ as a $\hat{\Lambda}$-bimodule (resp. right $\hat{\Lambda}$-module), where $\hat{?}$ denotes the completion.

If (R, J) has a normalized dualizing complex $\mathbb{I}$, then the lowest non-vanishing cohomology $\operatorname{Ext}_R^{-d}(\Lambda, \mathbb{I})$ is the canonical module of Λ, where $d = \dim \Lambda$.

Some basic properties

Let M and N be finite R-modules. We say that M satisfies the $(S_n^N)^R$-condition (or (S_n^N)-condition) if $\operatorname{depth} M_P \geq \min(n, \dim N_P)$ for each $P \in \operatorname{Spec} R$. $(S_n^R)^R$ is simply denoted by $(S_n)^R$ or (S_n).

Lemma 1

Let (R, J) be semilocal, and assume that Λ has a right canonical module K. Then

(1) $\operatorname{Ass}_R K = \mathrm{Assh}_R \Lambda := \{P \in \operatorname{Min}_R \Lambda \mid \dim R/P = \dim \Lambda\}$.

(2) K satisfies the $(S_2^\Lambda)^R$ condition.

(3) $R/\operatorname{ann}_R K_\Lambda$ is quasi-unmixed, and hence is universally catenary.

Globally Cohen–Macaulay modules

Let (R, J) be semilocal, and M a finite R-module. We denote $\operatorname{depth}(J, M)$ by $\operatorname{depth} M$. We say that M is globally Cohen–Macaulay (GCM) if $\dim M = \operatorname{depth} M$. This is equivalent to say that M is a Cohen–Macaulay R-module, and $\dim M_{\mathfrak{m}}$ is constant on $\operatorname{Max}(R)$. Let M be a finite right Λ-module. We say that M is globally maximal Cohen–Macaulay (GMCM) if $\dim \Lambda = \operatorname{depth} M$. This is equivalent to say that $M_{\mathfrak{m}}$ is a maximal Cohen–Macaulay $(R/\operatorname{ann}_R \Lambda)_{\mathfrak{m}}$-module for each $\mathfrak{m} \in \operatorname{Max}(R)$, and $\dim \Lambda_{\mathfrak{m}}$ is independent of $\mathfrak{m}$. Note that $\operatorname{depth} M$, GCM and GMCM property are determined only by M and Λ, and is independent of the choice of R.

GCM case

Let (R, J) be semilocal with a normalized dualizing complex $\mathbb{I}_R$. Set $\mathbb{I}_\Lambda = \mathbf{R}\mathrm{Hom}_R(\Lambda, \mathbb{I}_R)$. Let M be a finite right Λ-module. Note that Λ is GCM if and only if $K_\Lambda[d] \cong \mathbb{I}_\Lambda$ in $D(\Lambda \otimes_R \Lambda^{\mathrm{op}})$.

Lemma 2

Assume that Λ is GCM. The following are equivalent.

(1) M is GMCM.

(2) $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(M, K_\Lambda) = 0$ for $i > 0$.

If so, then $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(M, K_\Lambda)$ is again a GMCM left Λ-module, and the map

$$M \to \mathrm{Hom}_\Lambda(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(M, K_\Lambda), K_\Lambda)$$

is an isomorphism. In particular, K_Λ is GMCM and $\Lambda \to \mathrm{End}_\Lambda K_\Lambda$ is an isomorphism.

Non-commutative Aoyama's theorem (1)

Returning to the general (non-GCM) case, we reprove some classical theorems on canonical modules in non-commutative versions.

Proposition 3

Let $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ be a flat local homomorphism between Noetherian local rings. Let M be a right Λ-module. Assume that $R'/\mathfrak{m}R'$ is zero-dimensional, and $M' := R' \otimes_R M$ is the right canonical module of $\Lambda' := R' \otimes_R \Lambda$. If $\Lambda \neq 0$, then $R'/\mathfrak{m}R'$ is Gorenstein.

Non-commutative Aoyama's theorem (2)

Theorem 4 (Non-commutative Aoyama's theorem)

Let $(R, \mathfrak{m}) \to (R', \mathfrak{m}')$ be a flat local homomorphism between Noetherian local rings, and Λ a module-finite R-algebra.

(1) If M is a Λ-bimodule and $M' = R' \otimes_R M$ is the canonical module of $\Lambda' = R' \otimes_R \Lambda$, then M is the canonical module of Λ.

(2) If M is a right Λ-module such that M' is the right canonical module of Λ', then M is the right canonical module of Λ.

A corollary

Corollary 5

Let $(R, \mathfrak{m})$ be a Noetherian local ring, and suppose that K is the canonical (resp. right canonical) module of Λ. If $P \in \operatorname{supp}_R K$, then K_P is the canonical (resp. right canonical) module of Λ_P.

Semicanonical module

Let R be a Noetherian ring. Let ω be a finite Λ-bimodule (resp. right Λ-module). We say that ω is semicanonical (resp. right semicanonical) if for any $P \in \operatorname{supp}_R \omega$, ω_P is the canonical (resp. right canonical) module of Λ_P.

Example 6

(1) The zero module 0 is a semicanonical module.

(2) If R has a dualizing complex $\mathbb{I}$ and $\Lambda \neq 0$, then the lowest non-vanishing cohomology of $\mathbf{R}\mathrm{Hom}_R(\Lambda, \mathbb{I})$ is a semicanonical module.

(3) By non-commutative Aoyama's theorem, if R is semilocal and K_Λ is the (right) canonical module of Λ, then K_Λ is semicanonical.

n-canonical modules

Definition 7

Let R be a Noetherian ring and $n \geq 1$. A finite Λ-bimodule (resp. right Λ-module) C is called an n-canonical Λ-bimodule (resp. right Λ-module) over R if C satisfies $(S_n)^R$, and for each $P \in \operatorname{supp}_R C$ with $\operatorname{ht} P < n$, C_P is the canonical module of Λ_P.

Some examples

Example 1

(1) A semicanonical bimodule (resp. right module) is a 2-canonical bimodule (resp. right module) over $R/\operatorname{ann}_R \Lambda$.

(2) The R-module R is an n-canonical R-module (here $\Lambda = R$) if and only if R satisfies $(G_{n-1}) + (S_n)$, where we say that R satisfies (G_{n-1}) if for any $P \in \operatorname{Spec} R$ with $htP \leq n-1$, R_P is Gorenstein.

(3) If R is normal, then any rank-one reflexive R-module is 2-canonical.

Notation

Let C be a finite right Λ-module, and set $\Gamma := \operatorname{End}_{\Lambda^{\mathrm{op}}} C$. Let

$$(?)^{\dagger} = \mathsf{Hom}_{\Lambda^{\mathrm{op}}}(?, C) : \operatorname{mod}\Lambda \to \Gamma\operatorname{mod},$$

and

$$(?)^{\ddagger} = \mathsf{Hom}_{\Gamma}(?, C) : \Gamma\operatorname{mod} \to \operatorname{mod}\Lambda.$$

We have a natural map

$$\lambda_M : M \to M^{\dagger\ddagger}$$

for $M \in \operatorname{mod}\Lambda$.

(n, C)-TF property

Let $M \in \operatorname{mod}\Lambda$. We say that M is $(1, C)$-TF (resp. $(2, C)$-TF) if

$$\lambda_M : M \to M^{\dagger\ddagger}$$

is injective (resp. bijective).
For $n \geq 3$, we say that M is (n, C)-TF if it is $(2, C)$-TF and $\operatorname{Ext}^i_\Gamma(M^\dagger, C) = 0$ for $1 \leq i \leq n-2$.

Remark 8

The notion of (n, C)-TF property is a variant of n-C-torsionfreeness due to Takahashi.
Even if $R = \Lambda$ is commutative, Γ may not be commutative, and it is essential to consider non-commutative algebras.

(n, C)-universal pushforward

Let $M \in \operatorname{mod} \Lambda$. If there is an exact sequence

$$0 \to M \to C^0 \to \cdots \to C^{n-1} \tag{1}$$

with $C^i \in \operatorname{add} C$, then we say that M is an (n, C)-syzygy. If in addition, the sequence

$$0 \leftarrow M^\dagger \leftarrow (C^0)^\dagger \leftarrow \cdots \leftarrow (C^{n-1})^\dagger$$

derived from (1) is also exact, then we say that M has a universal (n, C)-pushforward.

(n, C)-TF and universal (n, C)-pushforward

Theorem 9

Let $M \in \operatorname{mod} \Lambda$ and $n \geq 1$. Then M is (n, C)-TF if and only if M has a universal (n, C)-pushforward.

Main theorem

Theorem 10

Let $M \in \operatorname{mod} \Lambda$ and $n \geq 1$. Let C be an n-canonical right Λ-module over R. Then the following are equivalent.

(1) M is (n, C)-TF (or equivalently, has a universal (n, C)-pushforward).

(2) M is an (n, C)-syzygy.

(3) M satisfies $(S_n)^R$, and $\operatorname{supp}_R M \subset \operatorname{supp}_R C$.

Remark 11

Considering the case that $R = \Lambda$, R is (S_n), and C is semidualizing, the theorem yields Araya–Iima theorem.

A corollary

Corollary 12

Let R be semilocal and assume that Λ has a right canonical module K.

(1) $\lambda_\Lambda : \Lambda \to \mathrm{End}_{\Lambda^{\mathrm{op}}} K$ is injective if and only if $\mathrm{Ass}_R \Lambda = \mathrm{Assh}_R \Lambda$.

(2) $\lambda_\Lambda : \Lambda \to \mathrm{End}_{\Lambda^{\mathrm{op}}} K$ is bijective if and only if Λ satisfies the $(S_2^\Lambda)^R$-condition.

Schenzel–Aoyama–Goto theorem

Theorem 13

Let (R, J) be semilocal, and suppose that Λ has a right canonical module K_Λ. Assume that Λ satisfies $(S_2^\Lambda)^R$, and assume that K_Λ is GCM. Then Λ is GCM.

Notation

Let X be a Noetherian scheme, U its open subscheme with $\operatorname{codim}(X \setminus U, X) \geq 2$, and Λ a coherent $\mathcal{O}_X$-algebra. A coherent right Λ-module C is said to be n-canonical if C satisfies (S_n) and C_x is the right canonical module of Λ_x or zero for each $x \in X$.
The full subcategory of $\mathrm{mod}\,\Lambda$ consisting of $\mathcal{M} \in \mathrm{mod}\,\Lambda$ satisfying (S_2) is denoted by $(S_2)^{\Lambda^{\mathrm{op}},X}$.

Codimension-two argument

Theorem 2

Suppose that X *has a* 2*-canonical module* C *such that* $\operatorname{supp} C = X$*. Let* $i : U \hookrightarrow X$ *be the inclusion. Then the restriction* $i^* : (S_2)^{\Lambda^{\mathrm{op}},X} \to (S_2)^{(i^*\Lambda)^{\mathrm{op}},U}$ *is an equivalence whose quasi-inverse is the direct image* i_**.*

Higher-dimensional analogue of quasi-Frobenius algebra

Lemma 3

Let (R, J) be semilocal. Then the following are equivalent.

(1) $(K_{\hat{\Lambda}})_{\hat{\Lambda}}$ *is projective in* $\operatorname{mod}\hat{\Lambda}$.

(2) ${}_{\hat{\Lambda}}(K_{\hat{\Lambda}})$ *is projective in* $\hat{\Lambda}\operatorname{mod}$.

If the equivalent conditions are satisfied, we say that Λ is pseudo-quasi-Frobenius. If, moreover, Λ is GCM, then we say that Λ is quasi-Frobenius.

Goto–Nishida's Gorensteinness (1)

Proposition 14

Let R be semilocal, and assume that Λ is GCM. Then the following are equivalent.

(1) Λ is quasi-Frobenius;

(2) $\dim \Lambda = \mathrm{idim}\,_\Lambda\Lambda$, where idim denotes the injective dimension.

(3) $\dim \Lambda = \mathrm{idim}\,\Lambda_\Lambda$.

Goto–Nishida's Gorensteinness (2)

Corollary 15

Let R be arbitrary. Then the following are equivalent.

(1) For any $P \in \operatorname{Spec} R$, Λ_P is quasi-Frobenius.

(2) For any $\mathfrak{m} \in \operatorname{Max} R$, $\Lambda_{\mathfrak{m}}$ is quasi-Frobenius;

(3) Λ is a Gorenstein in the sense that Λ is Cohen–Macaulay, and $\mathrm{idim}_{\Lambda_P} \Lambda_P = \dim \Lambda_P$ for $P \in \operatorname{Spec} R$.

Non-commutative version of quasi-Gorenstein property

Lemma 16

Let (R, J) be semilocal. Then the following are equivalent.

(1) ${}_\Lambda\Lambda$ is the left canonical module of Λ;

(2) Λ_Λ is the right canonical module of Λ.

Let (R, J) be semilocal. We say that Λ is quasi-symmetric if Λ is the canonical module of Λ. If, moreover, Λ is GCM, then we say that Λ is symmetric.

We say that Λ is pseudo-Frobenius if the equivalent conditions in Lemma 16 are satisfied. If, moreover, Λ is GCM, then we say that Λ is Frobenius.

Relative notions due to Scheja–Storch

Definition 17 (Scheja–Storch)

Let R be general. We say that Λ is symmetric (resp. Frobenius) relative to R if Λ is R-projective, and $\Lambda^* := \mathrm{Hom}_R(\Lambda, R)$ is isomorphic to Λ as a Λ-bimodule (resp. as a right Λ-module). It is called quasi-Frobenius relative to R if the right Λ-module Λ^* is projective.

Relative versus absolute notions (1)

Proposition 18

Let $(R, \mathfrak{m})$ be local.

(1) If $\dim \Lambda = \dim R$, R is quasi-Gorenstein, and $\Lambda^* \cong \Lambda$ as Λ-bimodules (resp. $\Lambda^* \cong \Lambda$ as right Λ-modules, Λ^* is projective as a right Λ-module), then Λ is quasi-symmetric (resp. pseudo-Frobenius, pseudo-quasi-Frobenius).

(2) If Λ is nonzero and R-projective, then Λ is quasi-symmetric (resp. pseudo-Frobenius, pseudo-quasi-Frobenius) if and only if R is quasi-Gorenstein and Λ is symmetric (resp. Frobenius, quasi-Frobenius) relative to R.

Relative versus absolute notions (2)

Corollary 19

Let $(R, \mathfrak{m})$ be local.

(1) If R is Gorenstein and Λ is symmetric (resp. Frobenius, quasi-Frobenius) relative to R, then Λ is symmetric (resp. Frobenius, quasi-Frobenius).

(2) If Λ is nonzero and R-projective, then Λ is symmetric (resp. Frobenius, quasi-Frobenius) if and only if R is Gorenstein and Λ is symmetric (resp. Frobenius, quasi-Frobenius) relative to R.

Thank you

This slide will soon be available at
http://www.math.okayama-u.ac.jp/~hashimoto/