F-rationality of rings of invariants

Mitsuyasu Hashimoto

Okayama University

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F-regularity and F-rationality of rings of invariants

Let $k = \bar{k}$ be an algebraically closed field of characteristic $p > 0$. Let $V = k^d$, and G be a finite subgroup of $\mathrm{GL}(V) = \mathrm{GL}_d$. We say that $g \in \mathrm{GL}(V)$ is a pseudo-reflection if $\operatorname{rank}(1_V - g) = 1$. Let $B = \operatorname{Sym} V = k[v_1, \ldots, v_d]$, where $v_1, \ldots, v_d$ is a basis of V, and $A = B^G$.

Question 1

Assume that G does not have a pseudo-reflection.

1. When is $A = B^G$ strongly F-regular?
2. When is $A = B^G$ F-rational?

Broer–Yasuda theorem

Theorem 2 (Broer, Yasuda)

Assume that G does not have a pseudo-reflection. The following are equivalent.

1. $A = B^G$ is strongly F-regular.
2. A is a direct summand subring of B.
3. p does not divide the order $\#G$ of G.

2$\Rightarrow$**1** is simply because strong F-regularity is inherited by a direct summand. **1**$\Rightarrow$**2** is because a weakly F-regular ring is a splinter (Hochster–Huneke). **3**$\Rightarrow$**2** is by the existence of the Reynolds operator. Broer and Yasuda proved **2**$\Rightarrow$**3**.

Today we consider the problem for F-rationality.

Frobenius twist

For a k-scheme $h : Z \to \operatorname{Spec} k$ and $e \in \mathbb{Z}$, the scheme Z with the new k-structure

$$Z \xrightarrow{h} \operatorname{Spec} k \xrightarrow{F^e} \operatorname{Spec} k$$

is denoted by ${}^e Z$. Note that the Frobenius map $F_Z^e : {}^e Z \to Z$ is a k-morphism for $e \geq 0$.
For a k-morphism $f : Z' \to Z$, the morphism $f : {}^e Z' \to {}^e Z$ is again a k-morphism. We denote this morphism ${}^e f$.

Frobenius twist of a group scheme

If H is a k-group scheme, then by the product

$$^{e}H \times {}^{e}H \cong {}^{e}(H \times H) \xrightarrow{{}^{e}\mu} {}^{e}H,$$

^{e}H is a k-group scheme, where $\mu : H \times H \to H$ is the product.

The Frobenius map $F_H^e : {}^{e}H \to H$ is a homomorphism of k-group schemes. If H is étale over k, F^e is an isomorphism. In particular, $F_G^e : {}^{e}G \to G$ is an isomorphism.

Frobenius twist of a vector space

For a k-vector space W and $e \in \mathbb{Z}$, the additive group W with the new k-structure given by

$$\alpha \cdot w := \alpha^{p^e} w$$

is denoted by ${}^e W$, where the right-hand side is by the original k-action of W, and the left hand side is by the new k-action of ${}^e W$. The vector w, viewed as an element of ${}^e W$ is denoted by ${}^e w$. Thus we have

$$ {}^e w + {}^e w' = {}^e (w + w'), \qquad \alpha({}^e w) = {}^e(\alpha^{p^e} w).$$

Frobenius twist of k-algebras

For a k-algebra A, the k-space ${}^eA$ with the ring structure of A is a k-algebra. That is,

$${}^ea\,{}^eb = {}^e(ab).$$

If A is commutative, then $\operatorname{Spec} {}^eA$ is identified with ${}^e(\operatorname{Spec} A)$, and the Frobenius map $F_A^e : A \to {}^eA$ is a k-algebra map.

Frobenius twist of a representation

Let H be a k-group scheme, and W an H-module. Then ${}^eW$ is an ${}^eH$-module in a natural way by the action

$$ {}^eH \times {}^eW \cong {}^e(H \times W) \to {}^eW. $$

When H is étale, as $F^e : {}^eH \to H$ is an isomorphism, ${}^eW$ is an H-module again.

Explicit description of Frobenius twist of representations

Let W be a G-module of dimension n and $w_1, \ldots, w_n$ be a k-basis of W. For $g \in G$, the representation matrix $\rho(g) = (a_{ij}(g))$ is given by

$$gw_j = \sum_{i=1}^{n} a_{ij}(g) w_i.$$

Then the representation matrix of ${}^e W$ with respect to the basis ${}^e w_1, \ldots, {}^e w_n$ is given by

$$g({}^e w_j) = \sum_{i=1}^{n} a_{ij}(g)^{p^{-e}} ({}^e w_i).$$

The frobenius twists of H-algebras and equivariant modules

Let C be an H-algebra. Then ${}^eC$ is an ${}^eH$-algebra.
Let M be an (H, C)-module. Then ${}^eM$ is an $({}^eH, {}^eC)$-module.
If, moreover, H is étale, then ${}^eC$ is an H-algebra, and ${}^eM$ is an $(H, {}^eC)$-module.

Review of the settings

Let k be an algebraically closed field of characteristic $p > 0$. Let $V := k^d$, and G a finite subgroup of $\mathrm{GL}(V)$. Let $B := \mathrm{Sym}\, V = k[v_1, \ldots, v_d]$ and $A := B^G$, where $v_1, \ldots, v_d$ is a basis of V.

Remarks

Remark 3

1. Let $B * G$ be the twisted group algebra. A (G, B)-module and a $B * G$-module are one and the same thing.
2. If $N \in \mathrm{Ref}(A)$, then $\mathrm{rank}_A N = \mathrm{rank}_B (B \otimes_A N)^{**}$. For any $M \in \mathrm{Ref}(G, B)$, $\mathrm{rank}_B M = \mathrm{rank}_A M^G$.
3. For $M \in \mathrm{Ref}(G, B)$, ${}^e M \in \mathrm{Ref}(G, B)$, and obviously
$$({}^e M)^G \cong {}^e(M^G).$$
4. For a (G, B)-module M and a G-module W, $M \otimes_k W$ is a B-module by $b(m \otimes w) = bm \otimes w$, and is a G-module $g(m \otimes w) = gm \otimes gw$, and it is in fact a (G, B)-module.

Graded (G, B)-modules

A $\mathbb{Z}$-graded (G, B)-module M is nothing but a $(\tilde{G}, B)$-module, where $\tilde{G} = \operatorname{Spec} k[t, t^{-1}] \times G$. So for $e \geq 0$, ${}^eM$ is a $({}^e\tilde{G}, B)$-module. As ${}^e\tilde{G} \cong \operatorname{Spec} k[t^{p^{-e}}, t^{-p^{-e}}] \times G$, it is a $p^{-e}\mathbb{Z}$-graded (G, B)-module. Let $\mathcal{M}$ be the category of $\mathbb{Z}[1/p]$-graded (G, B)-modules. Let $\mathcal{F}$ be its full subcategory consisting of B-finite B-free objects.
The Frobenius twist ${}^e(?)$ is an endofunctor of $\mathcal{M}$, and ${}^e\mathcal{F} \subset \mathcal{F}$.

kG is selfinjective

Let Λ be a finite dimensional k-algebra.

- We say that Λ is symmetric if ${}_\Lambda\Lambda_\Lambda \cong {}_\Lambda D(\Lambda)_\Lambda$, where $D(\Lambda) = \Lambda^*$ is the k-dual of Λ.
- The following are equivalent.
  1. ${}_\Lambda\Lambda$ is injective.
  2. Λ_Λ is injective.
  3. Any projective (left) Λ-module is injective.
  4. Any injective (left) Λ-module is projective.

  We say that Λ is selfinjective (or quasi-Frobenius) if these conditions are satisfied.
- If Λ is symmetric, then Λ is selfinjective.
- kG is symmetric, and hence is selfinjective.

$\mathcal{F}$ is Frobenius

Lemma 4 (Symonds–H)

We have $\mathrm{Hom}_B(B \otimes_k kG, B) \cong B \otimes kG$ in $\mathcal{F}$. The category $\mathcal{F}$ is a Frobenius category. Its full subcategory of projective injective objects agrees with

$$\mathcal{P} := \mathrm{add}\{(B \otimes_k kG)[\lambda] \mid \lambda \in \mathbb{Z}[1/p]\}.$$

Frobenius twists of objects of $\mathcal{F}$

Lemma 5 (Symonds–H)

There exists some $e_0 \geq 1$ such that for any $E \in \mathcal{F}$ of rank f, there exists a direct summand E_0 of ${}^{e_0}E$ in $\mathcal{F}$ such that $E_0 \cong (B \otimes_k kG)^f$ as (G, B)-modules.

Lemma 6 (Symonds–H)

${}^{e}(B \otimes_k kG) \cong (B \otimes_k kG)^{p^{de}}$ as (G, B)-modules.

Asymptotic behavior of Frobenius twists

Theorem 7 (Symonds–H)

There exists some $c > 0$ and $0 < \alpha < 1$ such that for any $E \in \mathcal{F}$ of rank f and any $e \geq 1$, there exists some decomposition

$$^{e}E \cong E_{0,e} \oplus E_{1,e}$$

in $\mathcal{F}$ such that $E_{0,e}$ is a direct sum of copies of $B \otimes_k kG$ as a (G, B)-module, and $E_{1,e}$ is an object of $\mathcal{F}$ whose rank less than or equal to $fcp^{de}\alpha^{e}$.

Some observations on $E_{0,e}$ and $E_{1,e}$

- $\lim_{e\to\infty} \frac{1}{p^{de}} \operatorname{rank} E_{1,e} = 0$. Hence $\lim_{e\to\infty} \frac{1}{p^{de}} \operatorname{rank} E_{0,e} = f$.
- Since $(B \otimes_k kG)^G \cong B$ as A-modules, we have $\displaystyle\lim_{e\to\infty} \frac{1}{p^{de}} \mu_{\hat{A}}(\hat{E}^G_{0,e}) = f\mu_{\hat{A}}(\hat{B})/|G| = fe_{\mathrm{HK}}(\hat{A})$ (by Watanabe–Yoshida theorem, as $[Q(\hat{B}) : Q(\hat{A})] = |G|$), where $\hat{A}$ and $\hat{B}$ are the completions of A and B, respectively.
- As $\displaystyle\lim_{e\to\infty} \frac{1}{p^{de}} \mu_{\hat{A}}({}^e\hat{E}^G) = e_{\mathrm{HK}}({}^e\hat{E}^G) = fe_{\mathrm{HK}}(\hat{A})$, we have

Corollary 8 (Symonds–H)

$$\lim_{e\to\infty} \frac{1}{p^{de}} \mu_{\hat{A}}(\hat{E}^G_{1,e}) = 0.$$

Interpretation to A-modules

Let $k = V_0, V_1, \ldots, V_n$ be the list of simple G-modules. Let P_i be the projective cover of V_i. Set $M_i := (B \otimes_k P_i)^G$.

Theorem 9 (Symonds–H)

There exists some sequence of non-negative integers $\{a_e\}$ such that

1. $\lim_{e\to\infty} a_e/p^{de} = 1/|G|$; and
2. For each B-finite B-free $\mathbb{Z}$-graded (G, B)-module E of rank f and $e \geq 1$, there is a decomposition

$$ {}^eE^G \cong \bigoplus_{i=0}^{n} M_i^{\oplus f a_e \dim V_i} \oplus M_{E,e} $$

as an A-module such that $\lim_{e\to\infty} \mu_{\hat{A}}(\hat{M}_{E,e})/p^{de} = 0$.

Sannai's dual F-signature

Let $(R, \mathfrak{m}, k)$ be a d-dimensional reduced F-finite local ring of prime characteristic p with k perfect. For finite R-modules M and N, define

$$\mathrm{surj}_R(M, N) := \max\{r \in \mathbb{Z}_{\geq 0} \mid \exists \text{ a surjection } M \to N^{\oplus r}\}.$$

We define

$$s(M) := \limsup_{e \to \infty} \frac{\mathrm{surj}_R({}^e M, M)}{p^{de}},$$

and call it the dual F-signature of M (Sannai). $s(R)$ is nothing but the F-signature of the ring R (defined by Huneke–Leuschke).

Characterizations of F-regularity and F-rationality

Theorem 10

Let $(R, \mathfrak{m}, k)$ be a reduced F-finite local ring with k perfect.

1. (Tucker) $s(R) := \limsup_{e\to\infty} \dfrac{\mathrm{surj}({}^eR, R)}{p^{de}} = \lim_{e\to\infty} \dfrac{\mathrm{surj}({}^eR, R)}{p^{de}}$.
2. (Aberbach–Leuschke) R is strongly F-regular if and only if $s(R) > 0$.
3. (Gabber) R is a homomorphic image of a regular local ring.
4. (Sannai) R is F-rational if and only if R is Cohen–Macaulay and $s(\omega_R) > 0$, where ω_R is the canonical module of R.

The group $[\mathcal{C}]$

Let $\mathcal{C}$ be an additive category. We define

$$[\mathcal{C}] := (\bigoplus_{M \in \mathcal{C}} \mathbb{Z} \cdot M)/(M - M_1 - M_2 \mid M \cong M_1 \oplus M_2).$$

The class of M in the group $[\mathcal{C}]$ is denoted by $[M]$.
The vector space $\mathbb{R} \otimes_{\mathbb{Z}} [\mathcal{C}]$ is denoted by $[\mathcal{C}]_{\mathbb{R}}$. If $\mathcal{C}$ is Krull–Schmidt and $\mathcal{C}_0$ is a complete set of representatives of $\operatorname{Ind} \mathcal{C}$, then $\{[M] \mid M \in \mathcal{C}_0\}$ is an $\mathbb{R}$-basis of $[\mathcal{C}]_{\mathbb{R}}$.

The metric of $[\mathrm{mod}(R)]$

Let R be a Henselian local ring, and $\mathcal{C} := \mathrm{mod}(R)$. For $\alpha \in [\mathcal{C}]_{\mathbb{R}}$, we can write

$$\alpha = \sum_{M \in \mathcal{C}_0} c_M [M].$$

We define $\|\alpha\| := \sum_M |c_M| \mu_R(M)$. Then $([\mathcal{C}]_{\mathbb{R}}, \|\cdot\|)$ is a normed space. So it is a metric space by the metric

$$d(\alpha, \beta) := \|\alpha - \beta\|.$$

The F-limit of a module

Let $\alpha = \sum_{M\in\mathcal{C}_0} c_M[M] \in [\mathcal{C}]_{\mathbb{R}} = [\mathrm{mod}(R)]_{\mathbb{R}}$.

- Define $\mu_R : [\mathcal{C}]_{\mathbb{R}} \to \mathbb{R}$ by $\mu_R(\alpha) := \sum_M c_M \mu_R(M)$.
- For $N \in \mathcal{C}_0$, define $\mathrm{sum}_N : [\mathcal{C}]_{\mathbb{R}} \to \mathbb{R}$ by $\mathrm{sum}_N(\alpha) = c_N$.
- Assume further that R is of characteristic $p > 0$ and F-finite with a perfect residue field.
- Define ${}^e\alpha = \sum_{M\in\mathcal{C}_0} c_M[{}^eM]$.
- Define $FL(\alpha) = \lim_{e\to\infty} \dfrac{1}{p^{de}}\,{}^e\alpha$ (if exists, the F-limit of α).
- Define $e_{\mathrm{HK}}(\alpha) = \sum_M c_M e_{\mathrm{HK}}(M)$.
- For $N \in \mathcal{C}_0$, define $FS_N(\alpha) = \sum_M c_M\, FS_N(M)$, where $FS_N(M) = \lim_{e\to\infty} \dfrac{1}{p^{de}} \mathrm{sum}_N({}^eM)$ (the generalized F-signature).

The Hilbert–Kunz multiplicity and F-signature

Lemma 11

$\mu_R : [\mathcal{C}]_{\mathbb{R}} \to \mathbb{R}$ is a short map. That is, $|\mu_R(\alpha) - \mu_R(\beta)| \leq \|\alpha - \beta\|$. Similarly for $\mathrm{sum}_N : [\mathcal{C}]_{\mathbb{R}} \to \mathbb{R}$ for $N \in \mathcal{C}_0$. In particular, they are uniformly continuous.

Corollary 12

Let $\alpha = \sum_{M \in \mathcal{C}_0} c_M[M] \in [\mathcal{C}]_{\mathbb{R}}$. If $FL(\alpha)$ exists, then

$$\mu_R(FL(\alpha)) = e_{\mathrm{HK}}(\alpha)$$

and

$$\mathrm{sum}_N(FL(\alpha)) = FS_N(\alpha).$$

The dual F-signature

For $\alpha = \sum_M c_M[M] \in [\mathcal{C}]_{\mathbb{R}}$ and $M, N \in \operatorname{mod} R$,

- Define $\langle \alpha \rangle := \sum_M \max\{0, \lfloor c_M \rfloor\}[M]$.
- Define
$$\operatorname{asn}(\alpha, N) := \lim_{t\to\infty} \frac{1}{t} \operatorname{surj}(\langle t\alpha \rangle, N)$$
(the limit exists, the asymptotic surjective number).
- In general, $\operatorname{surj}(M, N) \le \operatorname{asn}([M], N)$.
- $\operatorname{asn}(?, N)$ is a short map.
- We say that $\alpha \ge 0$ if $c_M \ge 0$ for any $M \in \mathcal{C}_0$.
- If $\alpha, \beta \ge 0$, then $\operatorname{asn}(\alpha + \beta, N) \ge \operatorname{asn}(\alpha, N) + \operatorname{asn}(\beta, N)$.
- If the F-limit of M exists, then $s(M) = \operatorname{asn}(FL([M]), M)$.

The restatement of Theorem 9

Theorem 13 (Symonds–H)

For each B-finite B-free $\mathbb{Z}$-graded (G, B)-module E of rank f,

$$FL([\hat{E}^G]) = \frac{f}{|G|}[\hat{B}] = \frac{f}{|G|}\bigoplus_{i=0}^{n}(\dim V_i)[\hat{M}_i]$$

in $[\mathrm{mod}\ \hat{A}]_{\mathbb{R}}$, where $M_i = (B \otimes_k P_i)^G$.

Remark 14

The theorem for the case that p does not divide $|G|$ is due to Nakajima–H.

The free locus U of the action

Set

$$\varphi : X := \operatorname{Spec} B \to \operatorname{Spec} A =: Y$$

be the canonical map, where $A = B^G$. Set
$U := X \setminus (\bigcup_{g \in G \setminus \{e\}} X_g) \subset X$, where

$$X_g = \{x \in X \mid gx = x\}.$$

We call U the free locus of the action.

From now, unless otherwise stated explicitly (in an example), assume that G has no pseudo-reflection.
Let $\varphi : X = \operatorname{Spec} B \to \operatorname{Spec} A = Y$ be as above, and U the free locus, and $U' := \varphi(U)$. We get the diagram

$$X \xleftarrow{i} U \xrightarrow{\rho} U' \xrightarrow{j} Y\ ,$$

where ρ is the restriction of φ.

Almost principal bundle

Lemma 15

1. Set $\tilde{G} := G \times \mathbb{G}_m$. Then $\varphi : X \to Y$ is a $\tilde{G}$-morphism.
2. (Since G does not have a pseudo-reflection) U is large in X. That is, $\operatorname{codim}_X(X \setminus U) \geq 2$.
3. U is a $\tilde{G}$-stable open subset of X, and agrees with the étale locus of φ.
4. $U' := \varphi(U)$ is a large $\mathbb{G}_m$-stable open subset of Y, and the restriction $\rho : U \to U'$ of φ is a principal G-bundle.
5. $\varphi : X \to Y$ is a $\tilde{G}$-enriched almost principal G-bundle with respect to U and U'.

The equivalences

Theorem 16

The functor $\gamma : i_*\rho^*j^* : \mathrm{Ref}(Y) \to \mathrm{Ref}(G, X)$ is an equivalence whose quasi-inverse is $\delta : (?)^G j_*\rho_* i^* : \mathrm{Ref}(G, X) \to \mathrm{Ref}(Y)$.

Corollary 17

The functor $(B \otimes_A ?)^{**} : \mathrm{Ref}(A) \to \mathrm{Ref}(G, B)$ is an equivalence whose quasi-inverse is $(?)^G : \mathrm{Ref}(G, B) \to \mathrm{Ref}(A)$.

Remark 18

Similarly, $(\hat{B} \otimes_{\hat{A}} ?)^{**} : \mathrm{Ref}(\hat{A}) \to \mathrm{Ref}(G, \hat{B})$ is an equivalence whose quasi-inverse is $(?)^G$.

The description of ω_A

Theorem 19 (Watanabe–Peskin–Broer–Braun)

Let $\det = \det_V$ denote the one-dimensional representation $\bigwedge^d V$ of G. Then

1. $\omega_A \cong (B \otimes_k \det)^G$.
2. Hence $B \otimes_k \det \cong (B \otimes_A \omega_A)^{**}$.
3. In particular, A is quasi-Gorenstein if and only if $\det \cong k$ as a G-module (or equivalently, $G \subset \mathrm{SL}(V)$).

Reproving Watanabe–Yoshida theorem and Broer–Yasuda theorem

Note that each $\hat{M}_i = (\hat{B} \otimes_k P_i)^G$ is an indecomposable $\hat{A}$-module, and $\hat{M}_i \not\cong \hat{M}_j$ for $i \neq j$. Moreover, $\hat{M}_i \cong \hat{A}$ if and only if $P_i \cong k$. This is equivalent to say that $i = 0$ and p does not divide $|G|$.

Corollary 20 (Watanabe–Yoshida, Broer, Yasuda)

The F-signature $s(\hat{A})$ of $\hat{A}$ is zero if p divides $|G|$, and is $1/|G|$ otherwise.

Proof.

$s(\hat{A}) = FS_{\hat{A}}(\hat{A}) = |G|^{-1} \sum_{i=0}^{n} (\dim V_i) \operatorname{sum}_{\hat{A}}([\hat{M}_i])$. $\square$

A lemma

Lemma 21

Let Λ be a selfinjective finite dimensional k-algebra, S a simple Λ-module, and $h : P \to S$ its projective cover. Let M be an indecomposable Λ-module (in this talk, an indecomposable Λ-module means a finitely generated one). Then the following are equivalent.

1. $\operatorname{Ext}^1_\Lambda(M, \operatorname{rad} P) = 0$.
2. $h_* : \operatorname{Hom}_\Lambda(M, P) \to \operatorname{Hom}_\Lambda(M, S)$ is surjective.
3. M is either projective, or $M/\operatorname{rad} M$ does not contain S.

Representation theoretic characterization of $s(\omega_{\hat{A}}) > 0$

Let ν be the number such that $V_\nu \cong \det$.

Theorem 22 (Main Theorem)

Assume that A is not strongly F-regular (or equivalently, p divides $|G|$). Then the following are equivalent.

1. $s(\omega_{\hat{A}}) > 0$;
2. The canonical map $M_\nu \to \omega_A$ is surjective.
3. $H^1(G, B \otimes_k \operatorname{rad} P_\nu) = 0$.
4. For any non-projective indecomposable G-summand M of B, $\operatorname{soc} M$ does not contain $\det^{-1}$ (the k-dual of $\det$).

If these conditions hold, then $s(\omega_{\hat{A}}) \geq 1/|G|$.

The proof of **2**$\Leftrightarrow$**3**

Let $B = \bigoplus_j M_j$ with M_j indecomposable. The map $M_\nu \to \omega_A$ in **2** is

$$(B \otimes P_\nu)^G \to (B \otimes \det)^G.$$

As $\mathrm{Ext}^i_G(M_j^*, ?) \cong H^i(G, M_j \otimes ?)$, **2** is equivalent to say that

$$\mathrm{Hom}_G(M_j^*, P_\nu) \to \mathrm{Hom}_G(M_j^*, \det)$$

is surjective for each j.
On the other hand, **3** is equivalent to say that

$$\mathrm{Ext}^1_G(M_j^*, \mathrm{rad}\, P_\nu) \cong H^1(G, M_j \otimes \mathrm{rad}\, P_\nu) = 0$$

for each j. The result follows from Lemma 21.

The proof of **3**⇔**4**

Similarly, **4** is equivalent to say that each M_j^* is injective (or equivalently, projective, as kG is selfinjective) or $M_j^* / \operatorname{rad} M_j^* \cong (\operatorname{soc} M_j)^*$ does not contain $\det$.
Again by Lemma 21, we have **3**⇔**4**.

Theorem 13 for $E = B \otimes \det$

Let $k = V_0, V_1, \ldots, V_n$ be the list of simple G-modules. Let P_i be the projective cover of V_i. Set $M_i := (B \otimes_k P_i)^G$.

Theorem 13 (Symonds–H)

$$FL([\omega_{\hat{A}}]) = \frac{1}{|G|}[\hat{B}] = \frac{1}{|G|}\bigoplus_{i=0}^{n}(\dim V_i)[\hat{M}_i],$$

where $M_i = (B \otimes P_i)^G$.

The proof of $\mathbf{2} \Rightarrow \mathbf{1}$

As we assume that there is a surjection $M_\nu \to \omega_A$, $\mathrm{surj}(\hat{M}_\nu, \omega_{\hat{A}}) \geq 1$. By Theorem 13 (applied to $E = B \otimes \det$),

$$s(\omega_{\hat{A}}) = \mathrm{asn}(FL([\omega_{\hat{A}}]), \omega_{\hat{A}}) = \frac{1}{|G|} \mathrm{asn}([\hat{M}_\nu] + \sum_{i \neq \nu} (\dim V_i)[\hat{M}_i], \omega_{\hat{A}})$$

$$\geq \frac{1}{|G|} \mathrm{asn}([\hat{M}_\nu], \omega_{\hat{A}}) \geq \frac{1}{|G|} \mathrm{surj}(\hat{M}_\nu, \omega_{\hat{A}}) \geq \frac{1}{|G|} > 0.$$

The proof of **1**⇒**2** (1)

By Theorem 13, we have that $\mathrm{asn}([\hat{B}], \omega_{\hat{A}}) > 0$. Or equivalently, there is a surjection $h : \hat{B}^r \to \omega_{\hat{A}}$ for $r \gg 0$. By the equivalence $\gamma = (\hat{B} \otimes_{\hat{A}} ?)^{**} : \mathrm{Ref}(\hat{A}) \to \mathrm{Ref}(G, \hat{B})$, there corresponds

$$\tilde{h} = \gamma(h) : (\hat{B} \otimes_k kG)^r \to \hat{B} \otimes_k \det .$$

As $\hat{B} \otimes_k kG$ is a projective object in the category of (G, B)-modules, $\tilde{h}$ factors through the surjection

$$\hat{B} \otimes_k P_\nu \to \hat{B} \otimes_k \det .$$

The proof of $\mathbf{1} \Rightarrow \mathbf{2}$ (2)

$$
\begin{tikzcd}
(\hat{B} \otimes_k kG)^r \arrow[r, "\tilde{h}"] \arrow[d, dotted] & \hat{B} \otimes_k \det \\
\hat{B} \otimes P_\nu \arrow[ur] &
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
\hat{B}^r \arrow[r, two heads, "h"] \arrow[d, dotted] & \omega_{\hat{A}} \\
\hat{M}_\nu \arrow[ur] &
\end{tikzcd}
$$

Returning to the category Ref $\hat{A}$, the surjection $h : \hat{B}^r \to \omega_{\hat{A}}$ factors through

$$\hat{M}_\nu = (\hat{B} \otimes_{\hat{A}} P_\nu)^G \to \omega_{\hat{A}}.$$

So this map is also surjective, and **2** follows. $\square$

A corollary

Corollary 23

Let $\det^{-1}$ denote the dual representation of $\det$. Assume that p divides $|G|$. If $s(\omega_{\hat{A}}) > 0$, then $\det^{-1}$ is not a direct summand of B.

Proof.

Note that the one-dimensional representation $\det^{-1}$ is not projective. Moreover, the socle of $\det^{-1}$ is $\det^{-1}$, which contains $\det^{-1}$ as a submodule. The result follows from **1**⇒**4** of the theorem. □

A lemma

Lemma 24

Let M and N be in $\mathrm{Ref}(G, B)$. There is a natural isomorphism

$$\gamma : \mathrm{Hom}_A(M^G, N^G) \to \mathrm{Hom}_B(M, N)^G$$

Proof.

This is simply because $\gamma = (B \otimes_A ?)^{**} : \mathrm{Ref}(A) \to \mathrm{Ref}(G, B)$ is an equivalence, and $\mathrm{Hom}_B(M, N)^G = \mathrm{Hom}_{G,B}(M, N)$. $\square$

Another criterion (1)

Theorem 25

A is F-rational if and only if the following three conditions hold.

1. A is Cohen–Macaulay.
2. $H^1(G, B) = 0$.
3. $(B \otimes_k (I/k))^G$ is a maximal Cohen–Macaulay A-module, where I is the injective hull of k.

Another criterion (2)

Proof.

Assume that A is F-rational. Then A is Cohen–Macaulay.
As $s(\omega_{\hat{A}}) > 0$, $H^1(B \otimes \operatorname{rad} P_\nu)^G = 0$, and

$$0 \to (B \otimes \operatorname{rad} P_\nu)^G \to (B \otimes P_\nu)^G \to (B \otimes \det)^G \to 0 \qquad (1)$$

is exact. As $(B \otimes P_\nu)^G$ is a direct summand of $(B \otimes kG)^G = B$, it is an MCM module. As $(B \otimes \det)^G = \omega_A$, it is an MCM module. So the canonical dual of the exact sequence is still exact.

Another criterion (3)

Proof (continued).

As

$$\mathrm{Hom}_A((B\otimes_k ?)^G, \omega_A) = \mathrm{Hom}_B(B\otimes_k ?, B\otimes_k \det)^G = (B\otimes_k ?^* \otimes_k \det)^G,$$

we get the exact sequence of MCM A-modules

$$0 \to A \to (B \otimes P_\nu^* \otimes \det)^G \to (B \otimes (\mathrm{rad}\, P_\nu)^* \otimes \det)^G \to 0. \qquad (2)$$

As $(\mathrm{rad}\, P_\nu)^* \otimes \det \cong I/k$, $(B \otimes (I/k))^G$ is an MCM. As I is an injective G-module, $B \otimes I$ is also injective, and hence $H^1(G, B \otimes I) = 0$. By the long exact sequence of the cohomology, we get $H^1(G, B) = 0$.

The converse is similar. Dualizing (2), we have that (1) is exact. □

A corollary

Corollary 26

If A is F-rational, then $H^1(G,k)=0$.

Proof.

k is a direct summand of B, and $H^1(G,B)=0$. $\square$

Example 27

If $\mathrm{char}(k)=2$ and $G=S_2$ or S_3, then $H^1(G,k)\neq 0$. So $A=B^G$ is not F-rational (provided G does not have a pseudo-reflection).

An example (1)

- Let p be an odd prime number.
- Let us identify $\mathrm{Map}(\mathbb{F}_p, \mathbb{F}_p)^\times$ with the symmetric group S_p.
- Let $Q := \mathbb{F}_p \subset S_p$, acting on $\mathbb{F}_p$ by addition. Q is generated by the cyclic permutation $\sigma = (1+) = (0\ 1\ \cdots\ p-1) \in S_p$.
- Let $\Gamma := \mathbb{F}_p^\times \subset S_p$, acting on $\mathbb{F}_p$ by multiplication. It is a cyclic group of order $p-1$ generated by $\tau = (\alpha\cdot) = (1\ \alpha\ \alpha^2\ \cdots\ \alpha^{p-2})$, where α is the primitive element.
- As $\tau\sigma\tau^{-1} = \sigma^\alpha$, Γ normalizes Q. Set $G = Q\Gamma$. $C_G(Q) = Q$.
- $G = \{\phi \in S_p \mid \exists a \in \mathbb{F}_p^\times\ \exists b \in \mathbb{F}_p\ \forall x \in \mathbb{F}_p\ \phi(x) = ax + b\} \subset S_p$.
- $\#G = p(p-1)$.

An example (2)

- The only involution of Γ is is $\tau^{(p-1)/2} = ((-1)\cdot) = (1\ (p-1))(2\ (p-2))\cdots((p-1)/2\ (p+1)/2)$, which is a transposition if and only if $p = 3$.
- As Γ contains a Sylow 2-subgroup, a transposition of G, if any, is conjugate to an element, which must be a transposition, of Γ. So G has a transposition if and only if $p = 3$.

An example (3)

- $G \subset S_p$ acts on $P = k^p = \langle w_0, w_1, \ldots, w_{p-1} \rangle$ by $\phi w_i = w_{\phi(i)}$ for $\phi \in G$ and $i \in \mathbb{F}_p$.
- Let $r \geq 1$, and set $V = P^{\oplus r}$.
- G acts on V by permutations of the obvious basis.
- A permutation in G is a pseudo-reflection on V if and only if it is a transposition (as a permutation on the basis of V).
- G has a pseudo-reflection on V if and only if $r = 1$ and $p = 3$.

An example (4)

- Let $S = \operatorname{Sym} P$.
- Let $\lambda \in \mathbb{Z}^p$, and let $w^\lambda = w_0^{\lambda_0} \cdots w_{p-1}^{\lambda_{p-1}}$ be the corresponding monomial of S.
- Unless $\lambda_0 = \lambda_1 = \cdots = \lambda_{p-1}$, Q acts freely on the orbit Gw^λ. So kGw^λ is a kQ-free module.
- For a G-module M, we have $H^i(G, M) \cong H^i(Q, M)^\Gamma$ (since the order of Γ is coprime to p, the Lyndon–Hochschild–Serre spectral sequence collapses).
- So kGw^λ is G-projective in this case.
- If $\lambda_0 = \lambda_1 = \cdots = \lambda_{p-1}$, $kGw^\lambda \cong k$ is trivial.
- So S is a direct sum of projectives and copies of k.

An example (5)

- Now consider $V = P^{\oplus r}$ and $B := \operatorname{Sym} V \cong S^{\otimes r}$.
- Let k^- be the sign representation of G. As $\tau \in G$ is an odd permutation, $k^- \ncong k$.
- $\det_V = (\det P)^{\otimes r} = (k^-)^{\otimes r} \cong \det_V^{-1}$. This is k if r is even and k^- if r is odd.
- If M is a projective G-module and N a G-module, then $M \otimes N$ is projective. So $B = S^{\otimes r}$ is again a direct sum of projectives and copies of k.
- If $r = 1$ and $p = 3$, then $A := B^G = k[e_1, e_2, e_3]$, the polynomial ring generated by the elementary symmetric polynomials.
- Otherwise, G does not have a pseudo-reflection. $s(\omega_{\hat{A}}) > 0$ if and only if r is odd.

Kemper's theorem

Let k be a field of characteristic $p > 0$, and G be a subgroup of the symmetric group of S_d acting on $B = k[v_1, \ldots, v_d]$ by permutation. Let Q be a Sylow p-subgroup of G. Assume that $|Q| = p$. Let $N = N_G(Q)$ be the normalizer. Let $X_1, \ldots, X_c$ be the Q-orbits of $\{v_1, \ldots, v_d\}$. Set

$$H := \{\sigma \in N \mid \forall i \ \sigma(X_i) \subset X_i\}.$$

Then Q is a normal subgroup of H. Set $m := [H : C_H(Q)]$.

Theorem 28 (Kemper)

$$\operatorname{depth} B^G = \min\{2m + c, d\}.$$

The depth of our example

- For our G, Q, and V, $H = N = G$. $C_H(Q) = Q$.
- So $m = p - 1$, and $c = r$.
- So $\operatorname{depth} A = \min\{2p - 2 + r, rp\}$ and $\dim A = d = rp$.
- So A is Cohen–Macaulay if and only if $r \leq 2$.
- It follows that A is F-rational if and only if $r = 1$.

Conclusion

Theorem 29

Let $p \geq 3$, r, G, V, $B = \operatorname{Sym} V$, and $A = B^G$ be as above.

1. $\#G = p(p-1)$.
2. If $p = 3$ and $r = 1$, then G is a reflection group and A is a polynomial ring. Otherwise, G does not have a pseudo-reflection, and A is not F-regular.
3. If $p \geq 5$ and $r = 1$, then A is F-rational but not F-regular.
4. If $r = 2$, then A is Gorenstein, but not F-rational.
5. If $r \geq 3$ and odd, then $s(\omega_{\hat{A}}) > 0$ but A is not Cohen–Macaulay.
6. If $r \geq 4$ and even, then A is quasi-Gorenstein, but not Cohen–Macaulay.

Thank you

This slide will soon be available at
http://www.math.okayama-u.ac.jp/~hashimoto/