Almost principal fiber bundles

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The purpose of the talk

Let G be an algebraic group acting on $X = \operatorname{Spec} B$. A principal G-bundle is a very good quotient, but the map $X = \operatorname{Spec} B \to \operatorname{Spec} B^G = Y$ is rarely a principal fiber bundle. However, if we remove closed subsets of codimension two or more from both X and Y, the remaining part is often a principal G-bundle. Thus we can compare the reflexive sheaves, class gropus, and the canonical modules of X and Y in this case.

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Modules over Krull rings

Let R be a Krull domain. An R module M is said to be torsionless if there exist some $n \geq 0$ and some injection $M \hookrightarrow R^n$. M is torsionless if and only if $\dim_{Q(R)} M \otimes_R Q(R) < \infty$ and M is a lattice in $M \otimes_R Q(R)$, where $Q(R)$ is the field of fractions of $Q(R)$. If M is torsionless and the canonical map $M \to M^{**}$ is an isomorphism, then we say that M is reflexive (or divisorial).

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Locally Krull schemes

A scheme is said to be locally Krull if it has an open covering consisting of the prime spectra of Krull domains. Note that a locally Krull scheme is a (possibly infinite) disjoint union of integral locally Krull closed open subschemes.

Let Z be a locally Krull scheme, and $\mathcal{M}$ a quasi-coherent sheaf over Z. We say that $\mathcal{M}$ is torsionless (resp. reflexive) if for any $z \in Z$, there exists some affine open neighborhood $U = \operatorname{Spec} R$ of z such that R is a Krull domain and $\Gamma(U, \mathcal{M})$ is torsionless (resp. reflexive).

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Fundamental settings

Throughout the talk, let S be a scheme, and G a flat, quasi-compact quasi-separated S-group scheme.

Equivariant class group (1)

Let Z be a locally Krull G-scheme. Then we define $\mathrm{Cl}(G, Z)$ (resp. $\mathrm{Pic}(G, Z)$) to be the set of isomorphism classes of $(G, \mathcal{O}_Z)$-modules which are rank-one reflexive (invertible sheaves) as $\mathcal{O}_Z$-modules. $\mathrm{Cl}(G, Z)$ and $\mathrm{Pic}(G, Z)$ are called the equivariant class group (resp. Picard group) of Z.

Equivariant class group (2)

$\mathrm{Pic}(G, Z)$ is an additive group by the sum

$$[\mathcal{L}] + [\mathcal{L}'] = [\mathcal{L} \otimes \mathcal{L}'].$$

$\mathrm{Cl}(G, Z)$ is an additive group by the sum

$$[\mathcal{M}] + [\mathcal{N}] = [(\mathcal{M} \otimes \mathcal{N})^{**}].$$

Forgetful map

There is an obvious map

$$\alpha : \mathrm{Cl}(G, Z) \to \mathrm{Cl}(Z),$$

forgetting the action of G.

The kernel of α (1)

Let

$$\mathcal{X}(G) := \{\chi \in \Gamma(G, \mathcal{O}_G)^\times \mid \chi(gg') = \chi(g)\chi(g')\}$$

be the character group of G.

Lemma 1

If $\Gamma(G \times Z, \mathcal{O}_{G\times Z})^\times = \mathrm{pr}_1^* \Gamma(G, \mathcal{O}_G)^\times$, then $\operatorname{Ker} \alpha \cong \mathcal{X}(G)$. In particular, if $S = \operatorname{Spec} R$, $G = \operatorname{Spec} H$, and $Z = \operatorname{Spec} B$ are all affine, and if $B = R[x_1, \ldots, x_n]$ is a polynomial ring, then $\operatorname{Ker} \alpha \cong \mathcal{X}(G)$.

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The kernel of α (2)

Lemma 1

Let $S = \operatorname{Spec} k$ with k a field. Let G be a smooth connected algebraic k-group scheme. Let Z be a quasi-compact quasi-separated locally Krull G-scheme such that k is algebraically closed in $\Gamma(Z, \mathcal{O}_Z)$. Then the kernel of $\alpha : \mathrm{Cl}(G, Z) \to \mathrm{Cl}(Z)$ is isomorphic to $\mathcal{X}(G)/\mathcal{X}(G, Z)$, where

$$\mathcal{X}(G, Z) = \{\chi \in \mathcal{X}(G) \mid \exists \phi \in \Gamma(Z, \mathcal{O}_Z)^\times \ \chi(g) = \phi(gz)/\phi(z)\}.$$

Finite generation

Lemma 2

Let $S = \operatorname{Spec} k$, and G an affine k-group scheme of finite type. Assume one of the following:

1. $\Gamma(G \times Z, \mathcal{O}_{G\times Z})^\times = \mathrm{pr}_1^* \Gamma(G, \mathcal{O}_G)^\times$;
2. G is connected smooth, Z is quasi-compact quasi-separated, and k is integrally closed in $\Gamma(Z, \mathcal{O}_Z)$.

If $\mathrm{Cl}(Z)$ is a finitely generated abelian group, then $\mathrm{Cl}(G, Z)$ is so.

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Affine quotient

Lemma 3

Let $S = \operatorname{Spec} R$ be affine, and $G = \operatorname{Spec} \Gamma$ a flat affine R-group scheme. Let $\varphi : X \to Y$ be a G-invariant morphism such that X is locally Krull and φ is affine. Assume that $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism. Then $\operatorname{Cl}(Y)$ is a subquotient of $\operatorname{Cl}(G, X)$.

Waterhouse type theorem

Theorem 4

Let $S = \operatorname{Spec} R$, $G = \operatorname{Spec} \Gamma$, and $\varphi : X \to Y$ be as in the lemma above. Assume one of the following.

1. $\Gamma(G \times X, \mathcal{O}_{G \times X})^\times = \mathrm{pr}_1^* \Gamma(G, \mathcal{O}_G)^\times$ (e.g., $X = \mathbb{A}^n_R$);
2. G is connected and smooth, X is quasi-compact quasi-separated, and k is integrally closed in $\Gamma(X, \mathcal{O}_X)$.

Then $\mathrm{Cl}(Y)$ is a finitely generated abelian group.

Remark 5

Case 2 above is proved by Waterhouse under the condition that $X = \operatorname{Spec} B$ is affine, and $\bar{k} \otimes_k B$ does not have any nontrivial idempotent or nonzero nilpotent element, where $\bar{k}$ is the algebraic closure of k.

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Principal fiber bundle

Let N be an S-flat closed normal subgroup scheme of G.

Definition 6

We say that $\pi : X \to Y$ is a G-equivariant principal N-bundle if

1. π is a G-morphism. That is, G acts on X and Y, and $\pi(gx) = g\pi(x)$.
2. N acts trivially on Y.
3. π is fpqc (i.e., π is faithfully flat, and for any quasi-compact open subset V of Y, there exists some quasi-compact open subset U of X such that $\pi(U) = V$).
4. $\Phi : N \times X \to X \times_Y X$ ($\Phi(g, x) = (gx, x)$) is an isomorphism.

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A remark

Remark 7

A principal N-bundle is locally trivial in the fpqc topology, and the converse is also true.

We set $H = G/N$

Let $q : G \to H$ be a homomorphism of S-group scheme, and assume that q is a G-equivariant principal N-bundle.

Remark 8

- We have $N = \operatorname{Ker} q$.
- Roughly speaking, $H = G/N$.

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Important properties of principal bundles

Lemma 9

Let $\pi : X \to Y$ be a G-equivariant principal N-bundle. Then

1. π is quasi-separated.
2. If G is of finite presentation (resp. separated, affine, finite), then so is π.
3. (Grothendieck) $\pi^* : \mathrm{Qch}(H, Y) \to \mathrm{Qch}(G, X)$ is an equivalence, and $(\pi_* ?)^N$ is its quasi-inverse.

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Affine quotients are rarely prinicipal fiber bundles

So principal fiber bundles are very good quotients. However, If $X = \operatorname{Spec} B$ is a spectrum of a G-algebra and $Y = \operatorname{Spec} B^N$, the canonical map $\pi : X \to Y$ is rarely a principal N-bundle.

Rational almost principal fiber bundles

Definition 10

We say that a diagram of S-schemes

$$X \xleftarrow{i} V \xrightarrow{\rho} U \xhookrightarrow{j} Y$$

is a G-equivariant rational almost principal N-bundle if

1. G acts on X and Y, and N acts trvially on Y.
2. V is a G-stable open subset of X, and $\operatorname{codim}_X(X \setminus V) \geq 2$.
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We say that $\pi : X \to Y$ is a G-equivariant almost principal N-bundle if

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Notation

From now on, we assume that G is of finite presentation.

Let Z be a locally Krull G-scheme. We denote the category of quasi-coherent $(G, \mathcal{O}_Z)$-modules which are reflexive as $\mathcal{O}_Z$-modules by $\mathrm{Ref}(G, Z)$.

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Main theorem (1)

Theorem 12

Let

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be a G-equivariant rational almost principal N-bundle such that X and Y are locally Krull. Then

1. $\mathcal{N} \mapsto i_*\rho^*j^*\mathcal{N} : \mathrm{Ref}(H, Y) \to \mathrm{Ref}(G, X)$ is an equivalence, and $\mathcal{M} \mapsto (j_*\rho_*i^*\mathcal{M})^N$ is its quasi-inverse.
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Settings for discussing canonical modules

When we discuss canonical modules, we assume the following.

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Assume that Assumption (#) is satisfied.

1. Let N be smooth of relative dimension d. Set $\Theta = \bigwedge^d \operatorname{Lie} N$. Then there are a $(G, \mathcal{O}_X)$-isomorphism $\omega_X \cong i_* \rho^* j^* \omega_Y \otimes_{\mathcal{O}_X} (f^* \Theta)^*$ and an $(H, \mathcal{O}_Y)$-isomorphism $\omega_Y \cong (j_* \rho_* i^* (\omega_X \otimes_{\mathcal{O}_X} f^*(\Theta)))^N$, where $f : X \to S$ is the structure map.
2. Let $S = \operatorname{Spec} k$, and N be a finite linearly reductive group scheme. Then there are a $(G, \mathcal{O}_X)$-isomorphism $\omega_X \cong i_* \rho^* j^* \omega_Y$ and an $(H, \mathcal{O}_Y)$-isomorphism $\omega_Y \cong (j_* \rho_* i^* \omega_X)^N$.

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Assume that Assumption $(\#)$ is satisfied.

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2. Let $S = \operatorname{Spec} k$, and N be a finite linearly reductive group scheme. Then there are a $(G, \mathcal{O}_X)$-isomorphism $\omega_X \cong i_*\rho^* j^* \omega_Y$ and an $(H, \mathcal{O}_Y)$-isomorphism $\omega_Y \cong (j_*\rho_* i^* \omega_X)^N$.

Main theorem (2)

Theorem 13

Assume that Assumption $(\#)$ is satisfied.

1. Let N be smooth of relative dimension d. Set $\Theta = \bigwedge^d \mathrm{Lie}\, N$. Then there are a $(G, \mathcal{O}_X)$-isomorphism $\omega_X \cong i_*\rho^*j^*\omega_Y \otimes_{\mathcal{O}_X} (f^*\Theta)^*$ and an $(H, \mathcal{O}_Y)$-isomorphism $\omega_Y \cong (j_*\rho_*i^*(\omega_X \otimes_{\mathcal{O}_X} f^*(\Theta)))^N$, where $f : X \to S$ is the structure map.
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A remark

Remark 14

If $S = \operatorname{Spec} k$ with k a field of characteristic zero, then Theorem 13 is due to Knop.
The idea of the theorem is based on his result.

A corollary

Corollary 15

If Assumption (#) is satisfied and $\Theta \cong \mathcal{O}_S$, then the following are equivalent.

1. $\omega_Y \cong \mathcal{O}_Y$ in $\mathrm{Ref}(H, Y)$;
2. $\omega_X \cong \mathcal{O}_X$ in $\mathrm{Ref}(G, X)$.

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When is Θ trivial?

Remark 16

If $S = \operatorname{Spec} k$ and G an affine algebraic group over k, then the following hold.

1. If G is connected reductive, then $\Theta \cong k$.
2. If G is finite, then $\Theta \cong k$.
3. (Knop) In general, Θ may not be trivial.

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The case of almost principal fiber bundles (1)

Corollary 17

Let $\pi : X \to Y$ be a G-equivariant almost principal N-bundle such that X and Y are locally Krull.

1. $\mathcal{N} \mapsto (\pi^*\mathcal{N})^{**} : \mathrm{Ref}(H, Y) \to \mathrm{Ref}(G, X)$ is an equivalence, and $\mathcal{M} \mapsto (\pi_*\mathcal{M})^N$ is its quasi-inverse.
2. The equivalence induces $\mathrm{Cl}(H, Y) \cong \mathrm{Cl}(G, X)$.

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The case of almost principal fiber bundles (2)

Corollary 18

Let $\pi : X \to Y$ be a G-equivariant almost principal N-bundle. Assume that $(\#)$ is satisfied. Then

1. if G is smooth of relative dimension d, there are a $(G, \mathcal{O}_X)$-isomorphism $\omega_X \cong (\pi^* \omega_Y)^{**} \otimes_{\mathcal{O}_X} (f^* \Theta)^*$ and an $(H, \mathcal{O}_Y)$-isomorphism $\omega_Y \cong (\pi_*(\omega_X \otimes_{\mathcal{O}_X} f^*(\Theta)))^N$.
2. If $S = \operatorname{Spec} k$ and N is finite linearly reductive, then there are a $(G, \mathcal{O}_X)$-isomorphism $\omega_X \cong (\pi^* \omega_Y)^{**}$ and an $(H, \mathcal{O}_Y)$-isomorphism $\omega_Y \cong (\pi_* \omega_X)^N$.

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Example of finite groups (1)

Let k be an algebraically closed field, $B = k[x_1, \ldots, x_n]$, $V = \bigoplus_i kx_i$, and $G \subset \mathrm{GL}(V)$ a finite subgroup. Set $N = G$ and $H = \{e\}$. Let $A = B^G$, and $\pi : X = \operatorname{Spec} B \to \operatorname{Spec} A = Y$ be the canonical map.

Definition 19

We say that $g \in \mathrm{GL}(V)$ is a pseudo-reflection if $\operatorname{codim}_V \{v \in V \mid gv = v\} = 1$.

Lemma 20

$\pi : X \to Y$ is an almost principal G-bundle if and only if G does not have a pseudo-reflection.

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Example of finite groups (2)

Lemma 21

Assume that G does not have a pseudo-reflection. Then

1. $\mathrm{Cl}(Y) \cong \mathrm{Cl}(G, X) \cong X(G)$.
2. $\omega_B \cong (B \otimes_A \omega_A)^{**}$ and $\omega_A \cong \omega_B^G$.
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Example of finite groups (3)

Corollary 22 (Watanabe–Braun)

The following are equivalent.

1. $\omega_B \cong B$;
2. $G \subset \mathrm{SL}(V)$;
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Example of finite groups (4)

If $n = \dim B = 2$, then the equivalence $(?)^G : \mathrm{Ref}(G, B) \to \mathrm{Ref}(A)$ has the following interpretation.

$$\mathrm{Ref}(G, B) = \mathrm{Proj}(G, B) = \{M \in \mathrm{Mod}(G, B) \mid M \text{ is a finite projective } B\text{-module}\}$$

and $\mathrm{Ref}(A) = \mathrm{MCM}(A)$. If, moreover, $\#G \neq 0$ in k, then indecomposable objects of $\mathrm{Proj}(G, B)$ and irreducible representations of G are in one-to-one correspondence, and hence A is of finite representation type (well-known).

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Example of multi-section ring (1)

Let Y be a separated connected Noetherian normal scheme, and $D_1, \ldots, D_r \in \mathrm{Div}(Y)$. Assume that $\sum_{i=1}^r \mathbb{Z}D_i$ contains an ample Cartier divisor. Set $U = Y_{\mathrm{reg}}$. Let

$$V := \underline{\mathrm{Spec}} \bigoplus_{\lambda \in \mathbb{Z}^r} \mathcal{O}_U(\lambda_1 D_1' + \cdots + \lambda_r D_r') \xrightarrow{\rho} U$$

be the canonical map, where $D_i' := D_i|_U$. Let

$$R := \bigoplus_{\lambda \in \mathbb{Z}^r} \Gamma(Y, \mathcal{O}_Y(\lambda_1 D_1 + \cdots + \lambda_r D_r))$$

and set $X = \mathrm{Spec}\, R$. Set $N = G = \mathbb{G}_m^r$.

Example of multi-section ring (2)

Lemma 23

Under the notation above,

1. R is a Krull domain.
2. The diagram
$$X \xleftarrow{i} V \xrightarrow{\rho} U \xhookrightarrow{j} Y$$
is a rational almost principal G-bundle.
3. The functor $\beta : \mathrm{Ref}(Y) \to \mathrm{Ref}(G, R)$ given by $\mathcal{M} \mapsto \bigoplus_{\lambda \in \mathbb{Z}^r} \Gamma(Y, (\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(\lambda_1 D_1 + \cdots + \lambda_r D_r))^{**})$ is an equivalence, and gives an isomorphism $\beta' : \mathrm{Cl}(Y) \cong \mathrm{Cl}(G, R)$.

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Theorem 24

1. (Elizondo–Kurano–Watanabe) The sequence

$$\mathbb{Z}^r \xrightarrow{\gamma} \mathrm{Cl}(Y) \xrightarrow{\alpha\beta'} \mathrm{Cl}(R) \to 0$$

is exact, where $\gamma(\lambda) = \sum_{i=1}^r \lambda_i D_i$ and $\alpha\beta'(D) = [\bigoplus_\lambda \Gamma(Y, \mathcal{O}_Y(D + \sum_{i=1}^r \lambda_i D_i))]$.

2. (Kurano–H) Assume (#). Then

$$\omega_R = \bigoplus_{\lambda \in \mathbb{Z}^r} \Gamma(Y, \mathcal{O}_Y(K_Y + \sum_{i=1}^r \lambda_i D_i)).$$

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Example of multi-section ring (4)

Example 25 (well-known)

Consider the case that $Y = \mathbb{P}^1$, $r = 1$, and $D_1 = \{0\}$. Then

$$\mathrm{vb}(\mathbb{P}^1) = \mathrm{Ref}(\mathbb{P}^1) \to \mathrm{Ref}(\mathbb{G}_m, k[x, y])$$

is an equivalence. Any finitely generated graded free $k[x, y]$-module is a direct sum of rank-one free modules $k[x, y](m)$ ($m \in \mathbb{Z}$). Thus any vector bundle of $\mathbb{P}^1$ is a direct sum of $\mathcal{O}_{\mathbb{P}^1}(m)$ ($m \in \mathbb{Z}$).

Example of determinantal ring (1)

Let $S = \operatorname{Spec} k$, $m, n, t \in \mathbb{Z}$, and $m, n \geq t \geq 2$. Set $V = k^n$, $W = k^m$, and $E = k^{t-1}$. Define $X = \operatorname{Hom}(E, W) \times \operatorname{Hom}(V, E)$ and $Y = \{\varphi \in \operatorname{Hom}(V, W) \mid \operatorname{rank} \varphi < t\}$. Then $\pi : X \to Y$ is defined by $\pi(f, g) = f \circ g$.

Lemma 26

$\pi : X \to Y$ is a $\mathrm{GL}(V) \times \mathrm{GL}(E) \times \mathrm{GL}(W)$-equivariant almost principal $\mathrm{GL}(E)$-bundle.

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Example of determinantal ring (2)

Corollary 27

1. (Bruns) $\mathrm{Cl}(Y) \cong X(\mathrm{GL}(E)) \cong \mathbb{Z}$.
2. (Svanes) The following are equivalent.
   1. $m = n$.
   2. $\omega_X \cong \mathcal{O}_X$ as $(\mathrm{GL}(E), \mathcal{O}_X)$-modules.
   3. $\omega_Y \cong \mathcal{O}_Y$ as $\mathcal{O}_Y$-modules.
   4. Y is Gorenstein.

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Example of Veronese subring (1)

Set $S = \operatorname{Spec} k$, $G = \mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, $N = \mu_m = \operatorname{Spec} k[t]/(t^m - 1) \hookrightarrow G$ $(m > 1)$. $H = \operatorname{Spec} k[t^m, t^{-m}]$. A G-algebra is a $\mathbb{Z}$-graded k-alebra. For a G-algebra B, a (G, B)-module is nothing but a graded B-module. For a (G, B)-module M, M^N is nothing but the Veronese submodule $M^{(m\mathbb{Z})} = \bigoplus_{i \in m\mathbb{Z}} M_i$.

Let B be a Noetherian normal $\mathbb{Z}$-graded algebra such that $B_0 = k$ and $B = k[B_1]$. Assume that $B \neq k$ and $B \neq k[x]$. Or equivalently, $\dim B \geq 2$. B^N is the Veronese subring $B^{(m\mathbb{Z})} = \bigoplus_{i \in m\mathbb{Z}} B_i$.

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Example of Veronese subring (2)

Lemma 28

Under the assumptions above,

1. $\pi : X = \operatorname{Spec} B \to \operatorname{Spec} B^N = Y$ is a G-equivariant almost principal N-bundle.
2. $\omega_{B^N} \cong \omega_B^N$ and $\omega_B \cong (B \otimes_{B^N} \omega_{B^N})^{**}$.
3. $\omega_B \cong B(rm) \Leftrightarrow \omega_{B^N} \cong B^N(rm)$. In particular, B^N is quasi-Gorenstein if and only if B is quasi-Gorenstein and $a(B)$ is divisible by m. A similar result (B is Cohen–Macaulay but may not be normal) is by Goto–Watanabe.
4. $\mathrm{Cl}(Y) \cong \mathrm{Cl}(N, X)$. If $B = k[x_1, \ldots, x_n]$, then $\mathrm{Cl}(Y) \cong X(N) \cong \mathbb{Z}/m\mathbb{Z}$.

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3. $\omega_B \cong B(rm) \Leftrightarrow \omega_{B^N} \cong B^N(rm)$. In particular, B^N is quasi-Gorenstein if and only if B is quasi-Gorenstein and $a(B)$ is divisible by m. A similar result (B is Cohen–Macaulay but may not be normal) is by Goto–Watanabe.
4. $\mathrm{Cl}(Y) \cong \mathrm{Cl}(N, X)$. If $B = k[x_1, \ldots, x_n]$, then $\mathrm{Cl}(Y) \cong X(N) \cong \mathbb{Z}/m\mathbb{Z}$.

Example of Veronese subring (2)

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Example of Veronese subring (3)

Consider the case that $G = N = \mu_m$, $H = \{e\}$, and $B = k[[x, y]]$. Then

$$\mathrm{MCM}(B^N) = \mathrm{Ref}(B^N) \cong \mathrm{Ref}(N, B).$$

The only indecomposables of $\mathrm{Ref}(N, B)$ are $B, B(-1), \ldots, B(-m+1)$. Hence B^N is of finite representation type.

Frobenius pushforward (1)

Let k be an algebraically closed field of characteristic $p > 0$, G an affine algebraic group over k, and X a normal G-variety of finite type. Let

$$X \xleftarrow{i} V \xrightarrow{\rho} U \xhookrightarrow{j} Y$$

be a rational almost principal G-bundle.

When Y is affine, the decomposition of $F_*^e \mathcal{O}_Y$ is important to study the ring theoretic properties and invariants of Y, such as FFRT property, F-signature, dual F-signature, and so on.

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Frobenius pushforward (2)

Theorem 29 (Sannai–H)

Let $e > 0$. Under the equivalence $\mathrm{Ref}(Y) \cong \mathrm{Ref}(G, X)$, the $\mathcal{O}_Y$-module $F_*^e\mathcal{O}_Y$ corresponds to $(F_*^e\mathcal{O}_X)^{G_e}$, where G_e is the kernel of the Frobenius map $F^e : {}^eG \to G$.

Frobenius pushforward (3)

Example 30

Let $V = k^n$, G a finite subgroup of $\mathrm{GL}(V)$ without pseudo-reflection, and assume that $(|G|, p) = 1$. Set $B = \mathrm{Sym}\, V$ and $A = B^G$. Let $V_0 = k, V_1, \ldots, V_r$ be the set of irreducible representations of G, and set $M_i = (B \otimes_k V_i)^G$. Then G_e is trivial, and ${}^eA$ corresponds to the (G, B)-module ${}^eB$. Each M_i is an indecomposable maximal Cohen–Macaulay A-module, and the following are equivalent.

1. ${}^eA \cong M_0^{c_{0,e}} \oplus \cdots \oplus M_r^{c_{r,e}}$ as A-modules.
2. ${}^eB \cong (B \otimes V_0)^{c_{0,e}} \oplus \cdots \oplus (B \otimes V_r)^{c_{r,e}}$ as (G, B)-modules.
3. ${}^e(B/\mathfrak{m}^{[p^e]}) \cong V_0^{c_{0,e}} \oplus \cdots \oplus V_r^{c_{r,e}}$ as G-modules,

where $\mathfrak{m} = \bigoplus_{i>0} B_i$ is the irrelevant ideal.

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Frobenius pushforward (4)

Let Y be a smooth projective toric variety associated with a fan Δ. Then letting $X := \operatorname{Spec} \operatorname{Cox}(Y)$, there is a rational almost principal G-bundle of the form

$$X = \mathbb{A}^{\#\Delta(1)} \xhookleftarrow{i} V \xrightarrow{\rho} Y \xrightarrow{1_Y} Y,$$

where $G = \operatorname{Spec} k \operatorname{Cl}(Y)$.

Frobenius pushforward (5)

Example 31 (Thomsen)

Let Y be a toric variety. Then there exists some finitely many rank one reflexive sheaves $\mathcal{M}_1, \ldots, \mathcal{M}_r$ such that for any $e > 0$, there exists some decomposition

$$F_*^e \mathcal{O}_Y \cong \mathcal{M}_1^{\oplus c_{1,e}} \oplus \cdots \oplus \mathcal{M}_r^{\oplus c_{r,e}}.$$

Thank you

This slide will soon be available at
http://www.math.nagoya-u.ac.jp/˜hasimoto/