Equivariant sheaves and their applications to invariant theory

Mitsuyasu Hashimoto

Nagoya University

October 29, 2009

A (G, A)-module

Let k be an algebraically closed field and G an affine algebraic group over k.

Definition 1

Let A be a commutative G-algebra. We say that M is a (G, A)-module if

- M is a G-module;
- M is an A-module;
- the k-space structures of M coming from the above two items agree;
- The action $A \otimes M \to M$ $(a \otimes m \mapsto am)$ is G-linear.

An application

Let k be an algebraically closed field of characteristic $p > 0$. Let G be a reductive group over k. Let U be the unipotent radical of a Borel subgroup of G. Let V be a finite dimensional G-module. Let $C := \operatorname{Sym} V$ be the symmetric algebra.

Theorem 1 (H—)

Assume that C has a good filtration. Then

1. *C^G is strongly F-regular. In particular, C^G is Cohen–Macaulay.*
2. *C^U is (finitely generated and) F-pure. In particular, $\operatorname{Proj} C^U$ is Frobenius split.*

A diagram of S-schemes

Let S be the base scheme, and let $\mathrm{Sch}\,/S$ be the category of S-schemes. Let I be a small category. Let $X_\bullet$ be an I^{op}-diagram of S-schemes. That is, let $X_\bullet \in \mathrm{Func}(I^{\mathrm{op}}, \mathrm{Sch}\,/S)$ be a contravariant functor form I to $\mathrm{Sch}\,/S$.

Zariski site of a diagram of schemes (1)

Definition 2

We define a category $\mathrm{Zar}(X_\bullet)$ by:

$$\mathrm{ob}(\mathrm{Zar}(X_\bullet)) := \{(i, U) \mid i \in \mathrm{ob}\, I,\ U \in \mathrm{ob}(\mathrm{Zar}\, X_i)\};$$

$$\mathrm{Zar}(X_\bullet)((j, V), (i, U)) := \{(\phi, h) \mid \phi \in I(i, j),$$
$$h : V \to U, \quad \begin{tikzcd} V \arrow[r, "h"] \arrow[d, hook] & U \arrow[d, hook] \\ X_j \arrow[r, "X_\phi"] & X_i \end{tikzcd} \text{ is commutative}\}.$$

The composition is given by $(\phi, h) \circ (\phi', h') = (\phi'\phi, hh')$.

Zariski site of a diagram of schemes (2)

We introduce a Grothendieck topology into $\mathrm{Zar}(X_\bullet)$.

A class of morphisms $((i_\lambda, U_\lambda) \xrightarrow{(\phi_\lambda, h_\lambda)} (i, U))_{\lambda\in\Lambda}$ is said to be a covering if $\forall\lambda\ i_\lambda = i,\ \phi_\lambda = \mathrm{id}_i$, and $U = \bigcup_\lambda h_\lambda(U_\lambda)$.

Moreover, defining $\Gamma((i, U), \mathcal{O}_{X_\bullet}) := \Gamma(U, \mathcal{O}_{X_i})$, $\mathcal{O}_{X_\bullet}$ is a sheaf of commutative rings on $\mathrm{Zar}(X_\bullet)$. Thus $\mathrm{Zar}(X_\bullet)$ is a ringed site. We denote the category $\mathrm{Mod}(\mathrm{Zar}(X_\bullet))$ simply by $\mathrm{Mod}(X_\bullet)$.

The restriction and the β map

For $i \in \mathrm{ob}(I)$, we define $(?)_i : \mathrm{Mod}(X_\bullet) \to \mathrm{Mod}(X_i)$ by $\Gamma(U, \mathcal{M}_i) := \Gamma((i, U), \mathcal{M})$ for $\mathcal{M} \in \mathrm{Mod}(X_\bullet)$. $(?)_i$ is called the restriction functor. Note that $(?)_i$ has both a left adjoint and a right adjoint. In particular, $(?)_i$ preserves arbitrary limits and colimits. In particular, $(?)_i$ is exact.

For $\phi \in I(i, j)$, we define $\beta_\phi : (?)_i \to (X_\phi)_*(?)_j$ by

$$\Gamma(U, \mathcal{M}_i) = \Gamma((i, U), \mathcal{M}) \xrightarrow{\mathrm{res}_{(\phi, X_\phi|_{X_\phi^{-1}(U)})}} \Gamma((j, X_\phi^{-1}(U)), \mathcal{M}) = \Gamma(X_\phi^{-1}(U), \mathcal{M}_j) = \Gamma(U, (X_\phi)_* \mathcal{M}_j).$$

The α map and equivariant modules

For $\phi \in I(i,j)$, we define $\alpha_\phi : X_\phi^*(?)_i \to (?)_j$ to be the composite

$$X_\phi^*(?)_i \xrightarrow{\beta_\phi} X_\phi^*(X_\phi)_*(?)_j \xrightarrow{\varepsilon} (?)_j,$$

where ε is the counit of adjunction of the adjoint pair $(X_\phi^*, (X_\phi)_*)$.

Definition 3

$\mathcal{M} \in \mathrm{Mod}(X_\bullet)$ is said to be equivariant if $\alpha_\phi : X_\phi^* \mathcal{M}_i \to \mathcal{M}_j$ is an isomorphism for $\phi \in \mathrm{Mor}(I)$. The full subcategory of $\mathrm{Mod}(X_\bullet)$ consisting of equivariant $\mathcal{O}_{X_\bullet}$-modules is denoted by $\mathrm{EM}(X_\bullet)$.

Quasi-coherent and coherent modules

Definition 4

$\mathcal{M} \in \mathrm{Mod}(X_\bullet)$ is said to be:

1. locally quasi-coherent (resp. locally coherent) if $\mathcal{M}_i$ is quasi-coherent (resp. coherent) for any $i \in \mathrm{ob}(I)$.
2. quasi-coherent (resp. coherent) if it is locally quasi-coherent (resp. locally coherent) and equivariant.

The full subcategory of locally quasi-coherent (resp. quasi-coherent, coherent) modules in $\mathrm{Mod}(X_\bullet)$ is denoted by $\mathrm{Lqc}(X_\bullet)$ (resp. $\mathrm{Qch}(X_\bullet)$, $\mathrm{Coh}(X_\bullet)$).

Direct and inverse image

Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathrm{Func}(I^{\mathrm{op}}, \mathrm{Sch})$. Then a ringed continuous functor $f_\bullet^{-1} : \mathrm{Zar}(Y_\bullet) \to \mathrm{Zar}(X_\bullet)$ is defined by $f_\bullet^{-1}((i, U)) = (i, f_i^{-1}(U))$. Thus $(f_\bullet)_* : \mathrm{Mod}(X_\bullet) \to \mathrm{Mod}(Y_\bullet)$ is defined by $\Gamma((i, U), (f_\bullet)_*\mathcal{M}) = \Gamma(f_\bullet^{-1}(i, U), \mathcal{M})$. $(f_\bullet)_*$ has a left adjoint $f_\bullet^*$.

Note that $f_\bullet^*$ preserves equivariance, local quasi-coherence, and quasi-coherence. Note also that $(f_\bullet)_*$ preserves local quasi-coherence if f_i is quasi-compact quasi-separated for each $i \in \mathrm{ob}(I)$. If, moreover, $Y_\phi f_j = f_i X_\phi$ is a cartesian square for each $\phi \in \mathrm{Mor}(I)$ ($f_\bullet$ is cartesian), then $(f_\bullet)_*$ also preserves quasi-coherence.

The category Δ_M

Let $[n]$ denote the totally ordered set $\{0, 1, \ldots, n\}$ for $n \geq -1$.
Define (Δ^+) by $\mathrm{ob}(\Delta^+) = \{[n] \mid n \geq -1\}$ and
$(\Delta^+)([m], [n]) = \{\varphi \in \mathrm{Map}([m], [n]) \mid \varphi$ is a monotone map$\}$.
Define the subcategory Δ_M of (Δ^+) by $\mathrm{ob}(\Delta_M) = \{[0], [1], [2]\}$, and

$$\Delta_M([m], [n]) = \{\varphi \in (\Delta^+)([m], [n]) \mid \varphi \text{ is injective}\}.$$

Pictorially, Δ_M looks like

$$[2] \; \begin{array}{c} \xleftarrow{\delta_0(2)} \\ \xleftarrow{\delta_1(2)} \\ \xleftarrow{\delta_2(2)} \end{array} \; [1] \; \begin{array}{c} \xleftarrow{\delta_0(1)} \\ \xleftarrow{\delta_1(1)} \end{array} \; [0] \, .$$

The groupoid B_G^M

Let S be a scheme, G an S-group scheme, and X a G-scheme. We define $B_G^M(X) \in \mathrm{Func}(\Delta_M^{\mathrm{op}}, \mathrm{Sch}/S)$ by

$$B_G^M(X) = \begin{tikzcd} G \times G \times X \arrow[r, shift left=2, "1\times a"] \arrow[r, "\mu\times 1"] \arrow[r, shift right=2, "p_{23}"'] & G \times X \arrow[r, shift left, "a"] \arrow[r, shift right, "p_2"'] & X \end{tikzcd},$$

where $a : G \times X \to X$ is the action, $\mu : G \times G \to G$ is the product, and p_{23} and p_2 are projections.
We denote $\mathrm{Mod}(B_G^M(X))$ by $\mathrm{Mod}(G, X)$ and call its object a $(G, \mathcal{O}_X)$-module. $\mathrm{Lqc}(B_G^M(X))$, $\mathrm{Qch}(B_G^M(X))$, and $\mathrm{Coh}(B_G^M(X))$ are denoted by $\mathrm{Lqc}(G, X)$, $\mathrm{Qch}(G, X)$, and $\mathrm{Coh}(G, X)$, respectively.

Direct and inverse images

For a G-morphism $f : X \to Y$, $B_G^M(f) : B_G^M(X) \to B_G^M(Y)$ is a cartesian morphism, and the direct image $B_G^M(f)_* : \mathrm{Mod}(G, X) \to \mathrm{Mod}(G, Y)$ and the inverse image $B_G^M(f)^* : \mathrm{Mod}(G, Y) \to \mathrm{Mod}(G, X)$ are induced.

Why diagrams of schemes?

Lemma 5

$\mathrm{EM}(B_G^M(X))$ is equivalent to the category of G-linearized $\mathcal{O}_X$-modules by Mumford. The equivalence induces the equivalence between $\mathrm{Qch}(G, X)$ and the category of quasi-coherent G-linearized $\mathcal{O}_X$-modules.

What is the merit of considering diagrams of schemes?

- We can use induction on the number of objects of I.
- $\mathrm{Mod}(G, X) = \mathrm{Mod}(B_G^M(X))$ is a module category of a ringed site. So $\mathrm{Mod}(G, X)$ has $\underline{\mathrm{Hom}}$, $\otimes$, etc. and is flexible enough. The embedding $\mathrm{Qch}(G, X) \hookrightarrow \mathrm{Mod}(G, X)$ is a natural generalization of the embedding $\mathrm{Qch}(X) \hookrightarrow \mathrm{Mod}(X)$.
- The use of $\mathrm{Lqc}(G, X)$ is sometimes effective.

Local noetherian property

In the rest of this talk, let S be a noetherian scheme, G a flat S-group scheme of finite type, and X a noetherian G-scheme.

Lemma 6

The category $\mathrm{Qch}(G, X)$ is a locally noetherian abelian category, and $\mathcal{M} \in \mathrm{Qch}(G, X)$ is a noetherian object of $\mathrm{Qch}(G, X)$ if and only if $\mathcal{M} \in \mathrm{Coh}(G, X)$ if and only if $\mathcal{M}_{[0]}$ is coherent as an $\mathcal{O}_X$-module. The forgetful functor

$$(?)_{[0]} : \mathrm{Qch}(G, X) \to \mathrm{Qch}((B_G^M(X)_{[0]}) = \mathrm{Qch}(X)$$

given by $\mathcal{M} \mapsto \mathcal{M}_{[0]}$ is faithful exact, and admits a right adjoint.

An abuse of notation

Example 7

If k is a field, $S = \operatorname{Spec} k$ and G is affine, and $X = S$, then $\mathrm{Qch}(G, X)$ (resp. $\mathrm{Coh}(G, X)$) is equivalent to the category $\mathrm{Mod}(G)$ of G-modules (resp. finite dimensional G-modules). The functor $(?)_{[0]} : \mathrm{Qch}(G, S) \to \mathrm{Qch}(S) \cong \mathrm{Mod}(k)$ is identified with the forgetful functor, forgetting the G-action.

Usually, a G-module and its underlying vector space are expressed by the same symbol, say V. We use this abuse of notation, and express a $(G, \mathcal{O}_X)$-module $\mathcal{M}$ and its underlying $\mathcal{O}_X$-module $\mathcal{M}_{[0]}$ by the same symbol. For example, $\mathcal{O}_{B_G^M(X)}$ is simply denoted by $\mathcal{O}_X$ because $(\mathcal{O}_{B_G^M(X)})_{[0]}$ is $\mathcal{O}_X$. For a G-morphism $f : X \to Y$, the associated direct image $B_G^M(f)_*$ is simply denoted by f_*. Similarly for $B_G^M(f)^*$.

Operations on $\mathrm{Qch}(G, X)$

Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{L}$ be in $\mathrm{Qch}(G, X)$, $\mathcal{I}$ be a G-ideal, and $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_\lambda$ be quasi-coherent $(G, \mathcal{O}_X)$-submodules of $\mathcal{M}$. Let $\mathcal{L}$ and $\mathcal{M}_3$ be coherent. Then the following modules have structures of quasi-coherent G-linearized $\mathcal{O}_X$-modules.

- $\underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$, $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{L}, \mathcal{M})$,
- $\underline{H}^i_{\mathcal{I}}(\mathcal{M}) \cong \varinjlim \underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^n, \mathcal{M})$,
- The Fitting ideal $\underline{\mathrm{Fitt}}_j(\mathcal{L})$,
- $\mathcal{M}_1 \cap \mathcal{M}_2$, $\sum_\lambda \mathcal{M}_\lambda$, $\mathcal{I}\mathcal{M}_1$,
- $\mathcal{M}_1 : \mathcal{M}_3$, $\mathcal{M}_1 : \mathcal{I}$,...

G-dualizing complex

Definition 8

Let $\mathbb{F} \in D(G, X)$ $(= D(\mathrm{Mod}(G, X)))$. We say that $\mathbb{F}$ is *G-dualizing* if $\mathbb{F}$ has coherent cohomology groups, and the restriction $\mathbb{F}_{[0]} \in D(X)$ is a dualizing complex of X.

Example 9

If X is Gorenstein of finite Krull dimension, then $\mathcal{O}_X$ is a G-dualizing complex of X.

Definition 10

Let X be connected with a fixed G-dualizing complex $\mathbb{I}$. The lowest nonzero cohomology sheaf ω_X of $\mathbb{I}$ is called the *G-canonical sheaf* of X. Note that $\omega_X \in \mathrm{Coh}(G, X)$.

Equivariant twisted inverse (1)

Theorem 11 (H—)

Let $f : Y \to X$ be a G-morphism separated of finite type. Then there is a functor $f^! : D_{\mathrm{Lqc}}(G, X) \to D_{\mathrm{Lqc}}(G, Y)$, called the (equivariant) twisted inverse, which satisfies:

- $f^!$ is triangulated, $\mathrm{id}_X^! \cong \mathrm{Id}$, and $g^! f^! \cong (fg)^!$.
- $f^!(D_{\mathrm{Qch}}(G, X)) \subset D_{\mathrm{Qch}}(G, Y)$, and $f^!(D_{\mathrm{Coh}}(G, X)) \subset D_{\mathrm{Coh}}(G, Y)$.
- If $\mathbb{I}_X$ is G-dualizing, then $f^!(\mathbb{I}_X)$ is also G-dualizing.
- If f is proper, then $f^!$ is a right adjoint of $Rf_* : D_{\mathrm{Lqc}}(G, Y) \to D_{\mathrm{Lqc}}(G, X)$.

Equivariant twisted inverse (2)

Theorem 11 (continued)

- If f is an open immersion, then $f^!$ agrees with the restriction f^*.
- If f is of finite flat dimension, then $f^!(\mathbb{F}) \cong f^!(\mathcal{O}_X) \otimes^L Lf^*\mathbb{F}$.
- Let $f : Y \to X$ be a finite G-morphism, and let Z denote the ringed site $(\mathrm{Zar}(B_G^M(X)), f_*\mathcal{O}_Y)$. Let $g : Z \to \mathrm{Zar}(B_G^M(Y))$ be the obvious ringed continuous functor. Then
  $g_\# R\,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(\mathcal{O}_Z, ?)$ is isomorphic to $f^!$
  ($g_\# : \mathrm{Mod}(Z) \to \mathrm{Mod}(G, Y)$ is the canonical functor, which is exact).

Equivariant twisted inverse (3)

Theorem 11 (continued)

- If $f : Y \to X$ is a regular embedding of a well-defined codimension, say d, then $f^! \mathcal{O}_X \cong \bigwedge^d (f^* \mathcal{I})^\vee [-d]$, where $\mathcal{I}$ is the defining ideal sheaf of Y in X (it belongs to $\mathrm{Qch}(G, X)$).
- If $f : Y \to X$ is smooth of a well-defined relative dimension, say d, then $\Omega_{Y/X}$ has a canonical coherent $(G, \mathcal{O}_Y)$-module structure, and $f^! \mathcal{O}_X \cong \bigwedge^d \Omega_{Y/X}[d]$.

A remark

Remark 12

In the construction of $f^! : D_{\mathrm{Lqc}}(G, X) \to D_{\mathrm{Lqc}}(G, Y)$, we use the existence of a factorization $B_G^M(Y) \xrightarrow{\varphi} Z \xrightarrow{\psi} B_G^M(X)$, where $Z \in \mathrm{Func}(\Delta_M^{\mathrm{op}}, \mathrm{Sch}/S)$, ψ is (componentwise) proper, and φ is a (componentwise) image-dense open immersion. However, Z is not necessarily of the form $B_G^M(W)$ for some G-scheme W (we avoid the problem of equivariant compactification).

Duality of proper morphisms

Theorem 13 (Duality of proper morphisms, H–)

Let $f : X \to Y$ be a proper G-morphism of noetherian G-schemes. Then the canonical map

$$Rf_* R\,\underline{\mathrm{Hom}}_{\mathsf{Mod}(G,X)}(\mathbb{F}, f^!\mathbb{G}) \to R\,\underline{\mathrm{Hom}}_{\mathsf{Mod}(G,Y)}(Rf_*\mathbb{F}, \mathbb{G})$$

is an isomorphism for $\mathbb{F} \in D_{\mathsf{Qch}}(G, X)$ and $\mathbb{G} \in D^+_{\mathsf{Lqc}}(G, Y)$.

Serre duality for representations of reductive groups

Corollary 14

Let $S = \operatorname{Spec} k$, and G a reductive group over k. Let T be a maximal torus of G, and fix a base of the root system of G. Let B be the negative Borel subgroup. For any finite dimensional B-module M and any $i \in \mathbb{Z}$, there is an isomorphism of G-modules

$$R^{n-i} \operatorname{ind}_B^G (M^* \otimes (-2\rho)) \cong (R^i \operatorname{ind}_B^G M)^*,$$

where ρ is the half sum of positive roots, and $n = \dim G/B$.

An application to invariant theory

Theorem 15 (H—)

Let k be a field, G a linearly reductive finite k-group scheme. Let A be a finitely generated k-algebra with a G-action. If A is Gorenstein and $\omega_A \cong A$ as a (G, A)-module, then $B := A^G$ is Gorenstein and $\omega_B \cong B$.

Corollary 16

Let k and G be as above, and V a finite dimensional G-module. If the representation $G \to \mathrm{GL}(V)$ factors through $\mathrm{SL}(V)$, then $B := (\mathrm{Sym}\ V)^G$ is Gorenstein and $\omega_B \cong B$.

Remark 17

The case that G is a finite group is due to K.-i. Watanabe.

A proof

Proof.

Cohen–Macaulay property is trivial, since $B \to A$ is finite, and B is a direct summand of A. As $\dim A = \dim B$ and $\mathrm{Ext}^i_B(A, \omega_B) = 0$ for $i > 0$,

$$\omega_A \cong \pi^! \omega_B \cong \mathrm{Hom}_B(A, \omega_B)$$

as (G, B)-modules by the equivariant duality of finite morphisms, where $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is the canonical map.
Hence

$$\omega_B \cong \mathrm{Hom}_B(B, \omega_B) \cong \mathrm{Hom}_B(A, \omega_B)^G \cong \omega_A^G \cong A^G \cong B.$$

□

Z^* of a closed subscheme Z of X

Definition 18

Let Z be a closed subscheme of X. Then we denote the scheme theoretic image of the action $a : G \times Z \to X$ by Z^*.

The following hold:

- Z^* is the smallest G-stable closed subscheme of X containing Z.
- If Z is irreducible and G has connected geometric fibers, then Z^* is irreducible.
- If Z is reduced and G is S-smooth, then Z^* is reduced.

Some operations on G-closed subschemes

Lemma 19 (M. Miyazaki–H—)

Let Z be a G-stable closed subscheme of X. If $p_2 : G \times X \to X$ has regular fibers, then Z_{red} is G-stable. In particular, if G is S-smooth, then Z_{red} is G-stable.

Lemma 20 (H—)

Let $\mathcal{I}$ be a coherent G-ideal of $\mathcal{O}_X$. If G is S-smooth, then the integral closure $\overline{\mathcal{I}}$ of $\mathcal{I}$ is again a G-ideal.

Matijevic–Roberts type theorem

Theorem 21 (M. Miyazaki–H—)

Assume either

- G is S-smooth; or
- $S = \operatorname{Spec} k$, where k is a perfect field.

Let $\mathcal{C}$ and $\mathcal{D}$ be class of noetherian local rings, and assume that

- If $A \in \mathcal{C}$ and $A \to B$ be a local homomorphism which is regular and essentially of finite type, then $B \in \mathcal{D}$; and
- If $B \in \mathcal{D}$ and $A \to B$ is a regular essentially of finite type local homomorphism, then $A \in \mathcal{D}$.

Let y be a point of X, Y the closure of $\{y\}$, and let η be the generic point of an irreducible component of Y^*. If $\mathcal{O}_{X,\eta} \in \mathcal{C}$, then $\mathcal{O}_{X,y} \in \mathcal{D}$.

Some consequences

Corollary 22

Let A be a $\mathbb{Z}^n$-graded noetherian ring. Let P be a prime ideal of A, and let P^* be the prime ideal generated by the homogeneous elements of P.

- If A_{P^*} is *regular*, then A_P is regular.
- (Matijevic–Roberts, Hochster–Ratliff, Goto–Watanabe) If A_{P^*} is Cohen–Macaulay (resp. Gorenstein), then A_P is Cohen–Macaulay (resp. Gorenstein).
- (Miyazaki–H—) If A_{P^*} is of char. p, F-regular (resp. F-rational) and excellent, then A_P is F-regular (resp. F-rational).
- (H—) If A_{P^*} is of char. p, F-pure (resp. Cohen–Macaulay F-injective), then A_P is F-pure (resp. Cohen–Macaulay F-injective).

G-local G-schemes

Generalizing notions in algebraic geometry or commutative ring theory to notions in equivariant settings is an important problem.

What is the equivariant version of local rings?

Definition 23

A G-scheme Z is said to be G-local if there is a unique minimal non-empty G-stable closed subscheme P of Z. In this case, we say that (Z, P) is G-local.

If G is trivial, then a scheme Z is G-local if and only if $Z \cong \operatorname{Spec} A$ for some local ring A. For a general G, a G-local scheme need not be affine.

Some examples of G-local G-schemes (1)

Example 24

Let $S = \operatorname{Spec} \mathbb{Z}$, and $G = \mathbb{G}_m^n$. Let A be a G-algebra (so A is a $\mathbb{Z}^n$-graded ring). Then $X = \operatorname{Spec} A$ is G-local if and only if A is H-local (i.e., there is a unique maximal graded ideal) as a $\mathbb{Z}^n$-graded ring.

Example 25

Let k be a field, and G a reductive group over k. Let A be a finitely generated G-algebra. For $\mathfrak{p} \in \operatorname{Spec} A^G$, the G-scheme $X = \operatorname{Spec} A'$ with $A' := A^G_{\mathfrak{p}} \otimes_{A^G} A$ is G-local.

Some examples of G-local G-schemes (2)

Example 26

Let k be a field, G an affine algebraic k-group scheme, H a closed subgroup scheme of G, and $X = G/H$. Then (X, X) is G-local. So a G-local scheme need not be affine, even if G is so.

Example 27

Let k be an algebraically closed field, G a reductive group over k, and B a Borel subgroup. Then $(G/B, B/B)$ is B-local, since B/B is the smallest Schubert variety.

Geometric quotients are of finite type

The following was proved using Fogarty's idea.

Theorem 28 (H—)

Let the G-scheme X be of finite type over S. If $\varphi : X \to Y$ is a universally submersive geometric quotient, then Y is of finite type over S. If $\mathcal{M}$ is a coherent $(G, \mathcal{O}_X)$-module, then $(\varphi_* \mathcal{M})^G$ is a coherent $\mathcal{O}_Y$-module.

An application to invariant theory

Theorem 29 (M. Ohtani–H—)

Let k be a field, and G a linearly reductive algebraic group over k. Let Z be a noetherian, Cohen-Macaulay G-scheme with an affine geometric quotient $p : Z \to W$. Then W is (noetherian and) Cohen-Macaulay.

Remark 30

In the proof, we may assume that Z is G-local. The case that G is a finite group is due to Hochster–Eagon.

G-Nakayama's lemma

Equivariant versions of some theorems in local ring theory are obtained. Until the end of this talk, Let (X, Y) be G-local, and let η be the generic point of an irreducible component of Y. Let $i : Y \hookrightarrow X$ be the inclusion.

Lemma 31

The stalk functor $(?)_\eta : \mathrm{Qch}(G, X) \to \mathrm{Mod}(\mathcal{O}_{X,\eta})$ is faithful and exact.

Lemma 32 (G-Nakayama's lemma)

For $\mathcal{M} \in \mathrm{Coh}(G, X)$, if $i^*\mathcal{M} = 0$, then $\mathcal{M} = 0$.

Equivariant local cohomology

The functor $\underline{\Gamma}_Y = \mathrm{Ker}(\mathrm{Id} \to g_* g^*)$ is a functor from $\mathrm{Mod}(G, X)$ to itself, and preserves $\mathrm{Lqc}(G, X)$ and $\mathrm{Qch}(G, X)$, where $g : X \setminus Y \hookrightarrow X$ is the inclusion. The derived functor $R\underline{\Gamma}_Y : D(G, X) \to D(G, X)$ preserves $D^+_{\mathrm{Qch}}(G, X)$.

From now on, assume that X has a (fixed) G-dualizing complex $\mathbb{I}$.

Lemma 33

The cohomology group of $R\underline{\Gamma}_Y(\mathbb{I})$ is concentrated in one place.

If $\underline{H}^0_Y(\mathbb{I}) := R^0\Gamma_Y(\mathbb{I}) \neq 0$, then we say that $\mathbb{I}$ is G-normalized. From now on, we assume that $\mathbb{I}$ is G-normalized.

G-sheaf of Matlis

Definition 34

We set $\mathcal{E}_X := \underline{H}^0_Y(\mathbb{I})$, and call it the G-sheaf of Matlis.

Lemma 35

The stalk $\mathcal{E}_{X,\eta}$ is the injective hull of the residue field of the local ring $\mathcal{O}_{X,\eta}$.

$\mathcal{E}_X$ is the equivariant version of the injective hull of the residue field of a local ring.

Equivariant version of modules of finite length

Lemma 36

For $\mathcal{M} \in \mathrm{Qch}(G, X)$, TFAE.

- $\mathcal{M}$ is of finite length in $\mathrm{Qch}(G, X)$.
- $\mathcal{M}$ is coherent in $\mathrm{Qch}(X)$, and $\exists n\, \mathcal{I}^n \mathcal{M} = 0$, where $\mathcal{I}$ is the defining ideal of Y.
- $\mathcal{M}_\eta$ is an $\mathcal{O}_{X,\eta}$-module of finite length.

G-Matlis duality

Let $\mathcal{F}$ denote the full subcategory of $\mathrm{Qch}(G, X)$ consisting of objects of finite length.

Theorem 37 (Ohtani–H—, G-Matlis duality)

Let $\mathbb{D}$ denote the functor $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(?, \mathcal{E}_X) : \mathrm{Mod}(G, X) \to \mathrm{Mod}(G, X)$.

- $\mathbb{D}$ is an exact functor on $\mathrm{Coh}(G, X)$.
- $\mathbb{D}(\mathcal{F}) \subset \mathcal{F}$.
- The canonical map $\mathcal{M} \to \mathbb{D}\mathbb{D}\mathcal{M}$ is an isomorphism for $\mathcal{M} \in \mathcal{F}$.
- $\mathbb{D} : \mathcal{F} \to \mathcal{F}$ is an anti-equivalence.

Equivariant local duality

Theorem 38 (Equivariant local duality, M. Ohtani–H—)

Let $\mathbb{F}$ be a bounded below complex in $\mathrm{Mod}(G, X)$ with coherent cohomology groups. Then there is an isomorphism in $\mathrm{Qch}(G, X)$

$$\underline{H}^i_Y(\mathbb{F}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\underline{\mathrm{Ext}}^{-i}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{I}), \mathcal{E}_X).$$

Thank you for your attention!