Introduction to Commutative Algebra of Singularities *

Kei-ichi Watanabe
(College of Humanity and Sciences, Nihon Univ.)

1 Preface

In the theory of commutative rings, there are many cases where we must use methods of singularity theory or algebraic geometry. For example, if we want to classify commutative rings which satisfy certain condition, there frequently occur that we need classification of algebraic varieties of corresponding property. Or, we are given many kinds of examples from singularity theory or algebraic geometry.

In this lecture, I try to talk about some of most basic languages of singularity theory, especially relationship of graded rings and projective varieties and 2-dimernsional singularities and its corresponding graph of exceptional curves.

2 Resolution of singularities and Rational Singularities.

In this article, let $(A, \mathfrak{m})$ be a Noetherian local ring or graded ring over a field k and $\mathfrak{m}$ is the unique (graded) maximal ideal. We always assume that A is normal.

Definition 2.1. Assume that A is essentially of finite type over a field k of characteristic 0.

1. $f : X \to \mathrm{Spec}(A)$ is a resolution of singularities of A if f is a projective morphism, X is a regular scheme and f is an isomorphism outside the singular locus of $\mathrm{Spec}(A)$.

2. A is a rational singularity if A is normal and $H^i(X, O_X) = 0$ for every $i > 0$.

*This paper is a survey of singularity theory for commutative algebraists.

To construct a resolution of singularities, the concept of Rees algebra $\mathcal{R}_A(I) = \oplus_{n\geq 0} I^n t^n$ is seesntial.

By Grauert=Riemenschneider (GR) vanishing theorem, we get the implication that rational singularities are Cohen-Macaulay rings.

Theorem 2.2. *Let A be a normal local ring of $\dim A = d$. Let $f : X \to \mathrm{Spec}(A)$ be a resolution of singularities of $\mathrm{Spec}(A)$ and assume that GR vanishing theorem holds for f. That is, $H^i(X, \omega_X) = 0$ for all $i > 0$. Then the following conditions are equivalent* [1]*.*

1. *A is a rational singularity.*
2. *A is Cohen-Macaulay and $H^0(X, \omega_X)$ is a reflexive A module.*

Proof. Let $D_A^\bullet$ (resp. $D_X^\bullet$) be a dualizing complex of A (resp. X). Note that A (resp. X) is Cohen-Macaulay if and only if $D_A^\bullet \cong \omega_A[d]$ (resp. $D_X^\bullet \cong \omega_X[d]$), where ω_A (resp. ω_X) is the dualizing (canonical) module of A (resp. X). We use so called Grothendieck duality theorem

$$\mathcal{R}\mathrm{Hom}_A(\mathcal{R}f_*\mathcal{F}, D_A^\bullet) \cong \mathcal{R}f_*(\mathcal{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, D_X^\bullet)) \tag{2.2.1}$$

for coherent $\mathcal{O}_X$ module $\mathcal{F}$.

Now, assume the condition (1) and put $\mathcal{F} = \mathcal{O}_X$. This is equivalent to say that $\mathcal{R}f_*\mathcal{O}_X = A$. Since X is regular, $D_X^\bullet \cong \omega_X[d]$ and by GR vanishing, $\mathcal{R}f_*\omega_X \cong f_*\omega_X$. Hence we get $\mathcal{R}\mathrm{Hom}_A(\mathcal{R}f_*\mathcal{O}_X, D_A^\bullet) \cong D_A^\bullet$
$\cong \mathcal{R}f_*(\omega_X[d]) \cong f_*(\omega_X)[d]$ and we get condition (2).

Conversely, if we assume condition (2), putting $\mathcal{F} = \mathcal{O}_X$, we get RHS of (2.2.1) is $D_A^\bullet$. Then taking $\mathcal{R}\mathrm{Hom}_A(-, D_A^\bullet)$ of both sides, we gat $\mathcal{R}f_*\mathcal{O}_X = A$. $\square$

If $\mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ has at most rational singularities and if GR vanishing holds for f, then we have isomorphisms

$$H^i(X, O_X) \cong H^{i+1}_{\mathfrak{m}}(A) \quad \text{for} \quad 0 < i < \dim A - 2. \tag{2.2.2}$$

and natural inclusion $H^{d-1}(X, O_X) \subset H^d_{\mathfrak{m}}(A)$ ($d = \dim A$). Thus we can see that $H^i(X, O_X)$ is important for commutative ring theory.

The following Boutot's Theorem is very important for the theory of rational singularities and theory of invariants.

[1] It is known that GR vanishing theorem holds if A is essentially of finite type over a field of characteristic 0. There are counterexamples in characteristic $p > 0$ in general.

Theorem 2.3 (Boutot's Theorem). *([Bo]) Let B be essentially of finite type over a field of characteristic* 0 *and A be a pure subring of B* [2]. *If B is a rational singularity, then so is A.*

An important corollaly of Boutot's theorem is;

Corollary 2.4. *Let k be a field of cahracteristic* 0 *and G be a linearly reductive algebraic subgroup of $GL(n,k)$ acting linearly on the polynomial ring $S := k[X_1,\dots,X_n]$. Then the invariant subring S^G of S is a rational singularity and hence is Cohen-Mcaulay.*

3 Reduction modulo p.

In positive characteristic, there are concepts of F-rational, F-regular, F-pure rings, defined by the notion of tight closure introduced by C. Huneke and M. Hochster [HH] and splitting of Frobenius, which is connected to the concepts of algebraic geometry over a field of characteristic 0.

Definition 3.1. Let A be a commutative Noetherian ring containing a field of characteristic $p > 0$ and I be an ideal of A. We assume that A is an integral domain.

1. We denote $F : A \to A$, the Frobenius map defined by $F(a) = a^p$. Note that this map is a ring homomorphism. We say that A is F *finite* if F is a finite map. This is the case if A is seesntially of finite type over a perfect field.

2. For a power $q = p^e$ of p, $I^{[q]}$ is the ideal generated by q-th powers of elements of I.

3. The "tight closure" I^* of I is defined by $x \in I^* \iff \exists c \neq 0$, such that $cx^q \in I^{[q]} \forall q = p^e$. If $I = I^*$, I is called *tightly closed.*

4. A is *weakly F-regular* if every ideal I of A is tightly closed

5. A is *strongly F-regular* if A is F finite and for every $c \in A, c \neq 0, \exists q = p^e$ such that cF^e sending ca^q to $a \in A$ splits as A-module [3] .

[2] A subring A of B is a pure subring of B if for every A module M, the natural map $M = M \otimes_A A \to M \otimes_A B$ is injective. This condition implies that for every ideal I of A, we have $IB \cap A = I$.

[3] In many cases, the concepts "strongly F-regular" and "weakly F regular are known to be equivalent.

6. A is F *pure* if $F : A \to A$ is a pure map in the sense of 2.3. This condition is equivalent to say that A is F *split* in the sense that $F : A \to A$ splits, under a mild condition.

7. A local ring A is F *rational* if every parameter ideal of A is [4] tightly closed. If A is not local, we say that A is F rational if $A_{\mathfrak{p}}$ is F every prime ideal $\mathfrak{p}$ of A. If A is Gorenstein, then A is F rational if and only if A is F regular.

Example 3.2. *If the defining equations of A are of simple form, it is easy to know if A is F rational (F pure) or not.*

Let k be a field of characteristic p and put $A = k[X_0, \ldots, X_d]/(X_0^n + \ldots + X_d^n)$ and we assume that p does not divide n. We denote by x_i the image of X_i in A. then $I = (x_1, \ldots, x_d)$ is a parameter ideal and the socle of A/I is generated by x_0^{n-1}. Thus $I = I^*$ if and only if $x_0^{n-1} \notin I^*$. The condition is equivalent to say that $\bigcup_{q=p^e} I^{[q]} : x_0^{(n-1)q} = 0$ and we can conclude that A is F rational if and only if $n \leq d$. Also, we can show that A is F pure and not F regular if and only if $n = d+1$ and $p \equiv 1$ (mod n) by the following Fedder's criterion.

Theorem 3.3 (Fedder's criterion)**.** *([Fe]) Assume that $A = B/I$ is a local ring, where $(B, \mathfrak{n})$ is a regular local ring. Then*

1. *A is F pure if and only if $[I^{[p]} : I] \not\subset \mathfrak{n}^{[p]}$.*

2. *If A is F finite, then A is strongly F regular $\Longleftrightarrow \exists c \notin I$ such that*

   $c[I^{[q]} : I] \not\subset \mathfrak{n}^{[q]}$ $(\forall q = p^e \gg 1)$.

Assume that A is essentially of finite type over a field k of characteristic 0. Since A is defined by finitely many relations, we can take a finitely generated ring R over $\mathbb{Z}$ and a subring A_0 of A, which is a flat R algebra and such that $A \cong A_0 \otimes_R k$. In this case, for every maximal ideal $\mathfrak{n}$ of R, $R/\mathfrak{n}$ is a finite field. We call $A \otimes_R R/\mathfrak{n}$ reduction mod p if $R/\mathfrak{n}$ is a field of characteristic p.

We can characterize rational singularity via mod p reduction.

Theorem 3.4. *([Sm],[Ha], [MS]) If A is essentially of finite type over a field k of characteristic 0, then A is a rational singularity if and only if for sufficiently large p, reduction mod p of A is F-rational.*

4 Case of Graded Rings as Examples.

If $A = \oplus_{n \geq 0} A_n$ is a normal graded ring over a field $k = A_0$ and if $\mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ has at most rational singularity, then the weighted blowing-up

$$\mathcal{C}(A) = \mathrm{Spec}(\oplus_{n \geq 0} O_X(n))) \quad \text{where} \quad X = \mathrm{Proj}(A)$$

[4]It is equivalent to say, some parameter ideal of A is

is a partial resolution of $\mathrm{Spec}(A)$ with has at most rational singularities. Moreover, we have isomorphisms $H^i(X, O_X(n)) \cong H^{i+1}_{\mathfrak{m}}(A)_n$. Hence we have the following theorem.

Theorem 4.1. *([Fl],[Wa2]) A is a rational singularity if and only if*

1. *A is Cohen-Macaulay,*
2. $\mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$ *has at most rational singularities and*
3. *$a(R) < 0$, where $a(R) = \sup\{n \mid H^d_{\mathfrak{m}}(A)_N \neq 0\}$ is the a-invariant of A defined by Goto-Watanabe [GW], [GW0].*

Theorem of the same type also holds for F rational rings [FW]. But the "Boutot's Theorem for F rational ring" does not hold. Actually, there exist local rings A and ideal I of A, whose Rees algebra $\mathcal{R}_A(I) = \oplus_{n\geq 0} I^n$ is F rational, while A is not F rational ([Wa3]).

5 Normal 2-dimensional Singularities and Graph of Exceptional Sets.

In this section, we assume $(A, \mathfrak{m})$ is a normal local ring of dimension 2, which has a resolution of singularity.

If $f : X \to \mathrm{Spec}(A)$ is a resolution of singularity and $E = f^{-1}(\mathfrak{m}) = \cup_{i=1}^r E_i$ be the exceptional set of f, we have the intersection theory on X and the intersection matrix $(E_i.E_j)_{1\leq i,j\leq r}$ is a negative-definite symmetric matrix. We express the exceptional set E by the "dual graph" as follows.

Definition 5.1. Let $f : X \to \mathrm{Spec}(A)$ be a resolution of singularity and $E = f^{-1}(\mathfrak{m}) = \cup_{i=1}^r E_i$ be the exceptional set of f. We also assume that each E_i is smooth and E_i and E_j $(i \neq j)$ intersect transversally at at most one point [5]. Then the dual graph $\Gamma(X)$ of X is defined as follows.

1. $\Gamma(X)$ has r circles $v_1, \ldots, v_r$ corresponding $E_1, \ldots, E_r$. We write the self intersection number E_i^2 inside the circle v_i. If $E_i^2 = -2$, sometimes, we don't write -2 inside the circle. If the genus of E_i is g we write $[g]$ under the circle v_i.
2. We connect v_i and v_j if $E_iE_j = 1$.

Then the graph $\Gamma(X)$ is always a connected graph.

[5] Such resolution always exists if A is excellent ([Li2]) and called a "good resolution".

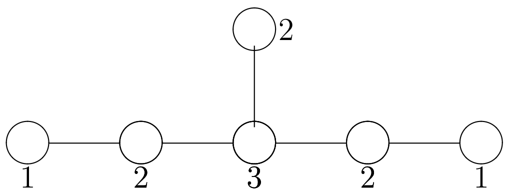
The examples of dual graphs are found in Figures (5.1) and (5.2).

We define some terminology concerning the resolution of singularity.

Definition 5.2. Let X be a resolution of $\mathrm{Spec}(A)$ with exceptional set $E = \cup_{i=1}^r E_i$.

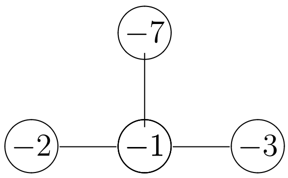
1. A cycle $Z = \sum_{i=1}^r n_i E_i$ with $(\forall n_i \geq 0)$ is called *anti-nef* if $Z.E_i \leq 0 (\forall i)$.
2. The *Fundamental cycle* Z_0 of E (or X) is the unique minimal anti-nef cycle [6] , where we write $Z = \sum_{i=1}^r n_i E_i \geq Z' = \sum_{i=1}^r n'_i E_i$ if $n_i \geq n'_i$ for every i.
3. For a cycle Z, we put $p_a(Z) = \dfrac{Z^2 + K_X.Z}{2} + 1$. We know (M. Artin) A is a rational singularity if and only if $p_a(Z_0) = 0$, where Z_0 is the fundamental cycle of X.
4. A resolution X of $\mathrm{Spec}(A)$ is said to be the minimal resolution if it contains no (-1) curve ($E_i \cong \mathbb{P}^1$ with $E_i^2 = -1$). Since (-1) curve can be contracted to a smooth point, the minimal resolution of $\mathrm{Spec}(A)$ is unique.

Example 5.3. *(1) If $A = k[X,Y,Z]/(X^2 + Y^3 + Z^4)$, then the dual graph of minimal resolution of* $\mathrm{Spec}(A)$ *is given by the following. The number attached to the cycles expresses the multiplicity of the fundamental cycle Z_0. Namely, $Z_0 = E_1 + 2E_2 + 3E_3 + 2E_4 + E_5 + 2E_6$.*



(5.1)

(2) If $A = k[X,Y,Z]/(X^2 + Y^3 + Z^7)$, then the dual graph of minimal good resolution of $\mathrm{Spec}(A)$ [7] *is given by the following.*



(5.2)

We can express integrally closed $\mathfrak{m}$ primary ideals by cycles on some resolution X of $\mathrm{Spec}(A)$.

[6] If Z, Z' are anti-nef cycles, so is $\inf(Z, Z')$. Thus the fundamental cycle is uniquely determined.

[7] Minimal good resolution is a resolution which is minimal among the good resolutions.

Definition 5.4. Let A be a Noetherian ring and $J \subset I$ be ideals of A.

1. We say that I is integral over J if $I^r = I^{r-1}J$ for some $r \geq 2$. We also say that J is a reduction of I.

2. We say that I is integrally closed if any $I' \supsetneq I$ is not integral over I.

Theorem 5.5. *([Li1], [Gi]) Assume that $(A, \mathfrak{m})$ is a 2 dimensional rational singularity and let I be an integrally closed $\mathfrak{m}$ primary ideal. Let X be a resolution of* $\mathrm{Spec}(A)$ *so that $I.O_X = O_X(-Z)$ is an invertible O_X- module* [8]*. We denote $I = I_Z$ in this case and say I is represented on X. Then the following properties hold.*

1. *Z is an anti-nef cycle and there is a one to one correspondence between anti-nef cycles on X and integrally closed $\mathfrak{m}$ primary ideals represented on X. This correspondence gives isomorphism of semigroups so that $I_Z I_{Z'} = I_{Z+Z'}$.*

2. *The multiplicity $e(I)$ and the colength I is calculated by*
$$e(I) = -Z^2 \quad \text{and} \quad \ell_A(A/I) = \frac{-Z^2 - K_X Z}{2},$$
where K_X is a canonical divisor on X (or a rational exceptional cycle determined by $p_a(E_i) = \dfrac{E_i^2 + K_X.E_i}{2} + 1$ for every E_i).

If A is regular local or $A \cong k[[X,Y,Z]]/(X^2+Y^3+Z^5)$, then the intersection matrix $(E_iE_j)_{1\leq i,j\leq r}$ is unimodular and we have the unique factorization of integrally closed ideals.

Corollary 5.6. *If A is regular local or $A \cong k[[X,Y,Z]]/(X^2 + Y^3 + Z^5)$, then the semigroup of integrally closed $\mathfrak{m}$ ideals has unique factorization. Namely, there are "prime" integrally closed ideals $\{P_\alpha\}$ such that any integrally closed ideal i is written as $I = \prod_{i=1}^r P_{\alpha_i}^{n_i}$ uniquely.*

Finally we show how to draw the graph $\Gamma(X)$ for a resolution X of $\mathrm{Spec}(A)$ in the case $A = \oplus_{n\geq 0} A_n$ of a normal graded ring over an algebraically closed field $k = A_0$. For that purpose, we recall so called DPD (Dolgachev-Pinkham-Demazure) construction of normal graded rings.

Theorem 5.7. *([Dem], [Wa0],[Wa1]) Let $R = \oplus_n R_n$ be a Noetherian normal graded ring with $X = \mathrm{Proj}(R)$. Then there exists an ample $\mathbb{Q}$-Cartier divisor D* [9] *on X such that $R \cong R(X,D) := \oplus_{n\geq 0} H^0(X, \mathcal{O}_X(nD))T^n$, where we denote*

[8] Such a resolution X exists.

[9] $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ such that ND is an ample Cartier divisor on X for some positive integer N.

$H^0(X, \mathcal{O}_X(nD)) = \{f \in k(X) \mid \mathrm{div}_X(f) + nD \geq 0\}$.

Actually, fix some homogeneous element $T = f/g$ *of degree* 1 *in the quotient field of* R *and let* $D = \mathrm{div}_X(T) = \mathrm{div}_X(f) - \mathrm{div}_X(g)$, *then* $R = \oplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))T^n$.

When $\dim R = 2$, then X is a smooth curve over k and $\mathrm{Div}(X) = \oplus_{P \in X} \mathbb{Z}P$, where P moves on closed points of X. Then the weighted blowing-up of R has only toric singularities and we can resolve $\mathrm{Spec}(R)$ by method of toric geometry.

Theorem 5.8. *Assume* $R = R(X, D)$, *where* X *is a smooth curve and* D *is an ample* $\mathbb{Q}$*-Cartier divisor on* X. *We put* $D = E - \sum_{i=1}^{r} \frac{p_i}{q_i} P_i$, *where* $E \in \mathrm{Div}(X)$ *with* $\deg E = a > 0$ *and* (q_i, p_i) *are relatively prime integers with* $0 < p_i < q_i$. *Then the graph* X *of minimal good resolution* X *of* $\mathrm{Spec}(R)$ *consists of the "central curve" corresponding to* X *and* r *"branches"* $B_1, \ldots, B_r$ *intersecting with* X *at point* P_i $(1 \leq i \leq r)$ *determined by continued fraction expression of* $\frac{p_i}{q_i}$. *Namely, if* $q_i/p_i = [b_1, \ldots, b_{m_i}]$, *then the branch* B_i *consists of chain of* m_i $\mathbb{P}^1$*'s of self-intersection number* $-b_1, \ldots, -b_{m_i}$, *where the first curve with self-interstction number* $-b_1$ *intersecting with* X *at* P_i. *Here, we denote*

$$[b_1, \ldots, b_m] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_m}}}$$

Let us explain by the graph of Example 5.3.

Example 5.9. *(1) Let* $R = k[X, Y, Z]/(X^2 + Y^3 + Z^4)$, *with* $\deg(x) = 6, \deg(y) = 4, \deg(z) = 3$. *Then we can calculate* $a(R) = 12 - (6 + 4 + 3) = -1$ *and hence* $X = \mathrm{Proj}(R) = \mathbb{P}^1$. *Then we can take* $T = y/z$. *Then since* $A/yA \cong k[x, z]/(x^2 + z^4)$ *with* $x^2 + z^4 = (x + iz)(x - iz)$, *then* $\mathrm{div}_X(y) = (P_1 + P_2)/3$, *where* $1/3$ *comes from the fact that* $k[x, z]/(x + iz)$ *is generated by element of deg*3. *In general, for irreducible variety* V *of codimension 1 of* X *corresponding* $\mathfrak{p}_V$, *then we take* $q_V = \mathrm{GCD}\{i \mid (R/\mathfrak{p}_V)_i \neq 0\}$. *Likewise, we can see* $\mathrm{div}_X(z) = P_3/2$. *Thus we see that*

$$R = R(\mathbb{P}^1, D), \quad D = \frac{1}{3}(P_1 + P_2) - \frac{1}{2}P_3 = (P_1 + P_2) - \frac{2}{3}(P_1 + P_2) - \frac{1}{2}P_3.$$

Since $\frac{3}{2} = [2, 2]$, B_1, B_2 *consists of* 2 $\mathbb{P}^1$*'s with self-intersection number* -2 *and also since* $E = P_1 + P_2$ *with* $\deg E = 2$, *we get the graph (5.1) of Example 5.3.*

(2) Let $A = k[X, Y, Z]/(X^2 + Y^3 + Z^7)$, *with* $\deg(x) = 21, \deg(y) = 14, \deg(z) = 6$. *Then we can take* $T = x/(yz)$. *Similarly as in (1), we get*

$$R = R(\mathbb{P}^1, D), \quad D = \frac{1}{2}P_1 - \frac{1}{3}P_2 - \frac{1}{7}P_3 = P_1 - \frac{1}{2}P_1 - \frac{1}{3}P_2 - \frac{1}{7}P_3.$$

Thus we get the graph (5.2) of Example 5.3.

参考文献

[Ar1] M. Artin, Some numerical criteria for contractability of curves on algebraic surfaces. Amer. J. Math. 84 (1962), 485?496.

[Ar2] M. Artin, On isolated rational singularities of surfaces, Amer. J. Math. 88 (1966), 129–136.

[AV] M. Artin and J.-L. Verdier., Reflexive modules over rational double points, Math. Ann. 270 (1985), 79–82.

[Bo] J.-F. Boutot, Singularité rationnelles et quotients par les groupes réductifs, Invent. Math. **88** (1987), 65-68.

[Dem] Demazure, M.: Anneaux gradués normaux ; in Seminarire Demazure - Giraud -Teissier, 1979. Ecole Polytechnique In:Lê Dũng Tráng(ed.) Introduction a la théorie des singularités II; Méthodes algébriques et géométriques (Travaux En Cours vol.**37**, pp.35-68) Paris:Hermann 1988

[Fe] R.Fedder, F-purity and rational singularity, Trans.Amer.Math.Soc. **278** (1983), 461-480.

[Fl] H. Flenner, Quasihomogene Rationale Singularitäten, Arch. Math. **36** (1981), 35-44.

[FW] R. Fedder and K. Watanabe, *A characterization of F-regularity in terms of F-purity*, Commutative algebra, Math. Sci. Research Inst. Publ. bf 15, Springer (1989), 227-245.

[Gi] J. Giraud, *Improvement of Grauert-Riemenschneider's Theorem for a normal surface*, Ann. Inst. Fourier, Grebnoble **32** (1982), 13–23.

[GW] S. Goto and K.-i. Watanabe, *On graded rings, I*, J. Math. Soc. Japan **30** (1978), 179–213.

[GW0] S. Goto and K.-i. Watanabe, *Commutative Ring Theory* (in Japanese), Nihon Hyouron-Sha, 2012, (後藤四郎・渡辺敬一, 可換環論， 日本評論社, 2012).

[Ha] N. Hara, *A characterization of rational singularities in terms of injectivity of Frobenius map*, Amer. J. Math. **120** (1998), 981–996.

[HH] M. Hochster and C. Huneke, *Tight Closure, invariant theory, and Briançon-Skoda theorem,* J. Amer. Math. Soc. **3** (1990), 31–116.

[Is] S. Ishii, Introduction to Singularity Theory (in Japanese), Springer Tokyo, 1997; (石井　志保子，特異点論入門，シュプリンガー東京，1997).

[K] M. Kato, *Riemann-Roch theorem for strongly pseudoconvex manifolds of dimension 2*, Math. Ann. **222**, (1976), 243–250.

[KM] J. Kollár and S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998.

[Li1] J. Lipman, *Rational singularities with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. 36 (1969) 195–279.

[Li2] J. Lipman, *Desingularization of* 2*-dimensional schemes*, Ann. Math. 107 (1978), 151–207.

[MS] V. B. Mehta and V. Srinivas, *A characterization of rational singularities*, Asian J. Math. **1** (1997), 249–278.

[Sm] K.E.Smith, *F-rational rings have rational singularities*, Amer. J. Math. **119** (1997), 159–180.

[TW] S. Takagi and K.-i. Watanabe, F-singularities − Application of Methods of Positive Characteristic to Singularity Theory (in Japanese), Suugaku (Japanese Mathematical Society), **66**, 1–30, (高木 俊輔・渡辺 敬一, F - 特異点 (正標数の手法の特異点論への応用) ; 「数学」**66**, 1-30).

[Wa0] K.-i. Watanabe, Methods of Algebraic Geometry in Commutative Ring Theory (in Japanese, Notes by T. Shirato), Lecture Notes in Math. **13**, Dept. of Math., Nagoya University, 2013 , (渡辺　敬一 (ノート 白土　智彬)；可換環論における幾何学的手法；名古屋大学「多元数理講義録」volume 11, 2013.

[Wa1] K.-i. Watanabe, Some remarks concerning Demazure's construction of normal graded rings, Nagoya Math. J. **83** (1981), 203-211.

[Wa2] K.-i. Watanabe, Rational singularities with k^*-action, in "Commutative Algebra", Proc. Conf. Trento/Italy 1981, Lect. Notes in Pure, Appl. Math. **84** (1983), 339-351.

[Wa3] K.-i. Watanabe, F-rationality of certain Rees algebras and counterexamples to "Boutot's Theorem" for F-rational rings, J. Pure and Appl. Alg. **122** (1997), 323-328.

[WY] K.-i.Watanabe and K.Yoshida, *Hilbert-Kunz multiplicity, McKay correspondence and good ideals in two-dimensional rational singularities*, manuscripta math. 104 (2001), 275–294.

[Yo] Y. Yoshino, Cohen-Macaulay Modules over Cohen-Macaulay Rings, London Mathematical Society Lecture Note Series No. 146 (1990).