HUNEKE–WIEGAND CONJECTURE OF RANK ONE WITH THE CHANGE OF RINGS

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1. Introduction

Let M and N be finitely generated modules over an integral domain R and assume that these modules are torsionfree. The purpose of this research is to get an answer for the question of when the tensor product $M \otimes_R N$ is torsionfree. Our interest dates back to the following conjecture.

Conjecture 1.1 (Huneke–Wiegand conjecture [12])**.** Let R be a Gorenstein local domain. Let M be a maximal Cohen–Macaulay R-module. Then M is free, once $M \otimes_R \mathrm{Hom}_R(M, R)$ is torsionfree.

Conjecture 1.1 holds true when the base ring is integrally closed or hypersurface. C. Huneke and R. Wiegand [12] showed that Conjecture 1.1 is reduced to the one-dimensional case. But the problem is still open in general, and no one has a complete answer to the following Conjecture 1.2, even in the case where the base ring is a complete intersection or a numerical semigroup ring; see [2, 8, 9, 10].

Conjecture 1.2. Let R be a Gorenstein local domain of dimension one and I an ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is a principal ideal.

In this paper we are interested in the question of what happens if we replace $\mathrm{Hom}_R(I, R)$ with $\mathrm{Hom}_R(I, \mathrm{K}_R)$, where K_R stands for the canonical module of R.

Conjecture 1.3. Let R be a Cohen–Macaulay local ring of dimension one and assume that R possesses a canonical module K_R. Let I be a faithful ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, \mathrm{K}_\mathrm{R})$ is torsionfree, then $I \cong R$ or K_R as an R-module.

If the ring R is Gorenstein, then Conjecture 1.3 is the same as Conjecture 1.2. One of the advantages of such a modification is the usage of the symmetry between I and $\mathrm{Hom}_R(I, \mathrm{K}_R)$ and the other one is the possible change of rings (see Proposition 2.3). However we should say that Conjecture 1.3 is not true in $\mathrm{e}(R) = 9$; later we shall give a counterexample, where $\mathrm{e}(R)$ stands for the multiplicity of R. In the case where $\mathrm{e}(R) = 7$ or 8, still we do not know whether Conjecture 1.3 is true or not. Nevertheless, the investigation into the truth of Conjecture 1.3 will make a certain amount of progress also in the study of Conjecture 1.2, which we would like to report in this paper.

2010 *Mathematics Subject Classification.* 13C12, 13H10, 13H15.

This paper is an announcement of our result and the detailed version have been submitted to somewhere.

The first author was partially supported by JSPS Grant-in-Aid for Scientific Research (C) 25400051. The second author was partially supported by JSPS Grant-in-Aid for Scientific Research (C) 25400038.

The following is the main result of our paper, which leads to Corollary 1.5 of higher dimension.

Theorem 1.4. *Let R be a Cohen–Macaulay local ring of dimension one having a canonical module K_R. Let I be a faithful ideal of R. Set $r = \mu_R(I)$ and $s = \mu_R(\mathrm{Hom}_R(I, \mathrm{K}_R))$.*

(1) *Assume that the canonical map $I \otimes_R \mathrm{Hom}_R(I, \mathrm{K}_R) \to \mathrm{K}_R$ is an isomorphism. If $r, s \geq 2$, then $\mathrm{e}(R) > (r+1)s \geq 6$.*
(2) *Suppose that $I \otimes_R \mathrm{Hom}_R(I, \mathrm{K}_R)$ is torsionfree. If $\mathrm{e}(R) \leq 6$, then I is isomorphic to either R or K_R.*

Here, $\mu_R(*)$ denotes the number of elements in a minimal system of generators.

Corollary 1.5. *Let R be a Cohen–Macaulay local ring with $\dim R \geq 1$. Assume that for every height one prime ideal $\mathfrak{p}$ the local ring $R_\mathfrak{p}$ is Gorenstein and $\mathrm{e}(R_\mathfrak{p}) \leq 6$. Let I be a faithful ideal of R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is reflexive, then I is a principal ideal.*

We shall prove Theorem 1.4 in Section 3. Section 2 is devoted to some preliminaries, which we need to prove Theorem 1.4. In Section 4 and 5, we focus on the monomial ideals in numerical semigroup rings. In the last section, we will give concrete examples including a counterexample of Conjecture 1.3.

In what follows, let R be a Cohen–Macaulay local ring with maximal ideal $\mathfrak{m}$. We set $F = \mathrm{Q}(R)$, the total ring of fractions of R. For each finitely generated R-module M, let $\mu_R(M)$ and $\ell_R(M)$ denote, respectively, the number of elements in a minimal system of generators of M and the length of M. For each Cohen–Macaulay R-module M, we denote by $\mathrm{r}_R(M)$ the Cohen–Macaulay type of M.

2. Change of rings

The purpose of this section is to summarize some preliminaries, which we need throughout this paper. Let R be a Cohen–Macaulay local ring with maximal ideal $\mathfrak{m}$ and $\dim R = 1$. Let $F = \mathrm{Q}(R)$ stand for the total ring of fractions of R and let $\mathcal{F}$ denote the set of fractional ideals I of R such that $FI = F$. Assume that R possesses a canonical module K_R. For each R-module M we set $M^\vee = \mathrm{Hom}_R(M, \mathrm{K}_R)$.

Let $I \in \mathcal{F}$. Denote by

$$t : I \otimes_R I^\vee \to \mathrm{K}_R$$

the R-linear map given by $t(x \otimes f) = f(x)$ for $x \in I$ and $f \in I^\vee$. Then the diagram

$$\begin{tikzcd} F \otimes_R (I \otimes_R I^\vee) \arrow[r, "\cong"] & F \otimes_R \mathrm{K}_R \\ I \otimes_R I^\vee \arrow[u, "\alpha"] \arrow[r, "t"] & \mathrm{K}_R \arrow[u] \end{tikzcd}$$

is commutative, where α is the base change map. Hence the torsion part $\mathrm{T}(I \otimes_R I^\vee)$ of the R-module $I \otimes_R I^\vee$ is given by

$$T = \mathrm{T}(I \otimes_R I^\vee) = \mathrm{Ker}\, t$$

and we get the following.

Lemma 2.1. *The R-module $I \otimes_R I^\vee$ is torsionfree if and only if the map $t : I \otimes_R I^\vee \longrightarrow \mathrm{K}_R$ is injective.*

Let $L = \mathrm{Im}(I \otimes_R I^\vee \xrightarrow{t} \mathrm{K}_R)$. Then $T^\vee = (0)$ since $\ell_R(T) < \infty$. Taking the K_R-dual of the short exact sequence $0 \to T \to I \otimes_R I^\vee \xrightarrow{t} L \to 0$, we have $L^\vee = (I \otimes_R I^\vee)^\vee$. Hence the equalities

$$L^\vee = (I \otimes_R I^\vee)^\vee = \mathrm{Hom}_R(I, I^{\vee\vee}) = I : I$$

follow. Recall that $B = I : I$ forms a subring of F which is a module-finite over R.

We take an arbitrary intermediate ring $R \subseteq S \subseteq B$. Then I is also a fractional ideal of S. Then we have

$$L = L^{\vee\vee} = B^\vee = \mathrm{K}_B \subseteq S^\vee = \mathrm{K}_S \quad \text{and}$$
$$\mathrm{Hom}_S(I, \mathrm{K}_S) = \mathrm{Hom}_S(I, \mathrm{Hom}_R(S, \mathrm{K}_R)) \cong \mathrm{Hom}_R(I \otimes_S S, \mathrm{K}_R) = \mathrm{Hom}_R(I, \mathrm{K}_R).$$

Let us identify $I^\vee = \mathrm{Hom}_S(I, \mathrm{K}_S)$ and we consider the commutative diagram

$$\begin{tikzcd}
0 & & \\
I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S) \arrow[u] \arrow[r, "t_S"] & \mathrm{K}_S & \\
I \otimes_R I^\vee \arrow[u, "\rho"] \arrow[r, "t"] & L \arrow[u, "\iota"'] \arrow[r] & 0
\end{tikzcd}$$

where $\iota : L \to \mathrm{K}_S$ is the embedding and $\rho : I \otimes_R I^\vee \to I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S)$ denotes the R-linear map defined by $\rho(x \otimes f) = x \otimes f$ for $x \in I$ and $f \in I^\vee$.

Suppose now that $I \otimes_R I^\vee$ is torsionfree. Then since the map $t : I \otimes_R I^\vee \to L$ is bijective by Lemma 2.1, the map $\rho : I \otimes_R I^\vee \to I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S)$ is bijective, whence the S-module $I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S)$ is also torsionfree.

To sum up with this kind of arguments, we have the following.

Lemma 2.2. *Let $I \in \mathcal{F}$ and suppose that $I \otimes_R I^\vee$ is torsionfree. Let $R \subseteq S \subseteq B = I : I$ be an intermediate ring. Then $I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S)$ is a torsionfree S-module and the canonical map $\rho : I \otimes_R I^\vee \to I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S)$ is bijective. In particular, if $S = B$, then the map*

$$t_B : I \otimes_B \mathrm{Hom}_B(I, \mathrm{K}_B) \to \mathrm{K}_B, \ \ x \otimes f \mapsto f(x)$$

is an isomorphism of B-modules.

The following is the key in our arguments, which we call "change of rings".

Proposition 2.3 (Change of rings). *Let $I \in \mathcal{F}$ and assume that $I \otimes_R I^\vee$ is torsionfree. If there exists an intermediate ring $R \subseteq S \subseteq B$ such that $I \cong S$ or $I \cong \mathrm{K}_S$ as an S-module, then $I \cong R$ or $I \cong \mathrm{K}_R$ as an R-module.*

Proof. Suppose that $I \cong S$ as an S-module and consider the isomorphisms

$$I \otimes_R I^\vee \overset{\rho}{\cong} I \otimes_S \mathrm{Hom}_S(I, \mathrm{K}_S) \cong \mathrm{Hom}_S(I, \mathrm{K}_S) \cong I^\vee$$

of R-modules. We then have $\mu_R(I) \cdot \mu_R(I^\vee) = \mu_R(I^\vee)$, so that $I \cong R$ as an R-module, since $\mu_R(I) = 1$. We similary have $I \cong \mathrm{K}_R$, if $I \cong \mathrm{K}_S$. $\square$

3. Proof of Theorem 1.4

The purpose of this section is to prove Theorem 1.4. We maintain the same notation and terminology as in Section 2. In this report, we only prove the assertion (1).

Proof of assertion (1) of Theorem 1.4. Enlarging the residue class field $R/\mathfrak{m}$ of R, without loss of generality we may assume that the field $R/\mathfrak{m}$ is infinite. Choose $f \in \mathfrak{m}$ so that fR is a reduction of $\mathfrak{m}$. We set $S = R/fR$, $\mathfrak{n} = \mathfrak{m}/fR$, and $M = I/fI$. Hence $\mu_S(M) = r$ and $\mathrm{r}_S(M) = \ell_S((0) :_M \mathfrak{n}) = s$ by [11, Bemerkung 1.21 a), Satz 6.10] (here $\mathrm{r}_S(M)$ denotes the Cohen–Macaulay type of M). We write $M = Sx_1 + Sx_2 + \cdots + Sx_r$ with $x_i \in M$ and consider the following presentation

$$(\sharp_0) \quad 0 \to X \to S^{\oplus r} \xrightarrow{\varphi} M \to 0$$

of the S-module M, where φ denotes the S-linear map defined by $\varphi(\mathbf{e}_j) = x_j$ for $1 \leq \forall j \leq r$ (here $\{\mathbf{e}_j\}_{1 \leq j \leq r}$ is the standard basis of $S^{\oplus r}$). Then, taking the K_S-dual (denoted by $[*]^\vee$ again) and the M-dual respectively of the above presentation $(\sharp_0)$, we get the following two exact sequences

$$(\sharp_1) \quad 0 \to M^\vee \to \mathrm{K}_S^{\oplus r} \to X^\vee \to 0,$$

$$(\sharp_2) \quad 0 \to \mathrm{Hom}_S(M, M) \to M^{\oplus r} \to \mathrm{Hom}_S(X, M)$$

of S-modules. Remember that $I \otimes_R I^\vee \overset{t}{\cong} \mathrm{K}_R$ and we have

$$M \otimes_S M^\vee \cong S \otimes_R (I \otimes_R I^\vee) \overset{S\otimes_R t}{\cong} S \otimes_R \mathrm{K}_R = \mathrm{K}_S,$$

because $S \otimes_R I^\vee = M^\vee$ and $S \otimes_R \mathrm{K}_R = \mathrm{K}_S$ ([11, Lemma 6.5, Korollar 6.3]). Hence

$$S = \mathrm{Hom}_S(\mathrm{K}_S, \mathrm{K}_S) \cong \mathrm{Hom}_S(M \otimes_S M^\vee, \mathrm{K}_S) = \mathrm{Hom}_S(M, M^{\vee\vee}) = \mathrm{Hom}_S(M, M),$$

so that exact sequence $(\sharp_2)$ gives rise to the exact sequence

$$(\sharp_3) \quad 0 \to S \xrightarrow{\psi} M^{\oplus r} \to \mathrm{Hom}_S(X, M),$$

where $\psi = {}^t\varphi$ is the transpose of φ, satisfying $\psi(1) = (x_1, x_2, \ldots, x_r) \in M^{\oplus r}$.

We set $q = \mu_S(X^\vee)$ $(= \ell_S((0) :_X \mathfrak{n}))$ and $e = \mathrm{e}(R)$. Then by $(\sharp_0)$ we get

$$\ell_S(X) = r{\cdot}\ell_S(S) - \ell_S(M) = re - e = (r-1)e,$$

since $\ell_S(S) = \mathrm{e}(R)$ and $\ell_S(M) = \mathrm{e}^0_{fR}(I) = \mathrm{e}^0_{fR}(R) = \mathrm{e}(R)$, where $\mathrm{e}^0_{fR}(I)$ and $\mathrm{e}^0_{fR}(R)$ denote respectively the multiplicity of I and R with respect to fR. On the other hand, by exact sequence $(\sharp_1)$ we have

$$q = \mu_S(X^\vee) \geq \mu_S(\mathrm{K}_S^{\oplus r}) - \mu_S(M^\vee) = r{\cdot}\mu_S(\mathrm{K}_S) - \mathrm{r}_S(M).$$

Because $I \otimes_R I^\vee \cong \mathrm{K}_R$ and $\mu_S(\mathrm{K}_S) = \mathrm{r}(S) = \mathrm{r}(R) = \mu_R(\mathrm{K}_R)$ ([11, Korollar 6.11]), we get $\mu_S(\mathrm{K}_S) = rs$, whence

$$(r-1)e = \ell_S(X) \geq \ell_S((0) :_X \mathfrak{n}) = q \geq r^2 s - s = s(r^2 - 1).$$

Thus $e \geq s(r+1)$, since $r, s \geq 2$.

Suppose now that $e = s(r+1)$. Then since $\ell_S(X) = \ell_S((0) :_X \mathfrak{n})$, we get $\mathfrak{n}{\cdot}X = (0)$, so that $\mathfrak{n}{\cdot}\operatorname{Hom}_S(X, M) = (0)$. Therefore $\mathfrak{n}{\cdot}M^{\oplus r} \subseteq S{\cdot}(x_1, x_2, \ldots, x_r)$ by exact sequence ($\sharp_3$). Let $1 \le i \le r$, $f \in M$, and $z \in \mathfrak{n}$ and write

$$z{\cdot}(0, \ldots, 0, \overset{\overset{i}{\vee}}{f}, \ldots, 0) = v{\cdot}(x_1, x_2, \ldots, x_r)$$

with $v \in S$. Then since $zf = vx_i$ and $0 = vx_j$ if $j \ne i$, we get $\mathfrak{n}M \subseteq \mathfrak{a}_i M$, where $\mathfrak{a}_i = (0) : (x_j \mid 1 \le j \le r,\ j \ne i)$. Notice that $\mathfrak{a}_i \ne S$, since $r = \mu_S(M) \ge 2$. Therefore $\mathfrak{n}M = \mathfrak{a}_i M$ for all $1 \le i \le r$, so that $\mathfrak{n}^2 M = (\mathfrak{a}_1\mathfrak{a}_2)M$, whence $\mathfrak{n}^2 M = (0)$ because $\mathfrak{a}_1\mathfrak{a}_2 \subseteq (0) : (x_i \mid 1 \le i \le r) = (0)$ (remember that M is a faithful S-module; see exact sequence ($\sharp_3$)). Thus $\mathfrak{n}M \subseteq (0) :_M \mathfrak{n}$. Consequently

$$s = \mathrm{r}_S(M) = \ell_S((0) :_M \mathfrak{n}) \ge \ell_S(\mathfrak{n}M) = \ell_S(M) - \ell_S(M/\mathfrak{n}M) = e - r = s(r+1) - r.$$

Hence $0 \ge rs - r = r(s-1)$, which is impossible because $r, s \ge 2$. $\square$

The following is a direct consequence of Theorem 1.4 (2).

Corollary 3.1. *Let R be a Gorenstein local ring with $\dim R = 1$ and $\mathrm{e}(R) \le 6$. Let I be a faithful ideal of R. If $I \otimes_R \operatorname{Hom}_R(I, R)$ is torsionfree, then I is a principal ideal.*

We also prove the following theorems. Here $\overline{R}$ stands for the integral closure of R.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be Cohen-Macaulay local ring of dimension one and assume that $\mathfrak{m}\overline{R} \subseteq R$. Let I be a faithful fractional ideal of R. If $I \otimes_R I^\vee$ is torsionfree, then $I \cong R$ or $I \cong \mathrm{K_R}$ as an R-module.*

Theorem 3.3. *Let R be a Cohen–Macaulay local ring with maximal ideal $\mathfrak{m}$ and $\dim R = 1$. Assume that R possesses a canonical module K_R and $\mu_R(\mathfrak{m}) = \mathrm{e}(R)$. Let I be a faithful ideal of R. We set $r = \mu_R(I)$ and $s = \mu_R(\operatorname{Hom}_R(I, \mathrm{K}_R))$. If $rs = \mathrm{r}(R)$, then $I \cong R$ or $I \cong \mathrm{K}_R$ as an R-module.*

Let us examine numerical semigroup rings.

Proposition 3.4. *Let $R = k[[t^a, t^{a+1}, \ldots, t^{2a-1}]]$ $(a \ge 1)$ be the semigroup ring of the numerical semigroup $H = \langle a, a+1, \ldots, 2a-1\rangle$ over a field k. Let $I \ne (0)$ be an arbitrary ideal of R. If $I \otimes_R I^\vee$ is torsionfree, then $I \cong R$ or $I \cong \mathrm{K}_R$ as an R-module.*

Corollary 3.5. *Let $R = k[[t^a, t^{a+1}, \ldots, t^{2a-2}]]$ $(a \ge 3)$ be the semigroup ring of the numerical semigroup $H = \langle a, a+1, \ldots, 2a-2\rangle$ over a field k and let I be an ideal of R. If $I \otimes_R \operatorname{Hom}_R(I, R)$ is torsionfree, then I is principal.*

Proof. Notice that R is a Gorenstein local ring with $R : \mathfrak{m} = R + kt^{2a-1}$ (see [11, Satz 3.3, Korollar 3.4]). Suppose that $\mu_R(I) > 1$ and set $B = I : I$. Then $R \subsetneq B$. In fact, $I \otimes_B \operatorname{Hom}_B(I, \mathrm{K}_B) \cong \mathrm{K}_B$ by Lemma 2.2. Hence, if $B = R$, then I is invertible, so that I must be a principal ideal. Thus $R \subsetneq B$ and therefore $t^{2a-1} \in B$, whence

$$R \subseteq S = k[[t^a, t^{a+1}, \ldots, t^{2a-1}]] \subseteq B.$$

Then by Lemma 2.2 $I \otimes_S \operatorname{Hom}_S(I, \mathrm{K}_S)$ is S-torsionfree, so that by Proposition 3.4 $I \cong S$ or $I \cong \mathrm{K}_S$ as an S-module. Hence $I \cong R$ by Proposition 2.3, which is impossible. $\square$

Remark 3.6. Corollary 3.5 gives a new class of one-dimensional Gorenstein local domains for which Conjecture 1.2 holds true. For example, in Corollary 3.5 take $a = 5$. Then $R = k[[t^5, t^6, t^7, t^8]]$ is not a complete intersection.

4. Numerical semigroup rings and monomial ideals

We focus our attention on numerical semigroup rings. Let us fix some notation and terminology.

Setting 4.1. Let $0 < a_1 < a_2 < \cdots < a_\ell$ be integers such that $\gcd(a_1, a_2, \ldots, a_\ell) = 1$. We set $H = \langle a_1, a_2, \ldots, a_\ell \rangle = \{\sum_{i=1}^{\ell} c_i a_i \mid 0 \le c_i \in \mathbb{Z}\}$ and

$$R = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \ \subseteq \ k[[t]],$$

where $V = k[[t]]$ is the formal power series ring over a field k. Let $\mathfrak{m} = (t^{a_1}, t^{a_2}, \ldots, t^{a_\ell})$ be the maximal ideal of R. We set $\mathfrak{c} = R : V$ and $c = \mathrm{c}(H)$, the conductor of H, whence $\mathfrak{c} = t^c V$. Let $a = c - 1$.

Notice that R is a Cohen–Macaulay local ring with $\dim R = 1$ and V the normalization. We have $\mathrm{e}(R) = a_1 = \mu_R(V)$.

Definition 4.2. Let $I \in \mathcal{F}$. Then I is said to be a monomial ideal, if $I = \sum_{n \in \Lambda} Rt^n$ for some $\Lambda \subseteq \mathbb{Z}$.

We denote by $\mathcal{M}$ the set of monomial ideals $I \in \mathcal{F}$. We are now going to explore Conjecture 1.3 on $I \in \mathcal{M}$. For the purpose, passing to the monomial ideal $t^{-q} I$ with $q = \min \Lambda$, we may assume $R \subseteq I \subseteq V$.

For the rest of this section let us assume that $e = a_1 \ge 2$. We set

$$\alpha_i = \max\{n \in \mathbb{Z} \setminus H \mid n \equiv i \mod e\}$$

for each $0 \le i \le e-1$ and put $\mathcal{S} = \{\alpha_i \mid 1 \le i \le e-1\}$. Hence $\alpha_0 = -e$, $\sharp \mathcal{S} = e - 1$, $a = \max \mathcal{S}$, and $\alpha_i \ge i$ for all $1 \le i \le e-1$.

With this notation we have the following.

Theorem 4.3. *Let $b = \min \mathcal{S}$ and suppose $t^b \in R : \mathfrak{m}$. Let $I \in \mathcal{M}$ such that $R \subseteq I \subseteq V$. If the canonical map $t : I \otimes_R I^\vee \to \mathrm{K}_R$ is an isomorphism, then $I \cong R$ or $I \cong \mathrm{K}_R$.*

The following is a special case of Theorem 3.3. We note a proof in the present context.

Corollary 4.4. *Suppose that $\mathrm{v}(R) = e$. Let $I \in \mathcal{M}$ such that $R \subseteq I \subseteq V$. If the canonical map $t : I \otimes_R I^\vee \to \mathrm{K}_R$ is an isomorphism, then $I \cong R$ or $I \cong \mathrm{K}_R$.*

The condition $t^b \in R : \mathfrak{m}$ does not imply $\mathrm{v}(R) = e$, as the following example shows.

Example 4.5. Let $H = \langle 7, 22, 23, 25, 38, 40 \rangle$. Then $\mathcal{S} = \{15, 16, 18, 33, 41\}$. We have $a = 41, b = 15$, and $\mathfrak{m} {\cdot} t^{15} \subseteq R$, but $\mathrm{v}(R) = 6 < e = 7$.

5. The case where $\mathrm{e}(R) = 7$

In this section we explore two-generated monomial ideals in numerical semigroup rings. We maintain Settings 4.1 and the notation in Section 4. Let $I \in \mathcal{M}$ be a monomial ideal of R such that $R \subseteq I \subseteq V$ and set $J = \mathrm{K}_R : I$. Suppose that $\mu_R(I) = \mu_R(J) = 2$ and write $I = (1, t^{c_1})$ and $J = (1, t^{c_2})$, where $c_1, c_2 > 0$. Throughout this section we assume:

Condition 5.1. $IJ = \mathrm{K}_R$ and $\mu_R(\mathrm{K}_R) = 4$.

Note that Condition 5.1 is not that the canonical map is an isomorphism. Then we have the following.

Proposition 5.2. $e = a_1 \geq 8$.

The goal of this section is Theorem 5.3.

Theorem 5.3. *Let $R = k[[t^{a_1}, t^{a_2}, \cdots, t^{a_\ell}]]$ be a numerical semigroup ring over a field k and suppose that $e = a_1 \leq 7$. Let I be a monomial ideal of R. If $I \otimes_R I^\vee$ is torsionfree, then $I \cong R$ or $I \cong \mathrm{K}_R$ as an R-module.*

Corollary 5.4 ([8, Main Theorem])**.** *Let R be a Gorenstein numerical semigroup ring with $\mathrm{e}(R) \leq 7$ and let I be a monomial ideal in R. If $I \otimes_R \mathrm{Hom}_R(I, R)$ is torsionfree, then I is a principal ideal.*

6. Examples

When $e = a_1 = 8$, there exists monomial ideals I for which Condition 5.1 is satisfied. However, for these ideals I the R-modules $I \otimes_R I^\vee$ have non-zero torsions. Let us show one example.

Example 6.1. We consider $H = \langle 8, 11, 14, 15 \rangle$ and $R = k[[t^8, t^{11}, t^{14}, t^{15}]]$. Then $\mathrm{K}_R = (1, t, t^3, t^4)$. We take $I = (1, t)$ and set $J = \mathrm{K}_R : I$. Then $J = (1, t^3)$ and

$$\mathrm{T}(I \otimes_R J) = R(t \otimes t^{16} - 1 \otimes t^{17}) \cong R/\mathfrak{m}.$$

Remark 6.2. The ring R of Example 6.1 contains no monomial ideals I such that $I \ncong R, I \ncong \mathrm{K}_R$, and $I \otimes_R I^\vee$ is torsionfree.

The following ideals also satisfy Condition 5.1 but $I \otimes_R I^\vee$ is not torsionfree.

(1) $H = \langle 8, 9, 10, 13 \rangle, \mathrm{K}_R = (1, t, t^3, t^4), I = (1, t)$.
(2) $H = \langle 8, 11, 12, 13 \rangle, \mathrm{K}_R = (1, t, t^3, t^4), I = (1, t)$.
(3) $H = \langle 8, 11, 14, 23 \rangle, \mathrm{K}_R = (1, t^3, t^9, t^{12}), I = (1, t^3)$.
(4) $H = \langle 8, 13, 17, 18 \rangle, \mathrm{K}_R = (1, t, t^5, t^6), I = (1, t)$.
(5) $H = \langle 8, 13, 18, 25 \rangle, \mathrm{K}_R = (1, t^5, t^7, t^{12}), I = (1, t^5)$.

If $a_1 \geq 9$, then Theorem 5.3 is not true in general. Let us note one example.

Example 6.3. Let $H = \langle 9, 10, 11, 12, 15 \rangle$. Then $R = k[[t^9, t^{10}, t^{11}, t^{12}, t^{15}]]$. We have $\mathrm{K}_R = (1, t, t^3, t^4)$. Let $I = (1, t)$ and put $J = \mathrm{K}_R : I$. Then $J = (1, t^3)$, $\mu_R(I) = \mu_R(J) = 2$, and $\mu_R(\mathrm{K}_R) = 4$. We have $R : I = (t^9, t^{10}, t^{11})$, $J : I = (t^9, t^{10}, t^{11}, t^{12}, t^{13}, t^{14})$, and $(R : I)J = J : I$, so that $I \otimes_R I^\vee$ is torsionfree.

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