NILPOTENCY OF FROBENIUS AND DIVISOR CLASS GROUPS

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In this note, we will briefly summarize our results on two-dimensional $F$-nilpotent rings. See [7] for the details. All rings are excellent in this note.

Let $R$ be a ring of prime characteristic $p$ and $F : R \to R$ the Frobenius map which sends $x \in R$ to $x^p \in R$. If $(R, \mathfrak{m})$ is local, then the Frobenius map $F$ induces a $p$-linear map $H^i_m(R) \to H^i_m(R)$ for each $i$, which we denote by the same letter $F$. The $e$-th iteration of $F$ is denoted by $F^e$. Also, we denote by $R^e$ the set of elements of $R$ which are not in any minimal prime ideal.

**Definition 1.** Let $(R, \mathfrak{m})$ be a $d$-dimensional reduced local ring of characteristic $p > 0$.

(i) We say that $R$ is $F$-injective if $F : H^i_m(R) \to H^i_m(R)$ is injective for all $i$.

(ii) We say that $R$ is $F$-rational if $R$ is Cohen-Macaulay and if for any $c \in R^e$, there exists $e \in \mathbb{N}$ such that $cF^e : H^d_m(R) \to H^d_m(R)$ is injective.

**Remark 2.** $F$-rationality implies $F$-injectivity.

The tight closure $0^*_{H^d_m(R)}$ of the zero submodule in $H^d_m(R)$ is the submodule of $H^d_m(R)$ consisting of all elements $z \in H^d_m(R)$ for which there exists $c \in R^e$ such that $cF^e(z) = 0$ for all large $e \in \mathbb{N}$. When $R$ is analytically irreducible, $0^*_{H^d_m(R)}$ is the unique maximal proper $R$-submodule of $H^d_m(R)$ stable under the Frobenius action $F$ (see [6]). It follows from the definition of $F$-rational rings that $R$ is $F$-rational if and only if $R$ is Cohen-Macaulay and $0^*_{H^d_m(R)} = 0$.

**Definition 3.** Let $(R, \mathfrak{m})$ be a $d$-dimensional reduced local ring of characteristic $p > 0$. We say that $R$ is $F$-nilpotent\(^1\) if the natural Frobenius actions $F$ on $H^0_m(R), \ldots , H^{d-1}_m(R), 0^*_{H^d_m(R)}$ are all nilpotent, that is, there exists $e \in \mathbb{N}$ such that $F^e(H^0_m(R)) = \cdots = F^e(H^{d-1}_m(R)) = F^e(0^*_{H^d_m(R)}) = 0$.

**Remark 4.** (i) When a (not necessarily finitely generated) $R$-module $M$ has a Frobenius action $F$, we denote $M_{\text{nil}} := \{ z \in M \mid F^e(z) = 0 \text{ for some } e \in \mathbb{N} \}$.

By Hartshorne–Speiser–Lyubeznik Theorem, the definition of $F$-nilpotency is equivalent to saying that $H^i_m(R)_{\text{nil}} = H^i_m(R)$ for all $i \leq d - 1$ and $(0^*_{H^d_m(R)})_{\text{nil}} = 0^*_{H^d_m(R)}$.

(ii) $R$ is $F$-rational if and only if $R$ is $F$-injective and $F$-nilpotent.

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\(^1\)Blickle and Bondu [2] called such rings “rings close to $F$-rational”.

This paper is an announcement of our result and the detailed version will be submitted to somewhere.
Example 5. Let $k$ be a perfect field of characteristic $p > 0$.

1. $k[[x, y, z]]/(x^2 + y^3 + z^7)$ is $F$-nilpotent but not $F$-injective.
2. $k[[x, y, z]]/(x^2 + y^3 + z^7 + xyz)$ is not $F$-nilpotent but $F$-injective.
3. ([1, Example 5.28]) $k[[x, y, z]]/(x^4 + y^4 + z^4)$ is $F$-nilpotent if and only if $p \equiv 3 \mod 4$.

Using reduction from characteristic zero to positive characteristic, we can define the notion of $F$-singularities in characteristic zero.

Definition 6. Let $R = k[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$ be a ring of finite type over a field $k$ of characteristic zero. Let $A$ be a $\mathbb{Z}$-subalgebra of $k$ generated by the coefficients of the $f_i$, and put $R_A = A[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$. Then $R_A \otimes_A k \cong R$. By the generic freeness, after possibly localizing $A$ at a single element, we may assume that $R_A$ is flat over $A$. We refer to $R_A$ as a model of $R$.

We say that $R$ is of $F$-rational type (resp. $F$-nilpotent type) if there exists a model $R_A$ of $R$ over a finitely generated $\mathbb{Z}$-subalgebra $A \subseteq k$ and a dense open subset $S \subseteq \text{Spec } A$ such that $R_\mu := R_A \otimes_A A/\mu$ is $F$-rational (resp. $F$-nilpotent) for all closed points $\mu \in S$.

Example 7. By Example 5, $\mathbb{C}[x, y, z]/(x^2 + y^3 + z^7)$ is of $F$-nilpotent type, but $\mathbb{C}[x, y, z]/(x^4 + y^4 + z^4)$ is not.

As the name suggests, $F$-rational rings correspond to rational singularities.

Theorem 8 ([3], [5], [6]). Let $(R, \mathfrak{m})$ be a normal local ring essentially of finite type over an field of characteristic zero. $R$ is of $F$-rational type if and only if $\text{Spec } R$ has only rational singularities, that is, for every (some) resolution of singularities $\pi : Y \to X = \text{Spec } R$, $R^i \pi_* \mathcal{O}_Y = 0$ for all $i \geq 1$.

We obtain a characterization of two-dimensional rings of $F$-nilpotent type in terms of dual graphs of resolutions of singularities.

Theorem 9. Let $(R, \mathfrak{m})$ be a two-dimensional normal local ring essentially of finite type over an algebraically closed field of characteristic zero. Let $\pi : Y \to X = \text{Spec } R$ be a resolution of singularities such that the exceptional locus $E$ of $\pi$ is a simple normal crossing divisor and $\pi|_{Y \setminus E} : Y \setminus E \to X \setminus \{\mathfrak{m}\}$ is an isomorphism. Then $R$ is of $F$-nilpotent type if and only if $E$ is a tree of smooth rational curves.

A combination of a result of Lipman [4] with Theorem 8 gives a characterization of two-dimensional local rings of $F$-rational type in terms of divisor class groups.

Theorem 10 (cf. [4, Theorem 17.4]). Let $(R, \mathfrak{m})$ be a two-dimensional normal local ring essentially of finite type over an algebraically closed field of characteristic zero. Let $\hat{R}$ be the $\mathfrak{m}$-adic completion of $R$. Then $R$ is of $F$-rational type if and only if the divisor class group $\text{Cl}(\hat{R})$ is finite.
As a corollary of Theorem 9, we give a similar characterization of two-dimensional local rings of \( F \)-nilpotent type.

**Theorem 11.** Let \((R, \mathfrak{m})\) be a two-dimensional normal local ring essentially of finite type over an algebraically closed field of characteristic zero. Let \(\widehat{R}\) be the \(\mathfrak{m}\)-adic completion of \(R\). Then \(R\) is of \(F\)-nilpotent type if and only if the divisor class group \(\text{Cl}(\widehat{R})\) does not contain the torsion group \(\mathbb{Q}/\mathbb{Z}\).

For example, the divisor class group of \(\mathbb{C}[x, y, z]/(x^2 + y^3 + z^7)\) does not contain \(\mathbb{Q}/\mathbb{Z}\), whereas that of \(\mathbb{C}[x, y, z]/(x^2 + y^3 + z^7 + xyz)\) contain \(\mathbb{Q}/\mathbb{Z}\).

**References**


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