Regularity in invariant theory and cohomology of groups

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1 Introduction

This paper is a survey of what is known about the use of regularity in invariant theory and the cohomology of finite groups over a field of finite characteristic.

The Castelnuovo-Mumford regularity of a graded ring is an invariant that carries a lot of information about that ring, although here we will mostly be concerned with bounds on the degrees of the generators and relations.

For simplicity, we will always work with rings R that are finitely generated over a field k and are graded by the non-negative integers. We also require them to have just k in degree 0, so the unique maximal ideal is $\mathfrak{m} = R_+$. All modules are graded too.

Given a finitely generated R-module M, let $a_i(M)$ denote the maximum degree of a non-zero element of the local cohomology $H^i_{\mathfrak{m}}(M)$ (possibly ∞ if unbounded or $-\infty$ if $H^i_{\mathfrak{m}}(M) = 0$). The Castelnuovo-Mumford regularity (or just regularity) of M over R is, by definition,

$$\operatorname{reg} M = \max_i \{a_i(M) + i\}.$$

The regularity of the ring R is just its regularity as a module over itself.

Note that if we have a module finite homomorphism of rings, $S \to R$, the regularity remains the same if we calculate it regarding M as an S-module.

We will also want to be able to consider regularity for cohomology rings, which are only graded commutative. In this case we consider R as a module over the commutative subring $R^{\mathrm{ev}} = \oplus_i R_{2i}$. The previous remark shows that this would not change the regularity for a commutative ring.

Now suppose that $R = k[x_1, \dots, x_n]$ is a polynomial ring in which the generators have arbitrary positive degrees. We set $\sigma(R) = \sum_{i=1}^n (\deg x_i - 1)$.

Now consider the minimal projective resolution of M over R.

$$\cdots \to P_1 \to P_0 \to M \to 0$$

Let $\rho_i(M)$ be the maximum degree of a non-zero element of $(R/\mathfrak{m}) \otimes_R P_i$ (possibly ∞ or $-\infty$), which is equal to the maximum degree of a generator of P_i. It can be shown as a consequence of local duality or directly that

$$\operatorname{reg} M = \max_i \{\rho_i(M) - i\} - \sigma(R).$$

The usual formulation does not contain a σ-term, because all the d_i are supposed to be in degree 1, so $\sigma(R) = 0$.

Such a characterization of regularity easily leads to bounds on the degrees of the generators and relations, as follows.

Given a finitely generated graded k-algebra S, either commutative or graded commutative, and an integer N, let $\tau_N S$ be the k-algebra (commutative or graded commutative, the same as S) determined by the generators and relations of S that occur in degrees at most N. There is a canonical map $\tau_N S \to S$, which is an isomorphism in degrees up to and including N.

Let S be a commutative or graded commutative ring. Let $R = k[x_1, \ldots, x_m]$ and suppose that there is a map $R \to S$ such that S is finitely generated over R (e.g. if R is a Noether normalization of S). Then:

1. if $N \geq \max\{\mathrm{reg}(S) + \sigma(R),\ \deg(x_i)\}$, then $\tau_N S \to S$ is a surjection;

2. if $N \geq \max\{2(\mathrm{reg}(S) + \sigma(R)),\ \mathrm{reg}(S) + \sigma(R) + 1,\ \deg(x_i)\}$, then $\tau_N S \to S$ is an isomorphism;

3. if $N \geq \max\{\mathrm{reg}(S) + \sigma(R) + 1,\ \deg(x_i)\}$ and if $\tau_N S$, considered an R-module, is generated in degrees at most N, then $\tau_N S \to S$ is an isomorphism.

2 Rings of Invariants

Given a standard graded polynomial ring $S = k[x_1, \ldots, x_n]$ and an action of a finite group by homogeneous linear transformations, it is natural to ask for bounds on the degrees of the generators and relations.

That just $|G|$ is a bound on the degrees of the generators when k has characteristic 0 is a result of Noether. This was generalized to the case of coprime characteristic by Fleischmann [6, 7] and Fogarty [8], with a much simplified proof by Benson. However, in general, no bound depending only on $|G|$ is possible, as was shown by Richman [13, 14].

We have shown that $\operatorname{reg} S^G \leq 0$, see [15]. By Dade's Lemma, there is always a set of parameters (elements that generate a subring over which S^G is finitely generated as a module) in degree at most $|G|$ (take a set of basis elements in degree 1 in general position and then form their orbit products). It follows from the formulas in the previous section that S^G is generated as a k-algebra in degrees at most $n(|G| - 1)$ (provided that $n > 1$, $|G| > 1$). The relations between the generators are generated in degrees at most $2n(|G| - 1)$.

The proof that $\operatorname{reg} S^G \leq 0$ in [15] employs relative homological algebra in order to utilize the structure theorem of Karagueuzian and the author [10]. The latter is a partial description of S as a kG-module, and has a long and complicated proof.

An alternative proof, based on the Cech complex over S with respect to a system of parameters for S^G, is given in [17]. The homology of this complex is the local cohomology of the polynomial ring S, so the complex is exact in degrees greater than $-n$. The idea of the proof is to regard this Cech complex as a complex of kG-modules and then to show that it is split exact in degrees greater than $-n$. This way, the complex is still exact in these degrees after taking fixed points. But the fixed points compute the local cohomology of S^G, so $a_i(S^G) \leq -n$.

In fact, both of the proofs that $\mathrm{reg}(S^G) \le 0$ yield the slightly stronger result that $\mathrm{hreg}(S^G) := \max_i\{a_i(S^G)\} \le -n$.

3 Cohomology of Groups

D.J. Benson conjectured that any finite group G satisfies $\mathrm{reg}\, H^*(G, \mathbb{F}_p) = 0$, [1, 2, 3]; he also showed that $\mathrm{reg}\, H^*(G, \mathbb{F}_p) \ge 0$. We proved equality in [16]. In this case, a system of parameters can be obtained as the Chern classes of a faithful complex representation. Considering these representations leads to the result that the degrees of the generators are bounded by $|G| - 1$ and those of the relations by $2(|G| - 1)$.

In fact, the method shows that if G is not cyclic and $\oplus_i V_i$ is a sum of irreducible complex representations for G that forms a faithful representation, then $\sum_i (\dim V_i)^2$ is a bound on the degrees of the generators (twice this for the relations).

The regularity result generalizes to the case of a compact Lie group acting on a smooth manifold with finite mod-p homology, where we find that, for the equivariant cohomology, $\mathrm{reg}\, H^*_G(M, \mathbb{F}_p) \le \dim M - \dim G$. In particular, for a compact Lie group we have $\mathrm{reg}\, H^*(BG, \mathbb{F}_p) \le -\dim G$.

Benson also conjectured that orientable virtual Poincaré duality groups of dimension d satisfy $\mathrm{reg}\, H^*(G, \mathbb{F}_p) = d$. A slight variant of this was proved in [15]. He also had an analogous conjecture for p-adic Lie groups; this remains open.

In invariant theory, the bounds on the degrees are not particularly important for calculation, because there are various means, such as integral closure, for knowing when all the invariants have been found. But, in machine computations of the cohomology of a group, the beginning of a projective resolution of k over kG has to be computed and this can grow very quickly. It is, therefore, essential to have a good bound on the degrees of the generators and relations, because this tells you how much of the resolution you need to compute.

Better estimates than the one given above can be obtained by using different systems of parameters and some tricks. It is possible to determine whether a given set of elements is a system of parameters by considering the restrictions to elementary abelian subgroups, so this is a feasible method, but there is no obvious way of obtaining a best system. Part of the problem seems to be that regularity only sees generators and relations as a module, not as a ring.

Using these methods it has proved possible to compute the cohomology rings of all groups of order 128 (all 2,328 of them) as well as those of many other groups of interest. For more information, see the work of Carlson [4, 5] and that of Green and King [9, 11, 12].

A different approach to some of these results, along with a generalization to Chow rings of classifying spaces of finite groups can be found in the forthcoming book of Totaro [18].

Finally, let us point out that it would be natural to combine the invariant theory and the cohomology and bound the regularity of the ring $H^*(G, k[x_1, \dots, x_n])$.

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