THE RING OF DIFFERENTIAL OPERATORS OF AN AFFINE SEMIGROUP ALGEBRA

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1. Introduction and Definitions

The ring of differential operators was introduced by Grothendieck [2]. Although it may be ugly in general [1], the ring of differential operators of an affine semigroup algebra shares the computability with the other objects concerning a semigroup. The aim of this article is to demonstrate it by using simple examples. In particular, we exhibit a beautiful structure of the spectrum of its graded ring (with respect to the order filtration) when the semigroup is scored.

Let $A := (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n) = (a_{ij})$ be a $d \times n$ matrix with coefficients in $\mathbb{Z}$. We sometimes identify A with the set of its column vectors. We assume that $\mathbb{Z}A = \mathbb{Z}^d$, where $\mathbb{Z}A$ is the abelian group generated by A.

Let $\mathbb{N}A$ be the monoid generated by A, and R_A its semigroup algebra:

$$R_A = \mathbb{C}[\mathbb{N}A] = \bigoplus_{\boldsymbol{a} \in \mathbb{N}A} \mathbb{C}t^{\boldsymbol{a}} \subseteq \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}].$$

Then the ring of differential operators of R_A can be given as a subalgebra of the ring of differential operators of the Laurent polynomial ring:

$$D(R_A) = \{P \in \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d \rangle \,:\, P(R_A) \subset R_A\}.$$

Let $D_k(R_A)$ be the subspace of differential operators of order less or equal to k in $D(R_A)$. Then the graded ring with respect to the order filtration $\{D_k(R_A)\}$ is commutative:

$$\operatorname{Gr} D(R_A) = \bigoplus_{k=0}^{\infty} D_k(R_A)/D_{k-1}(R_A) \subseteq \mathbb{C}[t_1^{\pm}, \dots, t_d^{\pm}, \xi_1, \dots, \xi_d],$$

where ξ_i denotes the image of ∂_i.

2. Finiteness

In general, the ring of differential operators on an affine variety may be neither left or right Noetherian nor finitely generated as an algebra [1]. In this section, we give some results on finiteness of $D(R_A)$.

Theorem 2.1 ([8]). *$D(R_A)$ is a finitely generated $\mathbb{C}$-algebra.*

Theorem 2.2 ([6]). (1) *$D(R_A)$ is right Noetherian.*
(2) *$D(R_A)$ is left Noetherian if $\mathbb{N}A$ is S_2.*

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In [6], we also gave a necessary condition for $D(R_A)$ being left Noetherian.

Definition 2.3. A semigroup $\mathbb{N}A$ is S_2 if $\mathbb{N}A = \bigcap_{\sigma:\text{ facet of } \mathbb{R}_{\geq 0}A} [\mathbb{N}A + \mathbb{Z}(A \cap \sigma)]$.

The following is an example of $\mathbb{N}A$ that does not satisfy the S_2 condition.
Example 1 (non-S_2).
$A_1 = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4) = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Then

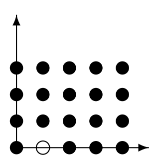

FIGURE 1. The semigroup $\mathbb{N}A_1$

In this case,

$$\mathbb{N}A_1 = \mathbb{N}^2 \setminus \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{whereas} \quad \bigcap_{\sigma:\text{ facet of } \mathbb{R}_{\geq 0}A_1} [\mathbb{N}A_1 + \mathbb{Z}(A_1 \cap \sigma)] = \mathbb{N}^2.$$

Theorem 2.4 ([7])**.**

$$\mathrm{Gr}\, D(R_A) \textit{ is Noetherian} \quad \Leftrightarrow \quad \mathbb{N}A \textit{ is } \textbf{scored}.$$

Let $\mathcal{F}$ be the set of facets of $\mathbb{R}_{\geq 0}A$. For a facet $\sigma \in \mathcal{F}$, we define the **primitive integral support function** F_σ of σ as the linear form on $\mathbb{R}^d$ uniquely determined by the conditions:

(1) $F_\sigma(\mathbb{R}_{\geq 0}A) \geq 0$,
(2) $F_\sigma(\sigma) = 0$,
(3) $F_\sigma(\mathbb{Z}^d) = \mathbb{Z}$.

Definition 2.5. The semigroup $\mathbb{N}A$ is said to be **scored** if

$$\mathbb{N}A = \bigcap_{\sigma:\text{facet}} \{\, \boldsymbol{a} \in \mathbb{Z}^d \;:\; F_\sigma(\boldsymbol{a}) \in F_\sigma(\mathbb{N}A) \,\}.$$

Remark 2.6.

$$\mathbb{N}A\text{: scored} \quad \Rightarrow \quad \mathbb{N}A\text{: } \mathrm{S}_2.$$

Proof. For each facet σ,

$$\mathbb{N}A \subseteq \mathbb{N}A + \mathbb{Z}(A \cap \sigma) \subseteq \{\, \boldsymbol{a} \in \mathbb{Z}^d \;:\; F_\sigma(\boldsymbol{a}) \in F_\sigma(\mathbb{N}A) \,\}.$$

Hence

$$\mathbb{N}A \subseteq \bigcap_{\sigma \in \mathcal{F}} (\mathbb{N}A + \mathbb{Z}(A \cap \sigma)) \subseteq \bigcap_{\sigma \in \mathcal{F}} \{\, \boldsymbol{a} \in \mathbb{Z}^d \;:\; F_\sigma(\boldsymbol{a}) \in F_\sigma(\mathbb{N}A) \,\}.$$

$\square$

Example 2 (Scored).
$A_2 = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}$. Then

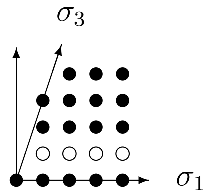

FIGURE 2. The semigroup $\mathbb{N}A_2$

$\mathcal{F} = \{ \sigma_1 = \mathbb{R}_{\geq 0}\boldsymbol{a}_1,\ \sigma_3 = \mathbb{R}_{\geq 0}\boldsymbol{a}_3 \}$,
$F_{\sigma_1}(s_1, s_2) = s_2,\ F_{\sigma_3}(s_1, s_2) = 3s_1 - s_2$.
$F_{\sigma_1}(\mathbb{N}A_2) = \mathbb{N} \setminus \{1\}, F_{\sigma_3}(\mathbb{N}A_2) = \mathbb{N}$.
We have

$$\mathbb{N}A_2 = \{\boldsymbol{a} \in \mathbb{Z}^2 \mid F_{\sigma_1}(\boldsymbol{a}) \in \mathbb{N} \setminus \{1\},\ F_{\sigma_3}(\boldsymbol{a}) \in \mathbb{N}\}.$$

Hence $\mathbb{N}A_2$ is scored.

Example 3 (S_2 but non-scored). $A_3 = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Then

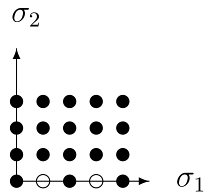

FIGURE 3. The semigroup $\mathbb{N}A_3$

$\mathcal{F} = \{ \sigma_1 = \mathbb{R}_{\geq 0}\boldsymbol{a}_1,\ \sigma_2 = \mathbb{R}_{\geq 0}\boldsymbol{a}_2 \}$,
$F_{\sigma_1}(s_1, s_2) = s_2,\ F_{\sigma_2}(s_1, s_2) = s_1$.
$F_{\sigma_1}(\mathbb{N}A_3) = \mathbb{N}, F_{\sigma_3}(\mathbb{N}A_3) = \mathbb{N}$.
We have

$$\mathbb{N}A_3 \subsetneq \{\boldsymbol{a} \in \mathbb{Z}^2 \mid F_{\sigma_1}(\boldsymbol{a}) \in \mathbb{N},\ F_{\sigma_3}(\boldsymbol{a}) \in \mathbb{N}\} = \mathbb{N}^2.$$

Hence $\mathbb{N}A_3$ is not scored.

Example 4 (scored).
$d = 1$, $n = 2$, $\quad A_4 = (2, 3)$.
This is the smallest non-trivial example; we use this as a running example.
We have the following:

- $\mathbb{N}A_4 = \{0, 2, 3, 4, \ldots\} = \mathbb{N} \setminus \{1\}$. $\quad \mathbb{R}_{\geq 0}A_4 = \mathbb{R}_{\geq 0}$.
- $\mathcal{F} = \{\{0\}\}$, $\quad F_{\{0\}}(s) = s$; $\quad \mathbb{N}A_4$ is scored.
- $R_{A_4} = \mathbb{C}[t^2, t^3]$.
- $D(R_{A_4}) = \{P \in \mathbb{C}[t^{\pm 1}]\langle \partial \rangle \ : \ P(\mathbb{C}[t^2, t^3]) \subseteq \mathbb{C}[t^2, t^3]\}$.

3. Weight Decomposition

It is easy to see $s_i := t_i\partial_i \in D(R_A) \quad (i = 1, \dots, d)$.

For $\boldsymbol{a} = {}^t(a_1, a_2, \dots, a_d) \in \mathbb{Z}^d$, set

$$D(R_A)_{\boldsymbol{a}} := \{P \in D(R_A) \,:\, [s_i, P] = a_i P \quad \text{for } i = 1, 2, \dots, d\}.$$

Then $t^{\boldsymbol{a}} \in D(R_A)_{\boldsymbol{a}}$ for $\boldsymbol{a} \in \mathbb{N}A$.

Lemma 3.1. (1) $D(R_A) = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} D(R_A)_{\boldsymbol{a}}$.

(2) $D_k(R_A) = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} D_k(R_A) \cap D(R_A)_{\boldsymbol{a}}$.

(3) $\operatorname{Gr} D(R_A) = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} \operatorname{Gr} D(R_A)_{\boldsymbol{a}}$.

Theorem 3.2 (Musson [4])**.**

$$D(R_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}}\mathbb{I}(\Omega(\boldsymbol{a})) \qquad \textit{for all } \boldsymbol{a} \in \mathbb{Z}^d,$$

where

$$\begin{aligned}
\Omega(\boldsymbol{a}) &:= \{\, \boldsymbol{b} \in \mathbb{N}A : \ \boldsymbol{b} + \boldsymbol{a} \notin \mathbb{N}A \,\} = \mathbb{N}A \setminus (-\boldsymbol{a} + \mathbb{N}A),\\
\mathbb{I}(\Omega(\boldsymbol{a})) &:= \{f(s) \in \mathbb{C}[s] := \mathbb{C}[s_1, \dots, s_d] \,:\, f \textit{ vanishes on } \Omega(\boldsymbol{a})\}.
\end{aligned}$$

In particular, $D(R_A)_{\mathbf{0}} = \mathbb{C}[s]$.

Example 1 Continued.

Put $D_{\boldsymbol{a}} := t^{\boldsymbol{a}} \prod_{a_i<0} \prod_{k=0}^{-a_i-1} (s_i - k) \in D(\mathbb{C}[t_1^{\pm1}, t_2^{\pm1}, \partial_1, \partial_2])$, $E_{\boldsymbol{a}} := D_{\boldsymbol{a}}(s_1 + a_1 - 1)$, and $F_{\boldsymbol{a}} := D_{\boldsymbol{a}}(s_2 + a_2)$. Then

$$\begin{aligned}
D(R_{A_1})_{\boldsymbol{a}} &= D_{\boldsymbol{a}}\mathbb{C}[s] && \text{if } \boldsymbol{a} \notin {}^t(1,0) - \mathbb{N}A_1,\\
D(R_{A_1})_{\boldsymbol{a}} &= E_{\boldsymbol{a}}\mathbb{C}[s] + F_{\boldsymbol{a}}\mathbb{C}[s] && \text{if } \boldsymbol{a} \in {}^t(1,0) - \mathbb{N}A_1.
\end{aligned}$$

Let $a_1, b_1 < 0$, and $\boldsymbol{a} = {}^t(a_1, 0), \boldsymbol{b} = {}^t(b_1, 0)$. Then

$$E_{\boldsymbol{a}} = \partial_1^{-a_1}(s_1 + a_1 - 1) = (s_1 - 1)\partial_1^{-a_1}, \quad F_{\boldsymbol{a}} = \partial_1^{-a_1} s_2 = s_2 \partial_1^{-a_1}.$$

We have

$$\begin{aligned}
E_{\boldsymbol{a}}E_{\boldsymbol{b}} &= (s_1 - a_1 - 1)E_{\boldsymbol{a}+\boldsymbol{b}},\\
F_{\boldsymbol{a}}E_{\boldsymbol{b}} &= (s_1 - a_1 - 1)F_{\boldsymbol{a}+\boldsymbol{b}},\\
E_{\boldsymbol{a}}F_{\boldsymbol{b}} &= s_2 E_{\boldsymbol{a}+\boldsymbol{b}} = (s_1 - 1)F_{\boldsymbol{a}+\boldsymbol{b}},\\
F_{\boldsymbol{a}}F_{\boldsymbol{b}} &= s_2 F_{\boldsymbol{a}+\boldsymbol{b}}.
\end{aligned}$$

Then, for $\boldsymbol{a}' = {}^t(a_1', 0), \boldsymbol{b}' = {}^t(b_1', 0)$ with $a_1', b_1' < 0$ and $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{a}' + \boldsymbol{b}'$, we have

$$F_{\boldsymbol{a}}E_{\boldsymbol{b}} - F_{\boldsymbol{a}'}E_{\boldsymbol{b}'} = (a_1' - a_1)F_{\boldsymbol{a}+\boldsymbol{b}}.$$

In this way, the right ideal $\sum_{a_1<0} F_{{}^t(a_1,0)} D(R_{A_1})$ is finitely generated. However, since $E_{\boldsymbol{a}}F_{\boldsymbol{b}} - E_{\boldsymbol{a}'}F_{\boldsymbol{b}'} = 0$, the left ideal $\sum_{a_1<0} D(R_{A_1})F_{{}^t(a_1,0)}$ is not finitely generated.

Example 4 Continued.

$A_4 = (2, 3), \quad \mathbb{N}A_4 = \mathbb{N} \setminus \{1\}$.

$a \in \mathbb{Z}. \quad \Omega(a) = \mathbb{N}A_4 \setminus (-a + \mathbb{N}A_4). \qquad D(R_{A_4})_a = t^a\mathbb{I}(\Omega(a))$.

- $\Omega(a) = \emptyset \quad (a \in \mathbb{N}A_4), \qquad D(R_{A_4})_a = t^a\mathbb{C}[s]$.
- $\Omega(1) = \{0\}, \qquad D(R_{A_4})_1 = ts\mathbb{C}[s] = t^2\partial\mathbb{C}[s]$.
- $\Omega(-1) = \{0, 2\}, \qquad D(R_{A_4})_{-1} = t^{-1}s(s-2)\mathbb{C}[s]$.
- $\Omega(-2) = \{0, 3\}, \qquad D(R_{A_4})_{-2} = t^{-2}s(s-3)\mathbb{C}[s]$.

- $\Omega(-k) = \{0, 2, \dots, k-1\} \cup \{k+1\} \quad (k \geq 3)$,
  $D(R_{A_4})_{-k} = t^{-k} s(s-2)\cdots(s-(k-1))(s-(k+1))\mathbb{C}[s]$.

Note that $|\Omega(-k)| = k$ if $k \in \mathbb{N}A_4$.

Example 3 Continued.
Since $\mathbb{N}A_3$ satisfies (S_2), each $D(R_{A_3})_{\mathbf{a}}$ is singly generated. For $\mathbf{a} = {}^t(a_1, a_2)$, put

$$Q_{\mathbf{a}} := \begin{cases} t_1^{a_1} t_2^{a_2} & (a_1 \geq 0, a_2 \geq 1, \text{ or } a_1 \geq 0 \text{ even}, a_2 = 0) \\ t_1^{a_1} t_2 \partial_2^{|a_2|+1} & (a_1 \geq 0, a_2 < 0, \text{ or } a_1 \geq 0 \text{ odd}, a_2 = 0) \\ t_2^{a_2} \partial_1^{|a_1|} & (a_1 < 0, a_2 \geq 1, \text{ or } a_1 < 0 \text{ even}, a_2 = 0) \\ t_2 \partial_1^{|a_1|} \partial_2^{|a_2|+1} & (a_1, a_2 < 0, \text{ or } a_1 < 0 \text{ odd}, a_2 = 0). \end{cases}$$

By computing $\mathbb{I}(\Omega(\mathbf{a}))$, we see that $D(R_{A_3})_{\mathbf{a}}$ is generated by $Q_{\mathbf{a}}$. The following is the list of some $Q_{\mathbf{a}}$:

-3	-2	-1	0	1	2	3	a_1/a_2
$t_2^2\partial_1^3$	$t_2^2\partial_1^2$	$t_2^2\partial_1$	t_2^2	$t_1t_2^2$	$t_1^2t_2^2$	$t_1^3t_2^2$	2
$t_2\partial_1^3$	$t_2\partial_1^2$	$t_2\partial_1$	t_2	t_1t_2	$t_1^2t_2$	$t_1^3t_2$	1
$t_2\partial_1^3\partial_2$	∂_1^2	$t_2\partial_1\partial_2$	1	$t_1t_2\partial_2$	t_1^2	$t_1^3t_2\partial_2$	0
$t_2\partial_1^3\partial_2^2$	$t_2\partial_1^2\partial_2^2$	$t_2\partial_1\partial_2^2$	$t_2\partial_2^2$	$t_1t_2\partial_2^2$	$t_1^2t_2\partial_2^2$	$t_1^3t_2\partial_2^2$	-1
$t_2\partial_1^3\partial_2^3$	$t_2\partial_1^2\partial_2^3$	$t_2\partial_1\partial_2^3$	$t_2\partial_2^3$	$t_1t_2\partial_2^3$	$t_1^2t_2\partial_2^3$	$t_1^3t_2\partial_2^3$	-2

Then we have

$$\mathrm{Gr}(D(R_{A_3}))/\langle \bigoplus_{a_1 \neq 0} \mathrm{Gr}(D(R_{A_3}))_{\boldsymbol{a}}, s, t_2 \rangle = \mathbb{C}\langle t_2\xi_2^2, t_2\xi_2^3, \cdots \rangle.$$

Since this is not a finitely generated algebra, neither is $\mathrm{Gr}(D(R_{A_3}))$.

4. The spectrum

By Theorem 2.4, the spectrum of $\mathrm{Gr}\, D(R_A)$ is in question, when $\mathbb{N}A$ is scored.

4.1. **$\mathbb{Z}^d$-graded Prime Ideals.** From now on, we assume that $\mathbb{N}A$ **is scored**, and set $G := \mathrm{Gr}\, D(R_A)$. By Lemma 3.1, we work on $\mathbb{Z}^d$-graded prime ideals of G.

Corollary 4.1 (to Theorem 3.2)**.**

$$G = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} \overline{t^{\boldsymbol{a}} \mathbb{I}(\Omega(\boldsymbol{a}))} = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} \overline{P}_{\boldsymbol{a}} \mathbb{C}[s],$$

where

$$\begin{aligned} p_{\boldsymbol{a}} &:= \prod_{\sigma} \prod_{m \in F_\sigma(\mathbb{N}A) \setminus (-F_\sigma(\boldsymbol{a}) + F_\sigma(\mathbb{N}A))} (F_\sigma(s) - m), \\ P_{\boldsymbol{a}} &:= t^{\boldsymbol{a}} \cdot p_{\boldsymbol{a}}(s), \\ \overline{P}_{\boldsymbol{a}} &= t^{\boldsymbol{a}} \cdot \prod_{\sigma} F_\sigma(s)^{\sharp(F_\sigma(\mathbb{N}A) \setminus (-F_\sigma(\boldsymbol{a}) + F_\sigma(\mathbb{N}A)))}. \end{aligned}$$

Since $G_{\mathbf{0}} = \mathbb{C}[s]$ is a subalgebra of G, the following lemma is immediate.

Lemma 4.2. *Let $\mathfrak{P} = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} \mathfrak{P}_{\boldsymbol{a}}$ be a $\mathbb{Z}^d$-graded prime ideal of G. Then $\mathfrak{P}_{\mathbf{0}}$ is a prime ideal of $G_{\mathbf{0}} = \mathbb{C}[s]$.*

Given a prime ideal $\mathfrak{p}$ of $\mathbb{C}[s]$, we shall classify all $\mathbb{Z}^d$-graded prime ideals $\mathfrak{P}$ of G with $\mathfrak{P}_{\mathbf{0}} = \mathfrak{p}$.

4.2. **Degree and Expected Degree.** For $\sigma \in \mathcal{F}$ and $\boldsymbol{a} \in \mathbb{Z}^d$, set

- $\deg_\sigma(\boldsymbol{a}) := {}^\sharp\left(F_\sigma(\mathbb{N}A) \setminus (-F_\sigma(\boldsymbol{a}) + F_\sigma(\mathbb{N}A))\right)$,
- $\operatorname{expdeg}_\sigma(\boldsymbol{a}) := \begin{cases} 0 & \text{if } F_\sigma(\boldsymbol{a}) \geq 0 \\ |F_\sigma(\boldsymbol{a})| & \text{if } F_\sigma(\boldsymbol{a}) \leq 0. \end{cases}$

Then
$$\overline{P_{\boldsymbol{a}}} = t^{\boldsymbol{a}} \prod_{\sigma \in \mathcal{F}} F_\sigma^{\deg_\sigma(\boldsymbol{a})}.$$

Example 4 Continued.
$A = (2,3), \quad \mathbb{N}A = \mathbb{N} \setminus \{1\}$.

$F_{\{0\}}(s) = s$.

a	$\cdots$	$-k$	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
$\operatorname{expdeg}_{\{0\}}(a)$	$\cdots$	k	$\cdots$	3	2	1	0	0	0	0	$\cdots$
$\deg_{\{0\}}(a)$	$\cdots$	k	$\cdots$	3	2	**2**	0	**1**	0	0	$\cdots$

$$G = \bigoplus_{a \in \mathbb{Z}} t^a s^{\deg_{\{0\}}(a)} \mathbb{C}[s] \subseteq \mathbb{C}[t^{\pm 1}, \xi], \qquad s = t\xi.$$

For a fixed prime ideal $\mathfrak{p}$ of $\mathbb{C}[s]$, we define

$$\begin{array}{rcl} \mathcal{F}(\mathfrak{p}) & := & \{\, \sigma \in \mathcal{F} \,:\, F_\sigma \in \mathfrak{p} \},\\ \Sigma(\mathfrak{p}) & : & \text{the fan determined by the hyperplane arrangement } \{\, \mathbb{R}\sigma \,:\, \sigma \in \mathcal{F}(\mathfrak{p})\,\},\\ S(\mathfrak{p}) & := & \{\, \boldsymbol{a} \in \mathbb{Z}^d \,:\, |F_\sigma(\boldsymbol{a})| \in F_\sigma(\mathbb{N}A) \quad (\text{for } \forall \sigma \in \mathcal{F}(\mathfrak{p}))\,\}. \end{array}$$

Example 4 Continued.
$A = (2,3), \quad \mathbb{N}A = \mathbb{N} \setminus \{1\}$.

$F_{\{0\}}(s) = s$.

- $\mathfrak{p} = (s - \beta)$: a fixed prime ideal of $\mathbb{C}[s]$
- $\mathcal{F}((s-\beta)) = \{\, \sigma \in \mathcal{F} \,:\, F_\sigma \in (s-\beta)\} = \begin{cases} \{0\} & (\beta = 0) \\ \emptyset & (\text{otherwise}). \end{cases}$
- $\Sigma((s-\beta)) = \begin{cases} \{\mathbb{R}_{\geq 0}, \{0\}, \mathbb{R}_{\leq 0}\} & (\beta = 0) \\ \{\mathbb{R}\} & (\text{otherwise}). \end{cases}$
- $S((s-\beta)) = \begin{cases} \mathbb{Z} \setminus \{\pm 1\} & (\beta = 0) \\ \mathbb{Z} & (\text{otherwise}). \end{cases}$
- $\mathcal{F}((0)) = \emptyset, \quad \Sigma((0)) = \{\mathbb{R}\}, \quad S((0)) = \mathbb{Z}$.

For $\boldsymbol{a} \in \mathbb{Z}^d$, put

- $\deg_{\mathfrak{p}}(\boldsymbol{a}) := \sum_{\sigma \in \mathcal{F}(\mathfrak{p})} \deg_\sigma(\boldsymbol{a})$.
- $\operatorname{expdeg}_{\mathfrak{p}}(\boldsymbol{a}) := \sum_{\sigma \in \mathcal{F}(\mathfrak{p})} \operatorname{expdeg}_\sigma(\boldsymbol{a})$.

Then $\deg_{\boldsymbol{m}}(\boldsymbol{a}) = \deg(p_{\boldsymbol{a}})$, where $\boldsymbol{m} = (s_1, \ldots, s_d)$.

Proposition 4.3. (1) $\deg_{\mathfrak{p}}(\boldsymbol{a}) \geq \operatorname{expdeg}_{\mathfrak{p}}(\boldsymbol{a})$.
(2) $\deg_{\mathfrak{p}}(\boldsymbol{a}) = \operatorname{expdeg}_{\mathfrak{p}}(\boldsymbol{a})$ *if and only if* $\boldsymbol{a} \in S(\mathfrak{p})$.

4.3. **Classification.** For a cone $\tau \in \Sigma(\mathfrak{p})$, we define an ideal $\mathfrak{P}(\mathfrak{p}, \tau) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} \mathfrak{P}(\mathfrak{p}, \tau)_{\boldsymbol{a}}$ of G by

$$\mathfrak{P}(\mathfrak{p}, \tau)_{\boldsymbol{a}} := \begin{cases} G_{\boldsymbol{a}}\mathfrak{p} & (\boldsymbol{a} \in \tau \cap S(\mathfrak{p})) \\ G_{\boldsymbol{a}} & (\text{otherwise}). \end{cases}$$

Proposition 4.4. *The* $\mathbb{Z}^d$*-graded ideal* $\mathfrak{P}(\mathfrak{p}, \tau)$ *is prime.*

Theorem 4.5 ([5])**.** *Let* $\mathfrak{P}$ *be a* $\mathbb{Z}^d$*-graded prime ideal with* $\mathfrak{P}_{\mathbf{0}} = \mathfrak{p}$*. Then there exists* $\tau \in \Sigma(\mathfrak{p})$ *such that* $\mathfrak{P} = \mathfrak{P}(\mathfrak{p}, \tau)$*.*

Proposition 4.6. $\mathfrak{P}(\mathfrak{p}, \tau) \subseteq \mathfrak{P}(\mathfrak{p}', \tau')$ *if and only if* $\mathfrak{p} \subseteq \mathfrak{p}'$ *and* $\tau \supseteq \tau'$*.*

Proposition 4.7. $\dim G/\mathfrak{P}(\mathfrak{p}, \tau) = \dim \mathbb{C}[s]/\mathfrak{p} + \dim \tau$*.*

Example 4 Continued.
$A = (2, 3)$, $\mathbb{N}A = \mathbb{N} \setminus \{1\}$. Let $a \in \mathbb{Z}$.

- $\mathfrak{P}((s), \mathbb{R}_{\geq 0})_a = \begin{cases} G_a s & (a \in \mathbb{N} \setminus \{1\}) \\ G_a & (\text{otherwise}). \end{cases}$
- $\mathfrak{P}((s), \{0\})_a = \begin{cases} G_a s & (a = 0) \\ G_a & (a \neq 0). \end{cases}$
- $\mathfrak{P}((s), \mathbb{R}_{\leq 0})_a = \begin{cases} G_a s & (-a \in \mathbb{N} \setminus \{1\}) \\ G_a & (\text{otherwise}). \end{cases}$

$$\mathfrak{P}((s), \mathbb{R}_{\geq 0}) \subseteq \mathfrak{P}((s), \{0\}) \supseteq \mathfrak{P}((s), \mathbb{R}_{\leq 0}).$$

- $\mathfrak{P}((s - \beta), \mathbb{R})_a = G_a(s - \beta)$ $(\forall a \in \mathbb{Z})$ for $\beta \neq 0$.
- $\mathfrak{P}((0), \mathbb{R})_a = G_a(0) = 0$ $(\forall a \in \mathbb{Z})$, i.e., $\mathfrak{P}((0), \mathbb{R}) = 0$.

5. Cohen-Macaulayness of $\operatorname{Gr} D(R_A)$

Theorem 5.1 (Musson [4])**.** *If* $\mathbb{N}A$ *is normal, then* $\operatorname{Gr} D(R_A)$ *is Gorenstein.*

Proof. Let Σ be the fan determined by $F_\sigma = 0$ $(\sigma \in \mathcal{F})$. For a facet $\tau \in \Sigma$, Let A_τ be a generating set of the semigroup $\tau \cap \mathbb{Z}^d$. Put $A_\Sigma := \cup_\tau A_\tau$. Then

$$\operatorname{Gr} D(R_A) = \mathbb{C}[s][\overline{P_{\boldsymbol{a}}} \,|\, \boldsymbol{a} \in A_\Sigma] = \mathbb{C}[\overline{F_\sigma}, \overline{P_{\boldsymbol{a}}} \,|\, \sigma \in \mathcal{F};\, \boldsymbol{a} \in A_\Sigma].$$

Replace $\overline{F_\sigma}$ by an indeterminate z_σ, and put

$$\widetilde{G} := \mathbb{C}[z_\sigma, \overline{P_{\boldsymbol{a}}} \,|\, \sigma \in \mathcal{F};\, \boldsymbol{a} \in A_\Sigma].$$

Then $\widetilde{G}$ is a normal affine semigroup algebra, and $\prod_{\sigma \in \mathcal{F}} z_\sigma$ represents the unique minimal positive element. (Indeed, in $\mathbb{Z}^d \oplus \mathbb{Z}^{\sharp \mathcal{F}}$, the corresponding semigroup has the primitive integral support functions $F_\sigma + z_\sigma$, z_σ $(\sigma \in \mathcal{F})$. $\prod_{\sigma \in \mathcal{F}} z_\sigma$ corresponds to $(\mathbf{0}, 1, \ldots, 1)$.)

Hence $\widetilde{G}$ is Gorenstein. The natural map

$$\pi : \widetilde{G} \to \operatorname{Gr} D(R_A)$$

defined by $\pi(z_\sigma) = \overline{F_\sigma}$ is surjective. Let $\{l_j\}$ be a basis of $\operatorname{Ker}(\pi_{|\langle z_\sigma \rangle})$. Then $\{l_j\}$ is a regular sequence, and generates $\operatorname{Ker}(\pi)$. Hence $\operatorname{Gr} D(R_A)$ is Gorenstein. If we consider $\operatorname{Gr} D(R_A)$ is $\mathbb{Z}^d \oplus \mathbb{Z}$-graded (the last one corresponds to the degree in s), then the a-invariant is $(\mathbf{0}, -\sharp\mathcal{F})$. □

However, if $\mathbb{N}A$ is not normal, then $\operatorname{Gr} D(R_A)$ is never Cohen-Macaulay:

Proposition 5.2 (Hsiao [3] $d = 1$)**.** *If $\mathbb{N}A$ is scored but not normal, then $\operatorname{Gr} D(R_A)$ is not Cohen-Macaulay.*

Proof. Let $G := \operatorname{Gr} D(R_A)$. Since $\mathbb{N}A$ satisfies (S_2), each $G_{\boldsymbol{a}}$ is a free $\mathbb{C}[s]$-module. Hence $s_1, \dots, s_d$ is a regular sequence of G. Let $\overline{G} := G/\langle s_1, \dots, s_d \rangle$. Then $\dim \overline{G} = d$, and G is Cohen-Macaulay if and only if so is $\overline{G}$. We have

$$\overline{G} = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} \mathbb{C}\overline{P_{\boldsymbol{a}}},$$

and

$$\overline{P_{\boldsymbol{a}}} \cdot \overline{P_{\boldsymbol{b}}} \neq 0 \Leftrightarrow \begin{cases} F_\sigma(\boldsymbol{a}) F_\sigma(\boldsymbol{b}) \geq 0 & (\forall \sigma \in \mathcal{F}) \\ \deg_\sigma(\boldsymbol{a}) - \operatorname{expdeg}_\sigma(\boldsymbol{a}) > 0 \Rightarrow F_\sigma(\boldsymbol{b}) = 0 & (\sigma \in \mathcal{F}) \end{cases} \tag{5.1}$$

by [5, Theorem 3.6]. Let

$$l := \max\{\deg(\boldsymbol{a}) - \operatorname{expdeg}(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d\},$$

$\deg(\boldsymbol{b}) - \operatorname{expdeg}(\boldsymbol{b}) = l$, and $\boldsymbol{b} \in \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d$. Put

$$\tau := \bigcap_{\deg_\sigma(\boldsymbol{b}) - \operatorname{expdeg}_\sigma(\boldsymbol{b}) > 0} \sigma.$$

If $\boldsymbol{x} = \sum_{\boldsymbol{a} \neq \mathbf{0}} c_{\boldsymbol{a}} \overline{P_{\boldsymbol{a}}}$ is not a zero-divisor, then, by (5.1), there exists $\mathbf{0} \neq \boldsymbol{a} \in \tau \cap \mathbb{R}_{\geq 0}A$ such that $c_{\boldsymbol{a}} \neq 0$, and

$$\boldsymbol{x} \cdot \overline{P_{\boldsymbol{b}}} = \sum_{\mathbf{0} \neq \boldsymbol{a} \in \mathbb{R}_{\geq 0}A \cap \tau} c_{\boldsymbol{a}} \overline{P_{\boldsymbol{a}+\boldsymbol{b}}}. \tag{5.2}$$

Let $\boldsymbol{b}$ be primitive in the sense that there exists no $\boldsymbol{b}' \in \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d$ and $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{R}_{\geq 0}A \cap \tau$ such that $\boldsymbol{b} = \boldsymbol{b}' + \boldsymbol{a}$ and $\deg(\boldsymbol{b}') - \operatorname{expdeg}(\boldsymbol{b}') = l$. Let $t := \dim \tau$. Suppose that $\boldsymbol{x}_1, \dots, \boldsymbol{x}_t$ forms a regular sequence of $\overline{G}$. Then, by the primitiveness and the equations (5.1), (5.2), we have $\overline{P_{\boldsymbol{b}}} \neq 0$ in $\overline{G}/(\boldsymbol{x}_1, \dots, \boldsymbol{x}_t)$. Since $t = \dim \tau$, for any $\overline{\boldsymbol{x}} \in \overline{G}/(\boldsymbol{x}_1, \dots, \boldsymbol{x}_t)$, we have $\boldsymbol{x}^m \cdot \overline{P_{\boldsymbol{b}}} \in (\boldsymbol{x}_1, \dots, \boldsymbol{x}_t)$ for some m by (5.2). Hence the length of a regular sequence of $\overline{G}$ cannot exceed t; $\operatorname{depth}(\overline{G}) \leq t \leq d - 1$. □

Example 4 Continued.
$A = (2, 3)$, $\quad \mathbb{N}A = \mathbb{N} \setminus \{1\}$.
Then $G = \operatorname{Gr} D(R_A)$ is again an affine semigroup algebra:

$$G = \mathbb{C}[t^3, t^2, ts, s, t^{-1}s^2, t^{-2}s^2, t^{-3}s^3].$$

Clearly, this semigroup does not satisfy S_2. Hence G is not Cohen-Macaulay.

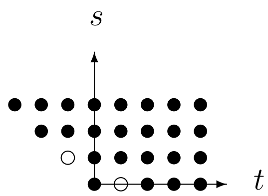

FIGURE 4. The semigroup for G

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