THE RING OF DIFFERENTIAL OPERATORS
OF AN AFFINE SEMIGROUP ALGEBRA

MUTSUMI SAITO

1. Introduction and Definitions

The ring of differential operators was introduced by Grothendieck [2]. Although it may be ugly in general [1], the ring of differential operators of an affine semigroup algebra shares the computability with the other objects concerning a semigroup. The aim of this article is to demonstrate it by using simple examples. In particular, we exhibit a beautiful structure of the spectrum of its graded ring (with respect to the order filtration) when the semigroup is scored.

Let $A := (a_1, a_2, \ldots, a_n) = (a_{ij})$ be a $d \times n$ matrix with coefficients in $\mathbb{Z}$. We sometimes identify $A$ with the set of its column vectors. We assume that $\mathbb{Z}A = \mathbb{Z}^d$, where $\mathbb{Z}A$ is the abelian group generated by $A$.

Let $N_A$ be the monoid generated by $A$, and $R_A$ its semigroup algebra:

$$R_A = \mathbb{C}[N_A] = \bigoplus_{\alpha \in \mathbb{N}_A} \mathbb{C}t^\alpha \subseteq \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}].$$

Then the ring of differential operators of $R_A$ can be given as a subalgebra of the ring of differential operators of the Laurent polynomial ring:

$$D(R_A) = \{ P \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] | \partial_1, \ldots, \partial_d : P(R_A) \subseteq R_A \}.$$

Let $D_k(R_A)$ be the subspace of differential operators of order less or equal to $k$ in $D(R_A)$. Then the graded ring with respect to the order filtration $\{D_k(R_A)\}$ is commutative:

$$\text{Gr } D(R_A) = \bigoplus_{k=0}^{\infty} D_k(R_A)/D_{k-1}(R_A) \subseteq \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, \xi_1, \ldots, \xi_d],$$

where $\xi_i$ denotes the image of $\partial_i$.

2. Finiteness

In general, the ring of differential operators on an affine variety may be neither left or right Noetherian nor finitely generated as an algebra [1]. In this section, we give some results on finiteness of $D(R_A)$.

**Theorem 2.1** ([8]). $D(R_A)$ is a finitely generated $\mathbb{C}$-algebra.

**Theorem 2.2** ([6]).

1. $D(R_A)$ is right Noetherian.
2. $D(R_A)$ is left Noetherian if $N_A$ is $S_2$.

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In [6], we also gave a necessary condition for $D(R_A)$ being left Noetherian.

**Definition 2.3.** A semigroup $NA$ is $S_2$ if $NA = \bigcap_{\sigma: \text{facet of } \mathbb{R}_{\geq 0}A} [NA + \mathbb{Z}(A \cap \sigma)]$.

The following is an example of $NA$ that does not satisfy the $S_2$ condition.

**Example 1 (non-$S_2$).**

$A_1 = (a_1, a_2, a_3, a_4) = \left( \begin{array}{cccc} 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right)$. Then

![Diagram](image)

**Figure 1.** The semigroup $NA_1$

In this case,

$$NA_1 = \mathbb{N}^2 \setminus \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{whereas} \quad \bigcap_{\sigma: \text{facet of } \mathbb{R}_{\geq 0}A_1} [NA_1 + \mathbb{Z}(A_1 \cap \sigma)] = \mathbb{N}^2.$$

**Theorem 2.4 ([7]).**

Gr $D(R_A)$ is Noetherian $\iff$ NA is scored.

Let $\mathcal{F}$ be the set of facets of $\mathbb{R}_{\geq 0}A$. For a facet $\sigma \in \mathcal{F}$, we define the **primitive integral support function** $F_{\sigma}$ of $\sigma$ as the linear form on $\mathbb{R}^d$ uniquely determined by the conditions:

1. $F_{\sigma}(\mathbb{R}_{\geq 0}A) \geq 0$,
2. $F_{\sigma}(\sigma) = 0$,
3. $F_{\sigma}(\mathbb{Z}^d) = \mathbb{Z}$.

**Definition 2.5.** The semigroup $NA$ is said to be scored if

$$NA = \bigcap_{\sigma: \text{facet}} \{ a \in \mathbb{Z}^d : F_{\sigma}(a) \in F_{\sigma}(NA) \}.$$

**Remark 2.6.**

NA: scored $\Rightarrow$ NA: $S_2$.

**Proof.** For each facet $\sigma$,

$$NA \subseteq NA + \mathbb{Z}(A \cap \sigma) \subseteq \{ a \in \mathbb{Z}^d : F_{\sigma}(a) \in F_{\sigma}(NA) \}.$$

Hence

$$NA \subseteq \bigcap_{\sigma \in \mathcal{F}} (NA + \mathbb{Z}(A \cap \sigma)) \subseteq \bigcap_{\sigma \in \mathcal{F}} \{ a \in \mathbb{Z}^d : F_{\sigma}(a) \in F_{\sigma}(NA) \}.$$

\[ \square \]
Example 2 (Scored).

\[ A_2 = (a_1, a_2, a_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}. \]

Figure 2. The semigroup \( N_{A_2} \)

\[ \mathcal{F} = \{ \sigma_1 = \mathbb{R}_{\geq 0} a_1, \sigma_3 = \mathbb{R}_{\geq 0} a_3 \}, \]

\( F_{\sigma_1}(s_1, s_2) = s_2, \ F_{\sigma_3}(s_1, s_2) = 3s_1 - s_2. \)

\( F_{\sigma_1}(N_{A_2}) = \mathbb{N} \setminus \{1\}, \ F_{\sigma_3}(N_{A_2}) = \mathbb{N}. \)

We have

\[ N_{A_2} = \{ a \in \mathbb{Z}^2 | F_{\sigma_1}(a) \in \mathbb{N} \setminus \{1\}, \ F_{\sigma_3}(a) \in \mathbb{N} \}. \]

Hence \( N_{A_2} \) is scored.

Example 3 (\( S_2 \) but non-scored). \( A_3 = (a_1, a_2, a_3) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \)

Figure 3. The semigroup \( N_{A_3} \)

\[ \mathcal{F} = \{ \sigma_1 = \mathbb{R}_{\geq 0} a_1, \sigma_2 = \mathbb{R}_{\geq 0} a_2 \}, \]

\( F_{\sigma_1}(s_1, s_2) = s_2, \ F_{\sigma_2}(s_1, s_2) = s_1. \)

\( F_{\sigma_1}(N_{A_3}) = \mathbb{N}, \ F_{\sigma_3}(N_{A_3}) = \mathbb{N}. \)

We have

\[ N_{A_3} \subseteq \{ a \in \mathbb{Z}^2 | F_{\sigma_1}(a) \in \mathbb{N}, \ F_{\sigma_3}(a) \in \mathbb{N} \} = \mathbb{N}^2. \]

Hence \( N_{A_3} \) is not scored.

Example 4 (scored).

\( d = 1, \ n = 2, \ A_4 = (2, 3). \)

This is the smallest non-trivial example; we use this as a running example.

We have the following:

- \( N_{A_4} = \{0, 2, 3, 4, \ldots\} = \mathbb{N} \setminus \{1\}. \ \mathbb{R}_{\geq 0} A_4 = \mathbb{R}_{\geq 0}. \)
- \( \mathcal{F} = \{\{0\}\}, \ F_{\{0\}}(s) = s; \ N_{A_4} \) is scored.
- \( R_{A_4} = \mathbb{C}[t^2, t^3]. \)
- \( D(R_{A_4}) = \{ P \in \mathbb{C}[t^{\pm 1}] : P(\mathbb{C}[t^2, t^3]) \subseteq \mathbb{C}[t^2, t^3] \}. \)
3. Weight Decomposition

It is easy to see \( s_i := t_i \partial_i \in D(R_{A}) \) \((i = 1, \ldots, d)\).

For \( a = t(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d \), set
\[
D(R_{A})_a := \{ P \in D(R_{A}) : [s_i, P] = a_i P \text{ for } i = 1, 2, \ldots, d \}.
\]
Then \( t^a \in D(R_{A})_a \) for \( a \in NA \).

**Lemma 3.1.**
\begin{enumerate}
\item \( D(R_{A}) = \bigoplus_{a \in \mathbb{Z}^d} D(R_{A})_a \).
\item \( D_k(R_{A}) = \bigoplus_{a \in \mathbb{Z}^d} D_k(R_{A}) \cap D(R_{A})_a \).
\item \( \text{Gr } D(R_{A}) = \bigoplus_{a \in \mathbb{Z}^d} \text{Gr } D(R_{A})_a \).
\end{enumerate}

**Theorem 3.2** (Musson [4]).
\[
D(R_{A})_a = t^a \mathbb{C}[\Omega(a)] \text{ for all } a \in \mathbb{Z}^d,
\]
where
\[
\Omega(a) := \{ b \in NA : b + a \not\in NA \} = NA \setminus (-a + NA),
\]
\[
\mathbb{C}[\Omega(a)] := \{ f(s) \in \mathbb{C}[s] : f \text{ vanishes on } \Omega(a) \}.
\]

In particular, \( D(R_{A})_a = \mathbb{C}[s] \).

**Example 1 Continued.**
Put \( D_a := t^a \prod_{a_i < 0} \prod_{k=0}^{-a_i-1} (s_i - k) \in D(\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \partial_1, \partial_2]) \), \( E_a := D_a(s_1 + a_1 - 1) \), and \( F_a := D_a(s_2 + a_2) \). Then
\[
D(R_{A_1})_a = D_a \mathbb{C}[s] \quad \text{if } a \not\in t(1, 0) - NA_1,
\]
\[
D(R_{A_1})_a = E_a \mathbb{C}[s] + F_a \mathbb{C}[s] \quad \text{if } a \in t(1, 0) - NA_1.
\]

Let \( a_1, b_1 < 0 \), and \( a = t(a_1, 0), b = t(b_1, 0) \). Then
\[
E_a = \partial_1^{-a_1}(s_1 + a_1 - 1) = (s_1 - 1) \partial_1^{-a_1}, \quad F_a = \partial_1^{-a_1}s_2 = s_2 \partial_1^{-a_1}.
\]

We have
\[
E_a E_b = (s_1 - a_1 - 1)E_{a+b},
\]
\[
F_a E_b = (s_1 - a_1 - 1)F_{a+b},
\]
\[
E_a F_b = s_2 E_{a+b} = (s_1 - 1)F_{a+b},
\]
\[
F_a F_b = s_2 F_{a+b}.
\]

Then, for \( a' = t(a'_1, 0), b' = t(b'_1, 0) \) with \( a'_1, b'_1 < 0 \) and \( a + b = a' + b' \), we have
\[
F_a E_b - F_{a'} E_{b'} = (a'_1 - a_1)F_{a+b}.
\]

In this way, the right ideal \( \sum_{a_1 < 0} F_{(a_1, 0)} D(R_{A_1}) \) is finitely generated. However, since \( E_a F_b - E_{a'} F_{b'} = 0 \), the left ideal \( \sum_{a_1 < 0} D(R_{A_1}) F_{(a_1, 0)} \) is not finitely generated.

**Example 4 Continued.**
\( A_4 = (2, 3), \quad NA_4 = N \setminus \{1\} \).
\( a \in \mathbb{Z}. \quad \Omega(a) = NA_4 \setminus (-a + NA_4). \quad D(R_{A_4})_a = t^a \mathbb{C}[\Omega(a)]. \)

- \( \Omega(0) = \emptyset \) \( (a \in NA_4) \), \( D(R_{A_4})_a = t^a \mathbb{C}[s] \).
- \( \Omega(1) = \{0\} \), \( D(R_{A_4})_1 = ts \mathbb{C}[s] = t^2 \partial \mathbb{C}[s] \).
- \( \Omega(-1) = \{0, 2\} \), \( D(R_{A_4})_{-1} = t^{-1}s(s-2)\mathbb{C}[s] \).
- \( \Omega(-2) = \{0, 3\} \), \( D(R_{A_4})_{-2} = t^{-2}s(s-3)\mathbb{C}[s] \).
• $\Omega(-k) = \{0, 2, \ldots, k-1\} \cup \{k+1\}$, $(k \geq 3)$, 
  $D(R_{A_4})_{-k} = t^{-k}s(s-2) \cdots (s-(k-1))(s-(k+1))\mathbb{C}[s]$.

Note that $|\Omega(-k)| = k$ if $k \in \mathbb{N}A_4$.

**Example 3 Continued.**

Since $\mathbb{N}A_3$ satisfies $(S_2)$, each $D(R_{A_3})_a$ is singly generated. For $a = t^i(a_1, a_2)$, put

\[Q_a := \begin{cases} 
  t_1^{a_1}t_2^{a_2} & (a_1 \geq 0, a_2 \geq 1, \text{ or } a_1 \geq 0 \text{ even}, a_2 = 0) \\
  t_1^{a_1}t_2^2t_3^{a_2+1} & (a_1 \geq 0, a_2 < 0, \text{ or } a_1 \geq 0 \text{ odd}, a_2 = 0) \\
  t_2^2p_1^{a_1} & (a_1 < 0, a_2 \geq 1, \text{ or } a_1 < 0 \text{ even}, a_2 = 0) \\
  t_2^2p_1^{a_1}p_2^{a_2+1} & (a_1, a_2 < 0, \text{ or } a_1 < 0 \text{ odd}, a_2 = 0).
\end{cases}\]

By computing $I(\Omega(a))$, we see that $D(R_{A_3})_a$ is generated by $Q_a$. The following is the list of some $Q_a$:

<table>
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<tr>
<th>$\ -3$</th>
<th>$\ -2$</th>
<th>$\ -1$</th>
<th>$\ 0$</th>
<th>$\ 1$</th>
<th>$\ 2$</th>
<th>$\ 3$</th>
<th>$a_1/a_2$</th>
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<td>$t_2^3p_1$</td>
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<td>$t_2^3p_1$</td>
<td>$t_1^2t_2$</td>
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<td>$1$</td>
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<td>$t_2^3p_1$</td>
<td>$t_2^3p_1$</td>
<td>$t_2^3p_1$</td>
<td>$t_2^3p_1$</td>
<td>$t_1^2t_2$</td>
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<td>$t_1^2t_2$</td>
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<td>$t_2^3p_1$</td>
<td>$t_1^2t_2$</td>
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<tr>
<td>$t_2^3p_1$</td>
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<td>$t_1^2t_2$</td>
<td>$-2$</td>
</tr>
</tbody>
</table>

Then we have

\[\text{Gr}(D(R_{A_3}))/\bigoplus_{a_1 \neq 0} \text{Gr}(D(R_{A_3}))(s, t_2) = \mathbb{C}(t_2^{s_2^2}, t_2^{s_2^3}, \ldots).\]

Since this is not a finitely generated algebra, neither is $\text{Gr}(D(R_{A_3}))$.

### 4. The Spectrum

By Theorem 2.4, the spectrum of $\text{Gr} D(R_A)$ is in question, when $\mathbb{N}A$ is scored.

#### 4.1. $\mathbb{Z}^d$-graded Prime Ideals

From now on, we assume that $\mathbb{N}A$ is scored, and set $G := \text{Gr} D(R_A)$. By Lemma 3.1, we work on $\mathbb{Z}^d$-graded prime ideals of $G$.

**Corollary 4.1** (to Theorem 3.2).

\[G = \bigoplus_{a \in \mathbb{Z}^d} t^aI(\Omega(a)) = \bigoplus_{a \in \mathbb{Z}^d} P_a \mathbb{C}[s],\]

where

\[p_a := \prod_{\sigma \in \mathbb{S}} \prod_{m \in F_\sigma(\mathbb{N}A) \setminus (F_\sigma(a) + F_\sigma(\mathbb{N}A))} (F_\sigma(s) - m),\]

\[P_a := t^a \cdot p_a(s),\]

\[P_a = t^a \prod_{\sigma \in \mathbb{S}} F_\sigma(s)^{(F_\sigma(\mathbb{N}A) \setminus (F_\sigma(a) + F_\sigma(\mathbb{A}))}.\]

Since $G_0 = \mathbb{C}[s]$ is a subalgebra of $G$, the following lemma is immediate.

**Lemma 4.2.** Let $\mathfrak{P} = \bigoplus_{a \in \mathbb{Z}^d} \mathfrak{P}_a$ be a $\mathbb{Z}^d$-graded prime ideal of $G$. Then $\mathfrak{P}_0$ is a prime ideal of $G_0 = \mathbb{C}[s]$. 

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Given a prime ideal $p$ of $\mathbb{C}[s]$, we shall classify all $\mathbb{Z}^d$-graded prime ideals $\mathfrak{P}$ of $G$ with $\mathfrak{P}_0 = p$.

4.2. Degree and Expected Degree. For $\sigma \in F$ and $a \in \mathbb{Z}^d$, set
\[ \deg_\sigma(a) := \dim (F_\sigma(\mathbb{N}A) \setminus (-F_\sigma(a) + F_\sigma(\mathbb{N}A))), \]
\[ \expdeg_\sigma(a) := \begin{cases} 0 & \text{if } F_\sigma(a) \geq 0 \\ |F_\sigma(a)| & \text{if } F_\sigma(a) \leq 0. \end{cases} \]
Then
\[ P_a = t^a \prod_{\sigma \in F \backslash \{0\}} F_\sigma^{\deg_\sigma(a)}. \]

Example 4 Continued.
$A = (2, 3)$, $\mathbb{N}A = \mathbb{N} \setminus \{1\}$.
\[ F((0)) = \emptyset; \quad F((0)) = \{\mathbb{R}\}; \quad S((0)) = \{\mathbb{R}\}; \quad S((0)) = \mathbb{Z}. \]

For a fixed prime ideal $p$ of $\mathbb{C}[s]$, we define
\[ F(p) := \{ \sigma \in F : F_\sigma(p) \} , \]
\[ \Sigma(p) := \text{the fan determined by the hyperplane arrangement } \{ \mathbb{R}\sigma : \sigma \in F(p) \}, \]
\[ S(p) := \{ a \in \mathbb{Z}^d : |F_\sigma(a)| \in F_\sigma(\mathbb{N}A) \text{ (for } \forall \sigma \in F(p)) \}. \]

Example 4 Continued.
$A = (2, 3)$, $\mathbb{N}A = \mathbb{N} \setminus \{1\}$.
\[ \deg_{(0)}(a) := \sum_{\sigma \in F((0))} \deg_\sigma(a). \]
\[ \expdeg_{(0)}(a) := \sum_{\sigma \in F((0))} \expdeg_\sigma(a). \]
Then $\deg_m(a) = \deg(p_a)$, where $m = (s_1, \ldots, s_d)$.6
Proposition 4.3. \(1\) \(\deg_p(a) \geq \expdeg_p(a)\).
\(2\) \(\deg_p(a) = \expdeg_p(a)\) if and only if \(a \in S(p)\).

4.3. Classification. For a cone \(\tau \in \Sigma(p)\), we define an ideal \(\mathfrak{P}(p, \tau) = \bigoplus_{a \in \mathbb{Z}^d} \mathfrak{P}(p, \tau)_a\) of \(G\) by
\[
\mathfrak{P}(p, \tau)_a := \left\{ \begin{array}{ll}
G_a p & (a \in \tau \cap S(p)) \\
G_a & (\text{otherwise}).
\end{array} \right.
\]

Proposition 4.4. The \(\mathbb{Z}^d\)-graded ideal \(\mathfrak{P}(p, \tau)\) is prime.

Theorem 4.5 ([5]). Let \(\mathfrak{P}\) be a \(\mathbb{Z}^d\)-graded prime ideal with \(\mathfrak{P}_0 = p\). Then there exists \(\tau \in \Sigma(p)\) such that \(\mathfrak{P} = \mathfrak{P}(p, \tau)\).

Proposition 4.6. \(\mathfrak{P}(p, \tau) \subseteq \mathfrak{P}(p', \tau')\) if and only if \(p \subseteq p'\) and \(\tau \geq \tau'\).

Proposition 4.7. \(\dim G/\mathfrak{P}(p, \tau) = \dim \mathbb{C}[s]/p + \dim \tau\).

Example 4 Continued.
Let \(a \in \mathbb{Z}\).

- \(\mathfrak{P}((s), \mathbb{R}_{\geq 0})_a = \left\{ \begin{array}{ll}
G_a s & (a \in \mathbb{N} \setminus \{1\}) \\
G_a & (\text{otherwise}).
\end{array} \right.
\)
- \(\mathfrak{P}((s), \{0\})_a = \left\{ \begin{array}{ll}
G_a s & (a = 0) \\
G_a & (a \neq 0).
\end{array} \right.
\)
- \(\mathfrak{P}((s), \mathbb{R}_{\leq 0})_a = \left\{ \begin{array}{ll}
G_a s & (-a \in \mathbb{N} \setminus \{1\}) \\
G_a & (\text{otherwise}).
\end{array} \right.
\)
- \(\mathfrak{P}((s), \mathbb{R}_{\geq 0}) \subseteq \mathfrak{P}((s), \{0\}) \supseteq \mathfrak{P}((s), \mathbb{R}_{\leq 0})\).
- \(\mathfrak{P}((s - \beta), \mathbb{R})_a = G_a(s - \beta) \quad (\forall a \in \mathbb{Z})\) for \(\beta \neq 0\).
- \(\mathfrak{P}((0), \mathbb{R})_a = G_a(0) = 0 \quad (\forall a \in \mathbb{Z})\), i.e., \(\mathfrak{P}((0), \mathbb{R}) = 0\).

5. COHEN-MACAULAYNESS OF \(\text{Gr} D(R_A)\)

Theorem 5.1 (Musson [4]). If \(NA\) is normal, then \(\text{Gr} D(R_A)\) is Gorenstein.

Proof. Let \(\Sigma\) be the fan determined by \(F_\sigma = 0\) \((\sigma \in \mathcal{F})\). For a facet \(\tau \in \Sigma\), let \(A_\tau\) be a generating set of the semigroup \(\tau \cap \mathbb{Z}^d\). Put \(A_\Sigma := \cup \tau A_\tau\). Then
\[
\text{Gr} D(R_A) = \mathbb{C}[s][\overline{F_a} | a \in A_\Sigma] = \mathbb{C}[\overline{F_\sigma}, \overline{P_a}] | \sigma \in \mathcal{F}; \ a \in A_\Sigma\).
\]
Replace \(\overline{F_a}\) by an indeterminate \(z_\sigma\), and put
\[
\overline{G} := \mathbb{C}[z_\sigma, \overline{P_a}] | \sigma \in \mathcal{F}; \ a \in A_\Sigma\).
\]
Then \(\overline{G}\) is a normal affine semigroup algebra, and \(\prod_{\sigma \in \mathcal{F}} z_\sigma\) represents the unique minimal positive element. (Indeed, in \(\mathbb{Z}^d \oplus \mathbb{Z}^d\), the corresponding semigroup has the primitive integral support functions \(F_\sigma + z_\sigma\), \(z_\sigma \quad (\sigma \in \mathcal{F})\). \(\prod_{\sigma \in \mathcal{F}} z_\sigma\) corresponds to \((0, 1, \ldots, 1)\).)
Hence \( \widetilde{G} \) is Gorenstein. The natural map

\[
\pi : \widetilde{G} \to \text{Gr } D(R_A)
\]

defined by \( \pi(z_\sigma) = \overline{F_\sigma} \) is surjective. Let \( \{l_i\} \) be a basis of \( \text{Ker}(\pi(z_\sigma)) \). Then \( \{l_i\} \) is a regular sequence, and generates \( \text{Ker}(\pi) \). Hence \( \text{Gr } D(R_A) \) is Gorenstein. If we consider \( \text{Gr } D(R_A) \) is \( \mathbb{Z}^d \oplus \mathbb{Z} \)-graded (the last one corresponds to the degree in \( s \)), then the \( a \)-invariant is \((0, -\sum F)\).

However, if \( \text{NA} \) is not normal, then \( \text{Gr } D(R_A) \) is never Cohen-Macaulay:

**Proposition 5.2** (Hsiao [3] \( d = 1 \)). If \( \text{NA} \) is scored but not normal, then \( \text{Gr } D(R_A) \) is not Cohen-Macaulay.

**Proof.** Let \( G := \text{Gr } D(R_A) \). Since \( \text{NA} \) satisfies \((S_2)\), each \( G_a \) is a free \( \mathbb{C}[s] \)-module. Hence \( s_1, \ldots, s_d \) is a regular sequence of \( G \). Let \( \overline{G} := G/(s_1, \ldots, s_d) \). Then \( \dim \overline{G} = d \), and \( G \) is Cohen-Macaulay if and only if so is \( \overline{G} \). We have

\[
\overline{G} = \bigoplus_{a \in \mathbb{Z}^d} \mathbb{C} \overline{P_a},
\]

and

\[
(5.1) \quad \overline{P_a} \cdot \overline{P_b} \neq 0 \iff \left\{ \begin{array}{l}
F_\sigma(a)F_\sigma(b) \geq 0 \\
\deg_\sigma(a) - \expdeg_\sigma(a) > 0 \Rightarrow F_\sigma(b) = 0
\end{array} \right. \quad (\forall \sigma \in \mathcal{F})
\]

by [5, Theorem 3.6]. Let

\[
l := \max\{\deg(a) - \expdeg(a) \mid a \in \mathbb{R}_0 A \cap \mathbb{Z}^d\},
\]

\[
\deg(b) - \expdeg(b) = l, \text{ and } b \in \mathbb{R}_0 A \cap \mathbb{Z}^d.
\]

Put

\[
\tau := \bigcap_{\deg_\sigma(b) - \expdeg_\sigma(b) > 0} \sigma.
\]

If \( \mathbf{x} = \sum_{a \neq 0} c_a \overline{P_a} \) is not a zero-divisor, then, by (5.1), there exists \( 0 \neq a \in \tau \cap \mathbb{R}_0 A \) such that \( c_a \neq 0 \), and

\[
(5.2) \quad \mathbf{x} \cdot \overline{P_b} = \sum_{0 \neq a \in \mathbb{R}_0 A \cap \tau} c_a \overline{P_{a+b}}.
\]

Let \( b \) be primitive in the sense that there exists no \( b' \in \mathbb{R}_0 A \cap \mathbb{Z}^d \) and \( 0 \neq a \in \mathbb{R}_0 A \cap \tau \) such that \( b = b' + a \) and \( \deg(b') - \expdeg(b') = l \). Let \( t := \dim \tau \). Suppose that \( x_1, \ldots, x_t \) forms a regular sequence of \( \overline{G} \). Then, by the primitiveness and the equations (5.1), (5.2), we have \( \overline{P_b} \neq 0 \) in \( \overline{G}/(x_1, \ldots, x_t) \). Since \( t = \dim \tau \), for any \( \mathbf{x} \in \overline{G}/(x_1, \ldots, x_t) \), we have \( \mathbf{x}^m \cdot \overline{P_b} \in (x_1, \ldots, x_t) \) for some \( m \) by (5.2). Hence the length of a regular sequence of \( \overline{G} \) cannot exceed \( t \); \( \text{depth}(\overline{G}) \leq t \leq d - 1 \). \( \square 

**Example 4 Continued.**

\( A = (2, 3) \), \( \text{NA} = \mathbb{N} \setminus \{1\} \).

Then \( G = \text{Gr } D(R_A) \) is again an affine semigroup algebra:

\[
G = \mathbb{C}[t^3, t^2, ts, s, t^{-1}s^2, t^{-2}s^2, t^{-3}s^3].
\]

Clearly, this semigroup does not satisfy \( S_2 \). Hence \( G \) is not Cohen-Macaulay.
Figure 4. The semigroup for $G$

REFERENCES


Department of Mathematics, Graduate School of Science, Hokkaido University, Sapporo, 060-0810, Japan

E-mail address: saito@math.sci.hokudai.ac.jp