SUBINTEGRALITY, INVERTIBLE MODULES AND POLYNOMIAL EXTENSIONS

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Let $A \subseteq B$ be a ring extension (of commutative rings).

This extension is an **elementary subintegral** extension if $B = A[b]$ with $b^2, b^3 \in A$. The extension $A \subseteq B$ is **subintegral** or B is subintegral over A if B is a union of subrings which are obtainable from A by a finite succession of elementary subintegral extensions. The **subintegral closure** of A in B, usually denoted by ${}_B^{+}\!A$, is the largest subintegral extension of A in B. This is simply the union of all intermediary subrings which are subintegral over A. The ring ${}_B^{+}\!A$ is integral over A. Further, if ${}_B^{+}\!A$ is an integral domain then ${}_B^{+}\!A$ and A have the same field of fractions. We say that A is **subintegrally closed** in B if ${}_B^{+}\!A = A$. This is equivalent to saying that whenever $b \in B$ and $b^2, b^3 \in A$ then $b \in A$. Without reference to B, the ring A is **seminormal** if the following condition holds: $b, c \in A$ and $b^3 = c^2$ imply that there exists $a \in A$ with $b = a^2$ and $c = a^3$. A seminormal ring is necessarily reduced and is subintegrally closed in every reduced overring.

The multiplicative group of those A-submodules of B which are invertible is denoted by $\mathcal{I}(A, B)$. The Picard group of A is denoted, of course, by $\mathrm{Pic\,A}$, while the group of units of A is denoted by $A^\times$. A relationship between these groups is given by the natural exact sequence

$$1 \to A^\times \to B^\times \to \mathcal{I}(A, B) \to \mathrm{Pic\,A} \to \mathrm{Pic\,B}.$$

We prove the following two theorems motivated by a well known result of Traverso and Swan which says that for a commutative ring A, A_{red} is seminormal if and only if the canonical map $\mathrm{Pic\,A} \to \mathrm{Pic\,A[X]}$ is an isomorphism. In the special case when A is reduced and Noetherian, the first of the two theorems yields Traverso-Swan's result as a corollary.

Theorem 1. *Let $A \subseteq B$ be a ring extension. Then A is subintegrally closed in B if and only if the canonical map $\mathcal{I}(A, B) \to \mathcal{I}(A[X], B[X])$ is an isomorphism.*

Theorem 2. *Let $A \subseteq B$ be a ring extension, and let $^{+}\!A$ denote the subintegral closure of A in B. Then:*

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(1) There exists a commutative diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathcal{I}(A,\ {}^{+}\!A) & \longrightarrow & \mathcal{I}(A,B) & \xrightarrow{\varphi(A,{}^{+}\!A,B)} & \mathcal{I}({}^{+}\!A,B) \\
 & & \Big\downarrow{\scriptstyle \theta(A,{}^{+}\!A)} & & \Big\downarrow{\scriptstyle \theta(A,B)} & & \approx\Big\downarrow{\scriptstyle \theta({}^{+}\!A,B)} \\
1 & \longrightarrow & \mathcal{I}(A[X],\ {}^{+}\!A[X]) & \longrightarrow & \mathcal{I}(A[X],B[X]) & \longrightarrow & \mathcal{I}({}^{+}\!A[X],B[X])
\end{array}$$

of canonical maps with exact rows and with $\theta({}^{+}\!A,B)$ *an isomorphism.*

(2) If B *is an integral domain and* $\dim A \leq 1$ *then the above diagram extends to the commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{I}(A,\ {}^{+}\!A) & \longrightarrow & \mathcal{I}(A,B) & \xrightarrow{\varphi(A,{}^{+}\!A,B)} & \mathcal{I}({}^{+}\!A,B) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \theta(A,{}^{+}\!A)} & & \Big\downarrow{\scriptstyle \theta(A,B)} & & \approx\Big\downarrow{\scriptstyle \theta({}^{+}\!A,B)} & & \\
1 & \longrightarrow & \mathcal{I}(A[X],\ {}^{+}\!A[X]) & \longrightarrow & \mathcal{I}(A[X],B[X]) & \longrightarrow & \mathcal{I}({}^{+}\!A[X],B[X]) & \longrightarrow & 1
\end{array}$$

with exact rows.

(3) If $\mathbb{Q} \subseteq A$ *then* $\mathcal{I}(A[X],\ {}^{+}\!A[X]) \cong \mathbb{Z}[X] \otimes_{\mathbb{Z}} M_0 \cong \bigoplus_{n=0}^{\infty} M_n$ *with* $M_0 = \operatorname{im}\theta(A,{}^{+}\!A) \cong \mathcal{I}(A,{}^{+}\!A)$ *and each* M_n *also isomorphic to* $\mathcal{I}(A,\ {}^{+}\!A)$.

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