A COMPUTATION OF BUCHSBAUM-RIM MULTIPLICITIES IN A SPECIAL CASE*

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1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ of dimension $d > 0$ and let C be a nonzero R-module of finite length. Let $\varphi : R^n \to R^r$ be an R-linear map of free modules with $C = \operatorname{Coker} \varphi$ as the cokernel of φ, and put $M := \operatorname{Im} \varphi \subset F := R^r$. Then one can consider the function

$$\lambda(p) := \ell_R([\operatorname{Coker} \operatorname{Sym}_R(\varphi)]_{p+1}) = \ell_R(S_{p+1}/M^{p+1}),$$

where S_p (resp. M^p) is a homogeneous component of degree p of $S = \operatorname{Sym}_R(F)$ (resp. $R[M] = \operatorname{Im} \operatorname{Sym}_R(\varphi)$). Buchsbaum-Rim [2] first introduced and studied the function of this type and proved that $\lambda(p)$ is eventually a polynomial of degree $d + r - 1$, which we call the *Buchsbaum-Rim polynomial.* Then they defined a multiplicity of C as

$$e(C) := (\text{The coefficient of } p^{d+r-1} \text{ in the polynomial}) \times (d + r - 1)!,$$

which we now call the *Buchsbaum-Rim multiplicity* of C. They also proved that the multiplicity is independent of the choice of φ. The multiplicity $e(C)$ coincides with the ordinary Hilbert-Samuel multiplicity when C is a cyclic module R/I.

Buchsbaum and Rim also introduced the notion of a parameter matrix, which generalizes the notion of a system of parameters. A matrix (a linear map of free modules) φ over R of size $r \times n$ is said to be *a parameter matrix* for R, if the following three conditions are satisfied: (i) $\operatorname{Coker} \varphi$ has finite length, (ii) $d = n - r + 1$, (iii) $\operatorname{Im} \varphi \subset \mathfrak{m} R^r$. Then it is known ([2, 4]) that there exists a formula

$$e(C) = \ell_R(C) = \ell_R(R/\operatorname{Fitt}_0(C))$$

for the Buchsbaum-Rim multiplicity, if R is Cohen-Macaulay and φ is a parameter matrix. Brennan, Ulrich and Vasconcelos observed in [1] that if R is Cohen-Macaulay and φ is a parameter matrix, then in fact

$$\lambda(p) = e(C) \binom{p + d + r - 1}{d + r - 1}$$

for all $p \geq 0$. In general, for any $p \geq 0$ the inequality

$$\lambda(p) \geq e(C) \binom{p + d + r - 1}{d + r - 1}$$

always holds true even if R is not Cohen-Macaulay, and moreover the equality for some $p \geq 0$ characterizes the Cohen-Macaulay property of the ring R ([3]).

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Kleiman-Thorup [7, 8] and Kirby-Rees [5, 6] introduced another kind of multiplicities associated to C, which is related to the Buchsbaum-Rim multiplicity. They consider the function of two variables

$$\Lambda(p,q) := \ell_R(S_{p+q}/M^{p+1}S_{q-1}),$$

and proved that $\Lambda(p,q)$ is eventually a polynomial of total degree $d+r-1$. Then they defined a sequence of multiplicities, for $j=0,1,\dots,d+r-1$,

$$e^j(C) := (\text{The coefficient of } p^{d+r-1-j}q^j \text{ in the polynomial}) \times (d+r-1-j)!j!$$

and proved that $e^j(C)$ is independent of the choice of φ. Moreover they proved that

$$e(C) = e^0(C) \geq e^1(C) \geq \cdots \geq e^{r-1}(C) > e^r(C) = \cdots = e^{d+r-1}(C) = 0,$$

where $r = \mu_R(C)$. Thus we call $e^j(C)$ *j-th Buchsbaum-Rim multiplicity* of C. Then it is natural to ask the following.

Problem 1.1. *Let $\varphi : R^n \to R^r$ be a parameter matrix with $C = \operatorname{Coker}\varphi$. Suppose that R is Cohen-Macaulay. Then*

(1) *does there exist a simple formula for the Buchsbaum-Rim multiplicities $e^j(C)$ for $j = 1,2,\dots,r-1$?*
(2) *Does the function $\Lambda(p,q)$ coincide with a polynomial function for all $p \geq 0$ and all $q > 0$?*

In this note, we will try to calculate the function $\Lambda(p,q)$ and multiplicities $e^j(C)$ in a special case where C is a direct sum of cyclic modules R/Q_i where Q_i is a parameter ideal in a one-dimensional Cohen-Macaulay local ring R. Especially, in the case $C = R/Q_1 \oplus R/Q_2$, we will determine when $\Lambda(p,q)$ is polynomial for all $p \geq 0$ and $q > 0$. As a consequence, we have that there exists the case where the function $\Lambda(p,q)$ does not coincide with the polynomial function. This should be contrasted with a result of Brennan-Ulrich-Vasconcelos [1] as stated above: the ordinary Buchsbaum-Rim function $\lambda(p) = \Lambda(p,1)$ coincides with the Buchsbaum-Rim polynomial for all $p \geq 0$ in the case where R is Cohen-Macaulay and φ is a parameter matrix.

2. A computation in a special case

In what follows, let $(R,\mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$. Let $r > 0$ be a fixed positive integer and let $Q_1, Q_2, \dots, Q_r$ be parameter ideals in R with $Q_i = (x_i)$ for $i = 1,2,\dots,r$. We put $a_i = \ell_R(R/Q_i) = e(R/Q_i)$ for $i = 1,2,\dots,r$. Let $\varphi : R^r \to R^r$ be an R-linear map represented by a parameter matrix

$$\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_r \end{pmatrix}.$$

Then we consider the module $C = \operatorname{Coker}\varphi = R/Q_1 \oplus R/Q_2 \oplus \cdots \oplus R/Q_r$ and compute the following:

- the multiplicities $e^j(C)$ for $j = 1,2,\dots,r-1$
- the polynomial $\Lambda(p,q) = \ell_R(S_{p+q}/N^{p+1}S_{q-1})$ for $p,q \gg 0$
- the function $\Lambda(p,q) = \ell_R(S_{p+q}/N^{p+1}S_{q-1})$ for $p \geq 0, q > 0$

where $S = \mathrm{Sym}_R(R^r)$ and $N = \mathrm{Im}\,\varphi = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_r$. If we fix a free basis $\{t_1, t_2, \ldots, t_r\}$ for R^r, then $S = R[t_1, t_2, \ldots, t_r]$ is a polynomial ring and $N = Q_1 t_1 + Q_2 t_2 + \cdots + Q_r t_r \subset S_1 = Rt_1 + Rt_2 + \cdots + Rt_r$. Then for any $p \geq 0, q > 0$,

$$\begin{aligned} N^{p+1}S_{q-1} &= \left(\sum_{\substack{|\boldsymbol{j}|=p+1 \\ \boldsymbol{j}\geq \boldsymbol{0}}} \boldsymbol{Q}^{\boldsymbol{j}} \boldsymbol{t}^{\boldsymbol{j}} \right) \left(\sum_{\substack{|\boldsymbol{k}|=q-1 \\ \boldsymbol{k}\geq \boldsymbol{0}}} R\boldsymbol{t}^{\boldsymbol{k}} \right) \\ &= \sum_{\substack{|\boldsymbol{\ell}|=p+q \\ \boldsymbol{\ell}\geq \boldsymbol{0}}} \left(\sum_{\substack{|\boldsymbol{k}|=q-1 \\ \boldsymbol{0}\leq\boldsymbol{k}\leq \boldsymbol{\ell}}} \boldsymbol{Q}^{\boldsymbol{\ell}-\boldsymbol{k}} \right) \boldsymbol{t}^{\boldsymbol{\ell}} \\ &\subset \; S_{p+q} = \sum_{\substack{|\boldsymbol{\ell}|=p+q \\ \boldsymbol{\ell}\geq \boldsymbol{0}}} R\boldsymbol{t}^{\boldsymbol{\ell}}. \end{aligned}$$

Here we use the multi-index notation: for a vector $\boldsymbol{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r_{\geq 0}$, we denote $\boldsymbol{Q}^{\boldsymbol{i}} = Q_1^{i_1} \cdots Q_r^{i_r}, \boldsymbol{t}^{\boldsymbol{i}} = t_1^{i_1} \cdots t_r^{i_r}$ and $|\boldsymbol{i}| = i_1 + \cdots + i_r$. For any vector $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_r) \in \mathbb{Z}^r_{\geq 0}$ such that $|\boldsymbol{\ell}| = p + q$, we define the ideal in R as follows:

$$J_{p,q}(\boldsymbol{\ell}) := \sum_{\substack{|\boldsymbol{k}|=q-1 \\ \boldsymbol{0}\leq\boldsymbol{k}\leq \boldsymbol{\ell}}} \boldsymbol{Q}^{\boldsymbol{\ell}-\boldsymbol{k}}.$$

Then for any $p \geq 0, q > 0$,

$$\Lambda(p, q) = \ell_A(S_{p+q}/N^{p+1}S_{q-1}) = \sum_{\substack{|\boldsymbol{\ell}|=p+q \\ \boldsymbol{\ell}\geq \boldsymbol{0}}} \ell_R(R/J_{p,q}(\boldsymbol{\ell})).$$

To compute the function $\Lambda(p, q)$, it is enough to compute the colength $\ell_R(R/J_{p,q}(\boldsymbol{\ell}))$ of the ideal $J_{p,q}(\boldsymbol{\ell})$. In the special case where the ideals $Q_1, Q_2, \ldots, Q_r$ becomes ascending chain, we can easily compute it as follows.

Proposition 2.1. *Suppose that $Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_r$. Then*

$$\Lambda(p, q) = (a_1 + \cdots + a_r)\binom{p+r}{r} + \sum_{i=1}^{r-1} (a_{i+1} + \cdots + a_r)\binom{p+r-i}{r-i}\binom{q-2+i}{i}$$

for all $p \geq 0$ and all $q > 0$, where $\binom{m}{n} = 0$ if $m < n$. In particular, the function $\Lambda(p, q)$ coincides with a polynomial function and

$$e^j(C) = \begin{cases} a_{j+1} + \cdots + a_r & (j = 0, 1, \ldots, r-1) \\ 0 & (j = r) \end{cases}$$

Proof. Let us fix any $p \geq 0$ and $q > 0$. We may assume that $r \geq 2$ and $q \geq 2$. Suppose $Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_r$. Then the ideal $J_{p,q}(\boldsymbol{\ell})$ coincides with the ideal of the product of last $(p+1)$-ideals of a sequence of ideals

$$\underbrace{\overbrace{Q_1, \ldots, Q_1}^{\ell_1}, \overbrace{Q_2, \ldots, Q_2}^{\ell_2}, \ldots, \overbrace{Q_r, \ldots, Q_r}^{\ell_r}}_{p+q}.$$

Hence its colength $\ell_R(R/J_{p,q}(\boldsymbol{\ell}))$ is the sum of last $(p+1)$-integers of a sequence of integers

$$\underbrace{\overbrace{a_1,\dots,a_1}^{\ell_1},\overbrace{a_2,\dots,a_2}^{\ell_2},\dots,\overbrace{a_r,\dots,a_r}^{\ell_r}}_{p+q}. \tag{1}$$

To compute the sum

$$\sum_{\substack{|\boldsymbol{\ell}|=p+q\\ \boldsymbol{\ell}\geq\mathbf{0}}}\ell_R(R/J_{p,q}(\boldsymbol{\ell})),$$

we divide the sequence (1) at the $(p+2)$th integer from the end. If the $(p+2)$th integer from the end is a_i, then the sum of all last $(p+1)$-integers of such sequences can be counted by

$$\binom{i+(q-2)-1}{i-1}\left(\sum_{\substack{u_1+\cdots+u_r=p+1\\ u_1,\dots,u_r\geq 0}}(u_ia_i+u_{i+1}a_{i+1}+\cdots+u_ra_r)\right).$$

Therefore

$$\begin{aligned}
\Lambda(p,q) &= \sum_{\substack{|\boldsymbol{\ell}|=p+q\\ \boldsymbol{\ell}\geq\mathbf{0}}}\ell_R(R/J_{p,q}(\boldsymbol{\ell}))\\
&= \sum_{i=1}^{r}\binom{i+(q-2)-1}{i-1}\left(\sum_{\substack{u_1+\cdots+u_r=p+1\\ u_1,\dots,u_r\geq 0}}(u_ia_i+u_{i+1}a_{i+1}+\cdots+u_ra_r)\right)\\
&= \sum_{i=1}^{r}\binom{i+(q-2)-1}{i-1}(a_i+\cdots+a_r)\binom{(r-i+1)+(p+1)-1}{r-i}\frac{p+1}{r-i+1}\\
&= \sum_{i=1}^{r}(a_i+\cdots+a_r)\binom{i+q-3}{i-1}\binom{r-i+p+1}{r-i}\frac{p+1}{r-i+1}\\
&= \sum_{i=1}^{r}(a_i+\cdots+a_r)\binom{r-i+p+1}{r-i+1}\binom{i+q-3}{i-1}\\
&= (a_1+\cdots+a_r)\binom{p+r}{r}+\sum_{i=1}^{r-1}(a_{i+1}+\cdots+a_r)\binom{p+r-i}{r-i}\binom{q-2+i}{i}.
\end{aligned}$$

□

Corollary 2.2. *Let $(R,\mathfrak{m})$ be a DVR and let C be a module of finite length. Then the function $\Lambda(p,q)$ associated to the module C coincides with a polynomial function. Moreover we have the formula*

$$e^j(C)=\ell_R(R/\operatorname{Fitt}_j(C))=e(R/\operatorname{Fitt}_j(C))$$

for any $j=0,1,\dots,r-1$.

Remark 2.3. In [5], Kirby and Rees computed the multiplicities $e^j(C)$ in the case where C is a module of finite length and R is a DVR. Proposition 2.1 and Corollary 2.2 gives more detailed information about the function $\Lambda(p,q)$.

The case where the ideals $Q_1, Q_2, \ldots, Q_r$ does not become an ascending chain is more complicated. However the case where $r = 2$ can be computed as follows.

Theorem 2.4. *Assume $r = 2$ and put $I := Q_1 + Q_2$. Then*

(1) *The Buchsbaum-Rim polynomial is*

$$\Lambda(p,q) = (a_1 + a_2)\binom{p+2}{2} + e(R/I)\binom{p+1}{1}\binom{q-1}{1} - e_1(I)(p+q) + c$$

for all $p, q \gg 0$, where $e_1(I)$ denotes the 1st Hilbert coefficient of I and c is a constant. In particular, we have that

$$\begin{cases} e^0(C) = \ell_R(R/\operatorname{Fitt}_0(C)) = \ell_R(R/Q_1Q_2) \\ e^1(C) = e(R/\operatorname{Fitt}_1(C)) = e(R/I) \\ e^2(C) = 0. \end{cases}$$

(2) *The function $\Lambda(p,q)$ coincides with a polynomial function if and only if the equality $\ell_R(R/I) = e(R/I) - e_1(I)$ holds true. When this is the case,*

$$\Lambda(p,q) = (a_1 + a_2)\binom{p+2}{2} + e(R/I)\binom{p+1}{1}\binom{q-1}{1} - e_1(I)(p+q) + e_1(I)$$

for all $p \geq 0$ and all $q > 0$.

(3) *The function $\Lambda(p,q)$ coincides with the following simple polynomial function*

$$\Lambda(p,q) = (a_1 + a_2)\binom{p+2}{2} + e(R/I)\binom{p+1}{1}\binom{q-1}{1}$$

if and only if there exits an inclusion between Q_1 and Q_2.

Proof. Let $p \geq 0$, $q > 0$ and let $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \mathbb{Z}^2_{\geq 0}$ such that $|\boldsymbol{\ell}| = p + q$. Let $\delta = \delta(\boldsymbol{\ell})$ be the number of elements of the set $\Delta = \Delta(\boldsymbol{\ell}) = \{\ell_i \mid \ell_i > q - 1\}$. Then the ideal $J_{p,q}(\boldsymbol{\ell})$ can be computed as follows directly.

Claim 1

$$J_{p,q}(\boldsymbol{\ell}) = \begin{cases} I^{p+1} & \text{if } \delta = 0 \\ Q_i^{\ell_i - q + 1} I^{\ell_j} \quad (i \neq j) & \text{if } \delta = 1 \text{ and } \Delta = \{\ell_i\} \\ Q_1^{\ell_1 - q + 1} Q_2^{\ell_2 - q + 1} I^{q-1} & \text{if } \delta = 2 \end{cases}$$

Let $h_n = \ell_R(R/I^n)$ be the Hilbert-Samuel function of the ideal I. Then, by Claim 1, the function $\Lambda(p,q)$ can be computed as follows.

Claim 2

$$\Lambda(p,q) = \begin{cases} (a_1 + a_2)\binom{p+2}{2} + 2(h_1 + \cdots + h_p) + (q - p - 1)h_{p+1} & \text{if } p + 1 \leq q - 1 \\ (a_1 + a_2)\binom{p+2}{2} + 2(h_1 + \cdots + h_{q-2}) + (p - q + 3)h_{q-1} & \text{if } p + 1 > q - 1 \end{cases}$$

Let p_0 be the postulation number of I, that is, $h_p = e(R/I)p - e_1(I)$ for all $p \geq p_0$. To compute the Buchsbaum-Rim polynomial, we may assume that $p \geq p_0$ and $q - 1 \geq p + 1$. Then, by Claim 2, we can compute the function $\Lambda(p,q)$ explicitly as follows.

$$\Lambda(p,q) = (a_1 + a_2)\binom{p+2}{2} + e(R/I)\binom{p+1}{1}\binom{q-1}{1} - e_1(I)(p+q) + c$$

where $c = 2(h_1 + \cdots + h_{p_0 - 1}) - e(R/I)p_0(p_0 - 1) + e_1(I)(2p_0 - 1)$ is a constant. This proves the assertion (1).

Suppose that the function $\Lambda(p,q)$ coincides with the polynomial function. Then, by substituting $p = 0$ in the polynomial, $\Lambda(0,q) = (e(R/I) - e_1(I))q + (a_1 + a_2 - e(R/I) + c)$ for any $q > 0$. On the other hand, by Claim 2, $\Lambda(0,q) = h_1 q + (a_1 + a_2 - h_1)$. By comparing the coefficient of q, we have $h_1 = e(R/I) - e_1(I)$. Conversely, suppose that $h_1 = e(R/I) - e_1(I)$. Then it is known that the Hilbert-Samuel function h_n coincides with the polynomial function for all $n > 0$ ([9]). Hence the function $\Lambda(p,q)$ also coincides with the polynomial function with the following form

$$\Lambda(p,q) = (a_1 + a_2)\binom{p+2}{2} + e(R/I)\binom{p+1}{1}\binom{q-1}{1} - e_1(I)(p+q) + e_1(I)$$

by Claim 2. Thus we have the assertion (2).

For the assertion (3), if the function $\Lambda(p,q)$ coincides with the following simple polynomial function

$$\Lambda(p,q) = (a_1 + a_2)\binom{p+2}{2} + e(R/I)\binom{p+1}{1}\binom{q-1}{1},$$

then $e_1(I) = 0$ and $h_1 = e(R/I)$. This implies that I is a parameter ideal for R and hence $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$. The other implication follows from Proposition 2.1. $\square$

Consequently, there exists the case where the Buchsbaum-Rim function $\Lambda(p,q)$ does not coincides with a polynomial function even if the ring R is Cohen-Macaulay and the module has a parameter matrix. This should be contrasted with a result on the classical Buchsbaum-Rim function of a parameter module due to Brennan-Ulrich-Vasconcelos [1].

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