The Picard and the class groups of an invariant subring

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1. Introduction

The purpose of this paper is to define equivariant class group of a locally Krull scheme (that is, a scheme which is locally a prime spectrum of a Krull domain) with an action of a flat group scheme, study its basic properties, and apply it to prove the finite generation of the class group of an invariant subring.

In particular, we prove the following.

Theorem 1.1. *Let k be a field, G a smooth k-group scheme of finite type, and X a normal variety over k on which G acts. Let $\varphi : X \to Y$ be a G-invariant morphism such that $\mathcal{O}_Y \cong (\varphi_* \mathcal{O}_X)^G$. Then*

(1) *If* $\mathrm{Pic}(X)$ *is a finitely generated abelian group, then so is* $\mathrm{Pic}(Y)$.

(2) *If* $\mathrm{Cl}(X)$ *is a finitely generated abelian group, then so is* $\mathrm{Cl}(Y)$.

If $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} B^G$, and $\varphi : X \to Y$ is the canonical map, then the condition $\mathcal{O}_Y \cong (\varphi_* \mathcal{O}_X)^G$ is satisfied. Results similar to (2) for *connected* G are proved by Magid and Waterhouse.

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2. Equivariant Picard group

The first part of Theorem 1.1 uses the equivariant Picard group.

Let Ord be the category of ordered sets and order-preserving maps. Let Δ be the full subcategory of Ord with $\mathrm{Ob}(\Delta) = \{[0], [1], [2], \ldots\}$, where $[n] = \{0 < 1 < \cdots < n\}$. Let Δ^+ be the subcategory of Δ such that $\mathrm{Ob}(\Delta^+) = \mathrm{Ob}(\Delta)$ and $\mathrm{Mor}(\Delta^+) = \{\phi \in \mathrm{Mor}(\Delta) \mid \phi \text{ is an injective map}\}$.

Thus Δ^+ looks like

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "\delta_0^0"] \arrow[r, shift right, "\delta_1^0"'] & {[1]} \arrow[r, shift left=2, "\delta_0^1"] \arrow[r, "\delta_1^1"] \arrow[r, shift right=2, "\delta_2^1"'] & {[2]} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots, \end{tikzcd}$$

where $\delta_i^n : [n] \to [n+1]$ is the unique injective monotone map such that $i \notin \mathrm{Im}\,\delta_i^n$.

Let S be a scheme, and Let G be an S-group scheme acting on X. Then we associate $B_G^+(X) \in \mathrm{Func}((\Delta^+)^{\mathrm{op}}, \underline{\mathrm{Sch}}/S)$ as

$$\begin{tikzcd} B_G^+(X) := X & G \times X \arrow[l, shift right, "d_1^0"'] \arrow[l, shift left, "d_0^0"] & G \times G \times X \arrow[l, shift right=2, "d_2^1"'] \arrow[l, "d_1^1"] \arrow[l, shift left=2, "d_0^1"] & \cdots \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3], \end{tikzcd}$$

where $\underline{\mathrm{Sch}}/S$ denotes the category of S-schemes, Func denotes the functor category, and

$$d_i^n = B_G^+(X)_{\delta_i^n} : B_G^+(X)_{[n+1]} = G^{n+1} \times X \to G^n \times X = B_G^+(X)_{[n]}$$

is defined by

$$d_i^n(g_n \ldots, g_0, x) = \begin{cases} (g_n, \ldots, g_1, g_0 x) & (i = 0) \\ (g_n, \ldots, g_i g_{i-1}, \ldots, g_0, x) & (0 < i < n+1) \\ (g_{n-1}, \ldots, g_0, x) & (i = n+1) \end{cases}.$$

The categories of modules $\mathrm{Mod}(\mathrm{Zar}(B_G^+(X)))$ and quasi-coherent modules $\mathrm{Qch}(\mathrm{Zar}(B_G^+(X)))$ are denoted by $\mathrm{Mod}(G, X)$ and $\mathrm{Qch}(G, X)$, respectively, where Zar denotes the Zariski site [Has, (4.3)]. An object of $\mathrm{Mod}(G, X)$ is called a *$(G, \mathcal{O}_X)$-module.*

If G is S-flat, then $\mathrm{Qch}(G, X)$ is closed under kernels, cokernels and extensions in $\mathrm{Mod}(G, X)$, and it is an abelian category and the inclusion $\mathrm{Qch}(G, X) \hookrightarrow \mathrm{Mod}(G, X)$ is exact.

Let $\mathcal{C}$ be a site. Let $\mathrm{Ps}(\mathcal{C})$ and $\mathrm{Sh}(\mathcal{C})$ denote the category of presheaves and sheaves over $\mathcal{C}$, respectively. For $\mathcal{M} \in \mathrm{Ps}(\mathcal{C})$ and $\mathcal{N} \in \mathrm{Sh}(\mathcal{C})$, we write $H^i_p(\mathcal{C}, \mathcal{M}) := \mathrm{Ext}^i_{\mathrm{Ps}(\mathcal{C})}(\underline{\mathbb{Z}}, \mathcal{M})$ and $H^i(\mathcal{C}, \mathcal{N}) := \mathrm{Ext}^i_{\mathrm{Sh}(\mathcal{C})}(a\,\underline{\mathbb{Z}}, \mathcal{N})$, where $\underline{\mathbb{Z}}$ is the constant presheaf and $a\,\underline{\mathbb{Z}}$ its sheafification.

For $\mathcal{M} \in \mathrm{Ps}(\mathrm{Zar}(B^+_G(X)))$, we denote $H^i_p(\mathrm{Zar}(B^+_G(X)), \mathcal{M})$ by $H^i_{\mathrm{alg}}(G, \mathcal{M})$, and call it the ith *algebraic G-cohomology group* of $\mathcal{M}$.

Lemma 2.1. *$H^i_{\mathrm{alg}}(G, \mathcal{M})$ is the cohomology group of the complex*

$$0 \to \Gamma(([0], X), \mathcal{M}) \xrightarrow{d_0 - d_1} \Gamma(([1], G \times X), \mathcal{M}) \xrightarrow{d_0 - d_1 + d_2} \Gamma(([2], G \times G \times X), \mathcal{M}) \to \cdots .$$

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. An $\mathcal{O}$-module $\mathcal{L}$ is called an *invertible sheaf* if for any $c \in \mathrm{Ob}(\mathcal{C})$, there exists some covering $(c_\lambda \to c)$ of c such that for each λ, $\mathcal{L}|_{c_\lambda} \cong \mathcal{O}|_{c_\lambda}$, where $(?)|_{c_\lambda}$ is the restriction to $\mathcal{C}/c_\lambda$. An invertible sheaf is quasi-coherent.

The set of isomorphism classes of invertible sheaves on $\mathcal{C}$ is denoted by $\mathrm{Pic}(\mathcal{C})$, and called the *Picard group* of $\mathcal{C}$. It is an additive group by the addition

$$[\mathcal{L}] + [\mathcal{L}'] := [\mathcal{L} \otimes_{\mathcal{O}} \mathcal{L}'].$$

Lemma 2.2. *There is an isomorphism* $\mathrm{Pic}(\mathcal{C}) \cong H^1(\mathcal{C}, \mathcal{O}^\times)$.

For the proof, see [dJ, (20.7.1)].

Definition 2.3. $\mathrm{Pic}(B^+_G(X))$ is denoted by $\mathrm{Pic}(G, X)$, and is called the *G-equivariant Picard group of X*.

There is an obvious map

$$\rho : \mathrm{Pic}(G, X) \to \mathrm{Pic}(X)$$

forgetting the G-action. The image of ρ is contained in

$$\mathrm{Pic}(X)^G := \mathrm{Ker}(\mathrm{Pic}(X) \xrightarrow{d_0 - d_1} \mathrm{Pic}(G \times X)) = \{[\mathcal{L}] \in \mathrm{Pic}(X) \mid a^*\mathcal{L} \cong p_2^*\mathcal{L}\},$$

where $a = d_0 : G \times X \to X$ is the action, and $p_2 = d_1 : G \times X \to X$ is the second projection.

From the five-term exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2$$

of the Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{alg}}(G, \underline{H}^q(\mathcal{O}^\times)) \Rightarrow H^{p+q}(\mathrm{Zar}(B_G^+(X)), \mathcal{O}^\times),$$

we get

Lemma 2.4. *There is an exact sequence*

$$0 \to H^1_{\mathrm{alg}}(G, \mathcal{O}^\times) \to \mathrm{Pic}(G, X) \xrightarrow{\rho} \mathrm{Pic}(X)^G \to \\ H^2_{\mathrm{alg}}(G, \mathcal{O}^\times) \to H^2(\mathrm{Zar}(B_G^+(X)), \mathcal{O}^\times).$$

Theorem 2.5. *Let k be a field, G a smooth k-group scheme of finite type, and X a reduced G-scheme which is quasi-compact and quasi-separated. Assume that there is a k-scheme Z of finite type and a dominating k-morphism $Z \to X$. Then $H^1_{\mathrm{alg}}(G, \mathcal{O}^\times) = \mathrm{Ker}(\rho : \mathrm{Pic}(G, X) \to \mathrm{Pic}(X))$ is a finitely generated abelian group.*

Note that a reduced k-scheme X of finite type is reduced, quasi-compact and quasi-separated, admitting a dominating map from a k-scheme of finite type, that is, $\mathrm{id} : Z = X \to X$.

Lemma 2.6. *Let $\varphi : X \to Y$ be a G-invariant morphism. If $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism, then $\varphi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(G, X)$ is injective.*

Proof. Note that the canonical map $\mathcal{L} \to (\varphi_* \varphi^* \mathcal{L})^G$ is an isomorphism for any invertible sheaf $\mathcal{L}$ on Y. Indeed, to check this, as the question is local on Y, we may assume that $\mathcal{L} \cong \mathcal{O}_Y$. But this case is nothing but the assumption itself. So if $\varphi^* \mathcal{L} \cong \mathcal{O}_X$, then

$$\mathcal{L} \cong (\varphi_* \varphi^* \mathcal{L})^G \cong (\varphi_* \mathcal{O}_X)^G \cong \mathcal{O}_Y,$$

and the assertion follows immediately. □

Combining Theorem 2.5 and Lemma 2.6, we immediately have the first part of Theorem 1.1.

Corollary 2.7. *Let k, G, X and $Z \to X$ be as in the theorem, and let $\varphi : X \to Y$ be a G-invariant morphism such that $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism. If* $\mathrm{Pic}(X)$ *is a finitely generated abelian group, then* $\mathrm{Pic}(G, X)$ *and* $\mathrm{Pic}(Y)$ *are also finitely generated.*

We outline the proof of Thereom 2.5.

Case **1** First, consider the case that G is a finite (constant) group, and $X = \operatorname{Spec} B$ is also finite.

(1) The case that $G \subset \mathrm{Aut}(B/k)$. Then $H^1_{\mathrm{alg}}(G, \mathcal{O}^\times) = H^1(G, B^\times) = 0$ (*Hilbert's Theorem 90*).

(2) The case that the action of G on X is trivial. Then $H^1(G, B^\times)$ is the group of homomorphisms from G to $B^\times$. This is finite.

(3) General case. Let N be the kernel of the map $G \to GL(B)$. Then there is an exact sequence

$$0 \to H^1(G/N, B^\times) \to H^1(G, B^\times) \to H^1(N, B^\times).$$

As $H^1(G/N, B^\times)$ and $H^1(N, B^\times)$ are finitely generated, $H^1(G, B^\times)$ is also finitely generated.

Case **2** Next, let G and X be finite (G is a finite group *scheme*, and is not a finite group in general). Let k' be a finite Galois extension of k such that $\Omega := k' \otimes_k G$ is a finite group (i.e., a disjoint union of $\operatorname{Spec} k'$). Let $\Gamma := \mathrm{Gal}(k'/k)$. Then there is an equivalence of categories

$$\mathrm{Mod}(G, B) \cong \mathrm{Mod}(\Theta, k' \otimes_k B),$$

where Θ is the semidirect product $\Gamma \ltimes \Omega$. Replacing G by Θ, the problem is reduced to case **1**.

Case **3**. The case that $G = \operatorname{Spec} H$ and $X = \operatorname{Spec} B$ are both affine. Let H_0 and B_0 be the integral closures of k in H and B, respectively. Then $G_0 := \operatorname{Spec} H_0$ is an affine k-group scheme acting on $X_0 := \operatorname{Spec} B_0$. Then the map of complexes

$$\begin{tikzcd}
0 \arrow[r] & B_0^\times \arrow[r] \arrow[d] & (H_0 \otimes B_0)^\times \arrow[r] \arrow[d] & (H_0 \otimes H_0 \otimes B_0)^\times \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B^\times \arrow[r] & (H \otimes B)^\times \arrow[r] & (H \otimes H \otimes B)^\times \arrow[r] & \cdots
\end{tikzcd}$$

is an isomorphism in the quotient category $\mathcal{A} := \mathrm{Mod}(\mathbb{Z})/\operatorname{mod}(\mathbb{Z})$ by the next lemma, and the problem is reduced to case **2**.

Lemma 2.8 (cf. [Ros]). *Let k be a field, and X be a reduced k-scheme. Assume that there is a k-scheme Z of finite type and a dominating k-morphism $Z \to X$. Then there is a short exact sequence of the form*

$$1 \to K^\times \xrightarrow{\iota} \Gamma(X, \mathcal{O}_X)^\times \to \mathbb{Z}^r \to 0,$$

where K is the integral closure of k in $k[X] = H^0(X, \mathcal{O}_X)$, and ι is the inclusion.

Proof. This is proved similarly to [Has2, (4.12)]. $\square$

Case **4** General case. Let $H = k[G]$ and $B = k[X]$. Then H is a commutative k-Hopf algebra, and B is an H-comodule algebra, as can be seen easily. The problem is reduced to that for $\operatorname{Spec} H$ and $\operatorname{Spec} B$, and we can invoke the result of case **3**. $\square$

Although Theorem 2.5 gives only the finite generation on $H^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times)$, we have more information on $H^i_{\mathrm{alg}}(G, \mathcal{O}_X^\times)$ in some cases.

Lemma 2.9. *Let k be a field, and G a quasi-compact quasi-separated k-group scheme such that $k[G]$ is geometrically reduced over k. Let X be a G-scheme. Assume that $\bar{k} \otimes_k X$ is integral, or X is quasi-compact quasi-separated and $\bar{k} \otimes_k k[X]$ is integral. If the unit group of $\bar{k} \otimes_k k[X]$ is $\bar{k}^\times$, then $H^i_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \cong H^i_{\mathrm{alg}}(G, k^\times)$. In particular, $H^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \cong \mathcal{X}(G) := \{\chi \in k[G]^\times \mid \chi(gg') = \chi(g)\chi(g')\}$.*

Example 2.10. If a smooth k-group scheme G acts on the affine space $X = \mathbb{A}^n$, then $H^1_{\mathrm{alg}}(G, \mathcal{O}_X^\times) \cong \mathcal{X}(G) \cong \mathrm{Pic}(G, \operatorname{Spec} k) \cong \mathrm{Pic}(G, X)$.

Proposition 2.11. *Let G be a connected smooth k-group scheme of finite type, and X a quasi-compact quasi-separated G-scheme such that $k[X]$ is reduced and k is integrally closed in $k[X]$. Then*

$$H^n_{\mathrm{alg}}(G, \mathcal{O}_X^\times) = \begin{cases} (k[X]^G)^\times & (n = 0) \\ \mathcal{X}(G)/\mathcal{X}(G, X) & (n = 1) \\ 0 & (n \geq 2) \end{cases},$$

where

$$\mathcal{X}(G, X) := \{\chi \in \mathcal{X}(G) \mid \exists \alpha \in k[X]^\times \; \forall g \in G \; x \in X \; \alpha(gx) = \chi(g)\alpha(x)\}.$$

The following is a slight refinement of Kamke's result [Kam].

Corollary 2.12. *In* Proposition 2.11*, assume that G and $X = \operatorname{Spec} B$ are affine. If f is a nonzerodivisor of B and Bf is a G-ideal of B, then f is a semiinvariant. That is, there exists some $\chi \in \mathcal{X}(G)$ such that $f(gx) = \chi(g)f(x)$ for $x \in X$ and $g \in G$.*

The following is more or less well-known. See [Dol].

Corollary 2.13. *Under the assumption of the proposition,*

$$\rho : \operatorname{Pic}(G, X) \to \operatorname{Pic}(X)^G$$

is surjective.

Proof. Follows imeediately by Lemma 2.4 and the proposition. □

Next, we introduce the notion of equivariant class group. It is defined for locally Krull schemes.

A *locally Krull scheme* is a shceme which is locally the prime spectrum of a Krull doain by definition.

Let A be a Krull domain. An A-module M is said to be *reflexive* (or *divisorial*), if M is a submodule of some finitely generated module, and the canonical map $M \to M^{**}$ is an isomorphism, where $(?)^* = \operatorname{Hom}_A(?, A)$.

Let Y be a locally Krull scheme. An $\mathcal{O}_Y$-module $\mathcal{M}$ is said to be *reflexive* if $\mathcal{M}$ is quasi-coherent, and $H^0(U, \mathcal{M})$ is a reflexive A-module for each affine open subset $U = \operatorname{Spec} A$ such that A is a Krull domain. If, moreover, $H^0(U, \mathcal{M})$ is of rank n for each U, then we say that $\mathcal{M}$ is of rank n.

Let Y be a locally Krull scheme. We denote the set of isomorphism classes of rank-one reflexive sheaves over Y by $\operatorname{Cl}(Y)$ and call it the *class group* of Y (again!). Note that $\operatorname{Cl}(Y)$ is an additive group by the addition

$$[\mathcal{M}] + [\mathcal{M}'] = [(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{M}')^{**}].$$

Almost by definition, $\operatorname{Pic}(Y)$ is a subgroup by $\operatorname{Cl}(Y)$. If Y is a non-singular variety, then $\operatorname{Pic}(Y) = \operatorname{Cl}(Y)$.

The definition above agrees with the usual one (the group freely generated by the set of prime divisors modulo the group of principal divisors) provided Y is quasi-compact. If this is the case, the map $[D] \mapsto [\mathcal{O}_Y(D)]$ gives an isomorphism from the "usual" class group to $\operatorname{Cl}(Y)$ defined above.

This definition is immediately generalized to that of the equivariant class group. Let G be S-flat and X be locally Krull. We say that a $(G, \mathcal{O}_X)$-module $\mathcal{M}$ is *reflexive* if $\mathcal{M}$ is quasi-coherent (as a $(G, \mathcal{O}_X)$-module), and is reflexive as an $\mathcal{O}_X$-module. The set of isomorphism classes of rank-one reflexive $(G, \mathcal{O}_X)$-modules is denoted by $\mathrm{Cl}(G, X)$, and we call it the *G-equivariant class group* of X.

Theorem 2.14. *Let G and X be as above, and $\mathcal{M}$ and $\mathcal{N}$ be reflexive $(G, \mathcal{O}_X)$-modules. Then*

1. *The $(G, \mathcal{O}_X)$-modules $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ and $(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})^{**}$ are reflexive, where $(?)^* = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(?, \mathcal{O}_X)$.*

2. *$\mathrm{Cl}(G, X)$ is an additive group with the sum*

$$[\mathcal{M}] + [\mathcal{N}] = [(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})^{**}].$$

There is an obvious map $\alpha : \mathrm{Cl}(G, X) \to \mathrm{Cl}(X)$, fogetting the G-action. We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} \rho \arrow[r] \arrow[d, "\cong"] & \mathrm{Pic}(G, X) \arrow[r, "\rho"] \arrow[d, hook] & \mathrm{Pic}(X) . \arrow[d, hook] \\
0 \arrow[r] & \operatorname{Ker} \alpha \arrow[r] & \mathrm{Cl}(G, X) \arrow[r, "\alpha"] & \mathrm{Cl}(X)
\end{tikzcd}$$

Lemma 2.15. *Let G be a flat S-group scheme, and X be a locally Krull G-scheme. Let U be its G-stable open subset. Let $\varphi : U \hookrightarrow X$ be the inclusion. Assume that $\mathrm{codim}_X(X \setminus U) \geq 2$. Then $\varphi^* : \mathrm{Ref}_n(G, X) \to \mathrm{Ref}_n(G, U)$ is an equivalence, and $\varphi_* : \mathrm{Ref}_n(G, U) \to \mathrm{Ref}_n(G, X)$ is its quasi-inverse. In particular, $\varphi^* : \mathrm{Cl}(G, X) \to \mathrm{Cl}(G, U)$ defined by $\varphi^*[\mathcal{M}] = [\varphi^* \mathcal{M}]$ is an isomorphism whose inverse is given by $\mathcal{N} \mapsto [\varphi_* \mathcal{N}]$.*

Proposition 2.16. *Let Y be a quasi-compact locally Krull scheme. Then $\mathrm{Cl}(Y) \cong \varinjlim \mathrm{Pic}(U)$, where the inductive limit is taken over all open subsets U such that $\mathrm{codim}_Y(Y \setminus U) \geq 2$.*

Lemma 2.17. *Let G be a flat S-group scheme. Let X be a quasi-compact quasi-separated locally Krull G-scheme, and let $\varphi : X \to Y$ be a G-invariant morphism such that $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism. Then Y is locally Krull, and the number of connected components of Y is finite. The class group $\mathrm{Cl}(Y)$ of Y is a subquotient of $\mathrm{Cl}(G, X)$.*

Thus we can prove the class group counterpart of Theorem 2.5.

Theorem 2.18. *Let k be a field, G a smooth k-group scheme of finite type, and X a quasi-compact quasi-separated locally Krull G-scheme. Assume that there is a k-scheme Z of finite type and a dominating k-morphism $Z \to X$. Let $\varphi : X \to Y$ be a G-invariant morphism such that $\mathcal{O}_Y \to (\varphi_* \mathcal{O}_X)^G$ is an isomorphism. If $\mathrm{Cl}(X)$ is finitely generated, then $\mathrm{Cl}(G, X)$ and $\mathrm{Cl}(Y)$ are also finitely generated.*

Even if X is a normal k-variety, Y may not be locally Noetherian.

Similar results for *connected* groups are proved by Magid and Waterhouse.

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