HILBERT POLYNOMIALS OF j-TRANSFORMS

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1. Introduction

This paper aims to explore the structure of certain graded objects associated to finitely generated modules over Noetherian local rings and ideals generated by partial systems of parameters. Let $(R, \mathfrak{m})$ be a Noetherian local ring and I $(\neq R)$ an ideal of R. Then for each finitely generated R-module M, we can associated two graded objects, which is the composition of the two functors

$$M \mapsto \mathrm{gr}_I(M) = \bigoplus_{n \geq 0} I^n M/I^{n+1}M \mapsto \mathrm{H}(M) = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_I(M)).$$

The study of $\mathrm{H} = \mathrm{H}(M)$ was initiated by Achilles and Manaresi [1] who made use of the fact that $\mathrm{H} = \bigoplus_{n \geq 0} \mathrm{H}_n$ has an associated numerical function $n \mapsto \psi_M(\mathbf{x}; n) = \sum_{k \leq n} \lambda(\mathrm{H}_k)$ that is a broad generalization of the classical Hilbert function – the case where I is an $\mathfrak{m}$-primary ideal. Its Hilbert polynomial

$$\sum_{i=0}^{r} (-1)^i j_i(I; M) \binom{n + r - i}{r - i}$$

will be referred to as the $\mathbf{j}$-polynomial of M relative to I. In general it is very difficult to predict properties of $\mathrm{H}(M) = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_I(M))$, beginning with their Krull dimensions or the coefficients $j_i(I; M)$. Nevertheless several authors have succeeded in applying the construction to extend the full array of classical integrality criteria for Rees algebras and modules ([1], [2], [3], [11] and [13]).

Our goal here is to study a different facet of these polynomials. The specific aim is to derive explicit formulas for $j_i(I; M)$ in terms of properties of M known a priori and explore the significance of their vanishing. For that we limit ourselves to ideals generated by partial systems of parameters of M or even special classes of modules. Thus we let $\mathbf{x} = \{x_1, x_2, \ldots, x_r\} \subset \mathfrak{m}$ be a partial system of parameters of M, that is $\dim M = r + \dim M/(\mathbf{x})M$, and set $I = (\mathbf{x})$ and $j_i(\mathbf{x}; M) = j_i(I; M)$. A general issue is what the values of $j_1(\mathbf{x}; M)$ say about M itself. In [4], [5] and [14] the authors, and colleagues, studied the values of a special class of these coefficients. For a Noetherian local ring R and a finitely generated R-module M, we considered the Hilbert coefficients $\mathrm{e}_i(\mathbf{x}; M)$ associated to filtrations defined by a system $\mathbf{x}$ of parameters of M, more precisely to the Hilbert functions

$$n \mapsto \lambda(M/(\mathbf{x})^{n+1}M)$$

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and made use of the values of $j_1(\mathbf{x}; M)$ as the means to detect various properties of M (e.g., Cohen-Macaulay, Buchsbaum, finite cohomology, etc.). Here we seek to extend these probes to cases when $r < \dim M$. The significant distinction between $\mathrm{gr}_{(\mathbf{x})}(M)$ and $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M))$ is that when $r < \dim M$, the latter may not be homogeneous and therefore the vanishing of some of its Hilbert coefficients does not place them entirely in the context of [4], [5] and [14].

We illustrate one of these issues with a series of questions. Let R be a Noetherian local ring and let $I = (x_1, x_2, \ldots, x_r)$, $r \leq \dim R$, be an ideal generated by a partial system $\mathbf{x} = \{x_1, x_2, \cdots, x_r\}$ of parameters of R. Let G be the associated graded ring of I, $\mathrm{G} = \mathrm{gr}_I(R)$. The module $\mathrm{H} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{G})$ has dimension $\leq r$. We list some questions similar to those raised in [14] for a full system of parameters:

(i) What are the possible values of $\dim \mathrm{H}$? Note that $\mathrm{H} = (0)$ may happen or $\mathrm{H} \neq (0)$ but of dimension zero.
(ii) What is the signature of $j_1(\mathbf{x}; R)$? If $r = \dim \mathrm{H}$, is $j_1(\mathbf{x}; R) \leq 0$? The answer is affirmative if H is generated in degree 0, because $j_1(\mathbf{x}; R) = \mathrm{e}_1(\mathbf{x}^*, \mathrm{H})$ and these coefficients are always non-positive according to [10], where $\mathbf{x}^* = \{x_1^*, x_2^*, \ldots, x_r^*\}$ denotes the initial forms of $x_i's$ relative to I.
(iii) If R is unmixed, $r = \dim \mathrm{H}$, and $j_1(\mathbf{x}; R) = 0$, then is $(\mathbf{x})$ a complete intersection? The answer is obviously no. What additional restriction is required?

The questions (ii) and (iii) were dealt with in [4], [5] and [14] for $r = \dim R$, but we do not know much in the other cases.

We shall focus on a special kind of partial systems of parameters which are shown to be ubiquitous. For a finitely generated R-module M, we call a partial system $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ of parameters of M *amenable* if the Koszul homology module $\mathrm{H}_1(\mathbf{x}; M)$ has finite support (and hence $\mathrm{H}_i(\mathbf{x}; M)$ have finite support for all $i \geq 1$).

Let us state with one of our main questions.

Conjecture 1.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated unmixed R-module. Let $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ be a partial system of parameters of M. Suppose that $\mathbf{x}$ is an amenable d-sequence relative to M, that $\dim \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M)) = r$, and that $j_1(\mathbf{x}; M) = 0$. Then $\mathbf{x}$ is a regular sequence on M.

2. Formulas for j-coefficients

Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R-module of dimension $d > 0$. Let $x_1, x_2, \cdots, x_d$ be a system of parameters of M. We fix an integer $0 \leq r \leq d$ and set $I = (x_1, x_2, \ldots, x_r)$. We assume the partial system $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ of parameters of M is an amenable d-sequence relative to M. Notice that when $r > 0$, the sequence $x_2, \cdots, x_r$ is naturally an amenable partial system of parameters of M/x_1M that is a d-sequence relative to M/x_1M.

Let $\mathrm{G} = \mathrm{gr}_I(R)$. For each R-module C we set $\mathrm{G}(C) = \mathrm{gr}_I(C)$ and let $\mathrm{H}(C) = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{G}(C))$ denote the $\mathbf{j}$-transform of C relative to I. We then have the following.

Theorem 2.1. *Suppose that $r > 0$. We set $\mathrm{H}_i = \mathrm{H}_i(\mathbf{x}; M)$ for $i \geq 0$ and let $S = R[T_1, T_2, \ldots, T_r]$ be the polynomial ring. Then there exists an exact sequence*

$$0 \to \mathrm{H}_r \otimes_R S[-r] \to \cdots \to \mathrm{H}_1 \otimes_R S[-1] \to \mathrm{H}_0 \otimes_R S \to \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_I(M)) \to 0$$

of graded S-modules.

Proof. Firstly we assume only that $\mathbf{x} = x_1, x_2, \ldots, x_r$ is a d-sequence relative to M and refer to [7] for details about the approximation complexes $\mathcal{M}(\mathbf{x}; M)$ used here. The complex is an acyclic complex of graded S-modules (unadorned tensor products are over R)

$$0 \to \mathrm{H}_r(\mathbf{x}; M) \otimes S[-r] \to \cdots \to \mathrm{H}_1(\mathbf{x}; M) \otimes S[-1] \to \mathrm{H}_0(\mathbf{x}; M) \otimes S \to \mathrm{gr}_{(\mathbf{x})}(M) \to 0.$$

Our complex arises from applying the functor $\mathrm{H}^0_{\mathfrak{m}}(*)$ to $\mathcal{M}(\mathbf{x}; M)$. We now assume $\mathbf{x}$ is amenable for M, so that $\mathrm{H}_i(\mathbf{x}; M) = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_i(\mathbf{x}; M)) = \mathrm{H}_i$ for all $i \geq 1$. We notice that the image L of $\mathrm{H}_1 \otimes S[-1]$ in $\mathrm{H}_0(\mathbf{x}; M) \otimes S$ has support in $\{\mathfrak{m}\}$, and therefore $\mathrm{H}^1_{\mathfrak{m}}(L) = 0$. Since all the $\mathrm{H}_i \otimes_R S[-i]$ ($i \geq 1$) are supported in $\{\mathfrak{m}\}$, we obtain the exact complex $\mathrm{H}^0_{\mathfrak{m}}(\mathcal{M}(\mathbf{x}; M))$

$$0 \to \mathrm{H}_r \otimes S[-r] \to \cdots \to \mathrm{H}_1 \otimes S[-1] \to \mathrm{H}^0_{\mathfrak{m}}(\mathrm{H}_0(\mathbf{x}; M)) \otimes S \to \mathrm{H}^0_{\mathfrak{m}}(\mathrm{gr}_{(\mathbf{x})}(M)) \to 0$$

as asserted. $\square$

We particularly notice that Theorem 2.1 shows $\mathrm{H}(M) = \mathrm{G}{\cdot}[\mathrm{H}(M)]_0$. Let us note below an elementary proof of this fact, which confirms where and how we make use of the assumption that $\mathbf{x}$ is amenable and a d-sequence relative to M. For an R-submodule N of M we put

$$N :_M \langle \mathfrak{m} \rangle = \bigcup_{\ell > 0} [N :_M \mathfrak{m}^\ell].$$

Hence $\mathrm{H}^0_{\mathfrak{m}}(M/N) = [N :_M \langle \mathfrak{m} \rangle]/N$.

Theorem 2.2. *$I^nM \cap [I^{n+1}M :_M \langle \mathfrak{m} \rangle] = I^n{\cdot}[IM :_M \langle \mathfrak{m} \rangle]$, whence $\mathrm{H}^0_{\mathfrak{m}}(I^nM/I^{n+1}M) = \{I^n{\cdot}[IM :_M \mathfrak{m}]\}/I^{n+1}M$ for all $n \geq 0$. Therefore $\mathrm{H}(M) = \mathrm{G}{\cdot}[\mathrm{H}(M)]_0$.*

Proof. We have only to show $I^nM \cap [I^{n+1}M :_M \langle \mathfrak{m} \rangle] \subseteq I^n{\cdot}[IM :_M \langle \mathfrak{m} \rangle]$. By induction, we may assume that $n, r > 0$ and the assertion holds true for $n-1$ and $r-1$. We set $\overline{M} = M/x_1M$ and consider the partial system $x_2, \cdots, x_r$ of parameters of $\overline{M}$. Let $f \in I^{n+1}M :_M \langle \mathfrak{m} \rangle$ and let $\overline{f}$ denote the image of f in $\overline{M}$. Then $\overline{f} \in I^n\overline{M} \cap [I^{n+1}\overline{M} :_{\overline{M}} \langle \mathfrak{m} \rangle]$ and hence $f \in I^n{\cdot}[IM :_M \langle \mathfrak{m} \rangle] + x_1M$ by the hypothesis on r. Therefore without loss of generality, we may assume that $f \in x_1M \cap I^nM$. Then, because $x_1M \cap I^nM = x_1I^{n-1}M$ (as $\mathbf{x}$ is a d-sequence relative to M; see [8, Proposition 2.2]), we get $f = x_1g$ for some $g \in I^{n-1}M$, so that for $\ell \gg 0$, $x_1(\mathfrak{m}^\ell g) \subseteq I^{n+1}M \cap x_1M = x_1I^nM$. Let $a \in \mathfrak{m}^\ell$ and write $x_1(ag) = x_1h$ with $h \in I^nM$. Then $ag - h \in [(0) :_M x_1] \cap I^{n-1}M$. If $n > 1$, then $ag = h$ because $[(0) :_M x_1] \cap IM = (0)$ ([8, Proposition 2.1]), and hence $g \in [I^nM :_M \langle \mathfrak{m} \rangle] \cap I^{n-1}M = I^{n-1}{\cdot}[IM :_M \langle \mathfrak{m} \rangle]$ by the hypothesis on n. Thus $f \in x_1I^{n-1}{\cdot}[IM :_M \langle \mathfrak{m} \rangle]$ as asserted. Suppose $n = 1$. Then $h \in IM$. Hence $\mathfrak{m}^q{\cdot}[(0) :_M x_1] = (0)$ for some $q \gg 0$, because $(0) :_M x_1 = (0) :_M I = \mathrm{H}_r(x_1, x_2, \ldots, x_r; M)$ that has finite length (remember that $\mathbf{x}$ is amenable and a d-sequence relative to M). Therefore $\mathfrak{m}^q{\cdot}[ag-h] \subseteq \mathfrak{m}^q{\cdot}[(0):_M x_1] = (0)$, so that $\mathfrak{m}^{q+\ell}g \subseteq IM$. Thus $g \in IM :_M \langle \mathfrak{m} \rangle$, whence $f \in x_1{\cdot}[IM :_M \langle \mathfrak{m} \rangle]$. $\square$

Let $W = \mathrm{H}^0_{\mathfrak{m}}(M)$. Remember that $W = (0) :_M x_1 = (0) :_M I$, when $r > 0$. Let $\mathrm{G}(M/W) = \mathrm{gr}_I(M/W)$ and let $\psi : \mathrm{G}(M) \to \mathrm{G}(M/W)$ be the canonical epimorphism of graded G-modules. We set $W^* = \mathrm{Ker}\ \psi$. Then because $W \cap IM = (0)$, we have $W^* = [W^*]_0 \cong W$ as an R-module.

Lemma 2.3. *There is an exact sequence*

$$0 \to W^* \xrightarrow{\iota} \mathrm{H}(M) \xrightarrow{\varphi} \mathrm{H}(M/W) \to 0$$

of graded G*-modules, where* φ *denotes the homomorphism induced from the canonical epimorphism* $\psi : \mathrm{G}(M) \to \mathrm{G}(M/W)$.

For each $1 \le i \le r$ let $f_i = x_i + I^2$ denote the image of x_i in $\mathrm{G}_1 = I/I^2$

Lemma 2.4. *Suppose that* $r > 0$ *and let* $\overline{M} = M/x_1M$. *Then there is an exact sequence*

$$0 \to W^*(-1) \xrightarrow{\iota} \mathrm{H}(M)(-1) \xrightarrow{f_1} \mathrm{H}(M) \xrightarrow{\varphi} \mathrm{H}(\overline{M}) \to 0$$

of graded G*-modules, where* φ *denotes the homomorphism induced from the canonical epimorphism* $\psi : \mathrm{G}(M) \to \mathrm{G}(\overline{M})$.

As a consequence of Lemmas 2.3, 2.4 we get the following.

Proposition 2.5. *The sequence* $f_1, f_2, \cdots, f_r$ *acts on* $\mathrm{G}(M)/\mathrm{H}(M)$ *as a regular sequence.*

We set $\mathrm{H} = \mathrm{H}(M)$. Then by Lemma 2.4 the induction on r readily gives the following description of the Hilbert series $[\![\mathrm{H}]\!]$ of H.

Theorem 2.6. $[\![\mathrm{H}]\!] = \dfrac{h^0(M/IM) + \sum_{i=1}^{r} (-1)^i {\cdot} \left[\sum_{j=i}^{r} h^0(M/I_{r-j}M){\cdot}\binom{j-1}{i-1} \right] \mathbf{t}^i}{(1-\mathbf{t})^{r+1}}.$

We put $\varphi_M(n) = \lambda(\mathrm{H}^0_{\mathfrak{m}}(I^nM/I^{n+1}M))$ and $\psi_M(n) = \sum_{\ell=0}^{n} \varphi_M(\ell)$ for $n \ge 0$. Let $I_i = (x_1, x_2, \ldots, x_i)$ $(0 \le i \le r)$. We set

$$k_i(M) = \begin{cases} h^0\left(M/[(I_{r-i-1}M :_M x_{r-i}) + x_{r-i}M]\right) & \text{if } \ 0 \le i \le r-1 \\ h^0(M) & \text{if } \ i = r. \end{cases}$$

Theorem 2.7. $\psi_M(n) = \sum_{i=0}^{r} k_i(M){\cdot}\binom{n+r-i}{r-i}$ *for all* $n \ge 0$.

We summarize partial answers to our main questions.

Corollary 2.8. *Suppose* $r > 0$. *Then*

$$-j_1(\mathbf{x}; M) = \begin{cases} h^0(M) & \textit{if } \ r = 1 \\ k_1(M) = h^0(M/I_{r-1}M) - h^0(M/I_{r-2}M) \ge 0 & \text{if } \ r > 1, \end{cases}$$

whence $j_1(\mathbf{x}, M) \le 0$.

Proposition 2.9. *The following assertions hold true.*

(1) *Suppose* $r = 1$. *Then* $j_1(\mathbf{x}; M) = 0$ *if and only if* $\operatorname{depth} M > 0$.
(2) *Suppose* $r > 1$. *Then* $j_1(\mathbf{x}; M) = 0$ *if and only if*

$$\mathrm{H}^0_{\mathfrak{m}}\left(M/[(I_{r-2}M :_M x_{r-1}) + x_{r-1}M]\right) = (0).$$

(3) *Suppose* $r > 0$. *Then* $j_1(\mathbf{x}; M) = 0$, *if* $x_1, x_2, \cdots, x_r$ *is an* M*-regular sequence. The converse is also true, when* $r \geq 2$ *and* $\operatorname{depth} M \geq r - 1$, *or* $r = 3$ *and* M *is unmixed, or* M *has FLC and* $\operatorname{depth} M > 0$.

Suppose $r > 0$ and set

$$\chi_1(x_1, x_2, \ldots, x_k; M) = \sum_{i=1}^{k} (-1)^{i+1} \cdot \lambda(\mathrm{H}_i(x_1, x_2, \ldots, x_k; M))$$

for each $1 \leq k \leq r$. The following result extends [6, Corollary 3.6] to partial amenable systems of parameters.

Theorem 2.10. *Suppose* $r \geq 2$. *Then*

$$\chi_1(\mathbf{x}; M) \geq -j_1(\mathbf{x}; M),$$

where the equality holds if and only if $x_1, x_2, \ldots, x_{r-1}$ *is an* M*-regular sequence.*

3. Boundedness of $\mathbf{j}$-coefficients

Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R-module with $d = \dim_R M \geq 2$. Let $0 < r < d$ be an integer.

Let $\Lambda(M)$ the set of non-negative integers

$$k_i(x_1, x_2, \ldots, x_r; M) = (-1)^i \cdot j_i(x_1, x_2, \ldots, x_r; M)$$

is finite, where $0 \leq i \leq r-1$ and $x_1, x_2, \ldots, x_r$ is a partial amenable system of parameters of M which is a d-sequence relative to M. We then have the following.

Theorem 3.1. *Assume that there exists a system of parameters of* M *which is a strong* d*-sequence relative to* M. *Then the following conditions are equivalent.*

(1) *The set* $\Lambda(M)$ *is finite.*
(2) $\mathrm{H}^i_{\mathfrak{m}}(M)$ *is a finitely generated* R*-module for every* $1 \leq i \leq r$.

When this is the case, one has $\mathfrak{m}^{\ell} \cdot \mathrm{H}^i_{\mathfrak{m}}(M) = (0)$ *for all* $1 \leq i \leq r$, *where* $\ell = \max \Lambda(M)$.

Proof. (2) $\Rightarrow$ (1) Let $\mathbf{x} = \{x_1, x_2, \ldots, x_r\}$ be an amenable partial system of parameters of M that is a d-sequence relative to M. We set

$$k_i(M) = k_i(\mathbf{x}; M) = (-1)^i \cdot j_i(\mathbf{x}; M)$$

for each $0 \leq i \leq r - 1$. Remember that

$$k_i(M) = h^0(M/I_{r-i}M) - h^0(M/I_{r-i-1}M),$$

where $I_j = (x_1, x_2, \ldots, x_j)$ for $0 \leq j \leq r$. We will show that

$$k_i(M) \leq \sum_{i=1}^{r-i} h^i(M) \cdot \binom{r-i-1}{i-1}.$$

If $r=1$, then by the exact sequence

$$(\sharp_0)\quad 0 \to \mathrm{H}^0_{\mathfrak m}(M) \to \mathrm{H}^0_{\mathfrak m}(M/x_1M) \to \mathrm{H}^1_{\mathfrak m}(M) \xrightarrow{x_1} \mathrm{H}^1_{\mathfrak m}(M)$$

we have $k_0(M) = \lambda((0) :_{\mathrm{H}^1_{\mathfrak m}(M)} x_1) \le h^1(M)$. Suppose that $r>1$ and that our assertion holds true for $r-1$. We consider $\overline{M} = M/x_1M$. Then thanks to the exact sequence

$$(\sharp_i)\qquad \mathrm{H}^i_{\mathfrak m}(M) \xrightarrow{x_1} \mathrm{H}^i_{\mathfrak m}(M) \to \mathrm{H}^i_{\mathfrak m}(\overline{M}) \to \mathrm{H}^{i+1}_{\mathfrak m}(M) \xrightarrow{x_1} \mathrm{H}^{i+1}_{\mathfrak m}(M),$$

$\mathrm{H}^i_{\mathfrak m}(\overline{M})$ is finitely generated and $h^i(\overline{M}) \le h^i(M) + h^{i+1}(M)$ for all $1 \le i \le r-1$. Therefore, since $k_i(M) = k_i(\overline{M})$ (here $k_i(\overline{M}) = k_i(x_2, \cdots, x_r; \overline{M})$) for all $0 \le i \le r-2$, we get

$$\begin{aligned} k_i(M) = k_i(\overline{M}) \quad &\le \quad \sum_{j=1}^{r-i-1} h^j(\overline{M})\cdot\binom{r-i-2}{j-1} \\ &\le \quad \sum_{j=1}^{r-i-1} \left[h^j(M) + h^{j+1}(M)\right]\cdot\binom{r-i-2}{j-1} \\ &= \quad \sum_{j=1}^{r-i} h^j(M)\cdot\binom{r-i-1}{j-1}, \end{aligned}$$

while we have

$$k_{r-1}(M) = h^0(M/x_1M) - h^0(M) \le h^1(M)$$

by exact sequence $(\sharp_0)$ above.

(i) $\Rightarrow$ (ii) We choose a system $a_1, a_2, \cdots, a_d$ of parameters of M which is a strong d-sequence relative to M. Let $\Lambda_0(M)$ denote the set of $k_i(a_1^{n_1}, a_2^{n_2}, \ldots, a_r^{n_r}; M)'s$, where $0 \le i \le r-1$ and $n_i's$ are positive integers. Then $\Lambda_0(M) \subseteq \Lambda(M)$ and hence $\Lambda_0(M)$ is finite. We will show by induction on r that $\mathfrak m^\ell\cdot\mathrm{H}^i_{\mathfrak m}(M) = (0)$ for all $1 \le i \le r$, where $\ell = \max \Lambda_0(M)$.

Let $n_1 > 0$ and $x_1 = a_1^{n_1}$. We set $\overline{M} = M/x_1M$ and consider exact sequence $(\sharp_0)$ above. We then have

$$\mathfrak m^\ell\cdot\left[(0) :_{\mathrm{H}^1_{\mathfrak m}(M)} a_1^{n_1}\right] = (0),$$

because

$$\lambda(\left[(0) :_{\mathrm{H}^1_{\mathfrak m}(M)} a_1^{n_1}\right]) = h^0(M/a_1^{n_1}M) - h^0(M) = k_0(a_1^{n_1}; M) \in \Lambda_0(M)$$

so that $\lambda(\left[(0) :_{\mathrm{H}^1_{\mathfrak m}(M)} a_1^{n_1}\right]) \le \ell$. Hence

$$\mathfrak m^\ell\cdot\mathrm{H}^1_{\mathfrak m}(M) = (0)$$

as $n_1 > 0$ is arbitrary, which proves the assertion when $r=1$. Assume that $r>1$ and that our assertion holds true for $r-1$. Then the set of

$$k_i(a_2^{n_2}, \ldots, a_r^{n_r}; \overline{M}) = k_i(a_1^{n_1}, a_2^{n_2}, \ldots, a_r^{n_r}; M)$$

where $0 \le i \le r-2$ and $n_i's$ are positive integers is a subset of Λ_0, whence the hypothesis of induction shows

$$\mathfrak m^\ell\cdot\mathrm{H}^i_{\mathfrak m}(\overline{M}) = (0)$$

for all $1 \leq i \leq r-1$. Therefore by exact sequence ($\sharp_i$) above we get

$$\mathfrak{m}^\ell \cdot \left[(0)_{\mathrm{H}^{i+1}_\mathfrak{m}(M)} a_1^{n_1}\right] = (0) \quad \text{for all} \quad n_1 > 0,$$

whence $\mathfrak{m}^\ell \cdot \mathrm{H}^{i+1}_\mathfrak{m}(M) = (0)$ if $2 \leq i \leq r$. Hence $\mathfrak{m}^\ell \cdot \mathrm{H}^i_\mathfrak{m}(M) = (0)$ for all $1 \leq i \leq r$, which completes the proof. $\square$

Let us describe a broad class of modules for which the existence of strong d-sequences is guaranteed. It is based on a result of T. Kawasaki [9, Theorem 4.2. (1)].

Proposition 3.2. *Let R be a homomorphic image of a Gorenstein local ring and let M ($\neq (0)$) be a finitely generated R-module. Then there is a system of parameters of M which is a strong d-sequence relative to M.*

We now consider the problem of when the set $\Gamma(M)$ of

$$\begin{aligned} \chi_1(x_1, x_2, \ldots, x_r; M) &= \sum_{i=1}^{r} (-1)^{i-1} \lambda(\mathrm{H}_i(x_1, x_2, \ldots, x_r; M)) \\ &= h^0(M/(x_1, x_2, \ldots, x_{r-1})M) \end{aligned}$$

is finite, where $x_1, x_2, \ldots, x_r$ is a partial amenable system of parameters of M which is a d-sequence relative to M. Let $h^i(M) = \ell_R(\mathrm{H}^i_\mathfrak{m}(M))$ for each $i \in \mathbb{Z}$.

Theorem 3.3. *Assume that there exists a system of parameters of M which is a strong d-sequence relative to M. Then the following conditions are equivalent.*

(1) *The set $\Gamma(M)$ is finite.*
(2) *$\mathrm{H}^i_\mathfrak{m}(M)$ is finitely generated for every $0 \leq i \leq r-1$.*

When this is the case,

$$\sup_{n_1, \ldots, n_r > 0} \chi_1(a_1^{n_1}, a_2^{n_2}, \ldots, a_r^{n_r}; M) = \sum_{i=0}^{r-1} \binom{r-1}{i} \cdot h^i(M)$$

and therefore $\max \Gamma(M) = \displaystyle\sum_{i=0}^{r-1} \binom{r-1}{i} \cdot h^i(M).$

4. The structure of some $\mathbf{j}$-transforms

The general outline of the $\mathbf{j}$-transform $\mathrm{H} = \mathrm{H}^0_\mathfrak{m}(\mathrm{gr}_I(M))$ is still unclear. In two cases however – Buchsbaum and sequentially Cohen-Macaulay modules – one has a satisfying vista.

Buchsbaum modules. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finitely generated R-module of dimension $d \geq 2$. Let $x_1, x_2, \ldots, x_d$ be a system of parameters of M. We fix an integer $0 < r < d$ and put $I = (x_1, x_2, \ldots, x_r)$, $\mathrm{G} = \mathrm{gr}_I(R)$, $\mathrm{G}(M) = \mathrm{gr}_I(M)$, and

$$\overline{\mathrm{G}} = \mathrm{G}/\mathfrak{m}\mathrm{G} = k[T_1, T_2, \ldots, T_r],$$

where T_i denotes the image of $f_i = x_i + I^2$ in $\overline{\mathrm{G}}$.

Let us consider the $\mathbf{j}$-transform $\mathrm{H} = \mathrm{H}^0_\mathfrak{m}(\mathrm{G}(M))$. We then have the following.

Theorem 4.1. *Suppose that M is a Buchsbaum R-module. Then the following assertions hold true.*

(1) $\mathrm{H} = (0)$ *if and only if* $\mathrm{depth} M > r$.

(2) *Suppose that* $\mathrm{H} \neq (0)$. *Then*

$$\dim \mathrm{H} = \begin{cases} 0 & \text{if} \quad h^1(M) = h^2(M) = \cdots = h^r(M) = 0, \\ r & \text{otherwise.} \end{cases}$$

(3) $\mathrm{H} \cong \bigoplus_{i=0}^{r} [Z_i(i)]^{\oplus h^i(M)}$ *as a graded* G*-module.*

Here $Z_i = \mathrm{Syz}^i_{\overline{G}}(k)$ *denotes the i-th syzygy module of the residue class field* $k = \overline{\mathrm{G}}/[\overline{\mathrm{G}}]_+$.

Sequentially Cohen-Macaulay modules. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module with $d = \dim_R M > 0$. Let

$$\mathcal{S} = \{\dim_R N \mid (0) \neq N \subseteq M, \text{an } R\text{-submodule of } M\}.$$

We set $\ell = \sharp \mathcal{S}$ and write $\mathcal{S} = \{d_1 < d_2 < \cdots < d_\ell = d\}$. Let $d_0 = 0$. We then have the dimension filtration

$$\mathcal{D}_0 = (0) \subsetneq \mathcal{D}_1 \subsetneq D_2 \subsetneq \cdots \subsetneq \mathcal{D}_\ell = M$$

of M, where each $\mathcal{D}_i$ $(1 \leq i \leq \ell)$ is the largest R-submodule of M with $\dim \mathcal{D}_i = d_i$. We put $\mathcal{C}_i = \mathcal{D}_i/\mathcal{D}_{i-1}$ $(1 \leq i \leq \ell)$ and assume that $\mathcal{C}_i$ is a Cohen-Macaulay R-module, necessarily of dimension d_i, for each $1 \leq i \leq \ell$. Hence M is a sequentially Cohen-Macaulay R-module.

We choose a system $x_1, x_2, \ldots, x_d$ of parameters of M so that

$$(x_j \mid d_i < j \leq d)M \cap \mathcal{D}_i = (0)$$

for all $1 \leq i \leq \ell$. Such a system of parameters exists and called a *good* system of parameters of M. Here we notice that the condition $(x_j \mid d_i < j \leq d)M \cap \mathcal{D}_i = (0)$ for all $1 \leq i \leq \ell$ is equivalent to saying that

$$(x_j \mid d_i < j \leq d)\mathcal{D}_i = (0)$$

for all $1 \leq i \leq \ell$, because M is a sequentially Cohen-Macaulay R-module.

Let $0 \leq r \leq d$ and $I = (x_1, x_2, \ldots, x_r)$. We are trying to find what the $\mathbf{j}$-transform $\mathrm{H}(M)$ of M relative to I is. The goal is the following.

Theorem 4.2. *Let* $q = \max\{0 \leq i \leq \ell \mid d_i \leq r\}$. *Then*

$$\mathrm{H}(M) \cong \mathrm{gr}_I(\mathcal{D}_q)$$

as a graded G*-module, where* $\mathrm{G} = \mathrm{gr}_I(R)$. *If $q > 0$, that is if $d_1 \leq r$, then* $\mathrm{H}(M) \neq (0)$ *and is a sequentially Cohen-Macaulay* G*-module with dimension filtration* $\{\mathrm{gr}_I(\mathcal{D}_i)\}_{0 \leq i \leq q}$*; hence* $\dim \mathrm{H}(M) = d_q \leq r$ *and the Hilbert function of* $\mathrm{H}(M)$ *is given by*

$$\sum_{k=0}^{n} \lambda([\mathrm{H}(M)]_k) = \sum_{i=1}^{q} \lambda(\mathcal{C}_i/\mathfrak{q}\mathcal{C}_i) \cdot \binom{n + d_i}{d_i}$$

for all $n \geq 0$, where $\mathfrak{q} = (x_1, x_2, \ldots, x_d)$.

Proof of Theorem 4.2. We apply the functor $\mathrm{H}^0_{\mathfrak{m}}(*)$ to the exact sequences

$$0 \to \mathrm{G}(\mathcal{D}_{i-1}) \to \mathrm{G}(\mathcal{D}_i) \to \mathrm{G}(\mathcal{C}_i) \to 0$$

$(1 \leq i \leq \ell)$ and get that

$$\mathrm{H}(M) \cong \mathrm{G}(\mathcal{D}_q),$$

since $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{G}(\mathcal{C}_i)) = (0)$ if $q < i$. Suppose $q > 0$, that is $d_1 \leq r$. Then $\{\mathrm{G}(\mathcal{D}_i)\}_{0 \leq i \leq q}$ gives rise to the dimension filtration of $\mathrm{G}(\mathcal{D}_i)$, since the graded module $\mathrm{G}(\mathcal{C}_i)$ $(1 \leq i \leq q)$ is Cohen-Macaulay and $\dim \mathrm{G}(\mathcal{C}_i) = d_i$. Hence $\dim \mathrm{H}(M) = d_q \leq r$, whose Hilbert function is given by

$$\begin{aligned} \sum_{k=0}^{n} \lambda([\mathrm{H}(M)]_n) &= \sum_{k=0}^{n} \left\{ \sum_{i=1}^{q} \lambda([\mathrm{G}(\mathcal{C}_i)]_n) \right\} \\ &= \sum_{i=1}^{q} \lambda(\mathcal{C}_i/\mathfrak{q}\mathcal{C}_i) \cdot \binom{n+d_i}{d_i} \end{aligned}$$

for all $n \geq 0$. $\square$

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