THICK SUBCATEGORIES OF STABLE CATEGORIES OVER GRADED GORENSTEIN RINGS

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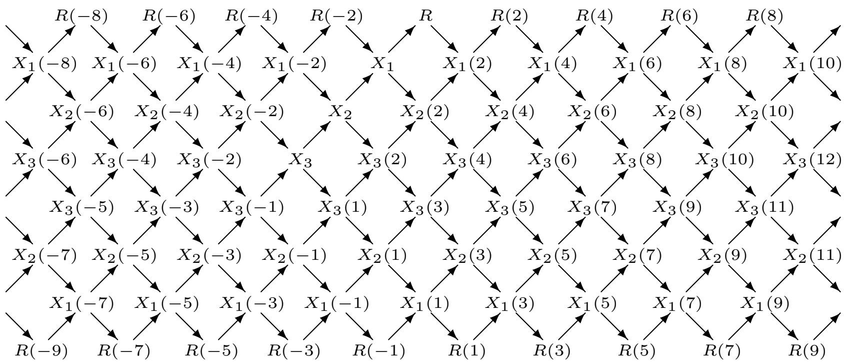
Abstract. Takahashi[6] classified the thick subcategories of the stable category of maximal Cohen-Macaulay modules over a hypersurface local ring. By his classification, we can see that if the base ring has an isolated singularity, then the thick subcategories are trivial. On the other hand, if the base ring is graded, then there exist non-trivial thick subcategories even if the base ring is of finite Cohen-Macaulay representation type. In this talk, we will classify the thick subcategories of the stable category of graded maximal Cohen-Macaulay modules over a graded Gorenstein rings of finite Cohen-Macaulay representation type.

1. Preliminaries

Throughout this talk, let k be an algebraically closed field of characteristic zero. We set $R = k[t^2, t^{2n+1}]$ with $\deg t = 1$. Then R is a graded Gorenstein ring of finite Cohen-Macaulay representation type (cf.[1] or [7]). Since R is isomorphic to $k[x, y]/(y^2 - x^{2n+1})$, R also has a type A singularity. We denote by $\operatorname{mod}^{\mathbb{Z}} R$ the category of finitely generated $\mathbb{Z}$-graded R-modules with degree preserving morphisms, by $\mathrm{CM}^{\mathbb{Z}}(R)$ the full subcategory of $\operatorname{mod}^{\mathbb{Z}} R$ consisting of all graded maximal Cohen-Macaulay modules, by $\operatorname{ind}\mathrm{CM}^{\mathbb{Z}}(R)$ the set of isomorphism classes of indecomposable graded maximal Cohen-Macaulay modules, and by Γ the Auslander-Reiten quiver of $\mathrm{CM}^{\mathbb{Z}}(R)$.

We remark that $\operatorname{ind}\mathrm{CM}^{\mathbb{Z}}(R) = \{R(j),\ X_i(j) \mid j \in \mathbb{Z},\ i = 1, 2, \ldots, n\}$ where $X_i := t^{2i}R + t^{2n+1}R$ $(i = 1, 2, \ldots, n)$ (cf.[1] or [7]).

Example 1.1. Let $R = k[t^2, t^7]$. We put $X_i = t^{2i}R + t^7 R$ for $i = 1, 2$ and 3. Then Γ is following:



In this case, we see Γ is a translation quiver $\mathbb{Z}A_8$. In general, Γ is $\mathbb{Z}A_{2n+2}$.

We write $X \prec Y$ if there exists a path from X to Y in Γ. One can easily check the following lemma by using the Auslander-Reiten theory.

Lemma 1.2. *For $X, Y \in \operatorname{ind}\mathrm{CM}^{\mathbb{Z}}(R)$, the followings hold.*

(1) *If $\operatorname{Hom}_R(X, Y) \neq 0$, then $X \preceq Y$.*
(2) $\dim_k \operatorname{Ext}^1_R(X, Y) \leq 1$.

This paper is an announcement of our result and the detailed version will be submitted to somewhere.

(3) $\mathrm{Ext}^1_R(X,Y) \neq 0$ *if and only if* $\Omega X \preceq Y \preceq \tau X$. *Here,* ΩX *is the syzygy module of* X *and* τ *is the Auslander-Reiten translation.*

The following proposition plays key role in this talk.

Proposition 1.3. *Let* $X, Y_1, Y_2, Z \in \mathrm{ind}\,\mathrm{CM}^{\mathbb{Z}}(R)$. *The exact sequence* $0 \to X \to Y_1 \oplus Y_2 \to Z \to 0$ *does not split if and only if* X, Y_1, Y_2 *and* Z *make a parallelogram in* Γ.

This propotision comes from the next lemma.

Lemma 1.4. *Let* $0 \to X \xrightarrow{\binom{f_1}{f_2}} Y_1 \oplus M \xrightarrow{(g_1\ \varphi)} N \to 0$ *and* $0 \to M \xrightarrow{\binom{\varphi}{f_3}} N \oplus Y_2 \xrightarrow{(g_2\ g_3)} Z \to 0$ *are exact sequences. Then* $0 \to X \xrightarrow{\binom{-f_1}{f_3 f_2}} Y_1 \oplus Y_2 \xrightarrow{(g_2 g_1\ g_3)} N \to 0$ *is exact.*

By watching Γ in Example 1.1, we can see that $X_3(-1), R(-1), R$ and $X_3(6)$ make a parallelogram. Therefore we have $0 \to X_3(-1) \to R(-1) \oplus R \to X_3(6) \to 0$ is an exact aequence. In particular, we see $\Omega X_3(6) = X_3(-1)$ and $\Omega^{-1} X_3(-1) = X_3(6)$. Thus, we can easily find the syzygy (cosyzygy) module of arbitrary maximal Cohen-Macaulay modules.

We denote by $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$ the stable category of $\mathrm{CM}^{\mathbb{Z}}(R)$ and by $\underline{\Gamma}$ the Auslander-Reiten quiver of $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$. Since the free modules are isomorphic to zero in $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$, we obtain $\underline{\Gamma}$ from Γ by deleting free modules and arrows whose one of end point is free module. Therefore, we have $\underline{\Gamma} = \mathbb{Z}A_{2n}$. We put Σ the suspension functor of $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$. We remark that $\Sigma X \cong \Omega^{-1} X$ for all $X \in \underline{\mathrm{CM}}^{\mathbb{Z}}(R)$. The following proposition comes from Lemma 1.2.

Proposition 1.5. *For* $X, Y \in \mathrm{ind}\,\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$ *and* $l \in \mathbb{Z}$, *the followings hold.*

(1) $\underline{\mathrm{Hom}}_R(X, \Sigma Y) \cong \mathrm{Ext}^1_R(X,Y)$

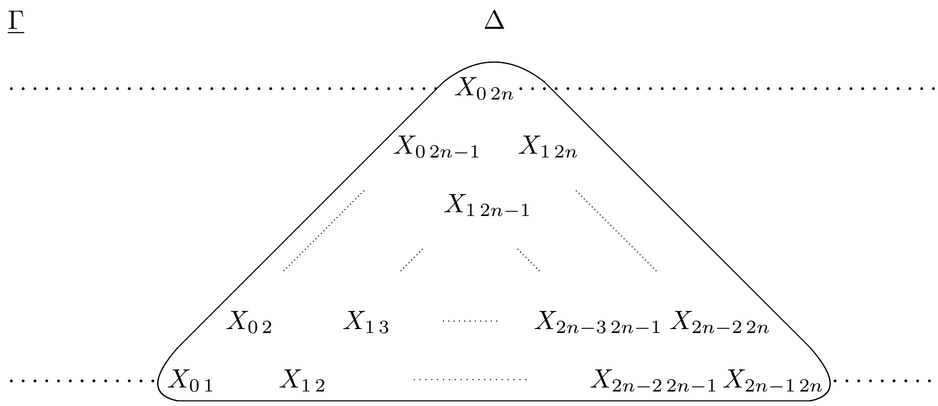
(2) *The following conditions are equivalent*

  (a) $\underline{\mathrm{Hom}}_R(X, \Sigma^l Y) \neq 0$,

  (b) $\Sigma^{-1} X \preceq \Sigma^{l-1} Y \preceq \tau X$ *in* $\underline{\Gamma}$,

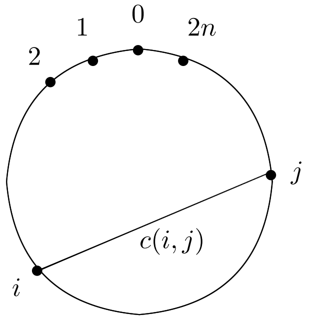
  (c) $\Sigma^{-l} X \preceq Y \preceq \tau \Sigma^{-l+1} X$ *in* $\underline{\Gamma}$.

We fix a triangle area Δ in $\underline{\Gamma}$ and add an index for each maximal Cohen-Macaulay modules as follows:



In this situation, for any $X \in \mathrm{ind}\,\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$, there exists i, j and l such that $X \cong \Sigma^l X_{i\,j}$. Therefore we get $\mathrm{ind}\,\underline{\mathrm{CM}}^{\mathbb{Z}}(R)/\Sigma \cong \Delta = \{\ X_{i\,j} \mid 0 \leq i < j \leq 2n\ \}$.

We consider the circle with $2n+1$ points labeled $0, 1, 2, \cdots, 2n$ counter clockwise on it. We put $c(i,j)(= c(j,i))$ the chord whose end points are i and j. We denote by $C_{2n} = \{c(i,j) \mid 0 \leq i < j \leq 2n\}$ the set of chords.



Note that the map $F : \operatorname{ind} \underline{\mathrm{CM}}^{\mathbb{Z}}(R) \ni \Sigma^l X_{i\,j} \mapsto c(i,j) \in C_{2n}$ is bijection. F gives a nice following properties.

Lemma 1.6. *Let $X, X' \in \operatorname{ind} \underline{\mathrm{CM}}^{\mathbb{Z}}(R)$.*

(1) *The following conditions are equivalent:*
  (a) $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X, \Sigma^l X') = 0$ *and* $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X', \Sigma^l X) = 0$.
  (b) $F(X)$ *does not meet* $F(X')$.
(2) *The following conditions are equivalent:*
  (a) $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X, \Sigma^l X') \neq 0$ *and* $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X', \Sigma^l X) = 0$.
  (b) $F(X)$ *meets* $F(X')$ *at the end point of chords.*
(3) *The following conditions are equivalent:*
  (a) $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X, \Sigma^l X') \neq 0$ *and* $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X', \Sigma^l X) \neq 0$.
  (b) $F(X)$ *meets* $F(X')$ *at interior of the circle.*

2. Main result

The following theorem is the main result of this talk.

Theorem 2.1. *For thick subcategory $\mathcal{X}$ in $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$, we set $\mathcal{F}(\mathcal{X}) = \{F(X) \mid X \in \operatorname{ind}\mathcal{X}\}$. Then there exists a following one-to-one correspondence:*

$$\begin{gathered}\{\textit{thick subcategories of } \underline{\mathrm{CM}}^{\mathbb{Z}}(R)\} \\ \mathcal{F} \downarrow\ \uparrow \\ \{\textit{disjoint union of complete subgraphs in } C_{2n}\}\end{gathered}$$

Proof. Let $X, X' \in \operatorname{ind} \underline{\mathrm{CM}}^{\mathbb{Z}}(R)$. We set $\mathcal{X}_{XX'}$ the smallest thick subcategory of $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$ which contains X and X'.

Case 1. The case of $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X, \Sigma^l X') = 0$ and $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X', \Sigma^l X) = 0$.

By Lemma 1.6 (1), we have $F(X)$ does not meet $F(X')$. Thus, we can easily check that $\mathcal{X}_{XX'} = \operatorname{add}\{\Sigma^l X,\ \Sigma^l X' \mid l \in \mathbb{Z}\ \}$ and $\mathcal{F}(\mathcal{X}_{XX'}) = \{\ F(X),\ F(X')\ \}$.

Case 2. $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X, \Sigma^l X') \neq 0$ and $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X', \Sigma^l X) = 0$.

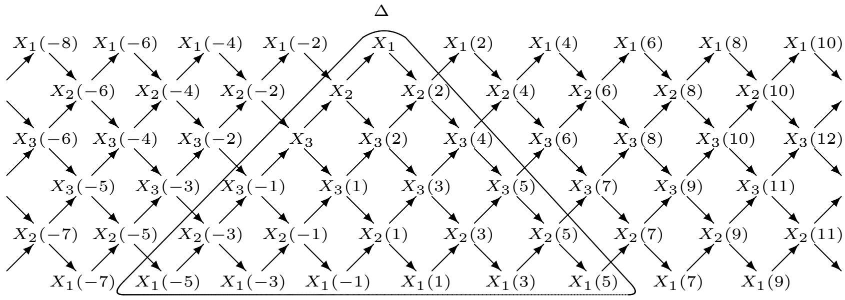
There exist integers $0 \leq i < j < t \leq 2n$ such that $\{\ F(X),\ F(X')\ \} = \{\ c(i,j),\ c(i,t)\ \}$, $\{\ c(i,j),\ c(j,t)\ \}$ or $\{\ c(i,t),\ c(j,t)\ \}$ by Lemma 1.6 (2). Thanks to Proposition 1.3, $X_{i\,j} \to X_{i\,t} \to X_{j\,t} \to \Sigma X_{i\,j}$ is an exact triangle. Thus, we have $\mathcal{X}_{XX'} = \operatorname{add}\{\Sigma^l X_{i\,j},\ \Sigma^l X_{i\,t},\ \Sigma^l X_{j\,t} \mid l \in \mathbb{Z}\ \}$ by Lemma 1.2 (2) and Proposition 1.5. Note that $\mathcal{F}(\mathcal{X}_{XX'})$ gives a triangle.

Case 3. $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X, \Sigma^l X') \neq 0$ and $\bigoplus_{l\in\mathbb{Z}} \underline{\mathrm{Hom}}_R(X', \Sigma^l X) \neq 0$.

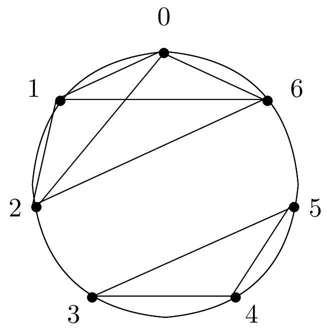
Lemma 1.6 (3) says that there exist integers $0 \leq i < s < j < t \leq 2n$ such that $\{\ F(X),\ F(X')\ \} = \{\ c(i,j),\ c(s,t)\ \}$. It comes from Proposition 1.3, we see that $X_{i\,j} \to X_{i\,t} \oplus X_{s\,j} \to X_{s\,t} \to \Sigma X_{i\,j}$ is an exact triangle. This gives that $\mathcal{X}_{XX'}$ contains $X_{i\,t}$ and $X_{s\,j}$. Moreover, $\mathcal{X}_{XX'}$ contains $X_{i\,s}$ and $X_{j\,j}$ by Case 2. Thus we have $\mathcal{X}_{XX'} = \operatorname{add}\{\Sigma^l X_{a\,b} \mid l \in \mathbb{Z},\ a, b \in \{i, s, j, t\}\ \}$ Lemma 1.2 (2) and Proposition 1.5. We remark that $\mathcal{F}(\mathcal{X}_{XX'})$ is a complete 4 graph.

For any thick subcategory $\mathcal{X}$ in $\underline{\mathrm{CM}}^{\mathbb{Z}}(R)$, we remark that $\mathcal{X} = \operatorname*{add}_{X, X' \in \operatorname{ind}\mathcal{X}} \mathcal{X}_{XX'}$. This yields a proof. $\square$

Example 2.2. Let $R = k[t^2, t^7]$. We shall find the smallest thick subcategory $\mathcal{X}$ which contains $X_1(8)$, $X_2(-3)$, $X_2(2)$ and $X_2(3)$. One can check that $\Sigma X = X(7)$ for any $X \in \underline{\mathrm{CM}}^{\mathbb{Z}}(R)$. Hence, $\mathcal{X}$ contains $\{X_1(8+7l), X_2(-3+7l), X_2(2+7l), X_2(3+7l) \mid l \in \mathbb{Z}\}$. We set a triangle area Δ with $X_{0\,1} = X_1(-5)$, $X_{0\,6} = X_1$ and $X_{5\,6} = X_1(5)$.



In this situation, we see that $\mathcal{X}$ contains $X_{3\,4} = X_1(1)$, $X_{0\,2} = X_2(-3)$, $X_{1\,6} = X_2(2)$ and $X_{3\,5} = X_2(3)$. On the other hand, $\mathcal{F}(\mathcal{X})$ is the smallest graph in C_{2n} such that $\mathcal{F}(\mathcal{X})$ is a disjoint union of complete subgraphs by Theorem 2.1 and that $\mathcal{F}(\mathcal{X})$ contains $c(3,4)$, $c(0,2)$, $c(1,6)$ and $c(3,5)$. Therefore on e can see that $\mathcal{F}(\mathcal{X}) = \{\ c(0,1),\ c(0,2),\ c(0,6),\ c(1,2),\ c(1,6),\ c(2,6),\ c(3,4),\ c(3,5),\ c(4,5)\ \}$.



Thus we have $\mathcal{X} = \mathrm{add}\,\{\ X_1(-5+7l), X_1(-3+7l), X_1(7l), X_1(1+7l), X_1(3+7l), X_2(-3+7l), X_2(2+7l), X_2(3+7l), X_3(4+7l) \mid l \in \mathbb{Z}\ \}$.

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