On the characterizations of cofinite complexes*

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1 Introduction

Our aim in this report is to make a summary of our results in [15]. Mainly we will introduce the following theorems:

Theorem 1 *Let A be a ring, which is a homomorphic image of a Gorenstein ring of finite Krull dimension, J an ideal of A. Suppose that A is complete with respect to a J-adic topology. Let $N^\bullet$ be a complex of A-modules in $\mathcal{D}^+(A)$, where $\mathcal{D}^+(A)$ is the derived category consisting of complexes bounded below. If the ideal J is of dimension one, then the complex $N^\bullet$ is J-cofinite if and only if $H^i(N^\bullet)$ is in $\mathcal{M}(A, J)_{cof}$ for all i, where $\mathcal{M}(A, J)_{cof}$ is the category of J-cofinite modules (see Definition 3 below).*

Theorem 2 *Let A and $N^\bullet$ be as in Theorem 1. Let J be an ideal of A. If J is principal up to radical, then the complex $N^\bullet$ is J-cofinite if and only if $H^i(N^\bullet)$ is in $\mathcal{M}(A, J)_{cof}$ for all i.*

In [9, Theorem 2, p. 588], the following result was proved (see also [4, Theorem 2, p. 537] for the result over complete local Gorenstein domains):

Theorem 3 (Theorem 2 of [9]) *Let R be a regular local ring complete with respect to an I-adic topology, where I is an ideal of R. Let $N^\bullet$ be in the derived category $\mathcal{D}^+(R)$. If the ideal I is of dimension one, then $N^\bullet$ is I-cofinite if and only if $H^i(N^\bullet)$ is in $\mathcal{M}(R, I)_{cof}$ for all i.*

Theorem 1 extends Theorem 3 to the result over a homomorphic image of a Gorenstein ring A of finite Krull dimension, where A is complete with respect to the J-adic topology. The same result for principal ideals is given as Theorem 2 in this note. It is well known

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that there is a counter example (cf. [7, § 3, An example, p. 149]). The ring R and the ideal I of the example are as follows:

$$R = k[x,y][[u,v]], \qquad I = (u,v).$$

The ring R is the formal power series ring over the polynomial ring $k[x,y]$. Notice that the ideal I is of dimension two and generated by two elements in R. Further the ring R is complete with respect to the ideal I.

2 Preliminaries

We shall collect the basic definitions in this section. Let A be a ring and $\mathcal{A}$ an abelian category.

First we introduce definitions on derived categories. In this report we mainly follow the notation by those of [8] (see also [17] and [5]):

$$\begin{aligned} \mathcal{M}(A) &: \text{ the category of } A\text{-modules}, \\ C^*(\mathcal{A}) &: \text{ the category of complexes consisting of objects in } \mathcal{A}, \\ K^*(\mathcal{A}) &: \text{ the homotopy category}, \\ \mathcal{D}^*(\mathcal{A}) &: \text{ the derived category}, \end{aligned}$$

where * stands for $+$, $-$, b or $\emptyset$.

Following the notation in [7, p. 149], we denote by $K^*_{ft}(A)$ (respectively $\mathcal{D}^*_{ft}(A)$) the homotopy category (respectively the derived category) consisting of all complexes whose cohomologies are A-modules of finite type.

We mainly treat the derived functor $\mathbb{R}\mathrm{Hom}^\bullet(\;-\;,\;-\;)$ on $\mathcal{D}^-(A)^\circ \times \mathcal{D}^+(A)$ in this report, following [8, p. 65]. On the other hand, the functor $\mathbb{R}\mathrm{Hom}^\bullet$ (respectively $\;-\otimes^{\mathbb{L}}-\;$) are defined over unbounded complexes by [20, Theorem C]. Here this functor is denoted by $\underset{=}{R}\,\mathrm{Hom}^\bullet$ (respectively $\;-\underset{=}{\otimes}-\;$) in [8]. In particular, for a bounded complex $\mathbf{D}^\bullet$ in $\mathcal{D}^b(A)$, we can handle $\mathbb{R}\mathrm{Hom}^\bullet(\;-\;,\mathbf{D}^\bullet)$ on $\mathcal{D}(A)^\circ$, which sometimes occurs in this report.

Before defining the J-cofiniteness on complexes, we introduce the following definition (cf. [17, § 4.3, p. 70]):

Definition 1 *Let A be a ring, equipped with a dualizing complex $\mathbf{D}^\bullet$ for A and J an ideal of A (see [19] for the definition of the dualizing complex). We denote by $D_J(\;-\;)$ the functor $\mathbb{R}\mathrm{Hom}^\bullet(\;-\;,\mathbb{R}\Gamma_J(\mathbf{D}^\bullet))$ on the derived category $\mathcal{D}(A)$. In this paper, we call this functor $D_J(\;-\;)$ the* J**-dualizing functor** *(cf. [17, p. 70]). Note that the J-dualizing functor is defined over unbounded complexes by [20, Theorem C], as we mentioned above.*

The J-cofiniteness on *complexes* is defined as follows (see [7, §2, p. 149] for the definition over regular rings):

Definition 2 *Let A be a ring, equipped with a dualizing complex $\mathbf{D}^\bullet$ for A and J an ideal of A. Let $N^\bullet$ be an object of the derived category $\mathcal{D}(A)$. We say $N^\bullet$ is* J**-cofinite***, if there exists $M^\bullet \in \mathcal{D}_{ft}(A)$, such that $N^\bullet \simeq D_J(M^\bullet)$ in $\mathcal{D}(A)$. Here $D_J(\;-\;)$ is the J-dualizing functor on $\mathcal{D}(A)$ defined as above.*

The J-cofiniteness on *modules* is defined as follows (cf. [7, p. 148] and [7, p. 159]):

Definition 3 *Let A be a ring, and J an ideal of A. We denote by $\mathcal{M}(A, J)_{cof}$ the full subcategory of $\mathcal{M}(A)$ consisting of all A-modules N satisfying the conditions*

$$(*) \qquad \mathrm{Supp}_A(N) \subseteq V(J) \quad \text{and}$$

$$\mathrm{Ext}^j_A(A/J, N) \quad \text{is of finite type, for all } j.$$

An A-module in $\mathcal{M}(A, J)_{cof}$ is called J-cofinite.

3 Some results

We propose the following results. To prove Theorem 4, we proceed by the standard argument.

Theorem 4 *Let A be a ring equipped with a dualizing complex $\mathbf{D}^\bullet$ for A, and J an ideal of A. Then there is a spectral sequence:*

$$E_2^{p,q} = H_J^q(H^p(\mathbf{D}^\bullet)) \Longrightarrow H^n = H^n(\Gamma_J(\mathbf{D}^\bullet)),$$

where $H_J^p(H^q(\mathbf{D}^\bullet))$ denotes the p-th local cohomology module for the ideal J of the q-th cohomology module $H^q(\mathbf{D}^\bullet)$ of the dualizing complex $\mathbf{D}^\bullet$ for A.

Lemma 5 *Let A be a ring, J an ideal of A, and $\mathcal{M}(A)$ the category of A-modules. Let $\mathcal{M}(A, J)_{cof}$ be the full subcategory of $\mathcal{M}(A)$ consisting of J-cofinite modules. Suppose that there is a convergent spectral sequence of A-modules:*

$$E_2^{p,q} \Longrightarrow H^{p+q}$$

in the first quadrant. The the following hold:

(a) *If $\mathcal{M}(A, J)_{cof}$ is an Abelian full subcategory of $\mathcal{M}(A)$ and the A-module $E_2^{p,q}$ is J-cofinite for each p, q, then the A-module H^n is J-cofinite for each n;*

(b) *If $E_2^{p,q}$ is an A-module of finite type for each p, q, then H^n is an A-module of finite type for each n.*

4 The key result

The following theorem is a key result to prove our main theorems, which is a variant of Hartshorne's result. The following holds without the conditions (i) and (ii), provided A is a regular ring of finite dimension (cf. [7, Theorem 5.1, p. 154]). We will modify [7, Theorem 5.1] as follows.

Theorem 6 *Let A be a ring equipped with a dualizing complex $\mathbf{D}^\bullet$, which is complete with respect to a J-adic topology, where J is an ideal of A. Let $N^\bullet$ be a bounded-below complex consisting of A-modules. Suppose that the following conditions* (i) *and* (ii) *are satisfied:*

(i) *$\mathcal{M}(A,J)_{cof}$ is an Abelian full subcategory of $\mathcal{M}(A)$, and*

(ii) *the local cohomology module $H^p_J(W)$ is J-cofinite for each p and each A-module W of finite type.*

Then the following conditions are equivalent:

(1) *The complex $N^\bullet$ is J-cofinite.*

(2) *The complex $N^\bullet$ satisfies the following conditions:*

  a) $\mathrm{Supp} H^i(N^\bullet) \subseteq V(J)$ *for each i,*

  b) $\mathrm{Ext}^j(A/J, N^\bullet)$ *is of finite type for each j.*

Proof. The theorem is shown by modifying the proof in [7, Theorem 5.1] with the above results in section three and the Affine duality theorem.

Example 1 *Let N be a module over a complete local Gorenstein ring $(A, \mathfrak{m}, k)$ of dimension d and J an ideal of A. Suppose that N is J-cofinite as a complex. Then there is a complex $M^\bullet$ of $\mathcal{D}_{ft}(A)$, such that $N \simeq D_J(M^\bullet)$ in $\mathcal{D}(A)$ by definition. In this case, the j-th cohomology module of $M^\bullet$ is $\mathrm{Ext}^j_A(N, A)$ for each j. Indeed, let $I^\bullet$ be an injective resolution of A. Then $N \simeq D_J(M^\bullet) = \mathbb{R}\mathrm{Hom}^\bullet(M^\bullet, \Gamma_J(I^\bullet))$. Note that $\mathrm{Supp}_A(N) \subset V(J)$. By the affine duality theorem, $M^\bullet \simeq D_J(D_J(M^\bullet)) = D_J(N) = \mathbb{R}\mathrm{Hom}^\bullet(N, \Gamma_J(I^\bullet)) = \mathbb{R}\mathrm{Hom}^\bullet(N, I^\bullet)$, which is a bounded complex of length $\leq d+1$. So we have the equalities of the cohomology modules: $H^j(M^\bullet) = H^j(\mathbb{R}\mathrm{Hom}^\bullet(N, I^\bullet)) = H^j(\mathrm{Hom}(N, I^\bullet)) = \mathrm{Ext}^j_A(N, A)$ for each j. Consequently, if the module N is J-cofinite as a complex then $\mathrm{Ext}^j_A(N, A)$ is of finite type over A for each j.*

5 Outline of the proofs of our theorems

To prove one implication of Theorem 1 that if $N^\bullet$ is J-cofinite for all q, then $H^q(N^\bullet)$ is in $\mathcal{M}(A,J)_{cof}$ for all q, we will divide the proof into several steps. First we note that A has a dualizing complex $\mathbf{D}^\bullet$ for A, since A is a homomorphic image of Gorenstein ring of finite dimension. So we can define the J-dualizing functor $D_J(\,-\,)$ on $\mathcal{D}(A)$. We proceed by the three steps below:

(Step 1) First we prove that the cohomology module $H^n(\Gamma_J(\mathbf{D}^\bullet))$ is J-cofinite for all n,

(Step 2) Next, we prove that $H^n(D_J(M))$ is J-cofinite for all n and for all A-module M of finite type,

(Step 3) Finally we prove that $H^q(N^\bullet)$ is in $\mathcal{M}(A,J)_{cof}$ for all q. Therefore the proof of one implication of the theorem is done.

The converse statement of Theorem 1 follows form the following spectral sequence:

$$E_2^{p,q} = \mathrm{Ext}_A^p(A/J, H^q(N^\bullet)) \Longrightarrow H^{p+q} = \mathrm{Ext}^{p+q}(A/J, N^\bullet),$$

combined with part (b) of Lemma 5, [2, Theorem 1.1], [18, Theorem 2.6, p. 461] and Theorem 6. □

Remark 1 *The proof of Theorem 2 is straightforward, by repeating the same argument as in the proof of Theorem 1.*

With recent results (cf. [2, Theorem 2.6, p. 2001], [1, Theorem 1.1, p. 508] and [18, Theorem 2.6, p. 461]), we can give a short proof of the theorem below due to Cuong, Goto and Hoang. In [3, Theorem 1.2], the stronger result is proved by them, namely 'If the generalized local cohomology module $H_J^j(M,N)$ has dimension one for all j, then $H_J^j(M,N)$ is J-cofinite for all j'.

Theorem 7 *Let M, N be of finite type over a ring R. If the ideal J is of dimension one (or principal), then the generalized local cohomology module $H_J^j(M,N)$ is J-cofinite for all j.*

Here the j-th *generalized local cohomology module* is defined by

$$H_J^j(M,N) = \lim_{\substack{\rightarrow \\ \alpha}} \mathrm{Ext}_R^j(M/J^\alpha M, N),$$

where j is a non-negative integer. For the proof (cf. [16, Theorem 1]), we use the following spectral sequence:

$$E_2^{p,q} = \mathrm{Ext}_R^p(M, H_J^q(N)) \Longrightarrow H^{p+q} = H_J^{p+q}(M,N),$$

from the Grothendieck spectral sequence.

Example 2 *Let A be a ring as in Theorem 1, J an ideal of A of dimension one or principal, and M a module of finite type over A. Let $\mathbf{D}^\bullet$ be a dualizing complex for A. Suppose that A is complete with respect to a J-adic topology. Consider the n-th local cohomology module $H_J^n(M)$ for some integer n. Then the cohomology module of the complex $\mathbb{R}\mathrm{Hom}^\bullet(H_J^n(M), \mathbf{D}^\bullet)$ is of finite type over A for each j (replace N with $H_J^n(M)$ in Example 1).*

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