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序

これは第 28 回可換環論シンポジウムの報告集です．シンポジウムは 2006 年 11 月 13 日から 11 月 16 日の日程で，ウェルサンピア多摩に於いて開催されました。今回は，Claudia Polini 氏 (Nortre Dame 大学) と Alberto Corso 氏 (Kentucky 大学) を招待講演者として迎えました．総勢 72 名の参加の下，24 の講演が行われ，大変有意義なものでした．

シンポジウムの開催にあたり，平成 18 年度科学研究費補助金基盤研究 B (代表者：吉野雄二)，平成１７年度文部科学省「魅力ある大学院教育」イニシアティブ採択プログラム「社会との関りを重視した MTS 数理科学教育」，2006 年度明治大学科学技術研究所重点研究費 A (代表者：後藤四郎) からの援助を頂きました．ここにあらためてお礼申し上げます．

後藤 四郎，蔵野 和彦，中村 幸男，鴨井 祐二 (明治大学)
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Contents

An example of a local ring R such that the kernel of the map from $G_0(R)$ to $G_0(\hat{R})$ is not torsion

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This is a joint work with V. Srinivas from Tata Institute, India.

1 Introduction

For a Noetherian ring R, we set

$$G_0(R) = \frac{\bigoplus_{M:\text{ f. g. } R\text{-mod.}} \mathbb{Z}[M]}{\langle [L] + [N] - [M] \,|\, 0 \to L \to M \to N \to 0 \text{ is exact} \rangle},$$

that is called the *Grothendieck group* of finitely generated R-modules.

For a flat ring homomorphism $f : R \to A$, we have the induced map $G_0(R) \to G_0(A)$ defined by $[M] \mapsto [M \otimes_R A]$.

We are intersted in the following problem:

Problem 1 Let R be a Noetherian local ring. Is the map $G_0(R)_{\mathbb{Q}} \to G_0(\widehat{R})_{\mathbb{Q}}$ injective?

Let me explain motivation and applications.

Assume that S is a regular scheme and X is a scheme of finite type over S. Then, by the singular Riemann-Roch theorem [2], we obtain an isomorphism $\tau_{X/S} : G_0(X)_{\mathbb{Q}} \xrightarrow{\sim} A_*(X)_{\mathbb{Q}}$, where $G_0(X)$ (resp. $A_*(X)$) is the *Grothendieck group* of coherent sheaves on X (resp. *Chow group* of X). The map $\tau_{X/S}$ usually depends on the choice of S. In fact, for a field k, we have

$$\begin{aligned} \tau_{\mathbb{P}^1_k/\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k}) &= [\mathbb{P}^1_k] \in A_*(\mathbb{P}^1_k)_{\mathbb{Q}} = \mathbb{Q}[\mathbb{P}^1_k] \oplus \mathbb{Q}[t] \\ \tau_{\mathbb{P}^1_k/\mathrm{Spec}(k)}(\mathcal{O}_{\mathbb{P}^1_k}) &= [\mathbb{P}^1_k] + \chi(\mathcal{O}_{\mathbb{P}^1_k})[t] \\ &= [\mathbb{P}^1_k] + [t] \in A_*(\mathbb{P}^1_k)_{\mathbb{Q}}, \end{aligned}$$

where t is a rational closed point of $\mathbb{P}^1_k$. Hence, $\tau_{\mathbb{P}^1_k/\mathbb{P}^1_k}(\mathcal{O}_{\mathbb{P}^1_k}) \neq \tau_{\mathbb{P}^1_k/\mathrm{Spec}(k)}(\mathcal{O}_{\mathbb{P}^1_k})$. However, for a local ring R which is a homomorphic image of a regular local ring T, the map $\tau_{\mathrm{Spec}(R)/\mathrm{Spec}(T)}$ is sometimes independent of the choice of T. In fact, if R is a complete local ring or R is essentially of finite type over either a field or the ring of integers, it is proved in [5] that the map $\tau_{\mathrm{Spec}(R)/\mathrm{Spec}(T)}$ is independent of T.

From now on, we simply denote $\tau_{\mathrm{Spec}(R)/\mathrm{Spec}(T)}$ by $\tau_{R/T}$. It is natural to ask;

Problem 2 Is the map $\tau_{R/T}$ always independent of T?

Remark that the diagram

$$\begin{array}{ccc} G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_{R/T}} & A_*(R)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ G_0(\widehat{R})_{\mathbb{Q}} & \xrightarrow{\tau_{\widehat{R}/\widehat{T}}} & A_*(\widehat{R})_{\mathbb{Q}} \end{array}$$

is commutative, where the vertical maps are induced by the completion $R \to \widehat{R}$. We want to emphasise that the bottom map and the vertical maps are independent of the choice of T. Therefore, if the vertical maps are injective, then the top map is also independent of T.

Therefore, if Problem 1 is true, then so is Problem 2.

Roberts [7] and Gillet-Soulé [3] proved the vanishing theorem of intersection multiplicity for local rings which are complete intersection. If a local ring R is complete intersection, then $\tau_{R/T}(R) = [\mathrm{Spec}(R)]$ is satisfied. In [7], Roberts proved the vanishing property not only for complete intersections but also for local rings with $\tau_{R/T}(R) = [\mathrm{Spec}(R)]$. Inspired by his work, Kurano [5] started to study a local ring which satisfies $\tau_{R/T}(R) = [\mathrm{Spec}(R)]$, and call it a *Roberts ring*. If R is a Roberts ring, then the completion, the henselization and localizations of it are also Roberts rings [5]. However, the following problem remained open.

Problem 3 Is R a Roberts ring if so is $\widehat{R}$?

It is proved in [4] that Problem 1 is equivalent to Problem 3.

The following partial result on Problem 1 was given by [4]:

Theorem 4 (Kamoi-K, 2001 [4]) *Let R be a homomorphic image of an excellent regular local ring. Assume that R satisfies one of the following three conditions:*
(i) R is henselian,
(ii) $R = S_{S_+}$, where S is a standard graded ring over a field,
(iii) R has only isolated singularity.
Then, the induced map $G_0(R) \to G_0(\widehat{R})$ is injective.

The following example was given by Hochster.

Example 5 (Hochster) *Let k be a field. We set*

$$\begin{aligned} T &= k[x,y,u,v]_{(x,y,u,v)}, \\ P &= (x,y), \\ f &= xy + ux^2 + vy^2. \end{aligned}$$

Then, $\mathrm{Ker}(G_0(T/fT) \to G_0(\widehat{T/fT})) \ni [T/P] \neq 0$. *However,* $2 \cdot [T/P] = 0$ *in this case.*

Unfortunately, it is not a counterexample of Problem 1. The following is the main theorem.

Theorem 6 *There exists a 2-dimensional local ring B (which is essentially of finite type over $\mathbb{C}$) that satisfies*

$$\mathrm{Ker}(G_0(B)_{\mathbb{Q}} \to G_0(\widehat{B})_{\mathbb{Q}}) \neq 0.$$

Remark 7

1. *By Theorem 6, we know that Problem 1 and Problem 3 are negative.*
2. *Problem 2 is still open.*
3. *In [6], we defined the notion of numerical equivalence for $G_0(R)$ and $A_*(R)$. We set $\overline{G_0(R)} = G_0(R)/\sim_{\mathrm{num.}}$. Then,*
   (a) *$\overline{G_0(R)} \to \overline{G_0(\widehat{R})}$ is injective for any local ring R.*
   (b) *R is a numerically Roberts ring iff so is $\widehat{R}$. (The notion of a numerically Roberts ring is defined in [6].)*
   (c) *The induced map $\overline{\tau_{R/T}} : \overline{G_0(R)_{\mathbb{Q}}} \xrightarrow{\sim} \overline{A_*(R)_{\mathbb{Q}}}$ is independent of T.*
4. *The ring B constructed in the main theorem is not normal. We do not know any example of a normal local ring that does not satisfy Problem 1.*

2 Proof of the main theorem

We give an outline of the proof of the theorem.

The main theorem immediately follows from the following two lemmas.

Lemma 8 *Let K be an algebraically closed field, and let $S = \oplus_{n \geq 0} S_n$ be a standard graded ring over K, that is, a Noetherian graded ring generated by S_1 over $S_0 = K$. We set $X = \mathrm{Proj}(S)$, and assume that $\dim X \geq 1$ and X is smooth over K. Let h be the very ample divisor on X of this embedding. Let $\pi : Y \to \mathrm{Spec}(S)$ be the blow-up at $S_+ = \oplus_{n>0} S_n$.*

Assume the following two conditions:

1. $A_1(R)_{\mathbb{Q}} \xrightarrow{\sim} A_1(\widehat{R})_{\mathbb{Q}}$ *for* $R = S_{S_+}$.

2. A smooth curve C *in* Y *intersects with* $\pi^{-1}(S_+) \simeq X$ *at two points transversally. We denote the two points by* α_1 *and* α_2. *Here, they satisfies* $[\alpha_1] - [\alpha_2] \neq 0$ *in* $A_0(X)_{\mathbb{Q}}/h \cdot A_1(X)_{\mathbb{Q}}$.

Then, there exists a 2*-dimensional local ring* B *(which is essentially of finite type over* K*) such that* $\mathrm{Ker}(G_0(B)_{\mathbb{Q}} \to G_0(\widehat{B})_{\mathbb{Q}}) \neq 0$.

Lemma 9 *We set* $S = \mathbb{C}[x_0, x_1, x_2]/(f)$, *where* f *is a homogeneous polynomial of degree* 3. *Assume that* $X = \mathrm{Proj}(S)$ *is smooth over* $\mathbb{C}$.

Then, R *satisfies the assumption in Lemma 8.*

In the rest of this section, we shall prove the above two Lemmas.

2.1 An outline of the proof of Lemma 8

Here, we shall give an outline of the proof of Lemma 8.

Let P be the prime ideal of S that satisfies $\mathrm{Spec}(S/P) = \pi(C)$. Set $R = S_{S_+}$.

Then, C is the normalization of $\mathrm{Spec}(S/P)$. We denote by v_i the normalized valuation of $\alpha_i \in C$ for $i = 1, 2$.

By the Riemann-Roch theorem for smooth projective curves, one can show that there exists $s \in R/PR$ such that

1. $v_1(s) = v_2(s) > 0$,
2. $\{\alpha_1, \alpha_2\}$ is just the set of zeros of s.

Then, the composite map $K[s]_{(s)} \to R/PR \to \overline{(R/PR)}$ is finite, where $\overline{(R/PR)}$ is the normalization of R/PR. Therefore, R/PR is finite over $K[s]_{(s)}$.

Let $R \xrightarrow{\xi} R/PR$ be the natural surjective map. We put $B = \xi^{-1}(K[s]_{(s)})$. Then, $B \hookrightarrow R$ is finite birational, and B is essentially of finite type over K with $\dim B = 2$. (In particular, B is a homomorphic image of a regular local ring T.)

Remark that the diagram

$$\begin{array}{ccc} G_0(B)_{\mathbb{Q}} & \xrightarrow{\tau_{B/T}} & A_*(B)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ G_0(\widehat{B})_{\mathbb{Q}} & \xrightarrow{\tau_{\widehat{B}/\widehat{T}}} & A_*(\widehat{B})_{\mathbb{Q}} \end{array}$$

is commutative and the horizontal maps are isomorphisms. Therefore, if $\mathrm{Ker}(A_*(B)_{\mathbb{Q}} \to A_*(\widehat{B})_{\mathbb{Q}})$ is not 0, then $\mathrm{Ker}(G_0(B)_{\mathbb{Q}} \to G_0(\widehat{B})_{\mathbb{Q}})$ is not 0.

It is sufficient to prove that $\mathrm{Ker}(A_1(B)_{\mathbb{Q}} \to A_1(\widehat{B})_{\mathbb{Q}})$ is not 0.

The diagram

$$\begin{array}{ccc} R & \longrightarrow & \widehat{R} \\ \uparrow & & \uparrow \\ B & \longrightarrow & \widehat{B} \end{array}$$

induces the commutative diagram

$$\begin{array}{ccc} A_1(R)_{\mathbb{Q}} & \longrightarrow & A_1(\widehat{R})_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ A_1(B)_{\mathbb{Q}} & \longrightarrow & A_1(\widehat{B})_{\mathbb{Q}} \end{array} \qquad (1)$$

where the vertical maps are induced by the finite morphisms $B \to R$ and $\widehat{B} \to \widehat{R}$, and the horizontal maps are induced by the completions $B \to \widehat{B}$ and $R \to \widehat{R}$.

The map in the left-hand-side in the diagram (1) is an isomorphism since, for each prime ideal of B, there is only one prime ideal of R lying over it.

The top map in the diagram (1) is also an isomorphism by the assumption 1 of Lemma 8.

Therefore, it is enough to prove that $\mathrm{Ker}(A_1(\widehat{R})_{\mathbb{Q}} \to A_1(\widehat{B})_{\mathbb{Q}})$ is not 0.

It is well-known that there exists the natural bijective correspondence between the set of maximal ideals of $\overline{(R/PR)}$ and the set of minimal prime ideals of $\widehat{R/PR} = \widehat{R}/P\widehat{R}$. Let $\mathfrak{p}_i$ be the minimal prime ideal of $P\widehat{R}$ corresponding to the maximal ideal α_i of $\overline{(R/PR)}$. Since $\mathfrak{p}_1 \cap \widehat{B} = \mathfrak{p}_2 \cap \widehat{B}$ and $v_1(s) = v_2(s)$, we have

$$[\mathrm{Spec}(\widehat{R}/\mathfrak{p}_1)] - [\mathrm{Spec}(\widehat{R}/\mathfrak{p}_2)] \in \mathrm{Ker}(A_1(\widehat{R})_{\mathbb{Q}} \to A_1(\widehat{B})_{\mathbb{Q}}).$$

It is enough to show

$$[\mathrm{Spec}(\widehat{R}/\mathfrak{p}_1)] - [\mathrm{Spec}(\widehat{R}/\mathfrak{p}_2)] \neq 0$$

in $A_1(\widehat{R})_{\mathbb{Q}}$.

Let $\widehat{\pi} : \widehat{Y} \longrightarrow \mathrm{Spec}(\widehat{R})$ be the blow-up at $S_+\widehat{R}$. Since $\widehat{\pi}^{-1}(S_+\widehat{R}) \simeq X$,

$$A_1(X)_{\mathbb{Q}} \xrightarrow{i_*} A_1(\widehat{Y})_{\mathbb{Q}} \xrightarrow{\widehat{\pi}_*} A_1(\widehat{R})_{\mathbb{Q}} \to 0$$

is exact and

$$\widehat{\pi}_*\left([\mathrm{Spec}(\overline{\widehat{R}/\mathfrak{p}_1})] - [\mathrm{Spec}(\overline{\widehat{R}/\mathfrak{p}_2})]\right) = [\mathrm{Spec}(\widehat{R}/\mathfrak{p}_1)] - [\mathrm{Spec}(\widehat{R}/\mathfrak{p}_2)].$$

Assume the contrary. Then, there exists $\delta \in A_1(X)_{\mathbb{Q}}$ such that

$$i_*(\delta) = [\mathrm{Spec}(\overline{\widehat{R}/\mathfrak{p}_1})] - [\mathrm{Spec}(\overline{\widehat{R}/\mathfrak{p}_2})].$$

Here, consider the map $A_1(\widehat{Y})_{\mathbb{Q}} \xrightarrow{i^!} A_0(X)_{\mathbb{Q}}$, that is taking the intersection with X. Since $i^! i_*(\delta) = -h \cdot \delta$ and

$$i^! \left([\overline{\mathrm{Spec}(\widehat{R}/\mathfrak{p}_1)}] - [\overline{\mathrm{Spec}(\widehat{R}/\mathfrak{p}_2)}] \right) = [\alpha_1] - [\alpha_2],$$

we have $[\alpha_1] - [\alpha_2] = -h \cdot \delta$. It contradicts to $[\alpha_1] - [\alpha_2] \neq 0$ in $A_0(X)_{\mathbb{Q}}/h \cdot A_1(X)_{\mathbb{Q}}$.

We have completed the proof of Lemma 8.

2.2 An outline of the proof of Lemma 9

We shall give a proof of Lemma 9.

Assume that $S = \mathbb{C}[x_0, x_1, x_2]/(f)$ and $X = \mathrm{Proj}(S)$ satisfy the assumption in Lemma 9. Let Z be the projective cone of X, that is, $Z = \mathrm{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3]/(f))$.

We set $X_\infty = V_+(x_3) \simeq X$. Let $W \xrightarrow{\xi} Z$ be the blow-up at $(0;0;0;1)$, and set $X_0 = \xi^{-1}((0;0;0;1)) \simeq X$. Then, $W \xrightarrow{\eta} X$ is a $\mathbb{P}^1$-bundle.

Take any two closed points $\beta_1, \beta_2 \in X$. We set $\ell_i = \eta^{-1}(\beta_i)$ for $i = 1, 2$. Consider the Weil divisor $\ell_1 + \ell_2 + X_\infty$ on W. Here we can prove the following claim.

Claim 10 *The complete linear system $|\ell_1 + \ell_2 + X_\infty|$ is base point free, and the induced morphism $W \xrightarrow{f} \mathbb{P}^n$ satisfies* $\dim f(W) \geq 2$.

The proof of the claim is omitted here.

For a general element $a \in H^0(W, \mathcal{O}(\ell_1 + \ell_2 + X_\infty))$, the Weil divisor

$$\mathrm{div}(a) + \ell_1 + \ell_2 + X_\infty$$

is smooth connected by the above claim. We denote it by V. Then, V is linearly equivalent to $\ell_1 + \ell_2 + X_\infty$, and we may assume that V intersects with X_0 at two points, namely α_1 and α_2, transversally. Furthermore, we may assume that $(\alpha_1) - (\alpha_2)$ is not torsion in the elliptic curve X, where $(\alpha_1) - (\alpha_2) \in X$ is the difference in the elliptic curve X.

Let Y be the blow-up of $\mathrm{Spec}(S)$ at the origin. Then, Y is an open subvariety of W. We set $C = V \cap Y$.

It is well-known that $X_{\mathbb{Q}} \simeq A_0(X)_{\mathbb{Q}}/h \cdot A_1(X)_{\mathbb{Q}}$ as a group. Under this isomorphism, $(\alpha_1) - (\alpha_2)$ corresponds to $[\alpha_1] - [\alpha_2]$. Therefore, $[\alpha_1] - [\alpha_2] \neq 0$ in $A_0(X)_{\mathbb{Q}}/h \cdot A_1(X)_{\mathbb{Q}}$, and $\{\alpha_1, \alpha_2\}$ satisfies the assumption 2 in Lemma 8.

Since $H^1(X, \mathcal{O}_X(n)) = 0$ for $n > 0$, we have $\mathrm{Cl}(R) \simeq \mathrm{Cl}(\widehat{R})$ by Danilov's Theorem [1]. Therefore, R satisfies the assumption 1 in Lemma 8.

We have completed the proof the Lemma 9.

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QUASI-SOCLE IDEALS IN A GORENSTEIN LOCAL RING

SHIRO GOTO, NAOYUKI MATSUOKA, AND RYO TAKAHASHI

ABSTRACT. This paper explores the structure of quasi-socle ideals $I = Q : \mathrm{m}^2$ in a Gorenstein local ring A, where Q is a parameter ideal and m is the maximal ideal in A. The purpose is to answer the problems of when Q is a reduction of I and when the associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is Cohen-Macaulay. Wild examples are explored.

1. INTRODUCTION

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring with $\dim A > 0$ and assume that $\mathrm{e}^0_{\mathfrak{m}}(A) \geq 3$, where $\mathrm{e}^0_{\mathfrak{m}}(A)$ denotes the multiplicity of A with respect to the maximal ideal $\mathfrak{m}$. Then for every parameter ideal Q in A, one has the following, where $I = Q : \mathfrak{m}^2$.*

(1) $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$ *and* $I^3 = QI^2$.
(2) *The associated graded ring* $\mathrm{G}(I)$ *of* I *and the fiber cone* $\mathrm{F}(I)$ *of* I *are both Cohen-Macaulay rings.*

Hence, the Rees algebra $\mathcal{R}(I)$ of I is also a Cohen-Macaulay ring, if $\dim A \geq 3$.

Here we define

$$\mathcal{R}(I) = A[IT] \quad \subseteq A[T],$$
$$\mathcal{R}'(I) = A[IT, T^{-1}] \quad \subseteq A[T, T^{-1}],$$
$$\mathrm{G}(I) = \mathcal{R}'(I)/T^{-1}\mathcal{R}'(I), \text{ and}$$
$$\mathrm{F}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I) \quad (\cong \mathrm{G}(I)/\mathfrak{m}\mathrm{G}(I))$$

with T an indeterminate over A.

Our theorem 1.1 is a generalization of the following result of A. Corso, C. Polini, C. Huneke, W. V. Vasconcelos, and the first author.

Theorem 1.2 ([CHV], [CP], [CPV], [G]). *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A > 0$. Let Q be a parameter ideal in A and let $I = Q : \mathfrak{m}$. Then the following three conditions are equivalent to each other.*

(1) $I^2 \neq QI$.
(2) $\overline{Q} = Q$, *that is the parameter ideal Q is integrally closed in A.*
(3) *A is a regular local ring which contains a regular system $x_1, x_2, \cdots, x_d$ of parameters such that $Q = (x_1, \cdots, x_{d-1}, x_d^q)$ for some integer $q > 0$.*

Consequently, if $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring which is not regular, then $I^2 = QI$ for every parameter ideal Q in A, so that $\mathrm{G}(I)$ and $\mathrm{F}(I)$ are both Cohen-Macaulay *rings, where* $I = Q : \mathfrak{m}$. *The Rees algebra* $\mathcal{R}(I)$ *is also a Cohen-Macaulay ring, if* $d = \dim A \geq 2$.

Our present research aims at a natural generalization of Theorem 1.2 but here we would like to note that there might be other directions of generalization. In fact, the equality $I^2 = QI$ in Theorem 1.2 remains true in certain cases, even though the base local rings A are not Cohen-Macaulay. For example, the first author and H. Sakurai investigated the case where A is a Buchsbaum local ring and gave the following.

Theorem 1.3 ([GSa2], cf. [GN]). *Let* $(A, \mathfrak{m})$ *be a Buchsbaum local ring and assume that either* $\dim A \geq 2$ *or* $\dim A = 1$ *but* $\mathrm{e}^0_{\mathfrak{m}}(A) \geq 2$. *Then there exists an integer* $n > 0$ *such that for every parameter ideal* Q *of* A *which is contained in* $\mathfrak{m}^n$, *one has the equality* $I^2 = QI$, *so that the graded rings* $\mathrm{G}(I)$ *and* $\mathrm{F}(I)$ *are Buchsbaum rings, where* $I = Q : \mathfrak{m}$.

See [GSa1, GSa3] for further developments of this direction.

We now explain how this paper is organized. Section 2 is devoted to some preliminary steps, which we will need later to prove Theorem 1.1. Theorem 1.1 will be proven in Section 3. Our method of proof is, unfortunately, applicable only to the case where the local ring A is Gorenstein and the situation seems totally different, unless A is Gorenstein. In order to show that the non-Gorenstein case of dimension 1 is rather wild, we shall explore three examples in the last section 4. One of them will show the quasi-socle ideals $I = Q : \mathfrak{m}^2$ are never integral over parameter ideals Q in certain Cohen-Macaulay local rings A of dimension 1, even though $\mathrm{e}^0_{\mathfrak{m}}(A) \geq 2$. The other two will show that unless A is a Gorenstein ring, one can not expect that $\mathrm{r}_Q(I) \leq 2$, even if I is integral over Q, where

$$\mathrm{r}_Q(I) = \min\{n \geq 0 \mid I^{n+1} = QI^n\}$$

denotes the reduction number of ideals $I = Q : \mathfrak{m}^2$ with respect to Q.

Unless otherwise specified, in what follows, let $(A, \mathfrak{m})$ be a Gorenstein local ring with $\dim A = d$. We denote by $\mathrm{e}^0_{\mathfrak{m}}(A)$ the multiplicity of A with respect to the maximal ideal $\mathfrak{m}$. Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A generated by the system $a_1, a_2, \cdots, a_d$ of parameters in A. For each finitely generated A-module M we denote by $\mu_A(M)$ and $\ell_A(M)$, respectively, the number of elements in a minimal system of generators for M and the length of M. Let $\mathrm{v}(A) = \ell_A(\mathfrak{m}/\mathfrak{m}^2)$ stand for the embedding dimension of A.

2. Preliminaries

Let A be a Gorenstein local ring with the maximal ideal $\mathfrak{m}$. The purpose of this section is to summarize some preliminaries, which we need in Section 3 to prove Theorem 1.1. Let us begin with the case where $\dim A = 0$.

Suppose that $\dim A = 0$. Let $n = \mathrm{v}(A) > 0$ and let $x_1, x_2, \cdots, x_n$ be a system of generators for $\mathfrak{m}$. We choose a socle element z in A. Hence $0 \neq z \in \mathfrak{m}$ and $\mathfrak{m}z = (0)$. Let $I = (0) : \mathfrak{m}^2$. We then have the following.

Lemma 2.1. *There exist elements $y_1, y_2, \cdots, y_n \in A$ such that $x_i y_j = \delta_{ij} z$ for all integers $1 \le i, j \le n$. We furthermore have the following.*

(1) *$I = (y_1, y_2, \cdots, y_n)$, $\mu(I) = n$, and $\ell_A(I) = n+1$.*
(2) *If $n > 1$, then $I \subsetneq A$.*

Proof. The existence of elements $y_1, y_2, \cdots, y_n$ is exactly the dual basis lemma. Let us note a brief proof for the sake of completeness. Let $1 \le j \le n$ be an integer. We look at the following diagram

$$\begin{tikzcd}
\mathfrak{m} \arrow[rrr, "\iota"] \arrow[d, "\varepsilon"] & & & A \arrow[d, dashed, "f=\widehat{y_j}"] \\
\mathfrak{m}/\mathfrak{m}^2 \arrow[r, "p"] & A/\mathfrak{m} \arrow[r, "h", "\sim"'] & (z) \arrow[r, "\iota"] & A
\end{tikzcd}$$

of A-modules, where ε is the canonical epimorphism, p is the projection map such that $p(\overline{x_i}) = \delta_{ij}$ for all $1 \le i \le n$ where $\overline{x_i} = x_i \bmod \mathfrak{m}^2$ denotes the image of x_i in $\mathfrak{m}/\mathfrak{m}^2$ and δ_{ij} is Kronecker's delta, h is the isomorphism of vector spaces over $A/\mathfrak{m}$ defined by $h(1) = z$, and ι's denote the embedding maps. Then, since the ring A is self-injective, we have a homothety map $f = \widehat{y_j} : A \to A$ with $y_j \in A$ such that the above diagram is commutative. Hence $x_i y_j = \delta_{ij} z$ for all integers $1 \le i, j \le n$. We put $J = (y_1, y_2, \cdots, y_n)$. Then $J \subseteq I = (0) : \mathfrak{m}^2$, because $\mathfrak{m} z = (0)$ and $x_i y_j = \delta_{ij} z$. We have $\ell_A(I) = n+1$, since

$$I \cong \mathrm{Hom}_A(A/\mathfrak{m}^2, A) \quad \text{and} \quad \ell_A(A/\mathfrak{m}^2) = n+1.$$

Therefore, to see that $I = J$, we have only to show $\ell_A(J) = n+1$, or equivalently $\ell_A(J/(z)) = n$. Let $\{b_j\}_{1 \le j \le n}$ be elements in A and assume that $\sum_{j=1}^{n} b_j y_j \in (z)$. Then

$$b_i z = b_i(x_i y_i) = x_i \cdot \sum_{j=1}^{n} b_j y_j = 0.$$

Hence $b_i \in \mathfrak{m}$. Thus the images of $\{y_j\}_{1 \le j \le n}$ in $J/(z)$ form a basis of the vector space $J/(z)$ over $A/\mathfrak{m}$, so that $\mu_A(J/(z)) = \ell_A(J/(z)) = n$. Hence $\ell_A(J) = n+1$ and assertion (1) follows. Assertion (2) is now obvious. $\square$

For the rest of this section we throughout assume that $d = \dim A > 0$. Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A generated by a system $a_1, a_2, \cdots, a_d$ of parameters for A and let $I = Q : \mathfrak{m}^2$. We assume $n = \mathrm{v}(A/Q) > 0$ and write $\mathfrak{m} = Q + (x_1, x_2, \cdots, x_n)$ with $x_i \in A$. Then $\mathfrak{m} I \subseteq Q : \mathfrak{m}$ and $\mathfrak{m} I \not\subseteq Q$ (recall that $Q \ne \mathfrak{m}$, since $n > 0$). Let us choose $z \in \mathfrak{m} I$ so that $z \notin Q$, whence

$$Q : \mathfrak{m} = Q + \mathfrak{m} I = Q + (z).$$

Then, applying Lemma 2.1 to the Artinian local ring A/Q, we get the elements $y_1, y_2, \cdots, y_n \in A$ such that $x_i y_j \equiv \delta_{ij} z \bmod Q$ for all integers $1 \le i, j \le n$. Hence

$$I = Q + (y_1, y_2, \cdots, y_n), \quad \mu_A(I/Q) = n, \quad \text{and} \quad \ell_A(I/Q) = n+1,$$

so that we have $\mu_A(I) \le n + d$.

We now look at the following inclusions

$$\begin{tikzcd}[column sep=small]
I \arrow[dd, dash, "n+1"'] \arrow[dr, dash, "\mu_A(I)\leq n+d"] & \\
& \mathrm{m}I \arrow[dd, dash, "1"] \\
Q \arrow[dd, dash, "d"'] & \\
& \mathrm{m}I\cap Q \arrow[dl, dash] \\
\mathrm{m}Q &
\end{tikzcd}$$

and notice that $[Q+\mathrm{m}I]/Q \cong \mathrm{m}I/[\mathrm{m}I\cap Q]$. Then $\ell_A(\mathrm{m}I/[\mathrm{m}I\cap Q]) = 1$ since $Q:\mathrm{m} = Q+\mathrm{m}I$, so that we have

$$\mu_A(I) = n+d \iff \mathrm{m}I\cap Q = \mathrm{m}Q.$$

We furthermore have the following.

Proposition 2.2. *Suppose that $n = \mathrm{v}(A/Q) > 1$. Then the following four conditions are equivalent to each other.*

(1) $I \subseteq \overline{Q}$.
(2) $\mathrm{m}I\cap Q = \mathrm{m}Q$.
(3) $\mu_A(I) = n+d$.
(4) $\mathrm{m}^2I = \mathrm{m}^2Q$.

Here $\overline{Q}$ denotes the integral closure of Q.

Proof. The implication (1) $\Rightarrow$ (2) is clear, since Q is a minimal reduction of I. The equivalence (2) $\iff$ (3) follows from the above observation.

(4) $\Rightarrow$ (1) This is well-known (cf. [NR]). Use the determinantal trick.

(2) $\Rightarrow$ (4) Because $z\in\mathrm{m}I\subseteq Q:\mathrm{m} = Q+(z)$, we get

$$\begin{aligned}\mathrm{m}I &= (\mathrm{m}I\cap Q)+(z)\\ &= \mathrm{m}Q+(z).\end{aligned}$$

Therefore, in order to see the equality $\mathrm{m}^2I = \mathrm{m}^2Q$, we have only to show that

$$\mathrm{m}z\subseteq\mathrm{m}^2Q.$$

Since $z\in\mathrm{m}I\subseteq\mathrm{m}^2$ (recall that $I\neq A$; cf. Lemma 2.1 (2)), we get $Qz\subseteq\mathrm{m}^2Q$. Hence, because $\mathrm{m} = Q+(x_1,x_2,\cdots,x_n)$, it suffices to show that $x_\ell z\in\mathrm{m}^2Q$ for every $1\leq\ell\leq n$. Choose an integer $1\leq i\leq n$ so that $i\neq\ell$ and write $z = x_iy_i+q_i$ with $q_i\in Q$. Then

$x_\ell z = x_i(x_\ell y_i) + x_\ell q_i$. Because $q_i = z - x_i y_i \in \mathfrak{m}I \cap Q = \mathfrak{m}Q$ and $x_\ell y_i \in \mathfrak{m}I \cap Q = \mathfrak{m}Q$, we certainly have $x_\ell z \in \mathfrak{m}^2 Q$. Thus $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$. $\square$

As a consequence of Proposition 2.2 we have the following.

Corollary 2.3. *Suppose that $n = \mathrm{v}(A/Q) > 1$ and that I is integral over Q. Then*

(1) *$Q^i \cap I^{i+1} = Q^i I$ for all integers $i \geq 1$. Hence $I^2 = QI$ if $I \subseteq \mathfrak{m}^2$.*
(2) *$(a_1) \cap I^2 = a_1 I$.*
(3) *$I^2 = QI$ if $Q \subseteq \mathfrak{m}^2$.*

Proof. (1) The second assertion follows from the first, since $I^2 \subseteq \mathfrak{m}^2 I \subseteq Q$. To see the first assertion, notice that $\mathfrak{m}^2 I^{i+1} = \mathfrak{m}^2 Q^{i+1}$, since $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$ by Proposition 2.2. Let $f \in Q^i \cap I^{i+1}$ and write

$$f = \sum_{i_1+i_2+\cdots+i_d=i} a_1^{i_1} a_2^{i_2} \cdots a_d^{i_d} f_{i_1 i_2 \cdots i_d}$$

with $f_{i_1 i_2 \cdots i_d} \in A$. Let $\alpha \in \mathfrak{m}^2$. We then have

$$\alpha f = \sum_{i_1+i_2+\cdots+i_d=i} a_1^{i_1} a_2^{i_2} \cdots a_d^{i_d} (\alpha f_{i_1 i_2 \cdots i_d}) \in \mathfrak{m}^2 I^{i+1} \subseteq Q^{i+1}.$$

Hence $\alpha f_{i_1 i_2 \cdots i_d} \in Q$ because $a_1, a_2, \cdots, a_d$ is an A-regular sequence, so that $f_{i_1 i_2 \cdots i_d} \in I$. Thus $f \in Q^i I$, whence $Q^i \cap I^{i+1} = Q^i I$.

(2) Let $f \in (a_1) \cap I^2$ and write $f = a_1 g$ with $g \in A$. Then for all $\alpha \in \mathfrak{m}^2$, we have $\alpha f = a_1(\alpha g) \in \mathfrak{m}^2 I^2 \subseteq Q^2$. Hence $\alpha g \in Q$ so that $g \in I$, and so $f \in a_1 I$. Thus $(a_1) \cap I^2 = a_1 I$.

(3) Let us prove the assertion by induction on d. Assume that $d = 1$. Let $b \in \mathfrak{m}^2$ be a non-zerodivisor in A. Then, thanks to the isomorphisms

$$[(b) : \mathfrak{m}^2]/(b) \cong \mathrm{Hom}_A(A/\mathfrak{m}^2, A/(b)) \cong \mathrm{Ext}_A^1(A/\mathfrak{m}^2, A)$$

of A-modules, we see the length $\ell_A([(b) : \mathfrak{m}^2]/(b)) = \ell_A(\mathrm{Ext}_A^1(A/\mathfrak{m}^2, A))$ is independent of the choice of the element $b \in \mathfrak{m}^2$. We put $a = a_1$. Let $Q' = (a^2)$ and $I' = Q' : \mathfrak{m}^2$. Let

$$\varphi : A/(a) \to A/(a^2)$$

be the monomorphism defined by $\varphi(\overline{x}) = \overline{ax}$, where $\overline{*}$ denote the images of the corresponding elements x and ax. Then $\varphi(I/(a)) = I'/(a^2)$, since $\varphi(I/(a)) \subseteq I'/(a^2)$ and $\ell_A(I/(a)) = \ell_A(I'/(a^2))$ (recall that $a \in \mathfrak{m}^2$). Therefore

$$(\sharp) \qquad I' = aI + (a^2) = aI,$$

whence $\mu_A(I') = \mu_A(I) = n + 1$, where the last equality follows from Proposition 2.2. Hence I' is also integral over Q' by Proposition 2.2, because $\mathrm{v}(A/Q') = \mathrm{v}(A) = \mathrm{v}(A/Q) = n > 1$. Therefore $(I')^2 = a^2 I'$ by assertion (1), since $I' \subseteq \mathfrak{m}^2$. Hence by equality ($\sharp$) we get $a^2 I^2 = (I')^2 = a^2 I' = a^3 I$, so that $I^2 = aI$.

Assume now that $d \geq 2$ and that our assertion holds true for $d - 1$. Let $\overline{A} = A/(a_1)$, $\overline{\mathfrak{m}} = \mathfrak{m}/(a_1)$, $\overline{Q} = Q/(a_1)$, and $\overline{I} = I/(a_1)$. Then $\overline{Q} : \overline{\mathfrak{m}}^2 = \overline{I}$, $\mathrm{v}(\overline{A}/\overline{Q}) = \mathrm{v}(A/Q) =$

$n > 1$, and $\overline{I}$ is integral over $\overline{Q}$. Hence the hypothesis of induction on d yields that $\overline{I}^2 = \overline{Q}\,\overline{I}$, since $\overline{Q} \subseteq \overline{\mathfrak{m}}^2$. Thus $I^2 \subseteq QI + (a_1)$. Therefore

$$I^2 = [QI + (a_1)] \cap I^2 = QI + [(a_1) \cap I^2] = QI + a_1 I = QI$$

by assertion (2). $\square$

Corollary 2.4. *Suppose* $\mathrm{v}(A/Q) > 1$ *and* I *is integral over* Q. *Then* $I \subseteq \mathfrak{m}^2$ *if* $Q \subseteq \mathfrak{m}^2$.

Proof. Suppose $Q \subseteq \mathfrak{m}^2$. Then $I^2 \subseteq Q$ since $I^2 = QI$ by Corollary 2.3 (3). On the other hand we have $Q : (Q : \mathfrak{m}^2) = \mathfrak{m}^2$, because Q is a parameter ideal in the Gorenstein local ring A. Hence $I \subseteq Q : I = Q : (Q : \mathfrak{m}^2) = \mathfrak{m}^2$ as is claimed. $\square$

Unless $Q \subseteq \mathfrak{m}^2$, the equality $I^2 = QI$ does not necessarily hold true. Let us note one example.

Example 2.5. Let $H = \langle 6, 7, 15 \rangle$ be the numerical semi-group generated by $6, 7, 15$ and let $A = k[[t^6, t^7, t^{15}]] \subseteq k[[t]]$, where $k[[t]]$ denotes the formal power series ring with one indeterminate t over a field k . Then A is a Gorenstein local ring with $\dim A = 1$. Let $0 < s \in H = \langle 6, 7, 15 \rangle$, $Q = (t^s)$ in A, and $I = Q : \mathfrak{m}^2$. Then I is integral over Q and $\mathrm{r}_Q(I) \leq 2$. However, $I^2 = QI$ if and only if $s \neq 7$.

Proof. Let $n \in H$. Then it is direct to check that $t^n \in I$ if and only if $n = s, s + 6, s + 7, s + 8$, or $s + \ell$ for some $12 \leq \ell \in \mathbb{Z}$. Thanks to this observation, we get $I = (t^s, t^{s+8}, t^{s+16}, t^{s+17})$ if $s \geq 12$ but $s \neq 15$. We also have $I = (t^6, t^{14}, t^{22})$ if $s = 6$, $I = (t^7, t^{15}, t^{24})$ if $s = 7$, and $I = (t^{15}, t^{31}, t^{32})$ if $s = 15$. Hence $I \subseteq t^s k[[t]] \cap A$, so that I is integral over $Q = (t^s)$, in any case. It is routine to check that $I^2 = QI$ when $s \neq 7$. If $s = 7$, then $I^3 = QI^2$ but $I^2 = QI + (t^{30})$ and $\ell_A(I^2/QI) = 1$, whence $I^2 \neq QI$. $\square$

Here let us note one example to clarify our arguments.

Example 2.6. Let $(A, \mathfrak{m})$ be a regular local ring with $d = \dim A \geq 2$ and let $x_1, x_2, \cdots, x_d$ be a regular system of parameters of A. Let $c_i \geq 2$ $(1 \leq i \leq d)$ be integers and put $Q = (x_1^{c_1}, x_2^{c_2}, \cdots, x_d^{c_d})$. Let $I = Q : \mathfrak{m}^2$. We then have the following.

(1) The following conditions are equivalent.
  (i) $I \not\subseteq \overline{Q}$.
  (ii) $d = 2$ and $\min\{c_1, c_2\} = 2$.

(2) $I^2 = QI$ if $I \subseteq \overline{Q}$.

Here $\overline{Q}$ denotes the integral closure of Q.

Proof. Let $z = \prod_{i=1}^{d} x_i^{c_i - 1}$, $a_i = x_i^{c_i}$, and $y_i = \dfrac{z}{x_i}$ for each $1 \leq i \leq d$. Then $Q : \mathfrak{m} = Q + (z)$ and $x_i y_j \equiv \delta_{ij} z$ modulo Q for all integers $1 \leq i, j \leq d$. Hence $I = Q + (y_1, y_2, \cdots, y_d)$ and $\mu_A(I/Q) = d$ by Lemma 2.1. We put $J = (y_1, y_2, \cdots, y_d)$.

Suppose now that $I \not\subseteq \overline{Q}$. Then, since $\mathrm{v}(A/Q) = d > 1$, by Proposition 2.2 we have $\mu_A(I) < 2d$. Hence $a_i \in (a_j \mid 1 \leq j \leq d, j \neq i) + J$ for some $1 \leq i \leq d$, because

$\mu_A(I/Q) = d$. We may assume that $i = 1$. Let us write

$$a_1 = \sum_{j=2}^{d} a_j\xi_j + \sum_{j=1}^{d} y_j\eta_j$$

with ξ_j and $\eta_j \in A$. Then $\eta_j \in \mathfrak{m}$ for all $1 \le j \le d$, since $\sum_{j=1}^{d} y_j\eta_j \in Q$ and $\mu_A(I/Q) = d$. Let $c = \sum_{i=1}^{d} c_i$. Then

$$a_1 - \sum_{j=2}^{d} a_j\xi_j = \sum_{j=1}^{d} y_j\eta_j \in Q \cap \mathfrak{m}^{c-d} = \sum_{j=1}^{d} a_j\mathfrak{m}^{c-(d+c_j)}.$$

Hence

$$a_1 - \sum_{j=2}^{d} a_j\xi_j = \sum_{j=1}^{d} a_j\rho_j$$

for some $\rho_j \in \mathfrak{m}^{c-(d+c_j)}$, so that $a_1(1-\rho_1) \in (a_j \mid 2 \le j \le d)$. Therefore ρ_1 is a unit of A, since $a_1 \notin (a_j \mid 2 \le j \le d)$. Thus $d = 2$ and $c_2 = 2$, because $\rho_1 \in \mathfrak{m}^{(c_2+c_3+\cdots+c_d)-d}$ and $c_j \ge 2$ for all $2 \le j \le d$.

Conversely, assume that $d = 2$ and $c_2 = 2$. We then have

$$I = Q + J = (x_1^{c_1-1}, x_1^{c_1-2}x_2, x_2^2).$$

Hence $\mu_A(I) < 4 = 2d$ and so $I \not\subseteq \overline{Q}$ by Proposition 2.2. Thus assertion (1) is proven. Since $Q \subseteq \mathfrak{m}^2$, the second assertion readily follows from Corollary 2.3 (3). $\square$

The following result is the heart of this paper.

Theorem 2.7. *Let $n = \mathrm{v}(A/Q) > 1$ and assume that I is not integral over Q. Then $\mathrm{e}_{\mathfrak{m}}^0(A) \le 2$ and $n = 2$.*

Proof. Firstly, suppose that $d = 1$ and let $a = a_1$. Then $I = (a) + (y_1, y_2, \cdots, y_n)$ and $\mathfrak{m} = (a) + (x_1, x_2, \cdots, x_n)$. We have $\mu_A(I) \le n$ by Proposition 2.2, because I is not integral over Q, while $\mu_A(I/Q) = n$ by Lemma 2.1 (1). Hence $I = (y_1, y_2, \cdots, y_n)$ and $a \in \mathfrak{m}{\cdot}(y_1, y_2, \cdots, y_n) \subseteq \mathfrak{m}^2$. Therefore $\mathfrak{m} = (x_1, x_2, \cdots, x_n)$. We put

$$J := Q : \mathfrak{m} = Q + \mathfrak{m}I = Q + (z).$$

Then $\mathfrak{m}J = \mathfrak{m}Q$ (cf. [CP, Proof of Theorem 2.2]; recall that A is not a discrete valuation ring, because $n > 1$). Hence $\mu_A(J) = 2$, because $\ell_A(J/\mathfrak{m}J) = \ell_A(J/Q) + \ell_A(Q/\mathfrak{m}Q) = 2$. We have $J = \mathfrak{m}I = (x_iy_j \mid 1 \le i,j \le n)$, because $Q \subsetneq \mathfrak{m}I \subseteq J$.

We divide the proof into two cases.

Case 1. ($x_iy_j \notin \mathfrak{m}Q$ for some $1 \le i,j \le n$ such that $i \ne j$.)
Without loss of generality we may assume that $i = 1$ and $j = 2$. Then, because $x_1y_2 \in Q$ but $x_1y_2 \notin \mathfrak{m}Q$, we have $Q = (x_1y_2)$. Hence $J = (x_1y_1) + Q = (x_1y_1, x_1y_2) = x_1{\cdot}(y_1, y_2) \subseteq (x_1)$ because $z \equiv x_1y_1 \bmod Q$, whence x_1 is a non-zerodivisor in A. We have $x_1y_\ell \in \mathfrak{m}I = J = x_1{\cdot}(y_1, y_2)$, so that $y_\ell \in (y_1, y_2)$ for all $1 \le \ell \le n$. Thus $I = (y_1, y_2)$. Hence $n = 2$. Because $\mathfrak{m}I = x_1I$ and $\mu_A(I) = 2$, we have $\mathfrak{m}^2 = x_1\mathfrak{m}$,

just thanks to the determinantal trick (cf. [DGH, Proposition 5.1]). Hence $\mathrm{e}^0_{\mathfrak{m}}(A) = 2$, because A is a Gorenstein local ring of maximal embedding dimension.

Case 2. ($x_iy_j \in \mathfrak{m}Q$ for all $1 \le i,j \le n$ such that $i \ne j$.)
In this case, we have $J = (x_iy_i \mid 1 \le i \le n)$, because $J = \mathfrak{m}I = (x_iy_j \mid 1 \le i,j \le n)$ and $\mathfrak{m}J = \mathfrak{m}Q$. Since $\mu_A(J) = 2$, without loss of generality, we may assume that $J = (x_1y_1, x_2y_2)$. Because $x_1y_1 - x_2y_2 \notin \mathfrak{m}J = \mathfrak{m}Q$ and $x_1y_1 \equiv x_2y_2 \equiv z \bmod Q$, we have $x_1y_1 = x_2y_2 + a\varepsilon$ with a unit ε in A, while $x_1y_2 = a\alpha$ and $x_2y_1 = a\beta$ with $\alpha, \beta \in \mathfrak{m}$. Hence

$$(x_1 + x_2)(y_1 - y_2) = a(\varepsilon - \alpha + \beta)$$

with $\varepsilon - \alpha + \beta$ a unit of A. We put

$$X_i = \begin{cases} x_1 + x_2 & (i = 1) \\ x_i & (i \ne 1) \end{cases} \quad \text{and} \quad Y_i = \begin{cases} y_1 - y_2 & (i = 2) \\ y_i & (i \ne 2). \end{cases}$$

Then $\mathfrak{m} = (X_1, X_2, \cdots, X_n)$, $I = (Y_1, Y_2, \cdots, Y_n)$, and $X_1Y_2 \notin \mathfrak{m}Q$ clearly. Thus thanks to Case 1, we have $n = \mathrm{e}^0_{\mathfrak{m}}(A) = 2$.

Now assume that $d \ge 2$. Then, by Proposition 2.2, we have $\mu_A(I) < n + d$. Since $\mu_A(I/Q) = n$, we may assume that $I = (a_2, a_3, \cdots, a_d) + (y_1, y_2, \cdots, y_n)$. Let $L = (a_2, a_3, \cdots, a_d)$, $\overline{A} = A/L$, $\overline{\mathfrak{m}} = \mathfrak{m}/L$, $\overline{Q} = Q/L$, and $\overline{I} = I/L$. Then $\overline{I} = \overline{Q} : \overline{\mathfrak{m}}^2$ and $\overline{A}$ is a Gorenstein local ring of dimension 1 with $\mathrm{v}(\overline{A}/\overline{Q}) = \mathrm{v}(A/Q) = n > 1$. We have $\mu_{\overline{A}}(\overline{I}) \le n$, whence by Proposition 2.2, $\overline{I}$ is not integral over $\overline{Q}$. Therefore, thanks to the result of the case where $d = 1$, we have $n = \mathrm{e}^0_{\overline{\mathfrak{m}}}(\overline{A}) = 2$. We see $\mathrm{e}^0_{\mathfrak{m}}(A) \le 2$ because $\mathrm{e}^0_{\overline{\mathfrak{m}}}(\overline{A}) \ge \mathrm{e}^0_{\mathfrak{m}}(A)$, which completes the proof of Theorem 2.7. $\square$

The following assertion readily follows from Theorem 2.7.

Corollary 2.8. *Suppose that $\mathrm{e}^0_{\mathfrak{m}}(A) \ge 3$. Then I is integral over Q, if $n = \mathrm{v}(A/Q) > 1$.*

3. Proof of Theorem 1.1

Throughout this section let $(A, \mathfrak{m})$ be a Gorenstein local ring with $d = \dim A > 0$ and $Q = (a_1, a_2, \cdots, a_d)$ a parameter ideal in A. We put $I = Q : \mathfrak{m}^2$.

The purpose of this section is to prove Theorem 1.1. Let us begin with the following.

Theorem 3.1. *Suppose that $n = \mathrm{v}(A/Q) > 1$ and I is integral over Q. Then*

(1) $I^3 = QI^2$.

(2) $\mathrm{G}(I)$ *and* $\mathrm{F}(I)$ *are Cohen-Macaulay rings.*

Hence $\mathcal{R}(I)$ is also a Cohen-Macaulay ring, if $d \ge 3$.

Proof. The last assertion directly follows from assertions (1) and (2), because the a-invariant $\mathrm{a}(\mathrm{G}(I))$ of $\mathrm{G}(I)$ is at most $2 - d$ (cf. [GS, THEOREM (1.1), REMARK (3.10)]).

We may assume that $I^2 \nsubseteq Q$, thanks to Corollary 2.3 (1). Choose the element $z \in \mathfrak{m}I$ so that $z \in I^2$. Hence $Q : \mathfrak{m} = Q + I^2 = Q + (z)$ and so $I^2 = QI + (z)$, because

$Q \cap I^2 = QI$ by Corollary 2.3 (1). Thus $I^3 = QI^2 + zI$ and we get the required equality $I^3 = QI^2$ modulo the following claim, because

$$(Q^2 + zQ) \cap I^3 = (Q^2 \cap I^3) + zQ = Q^2I + zQ \subseteq QI^2$$

by Corollary 2.3 (1).

Claim 1. *$zI \subseteq Q^2 + zQ$.*

Proof of Claim 1. Since $I = Q + (y_1, y_2, \cdots, y_n)$, it suffices to show that $zy_\ell \in Q^2 + zQ$ for all integers $1 \le \ell \le n$. Let $1 \le i \le n$ be an integer such that $i \ne \ell$ and write $z = x_iy_i + q_i$ with $q_i \in \mathfrak{m}Q$. Then $zy_\ell = (x_iy_\ell)y_i + y_\ell q_i \in (\mathfrak{m}I)Q$. Since $\mathfrak{m}I \subseteq Q : \mathfrak{m} = Q + (z)$, we have $zy_\ell \in [Q + (z)]{\cdot}Q = Q^2 + zQ$. Thus $zI \subseteq Q^2 + zQ$. $\square$

As $I^3 = QI^2$ and $Q \cap I^2 = QI$ by Corollary 2.3 (1), we have $Q \cap I^{i+1} = QI^i$ for every $i \in \mathbb{Z}$, whence $\mathrm{G}(I)$ is a Cohen-Macaulay ring. To show that $\mathrm{F}(I)$ is a Cohen-Macaulay ring, we need the following. The equality $\mathfrak{m}I^2 = \mathfrak{m}QI$ in Claim 2 yields, since $I^3 = QI^2$, that the elements $a_1T, a_2T, \cdots, a_dT \in \mathcal{R}(I)$ constitute a regular sequence in $\mathrm{F}(I)$.

Claim 2. *$\mathfrak{m}I^2 = \mathfrak{m}QI$.*

Proof of Claim 2. Let $J = (y_1, y_2, \cdots, y_n)$. Hence $I^2 = QI + J^2$ because $I = Q + J$. It suffices to show that $\mathfrak{m}J^2 \subseteq \mathfrak{m}QI$. Since $\mathfrak{m} = Q + (x_1, x_2, \cdots, x_n)$ and $QJ^2 \subseteq \mathfrak{m}QI$, we have only to show $x_\ell y_i y_j \in \mathfrak{m}QI$ for all integers $1 \le \ell, i, j \le n$. Let us write $x_\ell y_i = \delta_{\ell i}z + q_{\ell i}$ with $q_{\ell i} \in \mathfrak{m}Q$. Then

$$x_\ell y_i y_j = (\delta_{\ell i}z + q_{\ell i})y_j = \delta_{\ell i}y_jz + q_{\ell i}y_j \in I^3 + \mathfrak{m}QI = \mathfrak{m}QI,$$

because $I^3 = QI^2$. Hence $\mathfrak{m}I^2 = \mathfrak{m}QI$. $\square$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.2, Corollary 2.8, and Theorem 3.1 we may assume that $n = \mathrm{v}(A/Q) = 1$. Hence $\mathrm{v}(A) = d + 1$. Let $\mathfrak{m} = Q + (x)$ with $x \in \mathfrak{m}$; hence $a_1, a_2, \cdots, a_d, x$ is a minimal basis of $\mathfrak{m}$. We put

$$\overline{A} = A/Q,\ \overline{\mathfrak{m}} = \mathfrak{m}/Q = (\overline{x}),\ \overline{I} = I/Q, \text{ and } \ell = \ell_A(\overline{A}),$$

where $\overline{x} = x \bmod Q$ be the image of x in $\overline{A}$. Then, since $\overline{\mathfrak{m}} = (\overline{x})$, we have

$$\ell - 1 = \max\{t \in \mathbb{Z} \mid \overline{\mathfrak{m}}^t \ne (0)\} \text{ and } x^\ell \in Q.$$

Hence $\overline{I} = (0) : \overline{\mathfrak{m}}^2 = \overline{\mathfrak{m}}^{\ell-2}$ so that $I = Q + \mathfrak{m}^{\ell-2} = Q + (x^{\ell-2})$. Notice that $\ell = \mathrm{e}_Q^0(A) \ge \mathrm{e}_\mathfrak{m}^0(A) \ge 3$, where $\mathrm{e}_Q^0(A)$ denotes the multiplicity of A with respect to Q. We then have

$$\mathfrak{m}^2I = [Q\mathfrak{m} + (x^2)]{\cdot}[Q + (x^{\ell-2})] \subseteq \mathfrak{m}^2Q + (x^\ell),$$

because $\mathfrak{m}^2 = Q\mathfrak{m} + (x^2)$ and $\ell \ge 3$. Consequently, in order to see that $\mathfrak{m}^2I = \mathfrak{m}^2Q$, it suffices to show the following.

Claim 1. *$x^\ell \in \mathfrak{m}^2Q$.*

Proof of Claim 1. Let us write $x^\ell = \sum_{i=1}^d a_i w_i$ with $w_i \in A$. Let $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A and take an epimorphism $\varphi : B \to \widehat{A}$, where $(B, \mathfrak{n})$ is a regular local ring of dimension $d+1$. Then Ker φ is a principal ideal in B generated by a single element $\xi \in \mathfrak{n}^e$ such that $\xi \notin \mathfrak{n}^{e+1}$ where $e = \mathrm{e}_\mathfrak{m}^0(A)$; hence Ker $\varphi \subseteq \mathfrak{n}^e$. Choose elements $\{A_i\}_{1 \le i \le d}, X$, and $\{W_i\}_{1 \le i \le d}$ of B such that they are the preimages of $\{a_i\}_{1 \le i \le d}, x$, and $\{w_i\}_{1 \le i \le d}$, respectively. Then we have $\mathfrak{n} = (A_1, A_2, \cdots, A_d, X)$ and $X^\ell - \sum_{i=1}^d A_i W_i \in$ Ker $\varphi \subseteq \mathfrak{n}^e$. Hence $\sum_{i=1}^d A_i W_i \in \mathfrak{n}^e$, because $\ell \ge e$. Consequently, since $(A_1, A_2, \cdots, A_d) \cap \mathfrak{n}^e = (A_1, A_2, \cdots, A_d) \cdot \mathfrak{n}^{e-1}$, we see that $\sum_{i=1}^d A_i W_i = \sum_{i=1}^d A_i V_i$ for some elements $V_i \in \mathfrak{n}^{e-1}$, whence $x^\ell = \sum_{i=1}^d a_i v_i$ where $v_i = \varphi(V_i)$. Thus $x^\ell \in Q\mathfrak{m}^{e-1} \subseteq Q\mathfrak{m}^2$ as is wanted, because $e \ge 3$. $\square$

Since $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$, we have $Q \cap I^2 = QI$ similarly as in the proof of Corollary 2.3 (1). Therefore, to finish the proof of Theorem 1.1, we may assume $I^2 \not\subseteq Q$. Since $x^\ell \in Q$ and $I^2 = QI + (x^{2\ell-4})$, we have $2\ell - 4 < \ell$ whence $\ell = e = 3$, so that $I = Q + (x) = \mathfrak{m}$. Thus $\mathfrak{m}^3 = \mathfrak{m}^2 I = Q\mathfrak{m}^2$ and so $\mathrm{G}(\mathfrak{m}) = \mathrm{F}(\mathfrak{m})$ is a Cohen-Macaulay ring. As $\mathrm{a}(\mathrm{G}(\mathfrak{m})) \le 2 - d$, $\mathcal{R}(\mathfrak{m})$ is a Cohen-Macaulay ring if $d \ge 3$. This completes the proof of Theorem 1.1. $\square$

4. Examples

In this section we explore three examples to show that the non-Gorenstein case is rather wild.

Example 4.1. Let $n \ge 2$ be an integer and let

$$A = k[[X_1, X_2, \cdots, X_n]]/(X_i X_j \mid 1 \le i < j \le n)$$

where $k[[X_1, X_2, \cdots, X_n]]$ denotes the formal power series ring over a field k. Then A is a one-dimensional reduced local ring with $\mathrm{e}_\mathfrak{m}^0(A) = n$. For every parameter ideal Q in A, we have

$$Q : \mathfrak{m}^2 \not\subseteq \overline{Q},$$

where $\overline{Q}$ denotes the integral closure of Q.

Proof. Let $I = Q : \mathfrak{m}^2$ and assume that $I \subseteq \overline{Q}$. We write $Q = (a)$. Then $a = \sum_{i=1}^n x_i^{c_i} \varepsilon_i$ for some units ε_i in A and some integers $c_i \ge 1$. Let $1 \le i \le n$ be an integer. If $c_i \ge 2$, we then have $x_i^{c_i - 1} \in I$ but $x_i^{c_i-1}$ is not integral over Q. Hence $c_i = 1$ for all $1 \le i \le n$ and so $a = \sum_{i=1}^n x_i \varepsilon_i$. Therefore $\mathfrak{m}^2 = Q\mathfrak{m}$ so that we have $I = A$, which is absurd. $\square$

Letting $n = 2$, this example 4.1 shows the assumption that $\mathrm{e}_\mathfrak{m}^0(A) \ge 3$ in Theorem 1.1 is not superfluous.

It seems natural and quite interesting to ask what happens in the case where A is a numerical semi-group ring. Let us explore one example.

Example 4.2. Let $H = \langle 4, 7, 9 \rangle$ be the numerical semi-group generated by 4, 7, and 9 and let $A = k[[t^4, t^7, t^9]] \subseteq k[[t]]$, where $V = k[[t]]$ denotes the formal power series ring with one indeterminate t over a field k. Then A is a one-dimensional non-Gorenstein Cohen-Macaulay local ring. Let $0 < s \in H$. We put $Q = (t^s)$ and $I = Q : \mathfrak{m}^2$. Then $I \subseteq \overline{Q}$. We have $I \subseteq \mathfrak{m}^2$ if $s \ge 11$, whence $I^2 \subseteq Q$. However

(1) $\mathrm{r}_Q(I) = \begin{cases} 1 & \text{if} \quad s = 9, \\ 2 & \text{if} \quad s = 4,\ 8, \mathit{or}\ s \geq 11, \\ 3 & \text{if} \quad s = 7. \end{cases}$

(2) $\mathrm{G}(I)$ is a Cohen-Macaulay ring if and only if $s = 4, 8,$ *or* 9.

(3) $\mathrm{F}(I)$ is a Cohen-Macaulay ring if and only if $s = 4, 9$.

(4) $\mathrm{F}(I)$ is always a Buchsbaum ring.

(5) $\mathrm{G}(I)$ is a Buchsbaum ring if and only if $s \neq 7$.

(6) $\mathfrak{m}^2 I \neq \mathfrak{m}^2 Q$ if $s = 8, 11$.

Proof. We have $n \in H$ for all integers $n \geq 11$ but $10 \notin H$. Hence the conductor of H is 11. Notice that $t^n \in \mathfrak{m}^2$ for all $n \in \mathbb{Z}$ such that $n \geq 11$, where $\mathfrak{m} = (t^4, t^7, t^9)$ denotes the maximal ideal in A. Hence $I \subseteq \overline{Q}$. In fact, let $n \in H$ and assume that $t^n \in I$ but $n < s$. Then $t^{s-n+10} \in \mathfrak{m}^2$ because $s - n + 10 \geq 11$, so that $t^{s+10} = t^n t^{s-n+10} \in Q = (t^s)$ whence $t^{10} \in A$, which is impossible. Thus, for every $n \in H$ with $t^n \in I$, we have $t^n \in t^s V \cap A = \overline{Q}$, whence $I \subseteq \overline{Q}$ (recall that I is a monomial ideal generated by the elements $\{t^n \mid n \in H \text{ such that } t^n \in I\}$). In particular we have $I \subseteq \mathfrak{m}^2$ if $s \geq 11$, whence $I^2 \subseteq Q$.

We note the following.

Claim 1. *Let $s_2 \geq s_1 \geq 11$ be integers and let $q = s_2 - s_1$. We put $Q_i = (t^{s_i})$ and $I_i = Q_i : \mathfrak{m}^2$ for $i = 1, 2$. Then we have the following.*

(1) $I_2 = t^q I_1$.

(2) $\mathcal{R}(I_1) \cong \mathcal{R}(I_2)$ *as graded A-algebras.*

(3) $\mathrm{F}(I_1) \cong \mathrm{F}(I_2)$ *as graded $A/\mathfrak{m}$-algebras.*

(4) $\mathrm{r}_{Q_1}(I_1) = \mathrm{r}_{Q_2}(I_2)$.

Proof of Claim 1. Let $\varphi = \widehat{t^q} : V \to V$ be the V-linear map defined by $\varphi(x) = t^q x$ for all $x \in V$. Then, since $\varphi(Q_1) = Q_2$ and $\varphi(I_1) \subseteq I_2$, the map φ induces a monomorphism

$$\xi : I_1/Q_1 \to I_2/Q_2, \quad x \bmod Q_1 \mapsto t^q x \bmod Q_2$$

of A-modules. As $I_i/Q_i \cong \mathrm{Ext}_A^1(A/\mathfrak{m}^2, A)$ (recall that $t^{s_i} \in \mathfrak{m}^2$), we see $\ell_A(I_1/Q_1) = \ell_A(I_2/Q_2)$, whence $\xi : I_1/Q_1 \to I_2/Q_2$ is an isomorphism, so that $\varphi(I_1) = I_2$. Thus assertion (1) follows. Notice that

$$\mathcal{R}(I_2) = A[(t^q I_1){\cdot}T] = A[I_1{\cdot}t^q T] \text{ and } \mathcal{R}(I_1) = A[I_1 T]$$

with T an indeterminate over A. Then, since $t^q T$ is also transcendental over the ring A, we get an isomorphism $\xi : \mathcal{R}(I_1) \to \mathcal{R}(I_2)$ of graded A-algebras such that $\xi(t^{s_1}T) = t^{s_2}T$. Hence we have assertion (2). Because $\mathrm{F}(I_i) = \mathcal{R}(I_i)/\mathfrak{m}\mathcal{R}(I_i)$, we readily have an isomorphism $\eta : \mathrm{F}(I_1) \to \mathrm{F}(I_2)$ of graded $A/\mathfrak{m}$-algebras such that $\eta(\overline{t^{s_1}T}) = \overline{t^{s_2}T}$, where $\overline{t^{s_i}T}$ denotes the image of $t^{s_i}T$ in $\mathrm{F}(I_i)$. Hence assertion (4) also follows, because

$$\mathrm{r}_{Q_i}(I_i) = \max\{n \in \mathbb{Z} \mid [\mathrm{F}(I_i)/(\overline{t^{s_i}T})]_n \neq (0)\},$$

where $[\mathrm{F}(I_i)/(\overline{t^{s_i}T})]_n$ denotes the homogeneous component of the graded ring $\mathrm{F}(I_i)/(\overline{t^{s_i}T})$ of degree n. $\square$

We put $\mathcal{R} = \mathcal{R}(I)$, $G = \mathrm{G}(I)$, and $F = \mathrm{F}(I)$. Let $M = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$ be the graded maximal ideal in $\mathcal{R}$ and we denote by $\mathrm{H}^0_M(*)$ the $0^{\underline{th}}$ local cohomology functor with respect to M. Let $a = t^s$ and $f = aT \in \mathcal{R} = A[IT]$. For each graded $\mathcal{R}$-module L, let $[\mathrm{H}^0_M(L)]_n$ $(n \in \mathbb{Z})$ denote the homogeneous component of $\mathrm{H}^0_M(L)$ of degree n.

Let $\widetilde{I} = \bigcup_{n \geq 0}[I^{n+1} : I^n]$ denote the Ratliff-Rush closure of I. The following assertions readily follow from the equalities that

$$\widetilde{I} = \bigcup_{n \geq 0}[I^{n+1} : a^n] \quad \text{and} \quad I^{n+\ell} = a^\ell I^n$$

for all integers $n \geq r = \mathrm{r}_Q(I)$ and $\ell \geq 1$, whose detail is left to the reader.

Claim 2. *Let $r = \mathrm{r}_Q(I)$. Then*

(1) $[\mathrm{H}^0_M(G)]_n = (0)$ *for all* $n \geq r - 1$.

(2) $[\mathrm{H}^0_M(F)]_n = (0)$ *for all* $n \geq r$.

(3) *Suppose that $r = 2$. Then $\widetilde{I} = I^2 : a$ and $[\mathrm{H}^0_M(G)]_0 \cong \widetilde{I}/I$ as A-modules.*

We now consider the case $s = 11$. We then have $I = (t^{11}, t^{12}, t^{14}, t^{17})$, $I^3 = QI^2$, and

$$I^2 = QI + (t^{24}) \neq QI$$

since $t^{24} \notin QI = (t^{22}, t^{23}, t^{25}, t^{28})$. Hence $\mathrm{r}_Q(I) = 2$. Because $\widetilde{I} = I : t^{11} = I + (t^{13}) \neq I$ and

$$\mathrm{H}^0_M(G) = [\mathrm{H}^0_M(G)]_0 \cong \widetilde{I}/I$$

by Claim 2 (3), we see that G is not a Cohen-Macaulay ring but a Buchsbaum ring with $\ell_A(\mathrm{H}^0_M(G)) = \ell_A(\widetilde{I}/I) = 1$. Notice that $\mathfrak{m}^2 I = (t^{19}, t^{20}, t^{22}, t^{25}) \neq \mathfrak{m}^2 Q = (t^{19}, t^{22}, t^{24}, t^{25})$. Since $t^{11}t^{17} = t^{28} = t^4 t^{24} \in \mathfrak{m}I^2$ but $t^{17} \notin \mathfrak{m}I = (t^{15}, t^{16}, t^{18}, t^{21})$, the element $f = t^{11}T \in \mathcal{R}$ is a zerodivisor in F, whence F is a Buchsbaum ring by Claim 2 (2) but not a Cohen-Macaulay ring.

If $s > 11$, then thanks to Claim 1 and the assertions in the case where $s = 11$, we have $\mathrm{r}_Q(I) = 2$ and F is a Buchsbaum ring but not Cohen-Macaulay. To see that G is a Buchsbaum ring, recall that

$$\mathrm{H}^0_M(G) = [\mathrm{H}^0_M(G)]_0 \cong \widetilde{I}/I$$

since $\mathrm{r}_Q(I) = 2$. Let $Q' = (t^{11})$ and $I' = Q' : \mathfrak{m}^2$. Then because $\widetilde{I'} = I'^2 : t^{11}$ and $\widetilde{I} = I^2 : t^s$ (see Claim 1 (4)), it is standard to check that $t^{s-11} \cdot \widetilde{I'} = \widetilde{I}$, so that we have $\ell_A(\widetilde{I}/I) = \ell_A(\widetilde{I'}/I') = 1$ (recall that $t^{s-11} \cdot I' = I$; cf. Claim 1 (1)), whence G is a Buchsbaum ring with $\ell_A(\mathrm{H}^0_M(G)) = 1$.

Let $s = 4$. Then $I = \mathfrak{m}$. The ring G $(= F)$ is a Cohen-Macaulay ring, since $\mathfrak{m}^3 = Q\mathfrak{m}^2$ and $Q \cap \mathfrak{m}^2 = Q\mathfrak{m}$.

Let $s = 7$, then $I = (t^7, t^8, t^{11}, t^{13})$, $I^2 = (t^{14}, t^{15}, t^{16}) \subseteq Q$, and $t^{16} \notin QI = (t^{14}, t^{15}, t^{20})$. Hence G is not a Cohen-Macaulay ring. We have $I^4 = QI^3$ but $I^3 = QI^2 + (t^{24}) \neq QI^2$. Hence $\mathrm{r}_Q(I) = 3$. Because $t^7 t^{13} = t^{20} = t^4 t^{16} \in \mathfrak{m}I^2$ but $t^{13} \notin \mathfrak{m}I = (t^{11}, t^{12}, t^{14}, t^{17})$, F is not a Cohen-Macaulay ring. To see that G is not a Buchsbaum ring, let $W = \mathrm{H}^0_M(G)$. Then $W = W_0 + W_1$ by Claim 2 (1). It is now direct to check that $W_0 = \{\overline{c} \mid c \in (t^9)\}$ and $W_1 = \{\overline{cT} \mid c \in (t^{17})\}$ where $\overline{*}$ denotes the image of the corresponding element of $\mathcal{R}$ in G. Because $\overline{t^9} \neq 0$ in G and $t^9{\cdot}I = (t^{16}, t^{17}, t^{20}, t^{22}) \not\subseteq I^2 = (t^{14}, t^{15}, t^{16})$, we see $MW_0 \neq (0)$, whence G is not a Buchsbaum ring. Similarly, one can directly check that

$$\mathrm{H}^0_M(F) = [\mathrm{H}^0_M(F)]_1 = \{\overline{cT} \mid c \in (t^{13})\} \cong A/\mathfrak{m},$$

so that F is a Buchsbaum ring but not Cohen-Macaulay.

Let $s = 8$. Then $I = (t^8, t^9, t^{11}, t^{14})$ and $I^3 = QI^2$. We have $\mathfrak{m}^2 I = (t^{16}, t^{17}, t^{19}, t^{22}) \neq \mathfrak{m}^2 Q = (t^{16}, t^{19}, t^{21}, t^{22})$ and $\ell_A(\mathfrak{m}^2 I/\mathfrak{m}^2 Q) = 1$. To see that G is a Cohen-Macaulay ring, we have only to show that $Q \cap I^2 = QI$. Since $I^2 = QI + (t^{18})$, we have $Q \cap I^2 = QI + [Q \cap (t^{18})]$. Let $\varphi \in Q \cap (t^{18})$ and write $\varphi = t^8\xi = t^{18}\eta$ with $\xi, \eta \in A$. Then $\xi = t^{10}\eta$. Because $10 \notin H = \langle 4, 7, 9\rangle$, we have $\eta \in \mathfrak{m}$ so that $\varphi = t^{18}\eta \in t^{18}\mathfrak{m} = (t^{22}, t^{25}, t^{27}) \subseteq QI = (t^{16}, t^{17}, t^{19}, t^{22})$. Hence $Q \cap I^2 = QI$ and G is a Cohen-Macaulay ring. The ring F is Buchsbaum by Claim 2 (2) but not a Cohen-Macaulay ring, because $t^8 t^{14} = t^{22} = t^4(t^9)^2 \in \mathfrak{m}I^2$ but $t^{14} \notin \mathfrak{m}I = (t^{12}, t^{13}, t^{15}, t^{18})$.

Let $s = 9$. Then $I = (t^9, t^{12}, t^{14}, t^{15})$ and $I^2 = QI$, whence G and F are both Cohen-Macaulay rings. This completes the proof of all the assertions. $\square$

Our last example shows that unless A is Gorenstein, the reduction number $\mathrm{r}_Q(I)$ can be arbitrarily large even if $I \subseteq \overline{Q}$, where $I = Q : \mathfrak{m}^2$ and $\overline{Q}$ denotes the integral closure of Q.

Example 4.3. Let $n \geq 2$ be an integer and let

$$a_i = \begin{cases} 2n - 1 & (i = 1), \\ (2n+1)i - 2n - 2 & (2 \leq i \leq n). \end{cases}$$

Let $H = \langle a_1, a_2, \cdots, a_n\rangle$ be the numerical semi-group generated by a_i's. Let $A = k[[t^{a_1}, t^{a_2}, \cdots, t^{a_n}]] \subseteq k[[t]]$ be the semi-group ring of H, where $k[[t]]$ denotes the formal power series ring with one indeterminate t over a field k. Then A is a one-dimensional Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m} = (t^{a_1}, t^{a_2}, \cdots, t^{a_n})$. Let $Q = (t^{2a_1})$ and $I = Q : \mathfrak{m}^2$. Then $I \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 2n - 2$.

Proof. Let $B = k[[X_1, X_2, \cdots, X_n]]$ $(n \geq 2)$ be the formal power series ring over the field k and let

$$\varphi : B \to k[[t^{a_1}, t^{a_2}, \cdots, t^{a_n}]]$$

be the homomorphism of k-algebras defined by $\varphi(X_i) = t^{a_i}$ for all $1 \leq i \leq n$. Let $\mathrm{I}_2(M)$ be the ideal in B generated by all the 2×2 minors of the following matrix

$$M = \begin{pmatrix} X_1 & X_2 & X_3 & \cdots & X_{n-1} & X_n \\ X_2^2 & X_3 & X_4 & \cdots & X_n & X_1^{n+1} \end{pmatrix}.$$

We then have $\mathrm{Ker}\varphi = \mathrm{I}_2(M)$, because $\ell_B(B/[\mathrm{I}_2(M) + (X_1)]) = 2n - 1 = a_1$. Let us identify $A = B/\mathrm{I}_2(M) = k[[t^{a_1}, t^{a_2}, \cdots, t^{a_n}]]$. Let $x_i = X_i \bmod \mathrm{I}_2(M)$ be the image of X_i in $B/\mathrm{I}_2(M)$ for each $1 \leq i \leq n$; hence $\mathfrak{m} = (x_1, x_2, \cdots, x_n)$. With this notation it is standard and easy to check that $\mathfrak{m}^2 = (x_1, x_2)\mathfrak{m}$, $I = x_1\mathfrak{m} + (x_2x_n)$, and $I^i = x_1^i\mathfrak{m}^i$ for all $i \geq 2$.

Recall now that $\mathfrak{m}^{2n-1} = x_1\mathfrak{m}^{2n-2}$, because (x_1) is a minimal reduction of $\mathfrak{m}$ and $\mathrm{e}^0_{\mathfrak{m}}(A) = 2n-1$. Hence

$$I^{2n-1} = x_1^{2n-1}\mathfrak{m}^{2n-1} = x_1^{2n-1}{\cdot}x_1\mathfrak{m}^{2n-2} = x_1^2{\cdot}x_1^{2n-2}\mathfrak{m}^{2n-2} = x_1^2I^{2n-2} = QI^{2n-2}.$$

We must show that $I^{2n-2} \neq QI^{2n-3}$. To see this, we explore the following system of generators of $\mathfrak{m}^{2n-3}$;

$$\begin{aligned}\mathfrak{m}^{2n-3} &= (t^{a_1}, t^{a_2}, \cdots, t^{a_n})^{2n-3}\\ &= (t^{\sum_{i=1}^n c_ia_i} \mid c_i \geq 0, \ \sum_{i=1}^n c_i = 2n-3)\\ &= (t^{(2n-i-3)a_1+ia_2} \mid 0 \leq i \leq 2n-3)\\ &\quad + (t^{\sum_{i=1}^n c_ia_i} \mid c_j > 0 \text{ for some } j \geq 3, \ \sum_{i=1}^n c_i = 2n-3).\end{aligned}$$

Notice that $\{(2n-i-3)a_1 + ia_2 = (4n^2-8n+3)+i\}_{0\leq i\leq 2n-3}$ are continuous integers and that

$$\sum_{i=1}^n c_ia_i \geq (2n-4)a_1 + a_3 = 4n^2-6n+5,$$

if $c_j > 0$ for some $j \geq 3$ and $\sum_{i=1}^n c_i = 2n-3$. Hence

$$\mathfrak{m}^{2n-3} \subseteq (t^i \mid 4n^2-8n+3 \leq i \leq 4n^2-6n) + (t^i \mid i \in H, \ i \geq 4n^2-6n+5).$$

Therefore

$$(\sharp) \quad t^{2n-1}\mathfrak{m}^{2n-3} \subseteq (t^i \mid 4n^2-6n+2 \leq i \leq 4n^2-4n-1) + (t^i \mid i \in H, \ i \geq 4n^2-4n+4).$$

Suppose now that $I^{2n-2} = QI^{2n-3}$. Then $\mathfrak{m}^{2n-2} = x_1\mathfrak{m}^{2n-3}$, since $I^i = x_1^i\mathfrak{m}^i$ for all $i \geq 2$; hence $x_2^{2n-2} \in x_1\mathfrak{m}^{2n-3}$. Recall that $x_1 = t^{2n-1}$ and $x_2 = t^{2n}$. Then, because $4n^2-4n-1 < 4n^2-4n < 4n^2-4n+4$, we get by $(\sharp)$ that

$$t^{4n^2-4n} \in \mathfrak{m}{\cdot}(t^i \mid 4n^2-6n+2 \leq i \leq 4n^2-4n-1),$$

which is however impossible, since

$$\mathfrak{m}{\cdot}(t^i \mid 4n^2-6n+2 \leq i \leq 4n^2-4n-1) \subseteq t^{4n^2-4n+1}k[[t]]$$

(recall that $a_i + (4n^2-6n+2) \geq a_1 + (4n^2-6n+2) = 4n^2-4n+1$ for all $1 \leq i \leq n$). This is the required contradiction and we conclude that $I^{2n-2} \neq QI^{2n-3}$. Thus $\mathrm{r}_Q(I) = 2n-2$. $\square$

Added in proof

We are very grateful to Alberto Corso and Claudia Polini. During this conference they told us that N.-J. Wang independently gave some results which are closely related to the topics of this paper. Among the other things Wang [W] proved the following. Assume that $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring with $\dim A \geq 3$ or that $(A, \mathfrak{m})$ is a two-dimensional Cohen-Macaulay local ring which is not regular. Let Q be a parameter ideal of A contained in m^t with $t \geq 2$ and put $I = Q : \mathrm{m}^t$. Then $I \subseteq \mathrm{m}^t$, $\mathrm{m}^t I = \mathrm{m}^t Q$, and $I^2 = QI$. (For more detail see also the report of Corso in this volume.) Wang's results however take care of neither the case where $Q \not\subseteq \mathrm{m}^t$ nor the case where $\dim A = 1$ and his method of proof is totally different from ours. For these reasons our research could have its own significance, dealing with arbitrary cases (although we assumed the stronger assumption that the base local ring A is a Gorenstein ring) and showing the case where A is a one-dimensional Cohen-Macaulay local ring is very wild.

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CASTELNUOVO-MUMFORD REGULARITY FOR PROJECTIVE CURVES ON A DEL PEZZO SURFACE

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ABSTRACT. This paper investigates the next extremal case for a Castelnuovo-type bound $\operatorname{reg} C \leq \lceil (\deg C - 1)/\operatorname{codim} C \rceil + \max\{k(C), 1\}$ for the Castelnuovo-Mumford regularity for a nondegenerate projective curve C, where $k(C)$ is an invariant which measures the deficiency of the Hartshorne-Rao module of C. We decribe a projective curve with next to the maximal regularity lies on either a Hirzeburch surface or a normal del Pezzo surface. The socle lemma by Huneke-Ulrich and a result from the Castelnuovo theory by Eisenbud-Harris plays an important role for the theory. The details of the proof is written in [18]

1. INTRODUCTION

Let k be an algebraically closed field. Let $\mathbb{P}_k^N = \operatorname{Proj} S$ be the projective N-space, where S is the polynomial ring of $N+1$ variables over k. For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_k^N$ and an integer $m \in \mathbb{Z}$, $\mathcal{F}$ is said to be m-regular if $\mathrm{H}^i(\mathbb{P}_k^N, \mathcal{F}(m-i)) = 0$ for all $i \geq 1$. For a projective scheme $X \subseteq \mathbb{P}_k^N$, X is said to be m-regular if the ideal sheaf $\mathcal{I}_X$ is m-regular. The Castelnuovo-Mumford regularity of $X \subseteq \mathbb{P}_k^N$ is the least such integer m and is denoted by $\operatorname{reg} X$. It is well-known that X is m-regular if and only if for every $p \geq 0$ the minimal generators of the pth syzygy module of the defining ideal $I (\subseteq S)$ of $X \subseteq \mathbb{P}_k^N$ occur in degree $\leq m + p$. In this sense, the Castelnuovo-Mumford regularity is one of the important invariants measuring a complexity of the defining ideal of a given projective scheme.

Throughout this paper, a curve is always assumed to be irreducible and reduced. For a rational number $m \in \mathbb{Q}$, we write $\lceil m \rceil$ for the minimal integer which is larger than or equal to m, and $\lfloor m \rfloor$ for the maximal integer which is smaller than or equal to m.

In this paper, we investigate a Castelnuovo-type bound for the Castelnuovo-Mumford regularity for projective curves. If a nondegenerate projective curve C is ACM, that is, the coordinate ring of C is Cohen-Macaulay, then there is a well-known inequality $\operatorname{reg} C \leq \lceil (\deg C - 1)/\operatorname{codim} C \rceil + 1$. The inequality follows from the fact that $\operatorname{reg} X \leq \lceil (\deg X - 1)/\operatorname{codim} X \rceil + 1$ for a generic hyperplane section X of C, which is an easy consequence of the Uniform Position Principle, see, e.g. [1, page 115] and [3, page 95], for characteristic zero and also works for general case, see, e.g., [17, (1.1)] from the property (2.1) of the h-vectors of X. The extremal case is described as a rational normal curve under the assumption $\deg C$

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large enough, see [20]. On the other hand, the extremal curve for the Gruson-Lazarsfeld-Peskine bound $\operatorname{reg} C \leq \deg C - \operatorname{codim} C + 1$ are appeared as a rational curve with certain condition and an elliptic normal curve, see [7]. How about a Castelnuovo-type bound for the Castelnuovo-Mumford regularity? In order to extend a result of Castelnuovo-type regularity bound for a (not necessarily ACM) curve, we introduce, as in [13, 14], an invariant $k(C)$ which measures how far the coordinate ring of C from the Cohen-Macaulay property. For a projective curve $C \subseteq \mathbb{P}_k^N$, a graded S-module $\mathrm{M}(C) = \mathrm{H}_*^1(\mathcal{I}_{C/\mathbb{P}_k^n}) = \oplus_{\ell \in \mathbb{Z}} \mathrm{H}^1(\mathbb{P}_K^N, \mathcal{I}_C(\ell))$ is called the Hartshorne-Rao module. Then we define $k(C)$ as the minimal nonnegative integer v such that $\mathfrak{m}^v \mathrm{M}(C) = 0$. A curve C is ACM if and only if $k(C) = 0$. On the other hand, the coordinate ring of C is a Buchsbaum ring if and only if $k(C) = 1$. The extremal bound for the Buchsbaum curve, even for higher dimensional case, is also described in [21, 23]. For the general case, that is, C is a (not necessarily smooth) nondegenerate projective curve, we have an inequality $\operatorname{reg} C \leq \lceil (\deg C - 1)/\operatorname{codim} C \rceil + \max\{k(C), 1\}$, see (2.5). Furthermore, the following result (1.1) describes the extremal curve with the Castelnuovo-type maximal regularity from [3, (3.2)], or see [15, (1.2)].

Proposition 1.1. *Let $C \subseteq \mathbb{P}_k^N$ be a nondegenerate projective curve over an algebraically closed field k with* $\operatorname{char} k = 0$. *Assume that C is not ACM. If* $\deg C \geq (\operatorname{codim} C)^2 + 2\operatorname{codim} C + 2$ *and* $\operatorname{reg} C = \lceil (\deg C - 1)/\operatorname{codim} C \rceil + k(C)$, *then C lies on a rational normal surface scroll, that is, a Hirzeburch surface.*

The purpose of this paper is to study projective curves with next to sharp bounds of Castelnuovo-type on the Castelnuovo-Mumford regularity.

Theorem 1.2. *Let C be a nondegenerate projective curve over an algebraically closed field k with* $\operatorname{char} k = 0$. *Assume that C is not ACM, and* $\deg C \geq \max\{(\operatorname{codim} C)^2 + 4\operatorname{codim} C + 2, 13\}$. *If*

$$\operatorname{reg} C = \left\lceil \frac{\deg C - 1}{\operatorname{codim} C} \right\rceil + k(C) - 1,$$

then C lies either on a rational normal surface scroll or a normal del Pezzo surface.

Section 2 is devoted to the sketch of the proof of (1.2). The theorem states that a curve with next to the maximal regularity of Castelnuovo-type corresponds with a divisor on either a rational normal surface scroll or a del Pezzo surface. Invariants of the divisor on a rational normal surface scroll concerning the inequality is calculated to describe the curve with maximal regularity in [15]. On the other hand, a classical del Pezzo surface is defined to be a smooth surface $V (\subseteq \mathbb{P}_k^N)$ with $\deg V = \operatorname{codim} V + 2$ such that $\omega_V \cong \mathcal{O}_V(-1)$ is either the blowups of general $d (\leq 6)$ points of $\mathbb{P}_k^2$ or the 2-uple embbeding of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ to $\mathbb{P}_k^8$, see, e.g. [8, (4.7.1)]. A (not necessarily smooth) del Pezzo surface is classified by Fujita [4] and [5, (1.9.14)], see, e.g., [5, (1.6.3)] for the definition. In Section 3, we study some examples of divisors on a del Pezzo surface satisfying the equality in (1.2).

2. Sketch of the proof of the main theorem

Let us introduce the terminology for the zero-dimensional scheme. Let $X \subseteq \mathbb{P}_k^N$ a reduced zero-dimensional scheme such that X spans $\mathbb{P}_k^N$ as k-vector space. Then X is said to be in uniform position if $\mathrm{H}_Z(t) = \max\{\deg Z, \mathrm{H}_X(t)\}$ for all t, for any subscheme Z of X, where H_Z and H_X denote the Hilbert function of Z

and X respectively. Let R be the coordinate ring of a zero-dimensional scheme $X \subseteq \mathbb{P}_k^N$. Let $\underline{h} = \underline{h}(X) = (h_0, \cdots, h_s)$ be the h-vector of $X \subseteq \mathbb{P}_k^N$, where $h_i = \dim_k[R]_i - \dim_k[R]_{i-1}$ and s is the largest integer such that $h_s \neq 0$. Note that $s = \operatorname{reg} X - 1$.

Remark 2.1. For a generic hyperplane section X of a projective curve, $h_1 + \cdots + h_i \geq ih_1$ for all $i = 1, \cdots, s-1$ by [2]. A generic hyperplane section of a nondegenerate projective curve is in uniform position if $\operatorname{char} k = 0$, see [1]. If X is in uniform position, then $h_i \geq h_1$ for $i = 1, \cdots, s-1$, see [11, Section 4].

In this section, from now on, let C be a nondegenerate projective curve of $\mathbb{P}_k^{N+1}$ and H be a generic hyplerplane and $X = C \cap H \subseteq H \cong \mathbb{P}_k^N$. The following result (2.2) describes an extremal bound for the Castelnuovo-Mumford regularity of the generic hyperplane section of a projective curve $\operatorname{reg} X \leq \lceil (\deg X - 1)/N \rceil + 1$.

Lemma 2.2. (See [15, (2.6)]). *Let $X \subseteq \mathbb{P}_k^N$ be a generic hyperplane section of a nondegenerate projective curve. Assume that X is in uniform position and* $\deg X \geq N^2 + 2N + 2$. *If the equality* $\operatorname{reg} X = \lceil (\deg X - 1)/N \rceil + 1$ *holds, then X lies on a rational normal curve in $\mathbb{P}_k^N$.*

The extremal bound of the Castelnuovo-Mumford regularity for the generic hyperplane section of projective curve corresponds with a rational normal curve. The following lemma, which is obtained from Castelnuovo theory [9, Section 3], yields that the next extremal one corresponds with a elliptic normal curve. This reminds us of an analogy for the extremal and next to the extremal bound for the Castelnuovo-Mumford regularity towards the Eisenbud-Goto conjecture, see, e.g., [12] and the references there.

Lemma 2.3. *Let $X \subseteq \mathbb{P}_k^N$ be a generic hyperplane section of a nondegenerate projective curve. Assume that X is in uniform position and* $\deg X \geq N^2 + 4N + 2$. *If the equality* $\operatorname{reg} X = \lceil (\deg X - 1)/N \rceil$ *holds, then X lies on either a rational normal curve or an elliptic normal curve in $\mathbb{P}_k^N$.*

Remark 2.4. In the statement of (1.1), we may take an assumption that $\operatorname{reg} X = \lceil (\deg X - 1)/\operatorname{codim} X \rceil + 1$ for a generic hyperplane section X of C in place of the equality $\operatorname{reg} C = \lceil (\deg C - 1)/\operatorname{codim} C \rceil + k(C)$.

Proposition 2.5 ([22]). *Let $C(\subseteq \mathbb{P}_k^{N+1})$ be a nondegenerate projective curve over an algebraically closed field. Assume that C is not ACM. Then*

$$\operatorname{reg} C \leq \left\lceil \frac{\deg C - 1}{\operatorname{codim} C} \right\rceil + k(C).$$

Proof of Theorem 1.2. Let C be a nondegenerate projective curve in $\mathbb{P}_k^{N+1} = \operatorname{Proj} S$, where S be the polynomial ring and $\mathfrak{m}$ is the irrelevant ideal. Let $X = C \cap H$ be a generic hyperplane section. From the last line of the proof of (2.5), the equality $\operatorname{reg} C = \lceil (\deg C - 1)/\operatorname{codim} C \rceil + k(C)$ gives either $\operatorname{reg} X = \lceil (\deg X - 1)/\operatorname{codim} X \rceil + 1$ or $\operatorname{reg} X = \lceil (\deg X - 1)/\operatorname{codim} X \rceil$. By (2.2) and (2.3), X lies on either (i) a rational normal curve, or (ii) an elliptic normal curve. For the case (i), C is contained in a rational normal surface scroll from (1.1) and (2.4). Thus we have done in this case. Let us consider the case (ii). We may assume that X is contained in an elliptic normal curve Z in $H(\cong \mathbb{P}_k^N)$. Let $c = \operatorname{codim} C$ and $d = \deg C$. Then $\deg X = d$, $\operatorname{codim} X = c + 1$ and $\deg Z = \operatorname{codim} Z + 2 = c + 2$. For $c = 1$, Z is a plane smooth cubic curve. For $c \geq 2$, Z is generated by quadric equations.

In this paper we describe the proof for the case $c \geq 2$. First, we will show that $\Gamma(\mathcal{I}_{Z/H}(2)) \cong \Gamma(\mathcal{I}_{X/H}(2))$ if $c \geq 2$. Indeed, if there exists a hyperquadric Q such that $X \subseteq Q$ and $Z \nsubseteq Q$, then $X \subseteq Z \cap Q$ and $d \leq 2(c+2)$ by Bezout theorem, which contradicts the assumption $d \geq c^2 + 4c + 2$.

Next, we will show that $\Gamma(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}(2)) \to \Gamma(\mathcal{I}_{X/H}(2))$ is surjective if $c \geq 2$. Indeed, let $\varphi : \mathrm{H}^1_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}})(-1) \xrightarrow{\cdot h} \mathrm{H}^1_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}})$, where $h (\in [S]_1)$ is a linear form defining the hyperplane H. From the exact sequence

$$\begin{array}{ccccc} & & \Gamma_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}) & \to & \Gamma_*(\mathcal{I}_{X/H}) \\ \to & \mathrm{H}^1_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}})(-1) & \xrightarrow{\varphi} & \mathrm{H}^1_*(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}) & \to & \mathrm{H}^1_*(\mathcal{I}_{X/H}), \end{array}$$

we need to prove that $[\operatorname{Ker}\varphi]_2 = 0$ if $c \geq 2$. Then we see that $\Gamma(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}(2)) \to \Gamma(\mathcal{I}_{X/H}(2))$ is surjective if $c \geq 2$. By Socle Lemma[10, (3.11)], for a generic linear form $h (\in [S]_1)$ we have $a_-(\operatorname{Ker}\varphi) > a_-(\operatorname{Coker}\varphi)$, where $a_-(N) = \min\{\ell \mid [N]_\ell \neq 0\}$ for a graded S-module N. Hence we have $a_-(\operatorname{Ker}\varphi) > a_-(\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{X/H})))$.

Now let us evaluate $a_-(\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{X/H})))$. Since Z is ACM, we have the short exact sequence

$$0 \to \mathrm{H}^1_*(\mathcal{I}_{X/H}) \to \mathrm{H}^1_*(\mathcal{I}_{X/Z}) \to \mathrm{H}^2_*(\mathcal{I}_{Z/H}) \to 0$$

from the short exact sequence $0 \to \mathcal{I}_{Z/H} \to \mathcal{I}_{X/H} \to \mathcal{I}_{X/Z} \to 0$. Note that $\mathrm{H}^2_*(\mathcal{I}_{Z/H}) \cong \mathrm{H}^1_*(\mathcal{O}_Z) \cong k$. Now we will investigate the structure of $\mathrm{H}^1_*(\mathcal{I}_{X/Z})$. By Serre duality, $\mathrm{H}^1_*(\mathcal{I}_{X/Z})$ is isomorphic to the dual of $\Gamma_*(\mathcal{O}_Z(X))$. Hence $\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{X/Z}))$ is isomorphic to the dual of $\Gamma_*(\mathcal{O}_Z(X))/\mathfrak{m}\Gamma_*(\mathcal{O}_Z(X))$. Let $\mathcal{F} = \mathcal{O}_Z(X)$. Since Z is a smooth elliptic curve, we see that $\mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_Z(m-1)) = 0$ if $-d-(m-1)(c+2) < 0$. In other words, $\mathcal{F}$ is m-regular for $m \geq (c-d+3)/(c+2)$. Let $m = \lceil (c-d+3)/(c+2) \rceil$. Then we see that

$$\Gamma(\mathcal{F} \otimes \mathcal{O}_Z(\ell)) \otimes \Gamma(\mathcal{O}_Z(1)) \to \Gamma(\mathcal{F}(\ell+1))$$

is surjective for $\ell \geq m$ by [19]. Hence we obtain $a_-(\operatorname{Soc}(\mathrm{H}^1_*\mathcal{I}_{X/Z})) \geq -m$. Therefore, if $d \geq 3c+7$, then $a_-(\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{X/H}))) \geq 2$, and if $d \geq 4c+9$, $a_-(\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{X/H}))) \geq 3$. Since $d \geq \max\{c^2+4c+2, 13\}$, we obtain $[\operatorname{Ker}\varphi]_2 = 0$.

So, we have a surjective map $\Gamma(\mathcal{I}_{C/\mathbb{P}_k^{N+1}}(2)) \to \Gamma(\mathcal{I}_{X/H}(2)) \cong \Gamma(\mathcal{I}_{Z/H}(2))$. Note that Z is the intersection of the hyperquadrics containing X. Let Y' be the intersection of the hyperquadrics containing C. Since $Y' \cap H = Z$, there is an irreducible component Y of Y' such that $Y \cap H = Z$. Thus there exists a surface Y containing C such that $Y \cap H = Z$ and $\deg Y = \operatorname{codim} Y + 2$. Since a hyperplane section is an elliptic normal curve, Y is a normal surface. By [5, (1.6.5)], Y must be a normal del Pezzo surface. $\square$

Remark 2.6. Although I do not have counterexamples for the main theorem without the degree condition, the assumption $\deg C \gg 0$ seems to be indispensable. In fact, a non-hyperelliptic curve of genus $g \geq 5$ with the canonical embedding satisfies the extremal bound for ACM case, but not in a surface of minimal degree, see [23, page 160]. Moreover, there is a counterexample for (2.2) without degree condition, see [15, (2.6)].

3. Examples

We will study projective curves on some smooth del Pezzo surfaces with next to the extremal regularity.

Example 3.1. Let $V = \mathbb{P}^1_k \times \mathbb{P}^1_k$. Let π_1 and π_2 be the first and second projection respectively. We write $\mathcal{O}_V(a,b)$ for $\pi_1^*\mathcal{O}_V(a) \otimes \pi_2^*\mathcal{O}_V(b)$. Let Z_1 and Z_2 be divisors corresponding to $\mathcal{O}_V(1,0)$ and $\mathcal{O}_V(0,1)$ respectively. We have a 2-uple embedding of V by $H = 2Z_1 + 2Z_2$. Then V is a del Pezzo surface of degree 8 in $\mathbb{P}^8_k$. Let C be a divisor on V linearly equivalent to $a \cdot Z_1 + b \cdot Z_2$. We may assume $a \leq b$. By calculating the cohomologies $\mathrm{H}^i(\mathcal{I}_{C/V}(\ell H)) \cong \mathrm{H}^i(\mathcal{O}_V(-a+2\ell, -b+2\ell))$, $i = 1, 2$, by Künneth formula, we see that $[\mathrm{H}^1]_\ell \neq 0$ if and only if $a/2 \leq \ell \leq (b-2)/2$, and $[\mathrm{H}^2]_\ell \neq 0$ if and only if $\ell \leq (a-2)/2$. Assume that C is not ACM. Then we have $b \geq a+2$. In this case, we have $k(C) = \lfloor b/2 \rfloor - \lceil a/2 \rceil$, and $\operatorname{reg} C = \lfloor b/2 \rfloor + 1$. Also, we have $\deg C = 2a + 2b$. Thus there exists a curve C on V satisfying $\operatorname{reg} C = \lceil (\deg C - 1)/7 \rceil + k(C) - 1$ by choosing a and b such that $\lceil (a+4)/2 \rceil = \lceil (2a+2b-1)/7 \rceil$, while there are no such curves for $k(C)$ large enough.

Example 3.2. Let $\pi : V = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1_k$ be a projective bundle, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)$. Let Z be a minimal section of π corresponding to the natural map $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1_k}(-1)$ and F be a fibre corresponding to π. We have an embedding of V in $\mathbb{P}^8_k$ by a very ample sheaf corresponding to a divisor $H = 2 \cdot Z + 3 \cdot F$. Then V is a del Pezzo surface of degree 8 in $\mathbb{P}^8_k$. Let C be a divisor on V linearly equivalent to $a \cdot Z + b \cdot F$. From [15, (2.12)], $\mathrm{H}^1(V, \mathcal{O}_V(\alpha \cdot Z + \beta \cdot F)) \neq 0$ if and only if either $\alpha \geq 0$ and $\beta \leq \alpha - 2$, or $\alpha \leq -2$ and $\beta \geq \alpha + 1$. Thus $\mathrm{H}^1(\mathcal{I}_{C/V}(\ell H)) \neq 0$ if and only if either $a/2 \leq \ell \leq -a + b - 2$ or $-a + b + 1 \leq \ell \leq (a-2)/2$. From [15, (2.14)], $\mathrm{H}^2(V, \mathcal{O}_V(\alpha \cdot Z + \beta \cdot F)) \neq 0$ if and only if $\alpha \leq -2$ and $\beta \leq -3$. Thus $\mathrm{H}^2(\mathcal{I}_{C/V}(\ell H)) \neq 0$ if and only if $-a + 2\ell \leq -2$ or $-b + 3\ell \leq -3$. Hence we have $k(C) = b - \lceil 3a/2 \rceil - 1$ for $b \geq 3a/2 + 2$, and $k(C) = \lfloor 3a/2 \rfloor - b + 3$ for $b \leq 3a/2 + 2$. On the other hand, we have $\operatorname{reg} C = b - a$ for $b \geq 3a/2 + 2$, $\operatorname{reg} C = \lfloor a/2 \rfloor + 2$ for $3a/2 \leq b \leq 3a/2 + 2$, and $\operatorname{reg} C = \lfloor b/3 \rfloor + 2$ for $b \leq 3a/2$. Also, we have $\deg C = a + 2b$. For $b \leq 3a/2$, the equality $\operatorname{reg} C = \lceil (\deg C - 1)/7 \rceil + k(C) - 1$ is equivalent to saying that $\lfloor 4b/3 \rfloor = \lceil (a + 2b + 6)/7 \rceil + \lfloor 3a/2 \rfloor$ which does not happen for this case. For $3a/2 < b < 3a/2 + 2$, the equality $\operatorname{reg} C = \lceil (\deg C - 1)/7 \rceil + k(C) - 1$ is equivalent to saying that $\lceil (8a - 5b - 1)/7 \rceil = 0$, which does not happen if $\deg C \geq 79$. For $b \geq 3a/2 + 2$, the equality $\operatorname{reg} C = \lceil (\deg C - 1)/7 \rceil + k(C) - 1$ is equivalent to saying that $\lceil a/2 \rceil = \lceil (a + 2b - 15)/7 \rceil$. In this case, there exists a curve C on V satisfying $\operatorname{reg} C = \lceil (\deg C - 1)/7 \rceil + k(C) - 1$ by choosing a and b with $b \geq 3a/2 + 2$ such that $\lceil a/2 \rceil = \lceil (a + 2b - 15)/7 \rceil$, while there are no such curves with $k(C)$ large enough.

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A generalization of the Shestakov-Umirbaev inequality and polynomial automorphisms

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1 Introduction

Let k be a field, and $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k for some $n \in \mathbf{N}$. For $g \in k[\mathbf{x}]$ and a polynomial $\Phi = \sum_{i \in \mathbf{Z}_{\geq 0}} \phi_i T^i$ in a variable T over $k[\mathbf{x}]$, we define $\Phi(g) = \sum_{i \in \mathbf{Z}_{\geq 0}} \phi_i g^i$, where $\mathbf{Z}_{\geq 0}$ is the set of nonnegative integers, and $\phi_i \in k[\mathbf{x}]$ for each $i \in \mathbf{Z}_{\geq 0}$. Then, it follows that

$$\deg^g \Phi := \max\{\deg \phi_i g^i \mid i \in \mathbf{Z}_{\geq 0}\} \geq \deg \Phi(g)$$

in general. Here, $\deg f$ denotes the total degree of f for each $f \in k[\mathbf{x}]$. Shestakov-Umirbaev [10, Theorem 3] proved an inequality which describes the difference between $\deg^g \Phi$ and $\deg \Phi(g)$. Using this, they solved the following famous problem in [11].

Let $\sigma : k[\mathbf{x}] \to k[\mathbf{x}]$ be a homomorphism of k-algebras. Then, σ is isomorphic if and only if $k[\sigma(x_1), \ldots, \sigma(x_n)] = k[\mathbf{x}]$. For example, σ is isomorphic if there exist $(a_{i,j})_{i,j} \in GL_n(k)$ and $(b_i)_i \in k^n$ such that $\sigma(x_i) = \sum_{j=1}^n a_{i,j} x_j + b_i$ for each i. One can also check that σ is isomorphic if there exists $l \in \{1, \ldots, n\}$ such that $\sigma(x_i) = x_i$ for each $i \neq l$ and $\sigma(x_l) = \alpha x_l + f$ for some $\alpha \in k^\times$ and $f \in k[x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n]$. An automorphism of $k[\mathbf{x}]$ of the former form is said to be *affine*, and one of the latter form is said to be *elementary*. By the fact that an invertible matrix is expressed as a product of elementary matrices, it easily follows that each affine automorphism can be obtained by the composition of elementary automorphisms. Then, a problem arises whether the automorphism group $\mathrm{Aut}_k k[\mathbf{x}]$ can be generated by elementary automorphisms. If $n = 1$, then every automorphism of $k[\mathbf{x}]$ is in fact elementary. If $n = 2$, then $\mathrm{Aut}_k k[\mathbf{x}]$ is generated by elementary automorphisms, which was shown by Jung [2] in 1942 when k is of characteristic zero, and by van der Kulk [3] in 1953 for any k. We note that this result is a consequence of the following characterization of automorphisms.

Proposition 1.1 *If* $n = 2$*, then either* $\deg \sigma(x_1) | \deg \sigma(x_2)$ *or* $\deg \sigma(x_2) | \deg \sigma(x_1)$ *holds for each* $\sigma \in \mathrm{Aut}_k k[\mathbf{x}]$.

Here, $a|b$ denotes that b is divisible by a for each $a, b \in \mathbf{N}$.

When $n \geq 3$, the problem becomes extremely difficult. In 1972, Nagata [8] conjectured that the automorphism $\tau \in \mathrm{Aut}_k k[\mathbf{x}]$ for $n = 3$ defined by

$$\tau(x_1) = x_1 - 2(x_1x_3 + x_2^2)x_2 - (x_1x_3 + x_2^2)^2 x_3,\ \tau(x_2) = x_2 + (x_1x_3 + x_2^2)x_3,\ \tau(x_3) = x_3$$

cannot be obtained by the composition of elementary automorphisms of $k[\mathbf{x}]$. In spite of being well-known, this conjecture had been open for a long time. However, in 2004, Shestakov-Umirbaev [11] finally showed that the Nagata conjecture is true when k is of characteristic zero. The inequality mentioned at the beginning plays a crucial role in the solution of the conjecture. The problem is thus settled for $n = 3$, whereas it is still open for $n \geq 4$. We note that the extension $\tilde{\tau} \in \mathrm{Aut}_k k[\mathbf{x}]$ of the Nagata automorphism τ for $n \geq 4$ defined by $\tilde{\tau}(x_i) = \tau(x_i)$ for $i = 1, 2, 3$ and $\tilde{\tau}(x_i) = x_i$ for $i = 4, \ldots, n$ is a composite of elementary automorphisms (see [9]).

The argument in [11] is indeed difficult, but employs no advanced facts other than those in [10]. Therefore, the results in [10] is of great importance. However, its argument is also difficult, and, consequently, the proof of this landmark work of Shestakov-Umirbaev is unfortunately not understood widely.

The purpose of the present paper is to generalize the results of [10]. Our proof is quite simple and elegant, where the underlying concepts are completely clear. It will help us not only to understand the real meaning of the theory of Shestakov-Umirbaev, but also to generalize it to higher-dimensions toward the solution of the problem in case of $n \geq 4$. As an application of our result, we also give a generalization of Proposition 1.1 in Theorem 4.3.

Section 2 is devoted to formulating our theory. We give some consequences in Section 3, and apply it to characterizations of automorphisms of $k[\mathbf{x}]$ in Section 4.

We note that the author was recently working on Hilbert's Fourteenth Problem on the basis of an idea similar to that of this paper, and successfully gave several remarkable new counterexamples (see for example [4], [5], [6], [7]). This can suggest a certain effectivity of our method in the study of affine algebraic geometry. It should be mentioned that Makar-Limanov [1] also gave another proof of [10, Theorem 3] in a different way.

2 Machinery

In what follows, we always assume that k is of characteristic zero. First, we define some terminologies on the grading of a polynomial ring.

Let Γ be a totally ordered additive group, and $\mathbf{w} = (w_1, \ldots, w_n)$ an element of Γ^n. We define the $\mathbf{w}$-*weighted grading* $k[\mathbf{x}] = \bigoplus_{\gamma \in \Gamma} k[\mathbf{x}]_\gamma$ by setting $k[\mathbf{x}]_\gamma$ to be the k-vector space generated by $x_1^{a_1} \cdots x_n^{a_n}$ for $a_1, \ldots, a_n \in \mathbf{Z}_{\geq 0}$ with $\sum_{i=1}^n a_i w_i = \gamma$ for each $\gamma \in \Gamma$. Here, $l\gamma$ denotes the sum of l copies of γ for each $l \in \mathbf{Z}_{\geq 0}$ and $\gamma \in \Gamma$. Then, it follows that $k[\mathbf{x}]_\gamma k[\mathbf{x}]_\mu \subset k[\mathbf{x}]_{\gamma+\mu}$ for each $\gamma, \mu \in \Gamma$. Assume that $f = \sum_{\gamma \in \Gamma} f_\gamma$ is an element of $k[\mathbf{x}]$, where $f_\gamma \in k[\mathbf{x}]_\gamma$ for each γ. If $f \neq 0$, then the $\mathbf{w}$-*degree* $\deg_{\mathbf{w}} f$ of f is defined to be the maximum among $\gamma \in \Gamma$ with $f_\gamma \neq 0$. If $f = 0$, then we set $\deg_{\mathbf{w}} f = -\infty$, i.e., a symbol less than any element of Γ whose addition is defined by $(-\infty) + \gamma = \gamma + (-\infty) = -\infty$ for each $\gamma \in \Gamma \cup \{-\infty\}$. The sum of l copies of $-\infty$ is also denoted by $l(-\infty)$ for each $l \in \mathbf{Z}_{\geq 0}$. We say that f is $\mathbf{w}$-*homogeneous* if $f = f_\gamma$ for some γ. In case $f \neq 0$, we set $f^{\mathbf{w}} = f_\delta$, where $\delta = \deg_{\mathbf{w}} f$. Then, $(f_1 f_2)^{\mathbf{w}} = f_1^{\mathbf{w}} f_2^{\mathbf{w}}$ and $\deg_{\mathbf{w}}(f - f^{\mathbf{w}}) < \deg_{\mathbf{w}} f$ naturally hold for each $f_1, f_2, f \in k[\mathbf{x}] \setminus \{0\}$. We denote by $\Gamma_{\geq 0}$ the set of $\gamma \in \Gamma$ with $\gamma \geq 0$, where 0 is the zero of the additive group Γ. We remark that $\deg_{\mathbf{w}} f \geq 0$ holds for each $f \in k[\mathbf{x}] \setminus \{0\}$ whenever $\mathbf{w}$ is an element of $(\Gamma_{\geq 0})^n$. If $\Gamma = \mathbf{Z}$ and $\mathbf{w} = (1, \ldots, 1)$, then the $\mathbf{w}$-degree is the same as the total degree.

Now, for $\Phi \in k[\mathbf{x}][T]$ and $g \in k[\mathbf{x}]$, we define

$$\deg_{\mathbf{w}}^{g}(\Phi) = \max\{\deg_{\mathbf{w}}(\phi_i g^i) \mid i \in \mathbf{Z}_{\geq 0}\}, \tag{2.1}$$

where $\phi_i \in k[\mathbf{x}]$ for each i with $\Phi = \sum_i \phi_i T^i$. Then, $\deg_{\mathbf{w}} \Phi(g)$ is at most $\deg_{\mathbf{w}}^{g} \Phi$ in general. The purpose of this section is to give an inequality which describes the difference between $\deg_{\mathbf{w}} \Phi(g)$ and $\deg_{\mathbf{w}}^{g} \Phi$.

Let $\partial_T^i \Phi$ denote the i-th order derivative of Φ in T for each $i \in \mathbf{Z}_{\geq 0}$. Then, $\deg_{\mathbf{w}}^{g} \partial_T^i \Phi = \deg_{\mathbf{w}}(\partial_T^i \Phi)(g)$ if i is at least the degree $\deg_T \Phi$ of Φ in T. Hence, there always exists

$$m_{\mathbf{w}}^{g}(\Phi) = \min\{i \in \mathbf{Z}_{\geq 0} \mid \deg_{\mathbf{w}}^{g} \partial_T^i \Phi = \deg_{\mathbf{w}}(\partial_T^i \Phi)(g)\}. \tag{2.2}$$

If $m_{\mathbf{w}}^{g}(\Phi) \geq 1$ and $g \neq 0$, then

$$m_{\mathbf{w}}^{g}(\Phi) = m_{\mathbf{w}}^{g}(\partial_T^1 \Phi) + 1 \quad \text{and} \quad \deg_{\mathbf{w}}^{g} \Phi = \deg_{\mathbf{w}}^{g} \partial_T^1 \Phi + \deg_{\mathbf{w}} g, \tag{2.3}$$

since k is of characteristic zero.

Let $\Omega_{k[\mathbf{x}]/k}$ be the differential module of $k[\mathbf{x}]$ over k, and $\bigwedge^r \Omega_{k[\mathbf{x}]/k}$ the r-th exterior power of the $k[\mathbf{x}]$-module $\Omega_{k[\mathbf{x}]/k}$ for $r \in \{1, \ldots, n\}$. Then, each $\omega \in \bigwedge^r \Omega_{k[\mathbf{x}]/k}$ is uniquely expressed as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_r \leq n} f_{i_1, \ldots, i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r},$$

where $f_{i_1, \ldots, i_r} \in k[\mathbf{x}]$ for each $i_1, \ldots, i_r$. Here, df denotes the differential of f for each $f \in k[\mathbf{x}]$. We define the $\mathbf{w}$-degree of ω by

$$\deg_{\mathbf{w}} \omega = \max\{\deg_{\mathbf{w}}(f_{i_1, \ldots, i_r}) + w_{i_1} + \cdots + w_{i_r} \mid 1 \leq i_1 < \cdots < i_r \leq n\}. \tag{2.4}$$

Since $df = \sum_{i=1}^n (\partial f / \partial x_i) dx_i$ and k is of characteristic zero, it follows that

$$\deg_{\mathbf{w}} df = \max\left\{\deg_{\mathbf{w}}\left(\frac{\partial f}{\partial x_i}\right) + w_i \mid i = 1, \ldots, n\right\} = \deg_{\mathbf{w}} f \tag{2.5}$$

for each $f \in k[\mathbf{x}]$. It is readily verified that $\deg_{\mathbf{w}}(\omega + \omega') \leq \max\{\deg_{\mathbf{w}} \omega, \deg_{\mathbf{w}} \omega'\}$,

$$\deg_{\mathbf{w}}(\omega \wedge \eta) \leq \deg_{\mathbf{w}} \omega + \deg_{\mathbf{w}} \eta \quad \text{and} \quad \deg_{\mathbf{w}}(f\omega) = \deg_{\mathbf{w}} f + \deg_{\mathbf{w}} \omega \tag{2.6}$$

for each $f \in k[\mathbf{x}]$, and $\omega, \omega' \in \bigwedge^r \Omega_{k[\mathbf{x}]/k}$ and $\eta \in \bigwedge^s \Omega_{k[\mathbf{x}]/k}$ for $r, s \in \{1, \ldots, n\}$.

In the notation above, we have the following.

Theorem 2.1 *Let $f_1, \dots, f_r$ be elements of $k[\mathbf{x}]$ for some $r \in \mathbf{N}$ which are algebraically independent over k, and set $\omega = df_1 \wedge \cdots \wedge df_r$. Then, the inequality*

$$\deg_{\mathbf{w}} \Phi(g) \geq \deg_{\mathbf{w}}^g \Phi + m_{\mathbf{w}}^g(\Phi)(\deg_{\mathbf{w}}(\omega \wedge dg) - \deg_{\mathbf{w}} \omega - \deg_{\mathbf{w}} g) \tag{2.7}$$

holds for each $\Phi \in k[f_1, \dots, f_r][T] \setminus \{0\}$, $g \in k[\mathbf{x}] \setminus \{0\}$ and $\mathbf{w} \in \Gamma^n$.

Proof. Recall that, in case k is of characteristic zero, $h_1, \dots, h_s$ are algebraically independent over k if and only if $dh_1 \wedge \cdots \wedge dh_s \neq 0$ for $h_1, \dots, h_s \in k[\mathbf{x}]$ for $s \in \mathbf{N}$. Therefore, $\omega \neq 0$, while $\omega \wedge df_i = 0$ for $i = 1, \dots, n$. By chain rule, we may write $d(\Phi(g)) = (\partial_T^1 \Phi)(g) dg + \sum_{i=1}^r \psi_i df_i$, where $\psi_i \in k[\mathbf{x}]$ for each i. Hence, we get

$$\omega \wedge d(\Phi(g)) = (\partial_T^1 \Phi)(g) \omega \wedge dg + \sum_{i=1}^r \psi_i \omega \wedge df_i = (\partial_T^1 \Phi)(g) \omega \wedge dg. \tag{2.8}$$

By (2.5), (2.6) and (2.8), we have

$$\begin{aligned} \deg_{\mathbf{w}} \omega + \deg_{\mathbf{w}} \Phi(g) &= \deg_{\mathbf{w}} \omega + \deg_{\mathbf{w}} d(\Phi(g)) \geq \deg_{\mathbf{w}}(\omega \wedge d(\Phi(g))) \\ &= \deg_{\mathbf{w}}((\partial_T^1 \Phi)(g) \omega \wedge dg) = \deg_{\mathbf{w}}(\partial_T^1 \Phi)(g) + \deg_{\mathbf{w}}(\omega \wedge dg). \end{aligned} \tag{2.9}$$

Since $\omega \neq 0$, by adding $-\deg_{\mathbf{w}} \omega$ to the both sides of (2.9), we get

$$\deg_{\mathbf{w}} \Phi(g) \geq \deg_{\mathbf{w}}(\partial_T^1 \Phi)(g) + \deg_{\mathbf{w}}(\omega \wedge dg) - \deg_{\mathbf{w}} \omega. \tag{2.10}$$

Now, we prove (2.7) by induction on $m_{\mathbf{w}}^g(\Phi)$. If $m_{\mathbf{w}}^g(\Phi) = 0$, then $\deg_{\mathbf{w}} \Phi(g) = \deg_{\mathbf{w}}^g \Phi$ follows from the definition of $m_{\mathbf{w}}^g(\Phi)$. In this case, (2.7) is clear. Assume that $m_{\mathbf{w}}^g(\Phi) \geq 1$. Then, $m_{\mathbf{w}}^g(\partial_T^1 \Phi)$ is less than $m_{\mathbf{w}}^g(\Phi)$ by (2.3). Hence, by induction assumption combined with (2.3), we get

$$\deg_{\mathbf{w}}(\partial_T^1 \Phi)(g) \geq \deg_{\mathbf{w}}^g \partial_T^1 \Phi + m_{\mathbf{w}}^g(\partial_T^1 \Phi) M = (\deg_{\mathbf{w}}^g \Phi - \deg_{\mathbf{w}} g) + (m_{\mathbf{w}}^g(\Phi) - 1) M, \tag{2.11}$$

where $M = \deg_{\mathbf{w}}(\omega \wedge dg) - \deg_{\mathbf{w}} \omega - \deg_{\mathbf{w}} g$. By (2.10) and (2.11), we obtain that

$$\begin{aligned} \deg_{\mathbf{w}} \Phi(g) &\geq \deg_{\mathbf{w}}(\partial_T^1 \Phi)(g) + \deg_{\mathbf{w}}(\omega \wedge dg) - \deg_{\mathbf{w}} \omega \\ &\geq (\deg_{\mathbf{w}}^g \Phi - \deg_{\mathbf{w}} g) + (m_{\mathbf{w}}^g(\Phi) - 1) M + \deg_{\mathbf{w}}(\omega \wedge dg) - \deg_{\mathbf{w}} \omega \\ &= \deg_{\mathbf{w}}^g \Phi + m_{\mathbf{w}}^g(\Phi)(\deg_{\mathbf{w}}(\omega \wedge dg) - \deg_{\mathbf{w}} \omega - \deg_{\mathbf{w}} g). \end{aligned}$$

Therefore, (2.7) holds for any $m_{\mathbf{w}}^g(\Phi) \geq 0$. $\square$

3 Initial algebras

In this section, we give some consequences of Theorem 2.1.

First, we remark that the element $\deg_{\mathbf{w}}^{g} \Phi$ of Γ defined as in (2.1) is equal to the $(\mathbf{w}, \deg_{\mathbf{w}} g)$-degree of Φ for each $\Phi \in k[\mathbf{x}][T] \setminus \{0\}$, $g \in k[\mathbf{x}] \setminus \{0\}$ and $\mathbf{w} \in \Gamma$, where we regard Φ as a polynomial in the $n+1$ variables $x_1, \ldots, x_n$ and T over k. We denote $\Phi^{(\mathbf{w}, \deg_{\mathbf{w}} g)}$ by $\Phi^{\mathbf{w},g}$, for short.

Lemma 3.1 *Let $\Phi \in k[\mathbf{x}][T] \setminus \{0\}$, $g \in k[\mathbf{x}] \setminus \{0\}$ and $\mathbf{w} \in \Gamma$.*

(i) *The following conditions are equivalent:*

(1) $m_{\mathbf{w}}^{g}(\Phi) = 0$.

(2) $\deg_{\mathbf{w}}^{g} \Phi = \deg_{\mathbf{w}} \Phi(g)$.

(3) $\Phi^{\mathbf{w},g}(g^{\mathbf{w}}) \neq 0$.

(4) $\Phi(g) \neq 0$ *and* $\Phi(g)^{\mathbf{w}} = \Phi^{\mathbf{w},g}(g^{\mathbf{w}})$.

(ii) *It follows that* $m_{\mathbf{w}}^{g}(\Phi) = \min\{i \in \mathbf{Z}_{\geq 0} \mid (\partial_T^i(\Phi^{\mathbf{w},g}))(g^{\mathbf{w}}) \neq 0\}$.

Proof. (i) The equivalence between (1) and (2) is an immediate consequence of the definition of $m_{\mathbf{w}}^{g}(\Phi)$. We will establish that

$$\deg_{\mathbf{w}}(\Phi(g) - \Phi^{\mathbf{w},g}(g^{\mathbf{w}})) < \deg_{\mathbf{w}}^{g} \Phi \tag{3.1}$$

below. Assuming this, we can readily check that (2), (3) and (4) are equivalent, since

$$\Phi(g) = \Phi^{\mathbf{w},g}(g^{\mathbf{w}}) + (\Phi(g) - \Phi^{\mathbf{w},g}(g^{\mathbf{w}})),$$

and $\Phi^{\mathbf{w},g}(g^{\mathbf{w}})$ is contained in $k[\mathbf{x}]_{\delta}$, where $\delta = \deg_{\mathbf{w}}^{g} \Phi$. Write $\Phi = \sum_i \phi_i T^i$ and $\Phi^{\mathbf{w},g} = \sum_i \phi_i' T^i$, where $\phi_i, \phi_i' \in k[\mathbf{x}]$ for each i. Then, $\deg_{\mathbf{w}}(\phi_i g^i) \leq \deg_{\mathbf{w}}^{g} \Phi$ for each i by definition. Note that $\phi_i' = \phi_i^{\mathbf{w}}$ if $\deg_{\mathbf{w}}(\phi_i g^i) = \deg_{\mathbf{w}}^{g} \Phi$, and $\phi_i' = 0$ otherwise for each i. Hence, we have

$$\phi_i g^i - \phi_i'(g^{\mathbf{w}})^i = \phi_i g^i - \phi_i^{\mathbf{w}}(g^{\mathbf{w}})^i = \phi_i g^i - (\phi_i g^i)^{\mathbf{w}}$$

in the former case, and $\phi_i g^i - \phi_i'(g^{\mathbf{w}})^i = \phi_i g^i$ in the latter case. In each case, $\deg_{\mathbf{w}}^{g} \Phi$ is greater than the $\mathbf{w}$-degree of $\phi_i g^i - \phi_i'(g^{\mathbf{w}})^i$, and hence greater than that of

$$\sum_i (\phi_i g^i - \phi_i'(g^{\mathbf{w}})^i) = \Phi(g) - \Phi^{\mathbf{w},g}(g^{\mathbf{w}}).$$

Thus, we obtain (3.1), thereby proving that (2), (3) and (4) are equivalent.

(ii) Observe that $(\partial_T^i \Phi)^{\mathbf{w},g} = \partial_T^i(\Phi^{\mathbf{w},g})$ for each $i \in \mathbf{Z}_{\geq 0}$. Hence, $\deg_{\mathbf{w}}^g \partial_T^i \Phi = \deg_{\mathbf{w}}(\partial_T^i \Phi)(g)$ if and only if $(\partial_T^i(\Phi^{\mathbf{w},g}))(g^{\mathbf{w}}) \neq 0$ by the equivalence between (2) and (3) above. Accordingly, the assertion follows from the definition of $m_{\mathbf{w}}^g(\Phi)$. □

Now, let A be a k-subalgebra of $k[\mathbf{x}]$, and K the field of fractions of A. We define the *initial algebra* $A^{\mathbf{w}}$ of A for $\mathbf{w}$ to be the k-subalgebra of $k[\mathbf{x}]$ generated by $f^{\mathbf{w}}$ for $f \in A \setminus \{0\}$. Then, $\Phi^{\mathbf{w},g}$ belongs to $A^{\mathbf{w}}[T] \setminus \{0\}$ for each $\Phi \in A[T] \setminus \{0\}$ for any $g \in k[\mathbf{x}] \setminus \{0\}$. We claim that the field of fractions of $B^{\mathbf{w}}$ is equal to that of $A^{\mathbf{w}}$ whenever B is a k-subalgebra of $k[\mathbf{x}]$ whose field of fractions is equal to K. Indeed, if $fg_1 = g_2$ for $f \in A$ (resp. $f \in B$) and $g_1, g_2 \in B$ (resp. $g_1, g_2 \in A$), then we have $f^{\mathbf{w}} g_1^{\mathbf{w}} = (fg_1)^{\mathbf{w}} = g_2^{\mathbf{w}}$, so $f^{\mathbf{w}}$ belongs to the field of fractions of $B^{\mathbf{w}}$ (resp. $A^{\mathbf{w}}$). For this reason, we may denote the field of fractions of $A^{\mathbf{w}}$ by $K^{\mathbf{w}}$.

For a domain R and an element s of an over domain S of R, we define $I(R, s)$ to be the kernel of the substitution map $R[T] \ni f \mapsto f(s) \in S$. When $I(R, s)$ is a principal ideal of $R[T]$, we denote its generator by $P(R, s)$. We remark that $I(R, s)$ is always principal if R is a unique factorization domain. If R is a field and s is algebraic over R, then $P(R, s)$ is equal to the minimal polynomial of s over R up to multiplication by units.

Proposition 3.2 *Let A be a k-subalgebra of $k[\mathbf{x}]$, and K the field of fractions of A. For each $\Phi \in A[T] \setminus \{0\}$, $g \in k[\mathbf{x}] \setminus \{0\}$ and $\mathbf{w} \in \Gamma^n$, we have the following:*

(i) *If $g^{\mathbf{w}}$ is transcendental over $K^{\mathbf{w}}$, then $m_{\mathbf{w}}^g(\Phi) = 0$ and $\deg_{\mathbf{w}} \Phi(g) = \deg_{\mathbf{w}}^g \Phi$.*

(ii) *If $g^{\mathbf{w}}$ is algebraic over $K^{\mathbf{w}}$, then $m_{\mathbf{w}}^g(\Phi)$ is at most the quotient of $\deg_T \Phi^{\mathbf{w},g}$ divided by $[K^{\mathbf{w}}(g^{\mathbf{w}}) : K^{\mathbf{w}}]$. If furthermore $I(A^{\mathbf{w}}, g^{\mathbf{w}})$ is a principal ideal, then there exists $H \in A^{\mathbf{w}}[T] \setminus I(A^{\mathbf{w}}, g^{\mathbf{w}})$ such that $\Phi^{\mathbf{w},g} = P(A^{\mathbf{w}}, g^{\mathbf{w}})^{m_{\mathbf{w}}^g(\Phi)} H$.*

Proof. (i) If $g^{\mathbf{w}}$ is transcendental over $K^{\mathbf{w}}$, then $\Phi^{\mathbf{w},g}(g^{\mathbf{w}}) \neq 0$, since $\Phi^{\mathbf{w},g}$ is a nonzero element of $K^{\mathbf{w}}[T]$. Hence, $m_{\mathbf{w}}^g(\Phi) = 0$ and $\deg \Phi(g) = \deg^g \Phi$ due to Lemma 3.1(i).

(ii) Set $m = m_{\mathbf{w}}^g(\Phi)$, $P_K = P(K^{\mathbf{w}}, g^{\mathbf{w}})$, $P_A = P(A^{\mathbf{w}}, g^{\mathbf{w}})$ and $I_A = I(A^{\mathbf{w}}, g^{\mathbf{w}})$. By Lemma 3.1(ii), we have $(\partial_T^{m-1} \Phi^{\mathbf{w},g})(g^{\mathbf{w}}) = 0$ and $(\partial_T^m \Phi^{\mathbf{w},g})(g^{\mathbf{w}}) \neq 0$. Since k is of characteristic zero, this implies that $\Phi^{\mathbf{w},g} = P_K^m H$ for some $H \in K^{\mathbf{w}}[T]$ with $H(g^{\mathbf{w}}) \neq 0$.

By the assumption that $g^{\mathbf{w}}$ is algebraic over $K^{\mathbf{w}}$, it follows that $\deg_T P_K = [K^{\mathbf{w}}(g^{\mathbf{w}}) : K^{\mathbf{w}}]$. Thus, we get $\deg_T \Phi^{\mathbf{w},g} = m[K^{\mathbf{w}}(g^{\mathbf{w}}) : K^{\mathbf{w}}] + \deg_T H$. Therefore, $m^g_{\mathbf{w}}(\Phi)$ is at most the quotient of $\deg_T \Phi^{\mathbf{w},g}$ divided by $[K^{\mathbf{w}}(g^{\mathbf{w}}) : K^{\mathbf{w}}]$. Assume that I_A is a principal ideal. Write $\Phi^{\mathbf{w},g} = {P_A}^{m'} H'$, where $m' \in \mathbf{Z}_{\geq 0}$ and $H' \in A^{\mathbf{w}}[T] \setminus I_A$. Then, m' must be at most m, since P_A belongs to $P_K K^{\mathbf{w}}[T]$. On the other hand, P_A does not belong to ${P_K}^2 K^{\mathbf{w}}[T]$, for otherwise $\partial^1_T P_A$ would belong to $P_K K^{\mathbf{w}}[T] \cap A^{\mathbf{w}}[T] = I_A = P_A A^{\mathbf{w}}[T]$, a contradiction. Hence, m' must be at least m, since $H'(g^{\mathbf{w}}) \neq 0$. Thus, $m' = m$. This proves the latter part. $\square$

The following theorem is considered as a generalization of [10, Theorem 3].

Theorem 3.3 *Let $f_1, \ldots, f_r$ and g be nonzero elements of $k[\mathbf{x}]$ for some r with $f_1, \ldots, f_r$ algebraically independent over k, and let $A = k[f_1, \ldots, f_r]$, $K = k(f_1, \ldots, f_r)$ and $\omega = df_1 \wedge \cdots \wedge df_r$. For $\mathbf{w} \in \Gamma^n$ such that $\deg_{\mathbf{w}} h \geq 0$ for each $h \in A \setminus \{0\}$, put $M = \deg_{\mathbf{w}}(\omega \wedge dg) - \deg_{\mathbf{w}} \omega - \deg_{\mathbf{w}} g$. Then, the following holds for each $\Phi \in A[T] \setminus \{0\}$:*

(i) *Assume that $g^{\mathbf{w}}$ is algebraic over $K^{\mathbf{w}}$, and a and b are the quotient and residue of $\deg_T \Phi$ divided by $[K^{\mathbf{w}}(g^{\mathbf{w}}) : K^{\mathbf{w}}]$, respectively. Then, it follows that*

$$\deg_{\mathbf{w}} \Phi(g) \geq (\deg_T \Phi) \deg_{\mathbf{w}} g + aM = a\left([K^{\mathbf{w}}(g^{\mathbf{w}}) : K^{\mathbf{w}}] \deg_{\mathbf{w}} g + M\right) + b \deg_{\mathbf{w}} g. \quad (3.2)$$

(ii) *If $I(A^{\mathbf{w}}, g^{\mathbf{w}})$ is a principal ideal and $\deg_{\mathbf{w}} g \geq 0$, then*

$$\deg_{\mathbf{w}} \Phi(g) \geq m^g_{\mathbf{w}}(\Phi)(\deg^g_{\mathbf{w}} P(A^{\mathbf{w}}, g^{\mathbf{w}}) + M). \quad (3.3)$$

Proof. (i) In (3.2), the equality can easily be checked. We only show the inequality. By Theorem 2.1, we get $\deg_{\mathbf{w}} \Phi(g) \geq \deg^g_{\mathbf{w}} \Phi + m^g_{\mathbf{w}}(\Phi) M$. Hence, it suffices to verify that $\deg^g_{\mathbf{w}} \Phi \geq (\deg_T \Phi) \deg_{\mathbf{w}} g$ and $m^g_{\mathbf{w}}(\Phi) M \geq qM$. Note that $\deg^g_{\mathbf{w}} \Phi \geq \deg_{\mathbf{w}} \phi_e g^e$ by the definition of $\deg^g_{\mathbf{w}} \Phi$, where $e = \deg_T \Phi$ and ϕ_e is the coefficient of T^e in Φ. Since ϕ_e belongs to $A \setminus \{0\}$, it follows that $\deg_{\mathbf{w}} \phi_e \geq 0$ by assumption. Thus, we get

$$\deg^g_{\mathbf{w}} \Phi \geq \deg_{\mathbf{w}} \phi_e g^e = \deg_{\mathbf{w}} \phi_e + (\deg_T \Phi) \deg_{\mathbf{w}} g \geq (\deg_T \Phi) \deg_{\mathbf{w}} g.$$

On the other hand, we obtain $M \leq 0$ from (2.5) and (2.6). Moreover, $m^g_{\mathbf{w}}(\Phi) \leq a$ by Proposition 3.2(ii). Therefore, $m^g_{\mathbf{w}}(\Phi) M \geq aM$, proving the inequality in (3.2).

(ii) First, we claim that $\deg_{\mathbf{w}}^g \Psi \geq 0$ whenever Ψ is a nonzero element of $A[T]$ or $A^{\mathbf{w}}[T]$. In fact, we may write $\deg_{\mathbf{w}}^g \Psi = \deg_{\mathbf{w}} a + l \deg_{\mathbf{w}} g$, where $a \in A \setminus \{0\}$ and $l \in \mathbf{Z}_{\geq 0}$. By assumption, we have $\deg_{\mathbf{w}} a \geq 0$, $\deg_{\mathbf{w}} g \geq 0$, and thus $\deg_{\mathbf{w}}^g \Psi \geq 0$. Now, assume that $g^{\mathbf{w}}$ is transcendental over $K^{\mathbf{w}}$. Then, $m_{\mathbf{w}}^g(\Phi^{\mathbf{w},g}) = 0$ and $\deg_{\mathbf{w}} \Phi(g) = \deg_{\mathbf{w}}^g \Phi$ by Proposition 3.2(i). Hence, the right-hand side of (3.3) is zero, while $\deg_{\mathbf{w}}(\Phi(g)) \geq 0$, since $\deg_{\mathbf{w}}^g \Phi \geq 0$ as claimed. Therefore, (3.3) is true in this case. Next, assume that $g^{\mathbf{w}}$ is algebraic over $K^{\mathbf{w}}$. By Proposition 3.2(ii), we get $\Phi^{\mathbf{w},g} = P^m H$ for some $H \in A^{\mathbf{w}}[T]$, where $P = P(A^{\mathbf{w}}, g^{\mathbf{w}})$ and $m = m_{\mathbf{w}}^g(\Phi)$. Since $\deg_{\mathbf{w}} H \geq 0$ as claimed, we obtain

$$\deg_{\mathbf{w}}^g \Phi = \deg_{\mathbf{w}}^g \Phi^{\mathbf{w},g} = m \deg_{\mathbf{w}}^g P + \deg_{\mathbf{w}}^g H \geq m \deg_{\mathbf{w}}^g P.$$

Then, with the aid of this inequality, (3.3) follows from Theorem 2.1. $\square$

The following fact is well-known. We prove it at the end of this section.

Lemma 3.4 *Let f and g be $\mathbf{w}$-homogeneous elements of $k[\mathbf{x}]$ with $\deg_{\mathbf{w}} f > 0$ and $\deg_{\mathbf{w}} g > 0$ for some $\mathbf{w} \in \Gamma^n$. If f and g are algebraically dependent over k, then there exist mutually prime natural numbers $l(f,g)$ and $l(g,f)$ as follows:*

(i) $g^{l(f,g)} = \alpha f^{l(g,f)}$ *for some* $\alpha \in k$.

(ii) $I(k[f], g) = \left(T^{l(f,g)} - \alpha f^{l(g,f)}\right) k[f][T]$.

(iii) $[k(f)(g) : k(f)] = l(f,g)$.

(iv) $l(f,g) = (\deg_{\mathbf{w}} f) \gcd(\deg_{\mathbf{w}} f, \deg_{\mathbf{w}} g)^{-1}$ *if* $\Gamma = \mathbf{Z}$.

The inequality [10, Theorem 3] of Shestakov-Umirbaev is obtained as a corollary to Theorem 3.3.

Corollary 3.5 (Shestakov-Umirbaev) *Assume that $f, g \in k[\mathbf{x}] \setminus k$ satisfy $\deg_{\mathbf{w}} f > 0$ and $\deg_{\mathbf{w}} g > 0$ for some $\mathbf{w} \in \mathbf{Z}^n$. Then, for each $\Phi \in k[f][T] \setminus \{0\}$, it follows that*

$$\deg_{\mathbf{w}} \Phi(g) \geq a(\operatorname{lcm}(\deg_{\mathbf{w}} f, \deg_{\mathbf{w}} g) + M) + b \deg_{\mathbf{w}} g \tag{3.4}$$

where $M = \deg_{\mathbf{w}}(df \wedge dg) - \deg_{\mathbf{w}} f - \deg_{\mathbf{w}} g$, and a and b are the quotient and residue of $\deg_T \Phi$ divided by $(\deg_{\mathbf{w}} f) \gcd(\deg_{\mathbf{w}} f, \deg_{\mathbf{w}} g)^{-1}$, respectively.

Proof. Note that $k[f]^{\mathrm{w}} = k[f^{\mathrm{w}}]$ and $\deg_{\mathrm{w}} h \geq 0$ holds for each $h \in k[f] \setminus \{0\}$. In fact, if $h = \sum_{i=0}^{e} c_{e-i} f^i$, where $e \in \mathbf{Z}_{\geq 0}$ and $c_i \in k$ for $i = 0, \ldots, e$ with $c_0 \neq 0$, then $\deg_{\mathrm{w}} h = e \deg_{\mathrm{w}} f \geq 0$ and $h^{\mathrm{w}} = c_0 (f^{\mathrm{w}})^e$, since $\deg_{\mathrm{w}} f > 0$ by assumption. Consequently, we have $k(f)^{\mathrm{w}} = k(f^{\mathrm{w}})$. First, assume that f^w and g^w are algebraically dependent over k, and put $N = [k(f)^{\mathrm{w}}(g^{\mathrm{w}}) : k(f)^{\mathrm{w}}]$. Then, Theorem 3.3(i) gives

$$\deg_{\mathrm{w}} \Phi(g) \geq a'(N \deg_{\mathrm{w}} g + M) + b' \deg_{\mathrm{w}} g, \tag{3.5}$$

where a' and b' are the quotient and residue of $\deg_T \Phi$ divided by N, respectively. By Lemma 3.4, we have

$$N = \frac{\deg_{\mathrm{w}} f^{\mathrm{w}}}{\gcd(\deg_{\mathrm{w}} f^{\mathrm{w}}, \deg_{\mathrm{w}} g^{\mathrm{w}})} = \frac{\deg_{\mathrm{w}} f}{\gcd(\deg_{\mathrm{w}} f, \deg_{\mathrm{w}} g)} = \frac{\mathrm{lcm}(\deg_{\mathrm{w}} f, \deg_{\mathrm{w}} g)}{\deg_{\mathrm{w}} g}.$$

Thereby, $a' = a$ and $b' = b$, and (3.4) follows from (3.5). If f^w and g^w are algebraically independent over k, then $\deg_{\mathrm{w}} \Phi(g) = \deg_{\mathrm{w}}^g \Phi$ by Proposition 3.2(i). As in the proof of Theorem 3.3, we get $\deg_{\mathrm{w}}^g \Phi \geq (\deg_T \Phi) \deg_{\mathrm{w}} g$. Besides, $M \leq 0$ and the right-hand side of (3.4) is equal to $(\deg_T \Phi) \deg_{\mathrm{w}} g + aM$. Therefore, (3.4) is true in this case. $\square$

In the original statement of [10, Theorem 3], there appears the so-called "Poisson bracket" $[f, g]$ instead of $df \wedge dg$, but their degrees are defined in exactly the same way.

To conclude this section, we prove Lemma 3.4. We may easily deduce (ii), (iii) and (iv) from (i). So, we only show that there exist mutually prime natural numbers l and m such that $f^{-m} g^l$ belongs to k. Without loss of generality, we may assume that k is algebraically closed. In fact, $f^{-m} g^l$ necessarily belongs to k if $f^{-m} g^l$ is algebraic over k, since the field of fractions of $k[\mathbf{x}]$ is a regular extension of k.

By the assumption that f and g are algebraically dependent over k, we may find a nontrivial algebraic relation $\sum_{i,j} \beta_{i,j} f_0^i g_0^j = 0$, where $\beta_{i,j} \in k$ for each $i, j \in \mathbf{Z}_{\geq 0}$. Let J be the set of $(i, j) \in (\mathbf{Z}_{\geq 0})^2$ such that $\beta_{i,j} \neq 0$, and (i_0, j_0) and (i_1, j_1) the elements of J such that $i_0 \leq i \leq i_1$ for each $i \in \mathbf{Z}_{\geq 0}$ with $(i, j) \in J$ for some j. Since f and g are w-homogeneous, we may assume that $i \deg_{\mathrm{w}} f + j \deg_{\mathrm{w}} g$ are the same for any $(i, j) \in J$. Then, $(i_1 - i_0) \deg_{\mathrm{w}} g = (j_0 - j_1) \deg_{\mathrm{w}} f$. We note that $i_1 - i_0$ must be positive, for otherwise $J = \{(i_0, j_0)\}$, and then $0 = \sum_{(i,j) \in J} \beta_{i,j} f^i g^j = \beta_{i_0, j_0} f^{i_0} g^{j_0} \neq 0$, a contradiction. Since $\deg_{\mathrm{w}} f > 0$ and $\deg_{\mathrm{w}} g > 0$ by assumption, we get $j_0 - j_1 > 0$.

Set $l' = i_1 - i_0$, $m' = j_0 - j_1$ and $l = l'/e$, $m = m'/e$, where $e = \gcd(l', m')$. Then, J is contained in the set of $(i_0, j_0) + p(l, -m)$ for $p = 0, \ldots, e$. By putting $\beta'_p = \beta_{i_0+lp, j_0-mp}$ for each p, we get

$$0 = \sum_{(i,j)\in J} \beta_{i,j} f^i g^j = f^{j_0} g^{i_0} \sum_{p=0}^{e} \beta'_p (f^{-m} g^l)^p = \beta'_e f^{j_0} g^{i_0} \prod_{p=1}^{e} (f^{-m} g^l - \alpha_p),$$

where $\alpha_1, \ldots, \alpha_e \in k$ are the solutions of the algebraic equation $\sum_{p=0}^{e} \beta'_p T^p = 0$ in T. Hence, $f^{-m} g^l = \alpha_p$ for some p. Therefore, $f^{-m} g^l$ is contained in k. This completes the proof of Lemma 3.4.

4 Characterizations of polynomial automorphisms

As an application of our result, we study features of elements of $\mathrm{Aut}_k\, k[\mathbf{x}]$. Namely, we characterize n-tuples $\mathbf{f} = (f_1, \ldots, f_n)$ of elements of $k[\mathbf{x}]$ such that $k[f_1, \ldots, f_n] = k[\mathbf{x}]$.

First, we prove the following lemma.

Lemma 4.1 *Let $g_1, \ldots, g_r$ be elements of $k[\mathbf{x}]$ for some r. If $g_1^{\mathbf{w}}, \ldots, g_r^{\mathbf{w}}$ are algebraically independent over k for $\mathbf{w} \in \Gamma^n$, then $k[g_1, \ldots, g_r]^{\mathbf{w}} = k[g_1^{\mathbf{w}}, \ldots, g_r^{\mathbf{w}}]$.*

Proof. Clearly, $k[g_1, \ldots, g_r]^{\mathbf{w}}$ contains $k[g_1^{\mathbf{w}}, \ldots, g_r^{\mathbf{w}}]$. We show the reverse inclusion by induction on r. The assertion is obvious for $r = 0$; assume that $r \geq 1$. It suffices to verify that $h^{\mathbf{w}}$ belongs to $k[g_1^{\mathbf{w}}, \ldots, g_r^{\mathbf{w}}]$ for each $h \in k[g_1, \ldots, g_r] \setminus \{0\}$. Take $H \in A[T]$ such that $h = H(g_r)$, where $A = k[g_1, \ldots, g_{r-1}]$. Then, $H^{\mathbf{w}, g_r}$ belongs to $A^{\mathbf{w}}[T]$, while $A^{\mathbf{w}} = k[g_1^{\mathbf{w}}, \ldots, g_{r-1}^{\mathbf{w}}]$ by induction assumption. Hence, $H^{\mathbf{w}, g_r}(g_r^{\mathbf{w}})$ belongs to $k[g_1^{\mathbf{w}}, \ldots, g_r^{\mathbf{w}}]$. Since $g_1^{\mathbf{w}}, \ldots, g_r^{\mathbf{w}}$ are algebraically independent over k by assumption, it follows that $g_r^{\mathbf{w}}$ is transcendental over $k(g_1^{\mathbf{w}}, \ldots, g_{r-1}^{\mathbf{w}})$. Hence, $H^{\mathbf{w}, g_r}(g_r^{\mathbf{w}}) \neq 0$, and so $H(g_r)^{\mathbf{w}} = H^{\mathbf{w}, g_r}(g_r^{\mathbf{w}})$ by Lemma 3.1(i). Thus, $h^{\mathbf{w}}$ belongs to $k[g_1^{\mathbf{w}}, \ldots, g_r^{\mathbf{w}}]$. Therefore, $k[g_1, \ldots, g_r]^{\mathbf{w}} = k[g_1^{\mathbf{w}}, \ldots, g_r^{\mathbf{w}}]$ holds for any r. $\square$

As an immediate consequence of Lemma 4.1, we have the following.

Proposition 4.2 *Let $f_1, \ldots, f_n$ be elements of $k[\mathbf{x}]$ such that $k[f_1, \ldots, f_n] = k[\mathbf{x}]$. Then, $f_1^{\mathbf{w}}, \ldots, f_n^{\mathbf{w}}$ are algebraically independent over k if and only if $k[f_1^{\mathbf{w}}, \ldots, f_n^{\mathbf{w}}] = k[\mathbf{x}]$ for $\mathbf{w} \in \Gamma^n$.*

Proof. The "if" part is clear, for $k[\mathbf{x}]$ has transcendence degree n over k. Assume that $f_1^{\mathbf{w}}, \dots, f_n^{\mathbf{w}}$ are algebraically independent over k. Then, $k[f_1^{\mathbf{w}}, \dots, f_n^{\mathbf{w}}] = k[f_1, \dots, f_n]^{\mathbf{w}}$ by Lemma 4.1. Since $k[f_1, \dots, f_n] = k[\mathbf{x}]$, we have $k[f_1, \dots, f_n]^{\mathbf{w}} = k[\mathbf{x}]^{\mathbf{w}} = k[\mathbf{x}]$. Thus, $k[f_1^{\mathbf{w}}, \dots, f_n^{\mathbf{w}}] = k[\mathbf{x}]$. This proves the "only if" part. $\square$

Next, we consider the case where $k(f_1^{\mathbf{w}}, \dots, f_n^{\mathbf{w}})$ is of transcendence degree $n-1$ over k for some $\mathbf{w} \in \Gamma^n$. We define an element $\Delta_{\mathbf{f}}^{\mathbf{w}}$ of Γ as follows: Let $\lambda_{\mathbf{f}}^{\mathbf{w}} : k[\mathbf{x}] \to k[\mathbf{x}]$ be the homomorphism defined by $\lambda(x_i) = f_i^{\mathbf{w}}$ for $i = 1, \dots, n$. Since $k[\mathbf{x}]$ is a unique factorization domain, and $\ker_{\mathbf{f}}^{\mathbf{w}} \lambda$ is a prime ideal of $k[\mathbf{x}]$ of hight one, there exists $Q \in k[\mathbf{x}] \setminus \{0\}$ such that $\ker \lambda = Qk[\mathbf{x}]$. Then, we define $\Delta_{\mathbf{f}}^{\mathbf{w}}$ to be the $\mathbf{w}_{\mathbf{f}}$-degree of Q, where

$$\mathbf{w}_{\mathbf{f}} = (\deg_{\mathbf{w}} f_1, \dots, \deg_{\mathbf{w}} f_n).$$

Note that $\Delta_{\mathbf{f}}^{\mathbf{w}}$ is uniquely determined by $\mathbf{f}$ and $\mathbf{w}$, since Q is unique up to multiplication by elements in $k \setminus \{0\}$.

In the notation above, we have the following.

Theorem 4.3 *Let $f_1, \dots, f_n$ be elements of $k[\mathbf{x}]$ such that $k[f_1, \dots, f_n] = k[\mathbf{x}]$, and $\mathbf{w} = (w_1, \dots, w_n)$ an element of $(\Gamma_{\geq 0})^n$. If the transcendence degree of $k(f_1^{\mathbf{w}}, \dots, f_n^{\mathbf{w}})$ over k is $n-1$, then*

$$\sum_{i=1}^{n} \deg_{\mathbf{w}} f_i \geq \Delta_{\mathbf{f}}^{\mathbf{w}} + \sum_{i=1}^{n} w_i - \max\{w_i \mid i = 1, \dots, n\}, \tag{4.1}$$

where $\mathbf{f} = (f_1, \dots, f_n)$.

Proof. Since $k(f_1^{\mathbf{w}}, \dots, f_n^{\mathbf{w}})$ is of transcendence degree $n-1$ over k, we may find l such that x_l is not contained in $k[f_1^{\mathbf{w}}, \dots, f_n^{\mathbf{w}}]$. Moreover, we may assume that $f_1^{\mathbf{w}}, \dots, f_{n-1}^{\mathbf{w}}$ are algebraically independent over k by changing the indices of $f_1, \dots, f_n$ if necessary. Set $A = k[f_1, \dots, f_{n-1}]$ and $g = f_n$. Then, there exists $\Phi \in A[T]$ such that $\Phi(g) = x_l$, since $A[g] = k[\mathbf{x}]$ by assumption. Furthermore, $A^{\mathbf{w}} = k[f_1^{\mathbf{w}}, \dots, f_{n-1}^{\mathbf{w}}]$ by Lemma 4.1, and so $A^{\mathbf{w}}$ is a polynomial ring over k. Accordingly, the ideal $I(A^{\mathbf{w}}, g^{\mathbf{w}})$ of $A^{w}[T]$ is principal. Since $w_i \geq 0$ for $i = 1, \dots, n$ by assumption, $\deg_{\mathbf{w}} h \geq 0$ holds for each $h \in k[\mathbf{x}] \setminus \{0\}$ as mentioned in Section 2. Thus, $f_1, \dots, f_{n-1}$, g and $\mathbf{w}$ satisfy the assumptions of Theorem 3.3(ii). Therefore, we obtain

$$\max\{w_i \mid i=1,\ldots,n\} \geq w_l = \deg_{\mathbf{w}} x_l = \deg_{\mathbf{w}} \Phi(g) \geq m_{\mathbf{w}}^g(\Phi)(\deg_{\mathbf{w}}^g P + M). \tag{4.2}$$

Here, $P = P(A^{\mathbf{w}}, g^{\mathbf{w}})$ and

$$M = \deg_{\mathbf{w}}(\omega \wedge dg) - \deg_{\mathbf{w}} \omega - \deg_{\mathbf{w}} g \qquad \text{with} \qquad \omega = df_1 \wedge \cdots \wedge df_{n-1}.$$

Note that $\omega \wedge dg = df_1 \wedge \cdots \wedge df_n = D dx_1 \wedge \cdots \wedge dx_n$, where D is the determinant of the n by n matrix $(\partial f_i/\partial x_j)_{i,j}$. The assumption $k[f_1,\ldots,f_n] = k[\mathbf{x}]$ implies that D belongs to $k \setminus \{0\}$. Hence, we have $\deg_{\mathbf{w}}(\omega \wedge dg) = \deg_{\mathbf{w}} D + \sum_{i=1}^n w_i = \sum_{i=1}^n w_i$. By (2.5) and (2.6), it follows that $\deg_{\mathbf{w}} \omega \leq \sum_{i=1}^{n-1} \deg_{\mathbf{w}} df_i = \sum_{i=1}^{n-1} \deg_{\mathbf{w}} f_i$. Thus, we obtain that

$$M \geq \sum_{i=1}^n w_i - \sum_{i=1}^n \deg_{\mathbf{w}} f_i. \tag{4.3}$$

Due to (4.2) and (4.3), it remains only to show that $m_{\mathbf{w}}^g(\Phi) \geq 1$ and $\deg_{\mathbf{w}}^g P = \Delta_{\mathbf{f}}^{\mathbf{w}}$. First, suppose to the contrary that $m_{\mathbf{w}}^g(\Phi) = 0$. Then, from Lemma 3.1, we get $\Phi^{\mathbf{w},g}(g^{\mathbf{w}}) = \Phi(g)^{\mathbf{w}} = x_l^{\mathbf{w}} = x_l$. Recall that x_l does not belong to $k[f_1^{\mathbf{w}},\ldots,f_n^{\mathbf{w}}]$, and that $k[f_1^{\mathbf{w}},\ldots,f_n^{\mathbf{w}}] = A^{\mathbf{w}}[g^{\mathbf{w}}]$. Nevertheless, $\Phi^{\mathbf{w},g}(g^{\mathbf{w}})$ belongs to $A^{\mathbf{w}}[g^{\mathbf{w}}]$, since $\Phi^{\mathbf{w},g}$ belongs to $A^{\mathbf{w}}[T]$. This is a contradiction, and thus $m_{\mathbf{w}}^g(\Phi) \geq 1$. Let $\iota : k[\mathbf{x}] \to A^{\mathbf{w}}[T]$ be the homomorphism defined by $\iota(x_i) = f_i^{\mathbf{w}}$ for $i = 1,\ldots,n-1$ and $\iota(x_n) = T$, and Q an element of $k[\mathbf{x}]$ such that $\ker \lambda_{\mathbf{f}}^{\mathbf{w}} = Qk[\mathbf{x}]$. Then, ι is isomorphic, since we are assuming that $f_1^{\mathbf{w}},\ldots,f_{n-1}^{\mathbf{w}}$ are algebraically independent over k. Besides, the $\mathbf{w}_{\mathbf{f}}$-degree of Q is equal to the $(\mathbf{w}, \deg_{\mathbf{w}}^g)$-degree of $\iota(Q)$, which equals $\deg_{\mathbf{w}}^g \iota(Q)$ as mentioned. Consequently, we get $\Delta_{\mathbf{f}}^{\mathbf{w}} = \deg_{\mathbf{w}}^g \iota(Q)$. Since $\lambda_{\mathbf{f}}^{\mathbf{w}}$ is the composite of ι and the substitution map $A^w[T] \ni \psi \mapsto \psi(g^w) \in k[\mathbf{x}]$, we have

$$\iota(Qk[\mathbf{x}]) = \iota(\ker \lambda_{\mathbf{f}}^{\mathbf{w}}) = I(A^{\mathbf{w}}, g^{\mathbf{w}}) = PA^{\mathbf{w}}[T].$$

Thus, $\iota(Q) = \alpha P$ for some $\alpha \in k \setminus \{0\}$. Therefore, we obtain $\deg_{\mathbf{w}}^g P = \Delta_{\mathbf{f}}^{\mathbf{w}}$, thereby completing the proof. $\square$

Theorem 4.3 is considered as a generalization of Proposition 1.1. In fact, we have the following corollary in case $n = 2$.

Corollary 4.4 *Assume that* $f_1, f_2 \in k[x_1, x_2]$ *satisfy* $k[f_1, f_2] = k[x_1, x_2]$. *If* $f_1^{\mathbf{w}}$ *and* $f_2^{\mathbf{w}}$ *are algebraically dependent over* k *for an element* $\mathbf{w} = (w_1, w_2)$ *of* $(\mathbf{Z}_{\geq 0})^2$, *then it follows that* $\deg_{\mathbf{w}} f_1 > 0$, $\deg_{\mathbf{w}} f_2 > 0$, *and*

$$\deg_{\mathbf{w}} f_1 + \deg_{\mathbf{w}} f_2 \geq \operatorname{lcm}(\deg_{\mathbf{w}} f_1, \deg_{\mathbf{w}} f_2) + \min\{w_1, w_2\}. \tag{4.4}$$

In particular, $\deg_{\mathbf{w}} f_1 | \deg_{\mathbf{w}} f_2$ *or* $\deg_{\mathbf{w}} f_2 | \deg_{\mathbf{w}} f_1$.

Proof. Since $w_i \geq 0$ for $i = 1, 2$ by assumption, we have $\deg_{\mathbf{w}} f_i \geq 0$ for $i = 1, 2$ as mentioned in Section 2. We show that $\deg_{\mathbf{w}} f_i \neq 0$ for $i = 1, 2$ by contradiction. Suppose the contrary, say $\deg_{\mathbf{w}} f_1 = 0$. Then, $w_i = 0$ for some $i \in \{1, 2\}$, since f_1 cannot be contained in k. We claim that $\mathbf{w} \neq 0$. In fact, if $\mathbf{w} = 0$, then $f_i^{\mathbf{w}} = f_i$ for $i = 1, 2$. Nevertheless, $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ are algebraically dependent over k, whereas the assumption $k[f_1, f_2] = k[x_1, x_2]$ implies that f_1 and f_2 are algebraically independent over k, a contradiction. Hence, we have $w_j > 0$ for $j \in \{1, 2\} \setminus \{i\}$. Since we suppose that $\deg_{\mathbf{w}} f_1 = 0$, this implies that f_1 belongs to $k[x_i]$, and besides $f_1^{\mathbf{w}} = f_1$. Then, $f_2^{\mathbf{w}}$ also belongs to $k[x_i]$, since $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ are algebraically dependent over k. Consequently, f_2 belongs to $k[x_i]$ due to the conditions $w_i = 0$ and $w_j > 0$. Thus, $k[f_1, f_2]$ is contained in $k[x_i]$, a contradiction. Therefore, we conclude that $\deg_{\mathbf{w}} f_i > 0$ for $i = 1, 2$.

Put $P = P(k[f_1^{\mathbf{w}}], f_2^{\mathbf{w}})$ and $\mathbf{f} = (f_1, f_2)$. Then, we get $\Delta_{\mathbf{f}}^{\mathbf{w}} = \deg_{\mathbf{w}}^{f_2} P$ as in the proof of Theorem 4.3. By Lemma 3.4, we may write $P = \beta\left(T^{l(f_1, f_2)} - \alpha(f_1^{\mathbf{w}})^{l(f_2, f_1)}\right)$, where $\alpha, \beta \in k \setminus \{0\}$. Then, it is readily checked that $\deg_{\mathbf{w}}^{f_2} P = \operatorname{lcm}(\deg_{\mathbf{w}} f_1, \deg_{\mathbf{w}} f_2)$. Thus, $\Delta_{\mathbf{f}}^{\mathbf{w}} = \operatorname{lcm}(\deg_{\mathbf{w}} f_1, \deg_{\mathbf{w}} f_2)$. By Theorem 4.3, we obtain

$$\deg_{\mathbf{w}} f_1 + \deg_{\mathbf{w}} f_2 \geq \Delta_{\mathbf{f}}^{\mathbf{w}} + w_1 + w_2 - \max\{w_1, w_2\} = \Delta_{\mathbf{f}}^{\mathbf{w}} + \min\{w_1, w_2\}, \tag{4.5}$$

which yields (4.4). The last statement is a consequence of (4.4), since $a + b < \operatorname{lcm}(a, b)$ holds for each $a, b \in \mathbf{N}$ with $a \nmid b$ and $b \nmid a$. □

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Generic alternating matrices

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1 Introduction

DeConcini and Procesi [DP] considered the ring of absolute invariants of various classical groups including the symplectic group. Let $X = (X_{ij})$ be the $n \times n$ alternating matrix of indeterminates, T the $2m \times n$ matrix of indeterminates. They showed that the ring of absolute symplectic invariants is the algebra generated by the entries of ${}^tT\tilde{J}T$ (see §2 for notation). They also showed that this ring is isomorphic to the quotient ring of the polynomial ring with variables $\{X_{ij}\}$ by the ideal generated by $(2m+2)$-Pfaffians of X.

T is the $2m \times n$ matrix of full universal property. And the algebra generated by its minors is the homogeneous coordinate ring of a Grassmannian. On the other hand, there are universal matrices under certain conditions of minors. Maximal minors of these matrices generate the homogeneous coordinate ring of the Schubert subvarieties of a Grassmannian.

In this article, we consider the action of the symplectic group to such matrices. And show that the ring of absolute invariants is the algebra generated by the entries of an alternating matrix with certain universal property. We also show that this ring is a normal Cohen-Macaulay ring and give a combinatorial criterion of the Gorenstein property of this ring.

2 Preliminaries

All rings and algebras considered in this article are commutative with identity element.

Let k be a field. For an $m \times n$ matrix $M = (m_{ij})$ with entries in a k algebra S, we denote by $I_t(M)$ the ideal generated by all the t-minors of M, by $M^{(\leq i)}$ the $i \times n$ matrix consisting of first i-rows of M, by $M_{\leq j}$ the $m \times j$ matrix consisting of first j-columns of M, by $\Gamma(M)$ the set of maximal minors of M and by $k[M]$ the k-algebra generated by the entries of M.

For a positive integer l, we set $H(l) := \{[c_1, \ldots, c_r] \mid 1 \leq c_1 < \cdots < c_r \leq l,\ c_i \in \boldsymbol{Z} \text{ for } i = 1, \ldots, r\}$ and define the order relation on $H(l)$ by

$$\begin{aligned} & [c_1, \ldots, c_r] \leq [d_1, \ldots, d_s] \\ \stackrel{\text{def}}{\iff}\ & r \geq s,\ c_i \leq d_i \text{ for } i = 1, 2, \ldots, s. \end{aligned}$$

We also define the size of an element $[c_1, \ldots, c_r]$ to be r and denote $\mathrm{size}([c_1, \ldots, c_r]) = r$. Note that $H(l)$ is a distributive lattice.

We define $P(l) := \{\alpha \in H(l) \mid \mathrm{size}\alpha \equiv 0 \pmod 2\}$. It is clear that $P(l)$ is a sublattice of $H(l)$. We also define $\Delta(m \times n) := \{[\alpha|\beta] \mid \alpha \in H(m), \beta \in H(n), \mathrm{size}\alpha = \mathrm{size}\beta\}$ and the order relation on $\Delta(m \times n)$ by

$$\begin{aligned} & [\alpha|\beta] \leq [\alpha'|\beta'] \\ \overset{\mathrm{def}}{\Longleftrightarrow} \; & \alpha \leq \alpha' \text{ in } H(m) \text{ and } \beta \leq \beta' \text{ in } H(n). \end{aligned}$$

Note also that $\Delta(m \times n)$ is a distributive lattice.

For an $m \times n$ matrix $M = (m_{ij})$ and $[a_1, \ldots, a_r | b_1, \ldots, b_r] \in \Delta(m \times n)$, we set $[a_1, \ldots, a_r | b_1, \ldots, b_r]_M := \det(m_{a_i b_j})$.

If γ is an element of $H(l)$, we set $H(l; \gamma) := \{\delta \in H(l) \mid \delta \geq \gamma\}$. $P(l;\gamma)$ and $\Delta(m \times n; \delta)$ are defined similarly.

Now we recall the following

Fact 2.1 *If $T = (t_{ij})$ is an $n \times n$ alternating matrix, then*

$$\det T = \begin{cases} \mathrm{Pfaff}(T)^2 & n \equiv 0 \pmod 2, \\ 0 & n \equiv 1 \pmod 2. \end{cases}$$

Where

$$\mathrm{Pfaff}(T) = \sum_{\sigma \in S_n / H} (-1)^\sigma t_{\sigma(1)\sigma(2)} \cdots t_{\sigma(n-1)\sigma(n)}$$

and

$$H = \{\tau \in S_n \mid \forall i \exists j; \tau(\{2i-1, 2i\}) = \{2j-1, 2j\}\}.$$

$\mathrm{Pfaff}(T)$ is called the Pfaffian of T. Note that $(-1)^\sigma t_{\sigma(1)\sigma(2)} \cdots t_{\sigma(n-1)\sigma(n)}$ is independent of the choice of a representative σ of the left coset of S_n/H.

For $[c_1, \ldots, c_{2r}] \in P(n)$, we define $\langle c_1, \ldots, c_{2r} \rangle_T$ to be the Pfaffian of the submatrix of T consisting of rows and columns indexed by $c_1, \ldots, c_{2r}$.

Now we fix integers m and n such that $0 < 2m \leq n$ and $\gamma = [b_1, \ldots, b_{2m}] \in P(n)$. We set

$$W := \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1,2m} \\ W_{21} & W_{22} & \cdots & W_{2,2m} \\ \vdots & \vdots & \ddots & \vdots \\ W_{2m,1} & W_{2m,2} & \cdots & W_{2m,2m} \end{pmatrix},$$

$$U_\gamma := \begin{pmatrix} 0 & \cdots & 0 & U_{1b_1} & \cdots & U_{1b_2} & \cdots & \cdots & U_{1b_{2m}} & \cdots & U_{1n} \\ 0 & \cdots & 0 & 0 & \cdots & U_{2b_2} & \cdots & \cdots & U_{2b_{2m}} & \cdots & U_{2n} \\ & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & U_{2m,b_{2m}} & \cdots & U_{2m,n} \end{pmatrix},$$

where W_{ij}, U_{ij} are independent indeterminates, and

$$Z_\gamma := W U_\gamma.$$

Then

Lemma 2.2 (**[Miy1]**) *Z_γ is the universal $2m\times n$ matrix with $I_i((Z_\gamma)_{\le b_i-1})=(0)$ for $i=1,\ldots,2m$. i.e.,*

(1) $I_i((Z_\gamma)_{\le b_i-1})=(0)$ for $i=1,\ldots,2m$.

(2) If $I_i(M_{\le b_i-1})=(0)$ for $i=1,\ldots,2m$, then there is a unique k-algebra homomorphism $k[Z_\gamma]\to k[M]$ sending Z_γ to M.

It is also known that $k[\Gamma(Z_\gamma)]$ is the homogeneous coordinate ring of the Schubert subvariety, corresponding to γ, of a Grassmannian.

We set

$$J:=\begin{pmatrix} 0 & \cdots & 0 & 0 & 1\\ 0 & \cdots & 0 & 1 & 0\\ & & \cdot & & \\ & \cdot & & & \\ 1 & \cdots & & 0 & 0\end{pmatrix}\quad (m\times m \text{ matrix}),$$

$$\tilde{J}:=\begin{pmatrix} O & J\\ -J & O\end{pmatrix}$$

and

$$T_\gamma:={}^tZ_\gamma\tilde{J}Z_\gamma.$$

Theorem 2.3 (**[DP, Theorem 6.5]**) *Let $X=(X_{ij})$ be the $n\times n$ alternating matrix of indeterminates, i.e., $\{X_{ij}\}_{1\le i<j\le n}$ are independent indeterminates, $X_{ii}=0$ and $X_{ji}=-X_{ij}$. Then $k[X]$ is an ASL on $P(n)$.*

We set $T(X;\gamma):=k[X]/(P(n)\setminus P(n;\gamma))k[X]$. Then, by the general theory of ASL, we see

Corollary 2.4 *$T(X;\gamma)$ is an ASL on $P(n;\gamma)$.*

Note that the image $\overline{X}$ of X in $T(X;\gamma)$ is the universal $n\times n$ alternating matrix with $\langle\delta\rangle_{\overline{X}}=0$ for any $\delta\in P(n)\setminus P(n;\gamma)$.

Example 2.5 If $\gamma=[1,2,\ldots,2t-2]$, then $(P(n)\setminus P(n;\gamma))k[X]$ is the ideal generated by all $2t$ Pfaffians of X. In particular, if n is odd and $\gamma=[1,2,\ldots,n-3]$, then $(P(n)\setminus P(n;\gamma))k[X]$ is the Buchsbaum-Eisenbud type ideal of $k[X]$.

3 Properties of $T(X;\gamma)$

In this section, we show that $T(X;\gamma)$ is isomorphic to a subalgebra of a polynomial ring over k whose initial algebra is normal.

We begin with a lemma which can be proved by a direct computation. To state this lemma, we define the following notation. For $\delta=[a_1,\ldots,a_s]\in H(n)$ and $l\in \boldsymbol{Z}$, we set $l-\delta:=[l-a_s,\ldots,l-a_1]$.

Lemma 3.1 *If M is a $2m\times 2r$ matrix with $r\le m$, then*

$$\mathrm{Pfaff}({}^tM\tilde{J}M)=\sum_{\substack{\delta\in H(m)\\ \mathrm{size}\,\delta=r}}[\delta,2m+1-\delta|1,2,\ldots,2r]_M.$$

Note that if $[c_1, \ldots, c_{2r}] \not\geq \gamma$, then $[\delta, 2m+1-\delta | c_1, \ldots, c_{2r}]_{Z_\gamma} = 0$ for any $\delta \in H(m)$ with size$\delta = r$. Therefore $\langle c_1, \ldots, c_{2r} \rangle_{T_\gamma} = 0$ by the above lemma. So the k-algebra homomorphism $\varphi \colon k[X] \to k[T_\gamma]$ sending X to T_γ factors through $T(X;\gamma)$. We denote the induced map $T(X;\gamma) \to k[T_\gamma]$ by $\overline{\varphi}$.

We introduce the degree lexicographic monomial order on $k[W, U_\gamma]$ by $W_{11} > W_{12} > \cdots > W_{1,2m} > W_{21} > \cdots > W_{2m,2m} > U_{1b_1} > U_{1,b_1+1} > \cdots > U_{1n} > U_{2b_2} > \cdots > U_{2m,n}$. Note that this is a diagonal monomial order, i.e., the leading monomial of any minor of W (resp. U_γ) is the product of entries of its main diagonal.

Lemma 3.2 *(1) If $[d_1, \ldots, d_r] \geq \gamma$, then*

$$\mathrm{lm}([c_1, \ldots, c_r | d_1, \ldots, d_r]_{Z_\gamma}) = W_{c_1 1} W_{c_2 2} \cdots W_{c_r r} U_{1d_1} U_{2d_2} \cdots U_{rd_r}.$$

(2) If $[c_1, \ldots, c_{2r}] \geq \gamma$, then

$$\begin{aligned} & \mathrm{lm}(\langle c_1, \ldots, c_{2r} \rangle_{T_\gamma}) \\ = \; & W_{11} W_{22} \cdots W_{rr} W_{2m-r+1,r+1} W_{2m-r+2,r+2} \cdots W_{2m,2r} U_{1c_1} U_{2c_2} \cdots U_{2r,c_{2r}}. \end{aligned}$$

proof (1) follows from the following formula of linear algebra.

$$[c_1, \ldots, c_r | d_1, \ldots, d_r]_{WU_\gamma} = \sum_{\delta \in H(2m)} [c_1, \ldots, c_r | \delta]_W [\delta | d_1, \ldots, d_r]_{U_\gamma}.$$

And (2) follows from (1) and Lemma 3.1. ∎

Now let $\mu = [c_{11}, \ldots, c_{1,2r_1}][c_{21}, \ldots, c_{2,2r_2}] \cdots [c_{s1}, \ldots, c_{s,2r_s}]$ be a standard monomial on the poset $P(n;\gamma)$. Then one can reconstruct μ from the leading monomial $\mathrm{lm}(\mu_{T_\gamma}) = \prod_{i=1}^{s} \mathrm{lm}(\langle c_{i1}, \ldots, c_{i,2r_i} \rangle_{T_\gamma})$ of μ_{T_γ}. In praticular, if μ and μ' are different standard monomials on $P(n;\gamma)$, then $\mathrm{lm}(\mu_{T_\gamma}) \neq \mathrm{lm}(\mu'_{T_\gamma})$. Therefore $\{\mathrm{lm}(\mu_{T_\gamma}) \mid \mu \text{ is a standard monomial on } P(n;\gamma)\}$ is linearly independent over k. So we have the following

Theorem 3.3 *$\{\mathrm{lm}(\mu_{T_\gamma}) \mid \mu$ is a standard monomial on $P(n;\gamma)\}$ is a k-free basis of* $\mathrm{in}\, k[T_\gamma]$. *In particular, $\overline{\varphi}$ is an isomorphism and $\{\langle \delta \rangle_{T_\gamma} \mid \delta \in P(n;\gamma)\}$ is a sagbi basis of $k[T_\gamma]$.*

Corollary 3.4 *T_γ is the universal $n \times n$ alternating matrix with $\langle \delta \rangle_{T_\gamma} = 0$ for any $\delta \in P(n) \setminus P(n;\gamma)$.*

Since $\mathrm{lm}(\langle \delta \rangle_{T_\gamma}) \mathrm{lm}(\langle \delta' \rangle_{T_\gamma}) = \mathrm{lm}(\langle \delta \wedge \delta' \rangle_{T_\gamma}) = \mathrm{lm}(\langle \delta \vee \delta' \rangle_{T_\gamma})$, we see the following

Proposition 3.5 $\mathrm{in}\, k[T_\gamma]$ *is the Hibi ring on $P(n;\gamma)$. In particular,* $\mathrm{in}\, k[T_\gamma]$ *is normal and Cohen-Macaulay.*

So by the result of Conca-Herzog-Valla [CHV, Corollary 2.3],

Theorem 3.6 *$T(X;\gamma) \simeq k[T_\gamma]$ is a normal Cohen-Macaulay ring and has rational singularities if* $\mathrm{char}\, k = 0$ *and is F-rational if* $\mathrm{char}\, k > 0$.

4 Gorenstein property

In this section, we state a combinatorial criterion when $T(X;\gamma) \simeq k[T_\gamma]$ is Gorenstein.

First we recall the following result of Stanley.

Theorem 4.1 ([Sta, 4.4 Theorem]) *Let $A = \oplus_{n\geq 0} A_n$ be a Cohen-Macaulay graded domain such that A_0 is a field. Then A is Gorenstein if and only if* $\mathrm{Hilb}(A, \lambda^{-1}) = (-1)^d \lambda^\rho \mathrm{Hilb}(A,\lambda)$ *for some d, $\rho \in \boldsymbol{Z}$, where* $\mathrm{Hilb}(A,-)$ *is the Hilbert series of A.*

Since the Hilbert function of an ASL depends only on the generationg poset, we see

Corollary 4.2 *Assume that A and A' are Cohen-Macaulay ASL domains on the same poset P. Then A is Gorenstein if and only if so is A'.*

Now let D be a finite distributive lattice. We say $x \in D$ is join-irreducible if $x = \alpha \vee \beta$ implies $x = \alpha$ or $x = \beta$. Note that we regard the minimal element of D to be join-irreducible. We denote the set of all join-irreducible elements of D by P.

We recall here the criterion of Gorenstein property of a Hibi ring by Hibi.

Theorem 4.3 ([Hib]) *The Hibi ring $\mathcal{R}_k(D)$ is Gorenstein if and only if P is pure.*

By this result and Corollary 4.2, we see that $T(X;\gamma) \simeq k[T_\gamma]$ is Gorenstein if and only if the set of all join-irreducible elements of $P(n;\gamma)$ is pure. By analyzing the poset structure of $P(n;\gamma)$, we see the following

Proposition 4.4 *Set $P_1 := \{[b_1, \ldots, b_u, n-2s+u+1, \ldots, n] \mid s < m\}$, $P_2 := \{[b_1, \ldots, b_u, n-2s+u, \ldots, n-1] \mid s < m\}$ and $P_3 := \{[c_1, \ldots, c_{2m}] \mid \exists! i; c_i > b_i, c_i > c_{i-1}+1\}$ (here we set $c_0 = 0$). Then the set of all join-irreducible elements of $P(n;\gamma)$ is $P_1 \cup P_2 \cup P_3 \cup \{\gamma\}$.*

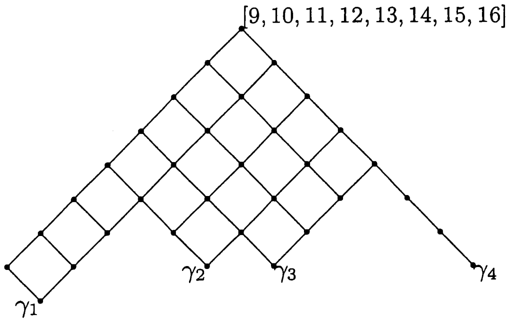
It is known that P_3 is anti-isomorphic to a finite poset ideal of $\boldsymbol{N} \times \boldsymbol{N}$, where the order of $\boldsymbol{N} \times \boldsymbol{N}$ is defined by componetwise [Miy2].

Now set $\{u \mid b_u + 1 < b_{u+1}\} = \{u_1, \ldots, u_t\}$ with $u_1 < \cdots < u_t$, where $b_{2m+1} := n+1$. Then the minimal elements of P_3 are $[b_1, \ldots, b_{u_i-1}, b_{u_i}+1, b_{u_i+1}, \ldots, b_{2m}]$ $(i = 1, 2, \ldots, t)$.

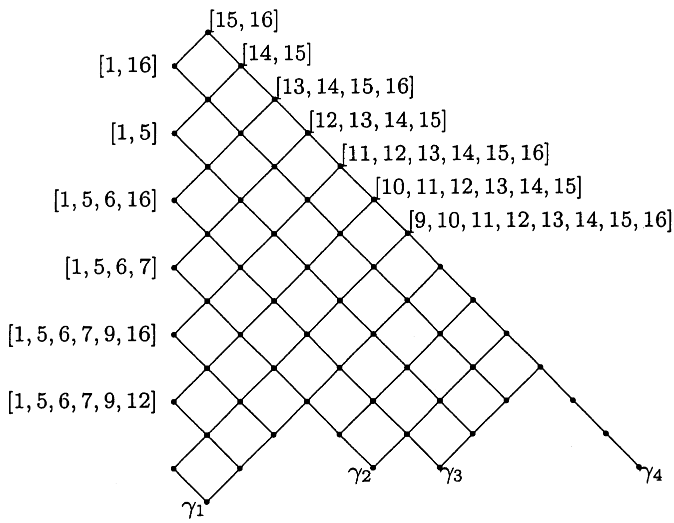
Example 4.5 (1) Let $m = 4$, $n = 16$ and $\gamma = [1,5,6,7,9,12,13,14]$. $[1,5,6,11,12,13,14,15]$ is an element of P_3 with $i = 4$.

Minimal elements of P_3 are $\gamma_1 = [1,5,6,7,9,12,13,15]$, $\gamma_2 = [1,5,6,7,10,12,13,14]$, $\gamma_3 = [1,5,6,8,9,12,13,14]$ and $\gamma_4 = [2,5,6,7,9,12,13,14]$.

The Hasse diagram of P_3 is



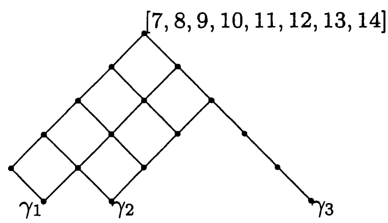
Since P_1 and P_2 are lined on upper left of P_3, the Hasse diagram of $P_1 \cup P_2 \cup P_3$ is



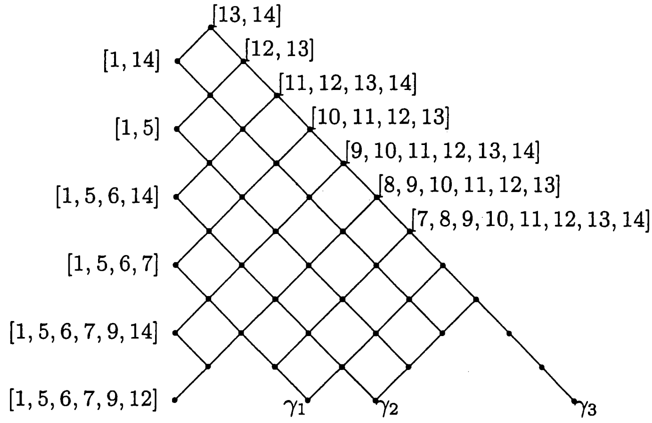
(2) Let $m = 4$, $n = 14$ and $\gamma = [1, 5, 6, 7, 9, 12, 13, 14]$.

Minimal elements of P_3 are $\gamma_1 = [1, 5, 6, 7, 10, 12, 13, 14]$, $\gamma_2 = [1, 5, 6, 8, 9, 12, 13, 14]$ and $\gamma_3 = [2, 5, 6, 7, 9, 12, 13, 14]$.

The Hasse diagram of P_3 is



and the Hasse diagram of $P_1 \cup P_2 \cup P_3$ is



We set $\chi_0 := \{1, 2, \ldots, b_1 - 1\}$, $B_1 := \{b_1, b_2, \ldots, b_{u_1}\}$, $\chi_1 := \{b_{u_1} + 1, b_{u_1} + 2, \ldots, b_{u_1+1} - 1\}$, $B_2 := \{b_{u_1+1}, b_{u_1+2}, \ldots, b_{u_2}\}$, $\chi_2 := \{b_{u_2} + 1, b_{u_2} + 2, \ldots, b_{u_2+1} - 1\}$, $B_3 := \{b_{u_2+1}, b_{u_2+2}, \ldots, b_{u_3}\}$, $\ldots$, $B_t := \{b_{u_{t-1}+1}, b_{u_{t-1}+2}, \ldots, b_{u_t}\}$, $\chi_t := \{b_{u_t} + 1, b_{u_t} + 2, \ldots, b_{u_t+1} - 1\}$ and $B_{t+1} := \{b_{u_t+1}, b_{u_t+2}, \ldots, b_{2m}\}$.

Then by the above observation, we see the following

Theorem 4.6 *(1) If $b_{2m} < n$ then $T(X;\gamma) \simeq k[T_\gamma]$ is Gorenstein if and only if $|B_i| = |\chi_{i-1}|$ for $i = 2, \ldots, t$.*

(2) If $b_{2m} = n$ and $m \geq 2$, then $T(X;\gamma) \simeq k[T_\gamma]$ is Gorenstein if and only if $|B_i| = |\chi_{i-1}|$ for $i = 2, \ldots, t$ and $|B_{t+1}| - 1 = |\chi_t|$.

(3) If $m = 1$ and $b_2 = n$, then $T(X;\gamma) \simeq k[T_\gamma]$ is isomorphic to the polynomial ring $k[X_{in} \mid b_1 \leq i < n]$.

5 Invariants

In this section, we consider the action of the symplectic group on $k[Z_\gamma]$ and prove that the ring of absolute invariants is $k[T_\gamma]$.

First we recall definitions.

Definition 5.1 $\mathrm{Sp}(2m, k) := \{A \in \mathrm{GL}(2m, k) \mid {}^t A \tilde{J} A = \tilde{J}\}$ is called the symplectic group.

Let A be an element of $\mathrm{Sp}(2m, k)$. Since

$$I_i((AZ_\gamma)_{\leq b_i - 1}) = (0) \quad \text{for } i = 1, 2, \ldots, 2m,$$

we see that there is a k-algebra automorphism of $k[Z_\gamma]$ mapping Z_γ to AZ_γ. So $\mathrm{Sp}(2m, k)$ acts on $k[Z_\gamma]$.

Definition 5.2 $f \in k[Z_\gamma]$ is called an absolute $\mathrm{Sp}(2m)$-invariant if for any k-algebra B, the image of f in $B[Z_\gamma]$ is an $\mathrm{Sp}(2m, B)$-invariant. The ring of absolute $\mathrm{Sp}(2m)$-invariants is denoted by $k[Z_\gamma]^{\mathrm{Sp}(2m,-)}$.

Now we state the following

Theorem 5.3 $k[Z_\gamma]^{\mathrm{Sp}(2m,-)} = k[T_\gamma]$.

For the proof of Theorem 5.3, we make some preparation. First we recall the following result.

Theorem 5.4 ([DP, Theorem 6.6]) *Let $Y = (Y_{ij})$ be a $2m \times u$ matrix of indeterminates. Then $k[Y]^{\mathrm{Sp}(2m,-)} = k[{}^tY\tilde{J}Y]$.*

Now let $W' = (W'_{ij})$ be a new $2m \times 2m$ matrix of indeterminates. Then it is well known that the k-algebra homomorphism $k[W] \to k[WW']$ mapping W to WW' is an isomorphism. So it follows from Theorem 5.4

Corollary 5.5 $k[WW']^{\mathrm{Sp}(2m,-)} = k[{}^t(WW')\tilde{J}(WW')]$.

Now set $Z'_\gamma := WW'U_\gamma$ and $T'_\gamma := {}^tZ'_\gamma\tilde{J}Z'_\gamma$. We introduce the degree lexicographic monomial order on $k[W, W', U_\gamma]$ by $W_{11} > W_{12} > \cdots > W_{1,2m} > W_{21} > \cdots > W_{2m,2m} > W'_{11} > W'_{12} > \cdots > W'_{1,2m} > W'_{21} > \cdots > W'_{2m,2m} > U_{1b_1} > U_{1,b_1+1} > \cdots > U_{1n} > U_{2b_2} > \cdots > U_{2m,n}$. Then

Theorem 5.6 *$\{\langle\delta\rangle_{T'_\gamma} \mid \delta \in P(n;\gamma)\}$ is a sagbi basis of $k[{}^t(WW')\tilde{J}(WW'), U_\gamma] \cap k[Z'_\gamma]$. In particular, $k[{}^t(WW')\tilde{J}(WW'), U_\gamma] \cap k[Z'_\gamma] = k[T'_\gamma]$.*

proof Set $\delta := [1, 2, \ldots, 2m|\gamma] \in \Delta(2m \times n)$. First note that $k[Z'_\gamma]$ is an ASL over k generated by $\Delta(2m \times n; \delta)$ and $k[{}^t(WW')\tilde{J}(WW'), U_\gamma]$ is an ASL over $k[U_\gamma]$ generated by $P(2m)$.

For $[c_1, \ldots, c_r|d_1, \ldots, d_r] \in \Delta(2m \times n; \delta)$,

$$\mathrm{lm}([c_1, \ldots, c_r|d_1, \ldots, d_r]_{Z'_\gamma}) = W_{c_11} \cdots W_{c_rr}W'_{11} \cdots W'_{rr}U_{1d_1} \cdots U_{rd_r},$$

and therefore, if ν is a standard monomial (in the sense of ASL) on $\Delta(2m \times n; \delta)$, then we can reconstruct ν from $\mathrm{lm}(\nu_{Z'_\gamma})$. In particular, if ν and ν' are different standard monomials, then $\mathrm{lm}(\nu_{Z'_\gamma}) \neq \mathrm{lm}(\nu'_{Z'_\gamma})$.

Now assume that $[c'_1, \ldots, c'_{2r}] \in P(2m)$. Then

$$\begin{aligned} & \mathrm{lm}(\langle c'_1, \ldots, c'_{2r}\rangle_{WW'}) \\ = \; & W_{11}W_{22} \cdots W_{rr}W_{2m-r+1,r+1}W_{2m-r+2,r+2} \cdots W_{2m,2r}W'_{1c'_1}W'_{2c'_2} \cdots W'_{2r,c'_{2r}} \end{aligned}$$

and therefore, if μ is a standard monomial on $P(2m)$, then we can reconstruct μ from $\mathrm{lm}(\mu_{WW'})$. In particular, if μ and μ' are different standard monomials, then $\mathrm{lm}(\mu_{WW'}) \neq \mathrm{lm}(\mu'_{WW'})$.

Now let f be an arbitrary non-zero element of $k[{}^t(WW')\tilde{J}(WW'), U_\gamma] \cap k[Z'_\gamma]$. If

$$f = \sum_\nu b_\nu \nu$$

is the standard representation of f in the ASL $k[Z'_\gamma]$, then there is a unique ν such that $\mathrm{lm}(f) = \mathrm{lm}(\nu)$. So the leading monomial of f is of the following form.

$$\prod_{i=1}^{t}\prod_{j=1}^{u_i} W_{c_{ij}j}W'_{jj}U_{jd_{ij}}$$

On the other hand, by considering the ASL $k[{}^t(WW')\tilde{J}(WW'), U_\gamma]$ over $k[U_\gamma]$, we see that the leading monomial of f is of the following form.

$$\prod_{i=1}^{s}(\prod_{j=1}^{r_i} W_{jj}W_{2m+1-j,2r_i+1-j})(\prod_{j=1}^{2r_i} W'_{jc'_{ij}}) \times (\text{a monomial of } U_{\bullet,\bullet})$$

Therefore, we see that $t = s$, $u_i = 2r_i$ for $i = 1, \ldots, s$, $c'_{ij} = j$ for $j = 1, \ldots, 2r_i$, $c_{ij} = j$ for $j = 1, \ldots, r_i$ and $c_{ij} = 2m - 2r_i + j$ for $j = r_i + 1, \ldots, 2r_i$.

This means that $\mathrm{lm}(f) = \mathrm{lm}(\langle v \rangle_{T'_\gamma})$ for some standard monomial v on $P(n;\gamma)$. The theorem follows. ∎

By Theorem 5.6 and Corollary 5.5, we see the following

Corollary 5.7 $k[Z'_\gamma]^{\mathrm{Sp}(2m,-)} = k[T'_\gamma]$.

Since $k[Z_\gamma] \simeq k[Z'_\gamma]$, Theorem 5.3 follows.

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The central simple modules and Lefschetz properties of Artinian Gorenstein algebras

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1. INTRODUCTION

In this report, we survey some recent results on the SLP Problem (see [6] and [7] for details).

A standard graded Artinian K-algebra $A = \oplus_{i=0}^{c} A_i$, where $A_c \neq (0)$, has the *Weak Lefschetz Property* (WLP) if there exists a linear form $g \in A_1$ such that the multiplication

$$\times g : A_i \longrightarrow A_{i+1}$$

is either injective or surjective for every $i = 0, 1, \ldots, c-1$. An algebra A has the *Strong Lefschetz Property* (SLP) if there exists a linear form $g \in A_1$ such that the multiplication

$$\times g^{c-2i} : A_i \longrightarrow A_{c-i}$$

is bijective for all $i = 0, 1, \ldots, [c/2]$.

We conjecture the following:

Conjecture 1.1. *ALL complete intersections have the SLP.*

A motivation to study the SLP Problem: There is a close relation between the SLP Problem and a problem on the classification of generic initial ideals. Here we use the reverse lexicographic order with $x_1 > x_2 > \cdots > x_n$. Wiebe's result is very important.

Proposition 1.2 ([10]). *Let $R = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables over a field K of characteristic zero, and let I be an Artinian homogeneous ideal of R. Then the following are equivalent.*

(1) *R/I has the SLP.*
(2) *$R/\mathrm{Gin}(I)$ has the SLP.*

In this case, the last variable x_n is a strong Lefschetz element of $R/\mathrm{Gin}(I)$.

We proved the following proposition a few years ago, but it is unpublished.

Proposition 1.3. *For any SI-sequence* $\mathbf{h} = (1, 3, h_2, \ldots, h_{c-2}, 3, 1)$*, there is only one Borel fixed ideal* I *such that* R/I *has the Hilbert function* $\mathbf{h}$ *and the SLP.*

Recently Cimpoeas [1] showed the same result as Proposition 1.3 for Hilbert functions of complete intersections of embedding dimension three .

Remark and Example 1.4. Proposition 1.3 does not hold in the case of embedding dimension grater than three. We would like to give such an example. Consider two Borel fixed ideals I and J of $R = K[x_1, x_2, x_3, x_4]$ as follows:

$$I = (x_1^2, x_1x_2, x_2^2, x_1x_3^2, x_2x_3^2, x_3^3, x_1x_3x_4, x_2x_3x_4, x_3^2x_4, x_1x_4^3, x_2x_4^3, x_3x_4^3, x_4^5),$$
$$J = (x_1^2, x_1x_2, x_2x_3, x_2^3, x_2^2x_3, x_2x_3^2, x_3^3, x_2^2x_4, x_2x_3x_4, x_3^2x_4, x_1x_4^3, x_2x_4^3, x_3x_4^3, x_4^5).$$

Then it is easy to show that R/I and R/J have the same Hilbert function $1, 4, 7, 4, 1$ and also have the SLP.

Noting that monomial complete intersections have the SLP ([9]), the following is an immediate consequence of Propositions 1.2 and 1.3.

Proposition 1.5. *Let* f_1, f_2, f_3 *be a homogeneous regular sequence of* $R = K[x_1, x_2, x_3]$*, where* $d_i = \deg(f_i)$ *for* $i = 1, 2, 3$*. Suppose that* $R/(f_1, f_2, f_3)$ *has the SLP. Then*

$$\mathrm{Gin}((f_1, f_2, f_3)) = \mathrm{Gin}((x_1^{d_1}, x_2^{d_2}, x_3^{d_3})).$$

Conjecture 1.6. *Let* $\{f_1, f_2, f_3\}$ *and* $\{g_1, g_2, g_3\}$ *be homogeneous regular sequences of* $R = K[x_1, x_2, x_3]$ *with the same degrees, i.e.,* $\deg(f_i) = \deg(g_i)$ *for* $i = 1, 2, 3$*. Then*

$$\mathrm{Gin}((f_1, f_2, f_3)) = \mathrm{Gin}((g_1, g_2, g_3)).$$

Remark 1.7. The following are equivalent.

(1) The conjectutre 1.6 is true.
(2) The conjecture 1.1 is true in the case of embedding dimension three.

2. Recent results

Let $A = \oplus_{i=0}^c A_i$ be a standard graded Artinian K-algebra. For a linear form $z \in A_1$, consider the associated graded ring

$$\mathrm{Gr}_{(z)}(A) = A/(z) \oplus (z)/(z^2) \oplus (z^2)/(z^3) \oplus \ldots \oplus (z^{p-1})/(z^p),$$

where p is the least integer such that $z^p = 0$. As is well known $\mathrm{Gr}_{(z)}(A)$ is endowed with a commutative ring structure. The multiplication in $\mathrm{Gr}_{(z)}(A)$ is given by

$$(a + (z^{i+1}))(b + (z^{j+1})) = ab + (z^{i+j+1}),$$

where $a \in (z^i)$ and $b \in (z^j)$.

For any homogeneous form $f \in R = K[x_1, \ldots, x_n]$, it is possible to write uniquely

$$f = f_0 + f_1x_n + f_2x_n^2 + \cdots + f_kx_n^k,$$

where f_i is a homogeneous form in $K[x_1, \ldots, x_{n-1}]$. Denote by $\mathrm{In}'(f)$ the term $f_jx_n^j$ for the minimal j such that $f_j \neq 0$. Furthermore we define $\mathrm{In}'(I)$ to be the homogeneous ideal of R generated by the set $\{\mathrm{In}'(f)\}$, where f runs over homogeneous forms of I. Suppose that $n = 2$. Then one notices that $\mathrm{In}'(I)$ coincides with the initial ideal $\mathrm{In}(I)$ of I with respect to the reverse lexicographic order with $x_1 > x_2$.

Let z be the image of x_n in $A = R/I$. Then it is easy to show that $\mathrm{Gr}_{(z)}(A) \cong R/\mathrm{In}'(I)$. So we consider that $\mathrm{Gr}_{(z)}(A)$ inherits a standard grading from R. Let $\mathrm{In}(I)$ be the initial ideal of I with respect to the reverse lexicographic order with $x_1 > \cdots > x_n$. Noting that $\mathrm{In}(\mathrm{In}'(I)) = \mathrm{In}(I)$, we have that all of $\mathrm{Gr}_{(z)}(A)$, $R/\mathrm{In}(I)$ and $A = R/I$ have the same Hilbert function.

Theorem 2.1. *Let K be a field of characteristic zero and let A be a standard graded Artinian K-algebra. Then A has the WLP (resp. SLP) if and only if $\mathrm{Gr}_{(z)}(A)$ has the WLP (resp. SLP) for some linear form $z \in A_1$.*

Remark 2.2. Let $\mathrm{In}(I)$ be the initial ideal of a homogeneous ideal I of $R = K[x_1, \dots, x_n]$ with respect to the reverse lexicographic order with $x_1 > \cdots > x_n$. Using a result of [2], it can be proved that if $R/\mathrm{In}(I)$ has the the SLP then so does R/I ([10], Proposition 2.9). The following example shows that Wiebe's result does not imply Theorem 2.1.

Let $R = K[x, y, z]$ and $I = (x^2, (x+y)^2, (x+y+z)^2)$. Put $A = R/I$. Then

$$\mathrm{Gr}_z(A) \cong R/(x^2, 2xy + y^2, xz + yz, y^3, y^2z, z^3)$$

has the SLP, but

$$R/\mathrm{In}(I) = R/(x^2, xy, xz, y^3, y^2z, z^3)$$

does not have the SLP (this can be checked, for instance, with the computer program Macaulay).

Let z be a linear form of A. The Jordan canonical matrix J of $\times z : A \longrightarrow A$ is of the following form,

$$J = \begin{bmatrix} J(0,n_1) & & & 0 \\ & J(0,n_2) & & \\ & & \ddots & \\ 0 & & & J(0,n_r) \end{bmatrix},$$

where $n_1 \geq n_2 \geq \cdots \geq n_r$ and $J(0, m)$ is the Jordan block of size $m \times m$. Let $(f_1, f_2, \dots, f_s)$ be the finest subsequence of $(n_1, n_2, \dots, n_r)$ such that $f_1 > f_2 > \cdots > f_s$. We rewrite the same sequence $(n_1, \dots, n_r)$ as

$$(n_1, \dots, n_r) = (\underbrace{f_1, \cdots f_1}_{m_1}, \underbrace{f_2, \dots, f_2}_{m_2}, \dots, \underbrace{f_s, \dots, f_s}_{m_s}).$$

Definition 2.3. We call the graded A-module

$$U_i = \frac{(0 : z^{f_i}) + (z)}{(0 : z^{f_{i+1}}) + (z)}$$

the *i-th central simple module* of (A, z), where $1 \leq i \leq s$ and $f_{s+1} = 0$. Note these are defined for a pair of the algebra A and a linear form $z \in A_1$.

Remark 2.4. By the definition, it is easy to see that the modules $U_1, U_2, \ldots, U_s$ are non-zero terms of the successive quotients of the descending chain of ideals

$$A = (0 : z^{f_1}) + (z) \supset (0 : z^{f_1 - 1}) + (z) \supset \cdots \supset (0 : z) + (z) \supset (z).$$

The Hilbert function of a graded vector space $V = \oplus_{i=a}^{b} V_i$ is the map $i \mapsto \dim V_i$. If V has finite dimension, then its Hilbert series is the polynomial

$$h_V(q) = \sum_{i=a}^{b} (\dim V_i) q^i.$$

Let $h(q)$ be a polynomial with coefficients of integers. We say that $h(q)$ is *symmetric* if $h(q) = q^d h(q^{-1})$ for some integer d. Then we call the half integer $d/2$ the *reflecting degree* of the symmetric polynomial $h(q)$.

Proposition 2.5. *Suppose that A is an Artinian Gorenstein K-algebra and let z be a linear form of A. Let $U_1, \ldots, U_s$ be the central simple modules of (A, z). Put*

$$\tilde{U}_i = U_i \otimes_K K[t]/(t^{f_i})$$

for $1 \leq i \leq s$. Then we have the following.

(1) $h_{\tilde{U}_i}(q) = h_{U_i}(q)(1 + q + q^2 + \cdots + q^{f_i - 1})$.
(2) $h_A(q) = \sum_{i=1}^{s} h_{\tilde{U}_i}(q)$.
(3) $h_{U_i}(q)$ *is symmetric for all $i = 1, 2, \ldots, s$.*
(4) $h_{\tilde{U}_i}(q)$ *is symmetric for all $i = 1, 2, \ldots, s$ with the same reflecting degree as that of $h_A(q)$.*
(5) *If all $h_{\tilde{U}_i}(q)$ are unimodal, then the Sperner number of A is the sum of the Sperner numbers of $\tilde{U}_i$.*

Definition 2.6. Let $A = \oplus_{i \geq 0} A_i$ be any graded K-algebra. Suppose that

$$V = \oplus_{i=a}^{b} V_i$$

is a graded Artinian A-module with $V_a \neq (0)$ and $V_b \neq (0)$.

(i) The A-module V has the *WLP* if there is a linear form $g \in A_1$ such that the multiplication $\times g : V_i \longrightarrow V_{i+1}$ is either injective or surjective for all $i = a, a+1, \ldots, b-1$.
(ii) The A-module V has the *SLP* if there is a linear form $g \in A_1$ such that the multiplication $\times g^{b-a-2i} : V_{a+i} \longrightarrow V_{b-i}$ is bijective for all $i = 0, 1, \ldots, [(b-a)/2]$.

We state the main theorem in this report.

Theorem 2.7. *Let K be a field of characteristic zero and let A be a standard graded Artinian Gorenstein K-algebra. Then the following conditions are equivalent.*

(i) *A has the SLP.*
(ii) *There exists a linear form z of A_1 such that all the central simple modules of (A, z) have the SLP.*

The following is an extension of Theorem 2.7.

Theorem 2.8. *Let K be a field of characteristic zero and let A be a standard graded Artinian K-algebra. Then the following conditions are equivalent.*

(i) *A has the SLP.*
(ii) *There exists a linear form z of A such that all the central simple modules U_i of (A,z) have the SLP and the reflecting degree of Hilbert function of $\widetilde{U}_i$ coincides with that of A for every $i = 1, 2, \ldots, s$.*

Theorem 2.9 is the main result in [4]. We can now give another simple proof using Theorem 2.8.

Theorem 2.9. *Let K be a field of characteristic zero, let B be a standard graded Artinian K-algebra and let A be a finite flat algebra over B such that the algebra map $B \longrightarrow A$ preserves grading. Assume that both B and A/mA have the SLP, where m is the maximal ideal of B. Then A has the SLP.*

To prove Theorem 2.9, we need a lemma.

Lemma 2.10. *We use the same notation as Theorem 2.9. Let z' be any linear form of B and put $z = \varphi(z')$. Let U_i' and U_i be the i-th central simple modules of (B, z') and (A, z), respectively. Then $U_i' \otimes_B A \cong U_i$.*

In the latest paper [7], we proved that the above theorems can be extended to Artinian algebras with non-standard grading.

3. Examples

Here are some examples of complete intersections which we can prove to have the SLP using Main Theorem 2.7.

Lemma 3.1. *Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K and let J be a homogeneous ideal of R such that R/J is a one dimensional Cohen-Macaulay K-algebra. Let g be a linear form of R which is not a zero divisor on R/J. Let d be a positive integer, and put $I = (J, g^d)$ and $A = R/I$. Let z be the image of g in A. Then (A, z) has only one central simple module which is isomorphic to $A/(z)$.*

Proof. It is easy to see that $I : g^j = (J, g^{d-j})$ for all $j = 0, 1, \ldots, d$. Hence, since $(0 : z^j) = (z^{d-j})$ for all $j = 0, 1, \ldots, d$, we have $(0 : z^j) + (z) = (z)$ for all $j = 0, 1, \ldots, d-1$. Thus (A, z) has only one central simple module

$$\frac{(0 : z^d) + (z)}{(0 : z^{d-1}) + (z)} = \frac{A}{(z)}.$$

□

Example 3.2. Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K of characteristic 0. Let r, s be positive integers. Put $f_i = x_i^r - x_{i+1}^r$ for $i = 1, 2, \ldots, n-1$. Let I be the ideal of R as follows:

$$I = (f_1, \ldots, f_{n-1}, x_n^s).$$

Then R/I has the strong Lefschetz property.

Proof. We have the isomorphism:

$$R/(f_1, \ldots, f_{n-1}, x_n) \cong K[x_1, \cdots, x_{n-1}]/(x_1^r, \ldots, x_{n-1}^r).$$

This shows that $R/(f_1, \ldots, f_{n-1})$ is a one-dimensional Cohen-Macaulay K-algebra and that the image of the element x_n is a non-zero-divisor. Moreover it is well known that the monomial complete intersection $K[x_1, \ldots, x_{n-1}]/(x_1^r, \ldots, x_{n-1}^r)$ has the strong Lefschetz property. Hence Lemma 3.1 immediately applies. $\square$

Example 3.3. As in the previous example, let $R = K[x_1, \cdots, x_n]$ be the polynomial ring over a field K of characteristic 0. We consider a complete intersection ideal as follows,

$$I = (f_1, f_2, g_3^{d_3}, g_4^{d_4}, \ldots, g_n^{d_n}),$$

where $g_3, \ldots, g_n$ are linear forms. Then $A = R/I$ has the SLP.

Proof. If $n \leq 2$, this is proved in Proposition 4.4 of [3]. Now we induct on n. Let $n \geq 3$. We may assume that $g_n = x_n$. Put $\overline{R} = R/x_nR$. Then

$$A/x_nA = R/(I, x_n) = \overline{R}/(\overline{f_1}, \overline{f_2}, \overline{g_3}^{d_3}, \ldots, \overline{g_{n-1}}^{d_{n-1}}),$$

where $\overline{f_i}$ and $\overline{g_j}$ are the images of f_i and g_j in $\overline{R}$. By the assumption of induction, we have that A/x_nA has the SLP. Hence, our assertion follows from Lemma 3.1 and Theorem 2.7. $\square$

Let $e_i = e_i(x_1, \ldots, x_n)$ be the elementary symmetric polynomial of degree i in the variables $x_1, \ldots, x_n$, i.e.,

$$e_i(x_1, \ldots, x_n) = \sum_{j_1 < j_2 < \cdots < j_i} x_{j_1} x_{j_2} \cdots x_{j_i}$$

for all $i = 1, 2, \ldots, n$. Let r and s be two positive integers. Put

$$\begin{cases} f_i = e_i(x_1^r, \ldots, x_n^r), \text{ for } i = 1, \ldots, n-1, \\ f_n = e_n(x_1^s, \ldots, x_n^s). \end{cases}$$

It is easy to see that the ideals $(e_1, e_2, \ldots, e_n)$ and $(f_1, f_2, \ldots, f_n)$ are complete intersections.

Example 3.4. With the same notation as above, let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K of characteristic 0. Put $I = (f_1, \ldots, f_n)$ and $A = R/I$. Suppose s is a multiple of r. Then the complete intersection A has the SLP.

Proof. Noting

$$(3.1) \qquad (-1)^{n+1}e_n + (-1)^n x_n e_{n-1} + (-1)^{n-1} x_n^2 e_{n-2} + \cdots + (-1)^2 x_n^{n-1} e_1 = x_n^n,$$

there exist polynomials $P_1, \ldots, P_n \in R$ such that

$$x_n^{rn} = P_1 f_1 + \cdots + P_{n-1} f_{n-1} + P_n x_1^r \cdots x_n^r.$$

Hence, since

$$(x_n^{rn})^{s/r} = (P_1 f_1 + \cdots + P_{n-1} f_{n-1} + P_n x_1^r \cdots x_n^r)^{s/r},$$

it follows that $x_n^{sn} \in I = (f_1, \ldots, f_{n-1}, f_n)$. Thus we obtain that

$$(f_1, \ldots, f_{n-1}, x_n^{sn}) \subset I.$$

On the other hand, noting that $R/(f_1, \ldots, f_{n-1}, x_n^{sn})$ and $A = R/I$ are two complete intersections with the same Hilbert function, we have

$$I = (f_1, \ldots, f_{n-1}, x_n^{sn}).$$

Let z be the image of x_n in A. Then we see that

$$A/(z) \cong K[x_1, \ldots, x_{n-1}]/(e'_1(x_1^r, \ldots, x_{n-1}^r), \ldots, e'_{n-1}(x_1^r, \ldots, x_{n-1}^r)),$$

where $e'_i(x_1, \ldots, x_{n-1})$ is the elementary symmetric function of degree i in the variables $x_1, \cdots, x_{n-1}$ for all $1 \leq i \leq n-1$. Here, inductively we may assume that $A/(z)$ has the SLP. Hence the SLP of A follows from Lemma 3.1 and Theorem 2.7. $\square$

Example 3.5. With the same notation as Example 3.4, suppose $s < r$. Then A has the SLP. In this case, taking the image z of x_n in A, it follows that (A, z) has only two central simple modules U_1 and U_2,

$$U_1 \cong K[x_1, \cdots, x_{n-1}]/(\overline{f}_1, \cdots, \overline{f}_{n-2}, (x_1 \cdots x_{n-1})^s),$$
$$U_2 \cong K[x_1, \cdots, x_{n-1}]/(\overline{f}_1, \cdots, \overline{f}_{n-2}, (x_1 \cdots x_{n-1})^{r-s}),$$

where $\overline{f}_j$ is the image of f_j in $K[x_1, \cdots, x_{n-1}]$.

Proof. First we show that

(1) $I : x_n^k = (f_1, \ldots, f_{n-1}, x_1^s \cdots x_{n-1}^s x_n^{s-k})$ for all $0 \leq k \leq s$,
(2) $I : x_n^k = (f_1, \ldots, f_{n-2}, x_n^{(n-1)r-(k-s)}, x_1^s \cdots x_{n-1}^s)$ for all $s < k < (n-1)r + s$,
(3) $I : x_n^k = R$ for all $k \geq (n-1)r + s$.

(1) is easy. So we give a proof of (2). Using the equation (3.1) we get the following relation

$$(-1)^{n+1} x_1^r \cdots x_n^r + (-1)^n x_n^r f_{n-1} + (-1)^{n-1} x_n^{2r} f_{n-2} + \cdots + (-1)^2 x_n^{(n-1)r} f_1 = x_n^{nr}.$$

Hence we have

$$(-1)^{n+1} x_1^r \cdots x_{n-1}^r + (-1)^n f_{n-1} + (-1)^{n-1} x_n^r f_{n-2} + \cdots + (-1)^2 x_n^{(n-2)r} f_1 = x_n^{(n-1)r}, \tag{3.2}$$

and

$$I : x_n^s = (f_1, \ldots, f_{n-1}, x_1^s \cdots x_{n-1}^s) = (f_1, \ldots, f_{n-2}, x_n^{(n-1)r}, x_1^s \cdots x_{n-1}^s).$$

Therefore, for all $s < k \leq (n-1)r + s$, it follows that

$$I : x_n^k = (I : x_n^s) : x_n^{k-s} = (f_1, \ldots, f_{n-2}, x_n^{(n-1)r-(k-s)}, x_1^s \cdots x_{n-1}^s).$$

Also, (3) is easy.

Next we calculate the central simple modules of (A, z). From (1), (2) and (3), we have

$$(I : x_n^k) + (x_n) = \begin{cases} (f_1, \ldots, f_{n-1}, x_n) & k = 0, 1, \ldots, s-1, \\ (f_1, \ldots, f_{n-1}, x_1^s \cdots x_{n-1}^s, x_n) & k = s, \\ (f_1, \ldots, f_{n-2}, x_1^s \cdots x_{n-1}^s, x_n) & k = s+1, s+2, \ldots, (n-1)r + s - 1, \\ R & k = (n-1)r + s, \ldots. \end{cases}$$

Here, noting the equality (3.2), it follows that

$$(f_1, \ldots, f_{n-1}, x_1^s \cdots x_{n-1}^s, x_n) = (f_1, \ldots, f_{n-2}, x_1^s \cdots x_{n-1}^s, x_n).$$

Hence we have

$$(0 : z^k) + (z) = \begin{cases} (z) & k = 0, 1, \ldots, s-1, \\ (\overline{x}_1^s \cdots \overline{x}_{n-1}^s, z) & k = s, s+1, \ldots, (n-1)r + s - 1, \\ A & k = (n-1)r + s, \ldots, \end{cases}$$

where $\overline{x}_j$ is the image of x_j in A. Thus we obtain that

$$U_1 = A/(\overline{x}_1^s \cdots \overline{x}_{n-1}^s, z) \cong K[x_1, \cdots, x_{n-1}]/(\overline{f}_1, \cdots, \overline{f}_{n-2}, (x_1 \cdots x_{n-1})^s)$$

and

$$\begin{aligned} U_2 &= (\overline{x}_1^s \cdots \overline{x}_{n-1}^s, z)/(z) \\ &\cong \frac{A/(z)}{(0 : \overline{x}_1^s \cdots \overline{x}_{n-1}^s) + (z)/(z)} \\ &\cong R/I + (x_n) : x_1^s \cdots x_{n-1}^s \\ &\cong K[x_1, \cdots, x_{n-1}]/(\overline{f}_1, \cdots, \overline{f}_{n-2}, (x_1 \cdots x_{n-1})^{r-s}). \end{aligned}$$

□

Finally we would like to give one more example. But we omit the proof.

Example 3.6. Let $p_d = x_1^d + x_2^d + \cdots + x_n^d$ be the power sum of degree d in the variables $x_1, x_2, \ldots, x_n$. Then $A = R/(p_a, p_{a+1}, \ldots, p_{a+n-1})$ is a complete intersection and has the SLP, for all positive integers a.

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Gorenstein property of a certain approximately Gorenstein local ring

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1 Introduction

This is a joint work with Thomas Marley and Mark W. Rogers.

It is well-known that a commutative Noetherian local ring $(A, \mathfrak{m})$ is Gorenstein if and only if A is Cohen-Macaulay and some ideal generated by a system of parameters (s.o.p.) is irreducible. Perhaps less widely known is a result of Northcott and Rees which states that if every ideal generated by an s.o.p. (henceforth called a *parameter ideal*) is irreducible then A is Cohen-Macaulay [NR, Theorem 1]. Thus, A is Gorenstein if and only if every parameter ideal is irreducible. There are, however, easy examples of non-Gorenstein rings possessing irreducible parameter ideals: $(y)A$ is irreducible in the local ring $A = K[[x, y]]/(x^2, xy)$, while $(y^n)A$ is reducible for all $n \geq 2$, where K is a field. In 1982, S. Goto showed that if there exists an s.o.p. $a_1, \ldots, a_d$ for A such that $(a_1^n, \ldots, a_d^n)$ is irreducible for all sufficiently large n then A is Gorenstein [G, Proposition 3.4]. Rephrasing Goto's result, if A is not Gorenstein then for every s.o.p. $a_1, \ldots, a_d$ of A there exists an integer n (depending on the s.o.p.) such that $(a_1^n, \ldots, a_d^n)$ is reducible. In light of this, given a non-Gorenstein local ring A it is natural to ask whether there exists a *uniform* exponent n such that $(a_1^n, \ldots, a_d^n)$ is reducible for all s.o.p.'s $a_1, \ldots, a_d$ of A. Our main result gives an affirmative answer to this question.

Theorem 1.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring, let M be a finitely generated A-module of dimension d. Then there exists an integer c such that M is Cohen-Macaulay with Cohen-Macaulay type $r(M) = 1$ if and only if some parameter ideal Q for M contained in $\mathfrak{m}^c$ has the property that QM is irreducible.*

Therefore when $M = A$ we have the following.

Corollary 1.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Then there exists an integer c such that A is Gorenstein if and only if some parameter ideal contained in $\mathfrak{m}^c$ is irreducible.*

Corollary 1.3. *A local ring A is Gorenstein if and only if every power of the maximal ideal contains an irreducible parameter ideal.*

A condition weaker than what is studied here was investigated by Hochster: A is called *approximately Gorenstein* if every power of $\mathfrak{m}$ contains an irreducible $\mathfrak{m}$-primary ideal. While approximately Gorenstein rings must have positive depth, they need not be Cohen-Macaulay. In fact, every complete Noetherian domain is approximately Gorenstein [Ho, Theorem 1.6].

2 Proof of Theorem 1.1.

Throughout let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and M be a finitely generated A-module with $d = \dim_A M$.

Let $a_1, a_2, \cdots, a_d \in \mathfrak{m}$ be a system of parameters for M. We denote by $\underline{a}^n$ the sequence $a_1^n, a_2^n, \cdots, a_d^n$. Let $\mathrm{K}_\bullet(\underline{a}^n)$ be the Koszul complex of A generated by the sequence $\underline{a}^n$ and let

$$\mathrm{H}^\bullet(\underline{a}^n; M) = H^\bullet(\mathrm{Hom}_A(\mathrm{K}_\bullet(\underline{a}^n), M))$$

be the Koszul cohomology module of M. Then for every $p \in \mathbb{Z}$ the family $\{\mathrm{H}^p(\underline{a}^n; M)\}_{n \geq 1}$ naturally forms an inductive system of A-modules, whose limit

$$\mathrm{H}^p_{\underline{a}}(M) = \lim_{n \to \infty} \mathrm{H}^p(\underline{a}^n; M)$$

is isomorphic to the local cohomology module

$$\mathrm{H}^p_{\mathfrak{m}}(M) = \lim_{n \to \infty} \mathrm{Ext}^p_A(A/\mathfrak{m}^n, M).$$

For each $n \geq 1$ and $p \in \mathbb{Z}$ let $\varphi^{p,n}_{\underline{a},M} : \mathrm{H}^p(\underline{a}^n; M) \to \lim\limits_{n \to \infty} \mathrm{H}^p_{\underline{a}}(M)$ denote the canonical homomorphism into the limit. With this notation we have the following.

Lemma 2.1 ([GSa2], Corollary 2.15). *There exists an integer $c > 0$ such that for all systems $a_1, a_2, \cdots, a_d \in \mathfrak{m}^c$ of parameters for M and for all $p \in \mathbb{Z}$ we have the canonical homomorphism*

$$\varphi^{p,1}_{\underline{a},M} : \mathrm{H}^p(\underline{a}; M) \to \mathrm{H}^p_{\underline{a}}(M) = \lim_{n \to \infty} \mathrm{H}^p(\underline{a}^n; M)$$

into the inductive limit is surjective on the socles, that is, the induced homomorphism

$$\left(\varphi^{p,1}_{\underline{a},M}\right)_* \ : \ \mathrm{Soc}_A(\mathrm{H}^p(\underline{a}; M)) \to \mathrm{Soc}_A(\mathrm{H}^p_{\underline{a}}(M))$$

is an epimorphism, where $\mathrm{Soc}_A(*)$ *denotes the socle* $\mathrm{Hom}_A(A/\mathfrak{m}, *) = (0) :_* \mathfrak{m}$.

Now we define $c(M)$ to be the least integer c with this property.

Proof. Passing to $A/\operatorname{Ann}_A M$, where $\operatorname{Ann}_A M$ denote the annihilator of M in A, we may assume that $\dim A = \dim_A M$. Then the assertion follows from [GSa1, Lemma (3.12)]. $\square$

We need one more Lemma to prove Theorem 1.1.

Lemma 2.2 ([St], Corollary 5.2.5). *Let $a_1, a_2, \cdots, a_r \in \mathfrak{m}$ $(r \geq 0)$. Assume that $(a_i^{n+1}, a_2^{n+1}, \cdots, a_r^{n+1})M :_M (\prod_{i=1}^r a_i)^n = (a_1, a_2, \cdots, a_r)M$ for every integer $n > 0$. Then $a_1, a_2, \cdots, a_r$ is an M-regular sequence.*

Proof. Take an integer $1 \leq i \leq r$ and fix it. It is enough to show the following.

Claim 1. *$(a_1, \cdots .a_{i-1})M :_M a_i \subseteq (a_i, \cdots, a_{i-1})M + (a_i, \cdots, a_r)^n M$ for every integer $n > 0$.*

Proof of Claim 1. It is easy to show

$$(\prod_{j=1}^{r} a_j)[(a_1, \cdots .a_{i-1})M :_M a_i] \subseteq (a_1^2, \cdots, a_{i-1}^2)M \subseteq (a_1^2, \cdots, a_r^2)M.$$

Hence we have

$$(a_1, \cdots .a_{i-1})M :_M a_i \subseteq (a_1^2, \cdots, a_r^2)M :_M (\prod_{j=1}^{r} a_j) = (a_1, \cdots, a_r)M$$

by our assumption. Now assume that $n \geq 2$ and our assertion holds true for $n-1$. Then

$$(a_1, \cdots .a_{i-1})M :_M a_i \subseteq (a_1, \cdots, a_{i-1})M + (a_i, \cdots, a_r)^{n-1}M.$$

Take an element $x \in (a_1, \cdots .a_{i-1})M :_M a_i$ and write

$$x = \sum_{j=1}^{i-1} a_j x_j + \sum_{\alpha \in \Lambda} a_i^{\alpha_i} \cdots a_r^{\alpha_r} x_\alpha$$

where $\Lambda = \{\alpha = (\alpha_i, \ldots, \alpha_r) \mid \alpha_i + \cdots + \alpha_r = n-1,\ \alpha_i, \ldots, \alpha_r \geq 0\}$ and $x_j, x_\alpha \in M$ $(1 \leq j \leq i-1,\ \alpha \in \Lambda)$. We want to show $x_\alpha \in (a_1, \ldots, a_r)M$ $(\alpha \in \Lambda)$. By the above equation we have

$$a_i x = \sum_{j=1}^{i-1} a_j a_i x_j + \sum_{\alpha \in \Lambda} a_i^{\alpha_i+1} a_{i+1}^{\alpha_{i+1}} \cdots a_r^{\alpha_r} x_\alpha.$$

On the other hand by the choice of x we can write

$$a_i x = \sum_{j=1}^{i-1} a_j y_j$$

where $y_j \in M$ $(1 \leq j \leq i-1)$. Now take $\alpha \in \Lambda$ and fix it. Then we have

$$a_i^{\alpha_i+1} a_{i+1}^{\alpha_{i+1}} \cdots a_r^{\alpha_r} x_\alpha = \sum_{j=1}^{i-1} a_j(y_j - a_i x_j) - \sum_{\alpha \neq \beta \in \Lambda} a_i^{\beta_i+1} a_{i+1}^{\beta_{i+1}} \cdots a_r^{\beta_r} x_\beta.$$

Set $b = (a_1 \cdots a_{i-1})^n a_i^{n-(\alpha_i+1)} a_{i+1}^{n-\alpha_{i+1}} \cdots a_r^{n-\alpha_r}$. For $1 \leq j \leq i-1$ we have

$$b(a_j(y_j - a_i x_j)) \in a_j^{n+1} M.$$

Take $\alpha \neq \beta \in \Lambda$. Then there is an integer $i \leq k \leq r$ such that $\beta_k \geq \alpha_k + 1$, and therefore we have

$$b(a_i^{\beta_i+1} a_{i+1}^{\beta_{i+1}} \cdots a_r^{\beta_r} x_\beta) \in a_k^{n+1} M.$$

Hence

$$\begin{aligned}
(a_1 \cdots a_r)^n x_\alpha &= b(a_i^{\alpha_i+1} a_{i+1}^{\alpha_{i+1}} \cdots a_r^{\alpha_r} x_\alpha) \\
&= \sum_{j=1}^{i-1} b(a_j(y_j - a_i x_j)) - \sum_{\alpha \neq \beta \in \Lambda} b(a_i^{\beta_i+1} a_{i+1}^{\beta_{i+1}} \cdots a_r^{\beta_r} x_\beta) \\
&\in (a_1^{n+1}, \ldots, a_r^{n+1}) M
\end{aligned}$$

and therefore $x_\alpha \in (a_1^{n+1}, a_2^{n+1}, \cdots, a_r^{n+1}) M :_M (\prod_{i=1}^r a_i)^n = (a_1, a_2, \cdots, a_r) M$ by our assumption. Then we have

$$x = \sum_{j=1}^{i-1} a_j x_j + \sum_{\alpha \in \Lambda} a_i^{\alpha_i} \cdots a_r^{\alpha_r} x_\alpha \in (a_1, \cdots, a_{i-1}) M + (a_i, \cdots, a_r)^n M$$

and proof is complete. $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $c = c(M)$. It is enough to show that if there exists a system of parameter $a_1, a_2, \ldots, a_d$ for M contained in $\mathfrak{m}^c$ such that $(a_1, \ldots, a_d)M$ is irreducible in M, then M is Cohen-Macaulay.

Thanks to Lemma 2.1 and our assumption, the canonical homomorphism

$$\varphi_{\underline{a},M}^{p,1} : \mathrm{H}^d(\underline{a}; M) \to \mathrm{H}_{\underline{a}}^d(M) = \lim_{n \to \infty} \mathrm{H}^d(\underline{a}^n; M)$$

into the inductive limit is surjective on the socles. Then we have

$$\dim_{A/\mathfrak{m}} \mathrm{Soc}_A(M/QM) = \dim_{A/\mathfrak{m}} \mathrm{Soc}_A(\mathrm{H}_{\mathfrak{m}}^d(M)) + \dim_{A/\mathfrak{m}} \mathrm{Soc}_A(\mathrm{Ker}\, \varphi_{\underline{a},M}^{d,1}),$$

where $Q = (a_1, \ldots, a_d)$. Since $\mathrm{H}_{\mathfrak{m}}^d(M)$ is nonzero Artinian, it has a nonzero socle. Since QM is irreducible in M, M/QM has a one-dimensional socle. Then we have $\dim_{A/\mathfrak{m}} \mathrm{Soc}_A(\mathrm{Ker}\, \varphi_{\underline{a},M}^{d,1}) = 0$. Hence $\mathrm{Ker}\, \varphi_{\underline{a},M}^{d,1} = (0)$ since $\mathrm{Ker}\, \varphi_{\underline{a},M}^{d,1}$ is Artinian. Then we have

$$(a_1^{n+1}, a_2^{n+1}, \cdots, a_d^{n+1}) M :_M (\prod_{i=1}^r a_i)^n = (a_1, a_2, \cdots, a_d) M$$

for every integer $n > 0$ and therefore $a_1, a_2, \ldots, a_d$ is an M-regular sequence by Lemma 2.2. Hence M is Cohen-Macaulay and $\mathrm{r}(M) = \dim_{A/\mathfrak{m}} \mathrm{Soc}_A(M/QM) = 1$. $\square$

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Gorenstein Rees algebras

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1 Introduction.

Let $(A, \mathfrak{m})$ be a Noetherian local ring and $d = \dim A$. Let $I(\subsetneq A)$ be an ideal of A. Assume that A is a homomorphic image of a Gorenstein local ring and that the field $A/\mathfrak{m}$ is infinite. In this paper we will discuss the Gorenstein Rees algebra $\mathrm{R}(I) := \bigoplus_{i \geq 0} I^i$. First of all, we state the following result that is concerned with the existence of Gorenstein Rees algebras of A.

Proposition 1.1. *Assume that A has finite local cohomology modules. Then there exists an ideal of A whose Rees algebra is Gorenstein if the ring A is quasi-Gorenstein.*

We say that the ring A has finite local cohomology modules when the ith local cohomology module of A with respect to $\mathfrak{m}$ is finitely generated for all integers $i < d$ and that the ring A is quasi-Gorenstein when the canonical module of A is a free A-module of rank 1.

For the existence of Cohen-Macaulay Rees algebras, we fortunately have Kawasaki's theorem on an arithmetic Cohen-Macaulayfication. In this paper we will show that a certain ideal of Kawasaki gives the Gorenstein Rees algebra under the conditions in Proposition 1.1. It will be proved in the last section 4 (see Theorem 4.1).

If an ideal Q is generated by a standard system of parameters, then the Rees algebra $\mathrm{R}(Q^k)$ is Cohen-Macaulay whenever $k \geq d-1$ and depth $A > 0$ (cf. [B2, GY, S1]). The following result whose proof will be given in the section 3 is concerned with such an ideal Q.

Proposition 1.2. *Let $\dim A \geq 2$ and let k be an integer with $k \geq d-1$. Assume that A is a quasi-Gorenstein ring has finite local cohomology modules. Let Q be an ideal generated by a standard system of parameters. Then the following two conditions are equivalent if the ring A satisfies Serre's condition* (S_3).

(1) $\mathrm{R}(Q^k)$ *is a Gorenstein ring.*

(2) A *is a Gorenstein ring and* $k = d-1$.

Now we consider a question of when the ring A is Gorenstein in case so is the Rees algebra $\mathrm{R}(I)$. In what follows let J be a minimal reduction of I. We denote the reduction number of I with respect to J by $\mathrm{r}_J(I)$. The analytic spread of I is $\ell(I) := \dim A/\mathfrak{m} \otimes_A \mathrm{R}(I)$. We put $r = \mathrm{r}_J(I)$ and $\ell = \ell(I)$. With this notation the third result of this paper can be stated as follows.

Theorem 1.3. *Assume that* $\mathrm{R}(I)$ *is a Gorenstein ring and* $\operatorname{grade} I \geq 2$. *Then the following two conditions are equivalent.*

(1) A *is a Gorenstein ring.*

(2) A *satisfies Serre's condition* (S_ℓ) *and* $r \leq \ell - 2$.

In general, if $\mathrm{R}(I)$ is a Cohen-Macaulay ring, then the inequality $r \leq \ell-1$ holds true. Therefore the result above means that if a non-Cohen-Macaulay ring A satisfies Serre's condition (S_ℓ), then $\mathrm{R}(I)$ is not a Gorenstein ring unless $r = \ell - 1$.

Our proof of the theorem above is based on a discussion of a^*-invariant formulas. To state our result of a^*-invariant formulas of the associated graded ring $\mathrm{G}(I) := \bigoplus_{i \geq 0} I^i/I^{i+1}$, let us set up a further notation. Set $G = \mathrm{G}(I)$ and $\mathfrak{M} = \mathfrak{m}G + G_+$. We denote by $\mathrm{H}^i_{\mathfrak{M}}(G)$ the ith graded local cohomology module of G with respect to $\mathfrak{M}$. Let $[\mathrm{H}^i_{\mathfrak{M}}(G)]_n$ be the homogeneous component of the graded module $\mathrm{H}^i_{\mathfrak{M}}(G)$ of degree n. We define that

$$\mathrm{a}_i(G) := \max\{n \in \mathbb{Z} \mid [\mathrm{H}^i_{\mathfrak{M}}(G)]_n \neq (0)\} \text{ and } \mathrm{a}^*(G) := \max\{\mathrm{a}_i(G) \mid i \in \mathbb{Z}\},$$

and call them the ith a-invariant and the a^*-invariant of G, respectively. We denote $\mathrm{a}_{\dim A}(G)$ simply by $\mathrm{a}(G)$. For each ideal L in A, let $\mathrm{r}(L) := \min\{\mathrm{r}_K(L) \mid K \text{ is a minimal reduction of } L\}$. Let $\mathrm{V}(I)$ be a set of prime ideals in A containing I. We put $\mathcal{A}(I) := \{\mathfrak{p} \in \mathrm{V}(I) \mid \lambda(I_\mathfrak{p}) = \dim A_\mathfrak{p}\}$ that is a finite set. The following fourth result is a generalization of a theorem due to [U].

Proposition 1.4. *Assume that* $\operatorname{depth} \mathrm{G}(I_\mathfrak{p}) \geq \min\{\dim A_\mathfrak{p}, \ell\}$ *for all* $\mathfrak{p} \in \mathrm{V}(I)$. *For each* $\mathfrak{p} \in \mathrm{V}(I)$, *let* $J(\mathfrak{p})$ *be a minimal reduction of* $I_\mathfrak{p}$ *and let* $\mathrm{r}_\mathfrak{p}(I) := \mathrm{r}_{J(\mathfrak{p})}(I_\mathfrak{p})$. *Then the following equalities hold true.*

$$\begin{aligned}
\mathrm{a}^*(G) &= \max\{\mathrm{r}_\mathfrak{p}(I) - \ell(I_\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{A}(I), \dim A_\mathfrak{p} < \ell\} \cup \{r - \ell\} \\
&= \max\{\mathrm{r}_\mathfrak{p}(I) - \ell(I_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{V}(I)\} \\
&= \max\{\mathrm{r}(I_\mathfrak{p}) - \ell(I_\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{A}(I), \dim A_\mathfrak{p} < \ell\} \cup \{r - \ell\} \\
&= \max\{\mathrm{r}(I_\mathfrak{p}) - \ell(I_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{V}(I)\}.
\end{aligned}$$

Moreover when the ring A *is quasi-unmixed, we have the equality*

$$\mathrm{a}^*(G) = \max\{\mathrm{a}(G),\ r - \ell\}.$$

Notice that even though $\operatorname{depth} \mathrm{G}(I_{\mathfrak{p}}) \geq \min\{\dim A_{\mathfrak{p}}, \ell\}$ for all $\mathfrak{p} \in \mathrm{V}(I)$, the equality $\mathrm{a}^*(G) = \max\{\mathrm{r}(I_{\mathfrak{p}}) - \ell(I_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathcal{A}(I)\}$ does not hold true in general (see the example below). If the Rees algebra $\mathrm{R}(I)$ is Cohen-Macaulay and if the base ring A satisfies Serre's condition (S_ℓ), then the assumption that $\operatorname{depth} \mathrm{G}(I_{\mathfrak{p}}) \geq \min\{\dim A_{\mathfrak{p}}, \ell\}$ for all $\mathfrak{p} \in \mathrm{V}(I)$ always holds true.

We will prove Theorem 1.3 by using the proposition above in the next section. We remark the condition that $r \leq \ell - 2$ cannot be removed in Theorem 1.3. There exists an example that the Rees algebra $\mathrm{R}(I)$ whose base ring A satisfies Serre's condition (S_ℓ) is Gorenstein but then A is not a Gorenstein ring. Let us close this section with such a example as follows.

Example 1.5. *Let k be a field of $\operatorname{char}(k) = 2$ and let $X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4$ be indeterminates over k. We consider the local ring $A = k[[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]]/L$ where L is an ideal generated by elements $X_1Y_1 + X_2Y_2 + X_3Y_3, Y_1Y_2 - X_3Y_4, Y_2Y_3 - X_1Y_4, Y_1Y_3 - X_2Y_4, Y_1^2, Y_2^2, Y_3^2, Y_4^2, Y_1Y_4, Y_2Y_4, Y_3Y_4$. Let $\mathfrak{m}$ denote the maximal ideal of A. Then $\mathrm{R}(\mathfrak{m})$ is Gorenstein, but A is not Cohen-Macaulay, see [I]. Thus A is a non-Gorenstein ring satisfies Serre's condition (S_2). Let x_i denote a reduction of X_i. Then we can find an ideal $I = (x_2, x_3) : x_1$ of A such that $\mathrm{R}(I)$ is a Gorenstein ring (see Theorem 4.1). And then $\ell(I) = \operatorname{grade} I = 2$. Moreover we have $\mathrm{a}^*(\mathrm{G}(I)) = -1$ and $\max\{\mathrm{r}(I_{\mathfrak{p}}) - \ell(I_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathcal{A}(I)\} = -2$, so that $\mathrm{a}^*(\mathrm{G}(I)) \neq \max\{\mathrm{r}(I_{\mathfrak{p}}) - \ell(I_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathcal{A}(I)\}$.*

2 A note on *-invariant formulas.

The purpose of this section is to prove Theorem 1.3. We should discuss a^*-invariant formulas here. Let $S = \bigoplus_{i \geq 0} S_i$ be a Noetherian graded algebra over the local ring $(S_0, \mathfrak{n})$ with infinite residue field, generated by elements of degree 1. Let $S_+ := \bigoplus_{i>0} S_i$ and put $\mathfrak{M} = \mathfrak{n}S + S_+$. For each graded ideal N in S and for each graded S-module M, we denote the ith graded local cohomology module of M with respect to N by $\mathrm{H}^i_N(M)$ and the homogeneous component of the graded module $\mathrm{H}^i_N(S)$ of degree n by $[\mathrm{H}^i_N(S)]_n$. We define

$$\mathrm{a}_i(S) := \max\{n \in \mathbb{Z} \mid [\mathrm{H}^i_{\mathfrak{M}}(S)]_n \neq (0)\} \text{ and}$$

$$\mathrm{a}^*(S) := \max\{\mathrm{a}_i(S) \mid i \in \mathbb{Z}\},$$

and call them respectively the ith a-invariant and the a^*-invariant of S. Write $\mathrm{a}(S) := \mathrm{a}_{\dim S}(S)$. Let $\mathrm{r}(S) := \min\{\mathrm{r}_L(S_+) \mid L$ is a minimal reduction of S_+ generated by elements of degree $1\}$ and let $\ell(S) := \dim S/\mathfrak{n}S$. We put $\mathcal{A}(S) := \{\mathfrak{p} \in \operatorname{Spec} S_0 \mid \ell(S_{\mathfrak{p}}) = \dim S_{\mathfrak{p}}\}$ that is a finite set. With this notation, we can state the following result that is a generalization of a theorem due to [U].

Proposition 2.1. *For each $\mathfrak{p} \in \operatorname{Spec} S_0$, let $Z(\mathfrak{p})$ be a minimal reduction of $[S_{\mathfrak{p}}]_+$ generated by elements of degree 1 and let $\mathrm{r}_{\mathfrak{p}}(S) := \mathrm{r}_{Z(\mathfrak{p})}([S_{\mathfrak{p}}]_+)$. Put*

$Z = Z(\mathfrak{n})$. *Assume that* $\operatorname{depth} S_\mathfrak{p} \geq \min\{\dim S_\mathfrak{p}, \ell(S)\}$ *for all* $\mathfrak{p} \in \operatorname{Spec} S_0$. *Then the following equalities hold true.*

$$\begin{aligned}
\mathrm{a}^*(S) &= \max\{\mathrm{r}_\mathfrak{p}(S) - \ell(S_\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{A}(S), \dim S_\mathfrak{p} < \ell(S)\} \cup \{\mathrm{r}_Z(S_+) - \ell(S)\} \\
&= \max\{\mathrm{r}_\mathfrak{p}(S) - \ell(S_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} S_0\} \\
&= \max\{\mathrm{r}(S_\mathfrak{p}) - \ell(S_\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{A}(S), \dim S_\mathfrak{p} < \ell(S)\} \cup \{\mathrm{r}_Z(S_+) - \ell(S)\} \\
&= \max\{\mathrm{r}(S_\mathfrak{p}) - \ell(S_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} S_0\}.
\end{aligned}$$

Moreover when the ring S is quasi-unmixed, we have the equality

$$\mathrm{a}^*(S) \quad = \quad \max\{\mathrm{a}(S), \mathrm{r}_Z(S_+) - \ell(S)\}.$$

To prove the proposition above, we begin by setting the following notation. For each $i \in \mathbb{Z}$, we put

$$\underline{a}_i(S) := \max\{n \in \mathbb{Z} \mid [\mathrm{H}^i_{S_+}(S)]_n \neq (0)\},$$

which we call the ith $\underline{a}$-invariant of S. The next lemma, which is an extension of results given by [JK] and [U] will play a key role in a proof of Proposition 2.1.

Lemma 2.2. *Assume that* $\operatorname{depth} S_\mathfrak{p} \geq \min\{\dim S_\mathfrak{p}, \ell(S)\}$ *for all* $\mathfrak{p} \in \operatorname{Spec} S_0$. *Let j be an integer with $j > \underline{a}_\ell(S)$. If* $[\mathrm{H}^{\dim S_\mathfrak{p}}_{[S_\mathfrak{p}]_+}(S_\mathfrak{p})]_j = (0)$ *for all* $\mathfrak{p} \in \operatorname{Spec} S_0$, *then* $[\mathrm{H}^i_N(S)]_j = (0)$ *whenever a graded ideal* $N \in \mathrm{V}(S_+)$ *and* $i \in \mathbb{Z}$ *with* $i \geq \ell(S)$.

Proof. Fix an integer $j > \underline{a}_\ell(S)$. Suppose that $\operatorname{depth} S_\mathfrak{p} \geq \min\{\dim S_\mathfrak{p}, \ell(S)\}$ and $[\mathrm{H}^{\dim S_\mathfrak{p}}_{[S_\mathfrak{p}]_+}(S_\mathfrak{p})]_j = (0)$ whenever $\mathfrak{p} \in \operatorname{Spec} S_0$. Then we have the following

Claim 2.3. $j > \underline{a}_{\ell(S_\mathfrak{p})}(S_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Spec} S_0$.

Proof of Claim 2.3. Let $\mathfrak{p} \in \operatorname{Spec} S_0$. We want to show $[\mathrm{H}^{\ell(S_\mathfrak{p})}_{[S_\mathfrak{p}]_+}(S_\mathfrak{p})]_j = (0)$. Notice that $\ell(S) \geq \ell(S_\mathfrak{p})$. When $\ell(S) = \ell(S_\mathfrak{p})$, this is clear because local cohomology commutes with localization. Let $\ell(S) > \ell(S_\mathfrak{p})$. By [U], 1.3 (a), it is enough to show that $[\mathrm{H}^{\ell(S_\mathfrak{p})}_{\mathfrak{q}S_\mathfrak{q}+[S_\mathfrak{q}]_+}(S_\mathfrak{q})]_j = (0)$ for all $\mathfrak{q} \in \operatorname{Spec} S_0$ with $\mathfrak{q} \subseteq \mathfrak{p}$. When $\dim S_\mathfrak{q} \geq \ell(S)$, this is clear because we have inequalities $\operatorname{depth} S_\mathfrak{q} \geq \ell(S)$ and $\ell(S) > \ell(S_\mathfrak{p})$. Let $\dim S_\mathfrak{q} < \ell(S)$. Then the ring $S_\mathfrak{q}$ is Cohen-Macaulay, so that we may assume $\ell(S_\mathfrak{p}) = \dim S_\mathfrak{q}$. We have $[\mathrm{H}^{\dim S_{\mathfrak{p}'}}_{[S_{\mathfrak{p}'}]_+}(S_{\mathfrak{p}'})]_j = (0)$ for all $\mathfrak{p}' \in \operatorname{Spec} S_0$ with $\mathfrak{p}' \subseteq \mathfrak{q}$ by our standard assumption and thus obtain that $[\mathrm{H}^{\dim S_\mathfrak{q}}_{\mathfrak{q}S_\mathfrak{q}+[S_\mathfrak{q}]_+}(S_\mathfrak{q})]_j = (0)$ from [U], 1.3 (b). So the proof of Claim 2.3 is completed. $\square$

Now, we take any graded ideal $N \in \mathrm{V}(S_+)$ and $i \in \mathbb{Z}$ with $i \geq \ell(S)$. We put $N_0 = N \cap S_0$ and $\alpha = \mu(N_0)$, which denote the number of generator of the ideal N_0 in S_0. We will show that $[\mathrm{H}^i_N(S)]_j = (0)$ by induction on α.

When $\alpha = 0$, we have $N = S_+$, so that we may assume $i = \ell(S)$. Then since $j > \underline{a}_\ell(S)$, there is nothing to prove. Let $\alpha > 0$ and assume that it holds ture for $\alpha - 1$. Then we may assume the ideal N_0 is not nilpotent. Take $c \in N_0$ such that $c \notin \sqrt{(0)} \cup \mathfrak{n}N_0$. We can find an ideal X of S_0 such that $N_0 = X + cS_0$ and $\mu(X) < \alpha$. Set $L = XS + S_+$. By [B1], 3.9, there exists an exact sequence

$$\cdots \to [\mathrm{H}^{i-1}_{L_c}(S_c)]_j \to [\mathrm{H}^i_N(S)]_j \to [\mathrm{H}^i_L(S)]_j \to \cdots$$

of graded S-modules. We have $[\mathrm{H}^i_L(S)]_j = (0)$ by the inductive hypothesis on α. We want to show that $[\mathrm{H}^{i-1}_{L_c}(S_c)]_j = (0)$. Take any $\mathfrak{p} \in \operatorname{Spec} S_0$ with $c \notin \mathfrak{p}$. It suffices to prove that $[\mathrm{H}^{i-1}_{L_\mathfrak{p}}(S_\mathfrak{p})]_j = (0)$. When $i - 1 \geq \ell(S)$, since $\mu(X_\mathfrak{p}) < \alpha$, it is true by the hypothesis of induction on α (recall that Claim 2.3). We must show $[\mathrm{H}^{\ell(S)-1}_{L_\mathfrak{p}}(S_\mathfrak{p})]_j = (0)$. By [U], 1.3 (a), it is enough to prove that $[\mathrm{H}^{\ell(S)-1}_{\mathfrak{q}S_\mathfrak{q}+[S_\mathfrak{q}]_+}(S_\mathfrak{q})]_j = (0)$ for all $\mathfrak{q} \in \operatorname{Spec} S_0$ with $\mathfrak{q} \subseteq \mathfrak{p}$. When $\dim S_\mathfrak{q} \geq \ell(S)$, there is nothing to prove because $\operatorname{depth} S_\mathfrak{q} \geq \ell(S)$ by our standard assumption. Let $\dim S_\mathfrak{q} < \ell(S)$. Then the ring $S_\mathfrak{q}$ is Cohen-Macaulay, so that we may assume $\ell(S) - 1 = \dim S_\mathfrak{q}$. We have $[\mathrm{H}^{\dim S_{\mathfrak{p}'}}_{[S_{\mathfrak{p}'}]_+}(S_{\mathfrak{p}'})]_j = (0)$ for all $\mathfrak{p}' \in \operatorname{Spec} S_0$ with $\mathfrak{p}' \subseteq \mathfrak{q}$ by our standard assumption, so that the result of [U], 1.3 (b) completes the proof of Lemma 2.2. □

Proof of Proposition 2.1. Lemma 2.2 implies an inequality

$$\mathrm{a}^*(S) \leq \max\{\underline{a}_{\ell(S_\mathfrak{p})}(S_\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{A}(S)\} \cup \{\underline{a}_\ell(S)\}.$$

Since local cohomology commutes with localization, we get an inequality $\underline{a}_{\ell(S)}(S_\mathfrak{p}) \leq \underline{a}_{\ell(S)}(S)$ for all $\mathfrak{p} \in \operatorname{Spec} S_0$. By [T1], 3.2, $\underline{a}_{\ell(S)}(S) \leq \mathrm{r}(S) - \ell(S)$. Therefore we obtain inequalities

$$\begin{aligned}
&\max\{\underline{a}_{\ell(S_\mathfrak{p})}(S_\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{A}\} \cup \{\underline{a}_\ell(S)\} \\
&\qquad \leq \max\{\mathrm{r}(S_\mathfrak{p}) - \ell(S_\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{A}(S), \dim S_\mathfrak{p} < \ell(S)\} \cup \{\mathrm{r}_Z(S_+) - \ell(S)\} \\
&\qquad \leq \max\{\mathrm{r}_\mathfrak{p}(S) - \ell(S_\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{A}(S), \dim S_\mathfrak{p} < \ell(S)\} \cup \{\mathrm{r}_Z(S_+) - \ell(S)\}.
\end{aligned}$$

In general, we have $\mathrm{a}^*(S_\mathfrak{p}) \leq \mathrm{a}^*(S)$ for all $\mathfrak{p} \in \operatorname{Spec} S_0$. By [T2], 2.2, $\mathrm{r}_L(S_+) - \ell(S) \leq \mathrm{a}^*(S)$ for all minimal reductions L of S_+ generated by elements of degree 1. Therefore an inequality

$$\max\{\mathrm{r}_\mathfrak{p}(S) - \ell(S_\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{A}(S), \dim S_\mathfrak{p} < \ell(S)\} \cup \{\mathrm{r}_Z(S_+) - \ell(S)\} \leq \mathrm{a}^*(S)$$

holds true. These inequalities above imply the first and the third equalities in the proposition. Since we have inequalities $\underline{a}_{\ell(S_\mathfrak{p})}(S_\mathfrak{p}) \leq \mathrm{r}(S_\mathfrak{p}) - \ell(S_\mathfrak{p}) \leq \mathrm{r}_\mathfrak{p}(S) - \ell(S_\mathfrak{p}) \leq \mathrm{a}^*(S_\mathfrak{p}) \leq \mathrm{a}^*(S)$ for all $\mathfrak{p} \in \operatorname{Spec} S_0$, the second and the forth equalities similarly follow from the first inequality in this proof.

Let us prove the last equality in the proposition. Take any $\mathfrak{p} \in \operatorname{Spec} S_0$ with $\dim S_\mathfrak{p} < \ell(S)$. Then the ring $S_\mathfrak{p}$ is Cohen-Macaulay by our standard assumption and thus $\mathrm{r}_{\mathfrak{p}}(S) - \ell(S_{\mathfrak{p}}) \leq \mathrm{a}(S_{\mathfrak{p}})$ by the a-invariant formula (see [U]). Therefore we have

$$\begin{aligned}\max\{\mathrm{r}_{\mathfrak{p}}(S) - \ell(S_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathcal{A}(S), \dim S_{\mathfrak{p}} < \ell(S)\} \cup \{\mathrm{r}_Z(S_+) - \ell(S)\} \\ \leq \max\{\mathrm{a}(S_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathcal{A}(S), \dim S_{\mathfrak{p}} < \ell(S)\} \cup \{\mathrm{r}_Z(S_+) - \ell(S)\}.\end{aligned}$$

Since the ring S is quasi-unmixed, $\mathrm{a}(S_{\mathfrak{p}}) \leq \mathrm{a}(S)$ by [HHK], 2.3. Therefore

$$\begin{aligned}\max\{\mathrm{a}(S_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathcal{A}(S), \dim S_{\mathfrak{p}} < \ell(S)\} \cup \{\mathrm{r}_Z(S_+) - \ell(S)\} \\ \leq \max\{\mathrm{a}(S), \mathrm{r}_Z(S_+) - \ell(S)\}.\end{aligned}$$

By [T2], 2.2, we have $\mathrm{r}_Z(S_+) - \ell(S) \leq \mathrm{a}^*(S)$, so that $\max\{\mathrm{a}(S), \mathrm{r}_Z(S_+) - \ell(S)\} \leq \mathrm{a}^*(S)$. These inequalities imply the last equality in the proposition. $\square$

Now, let us prove Theorem 1.3 by applying Proposition 2.1. We will use the last equality only. We note that in the case where $S = \mathrm{G}(I)$, an ideal Z that is generated by the initial forms of generators for J is a minimal reduction of S_+ such that $r = \mathrm{r}_Z(S_+)$ and $\ell = \ell(S)$.

Proof of Theorem 1.3. We set $S = \mathrm{G}(I)$. Suppose that the ring A satisfies Serre's condition (S_ℓ) and $r \leq \ell - 2$. Since the Rees algebra $\mathrm{R}(I)$ is Cohen-Macaulay, S is quasi-unmixed (see, e.g., [HIO], 18.23 and 18.24). Take any $\mathfrak{p} \in \mathrm{V}(I)$. Since the ring $\mathrm{R}(I_{\mathfrak{p}})$ is Cohen-Macaulay, if $i < \dim A_{\mathfrak{p}}$, then $\mathrm{H}^i_{\mathfrak{p}\mathrm{G}(I_{\mathfrak{p}})+\mathrm{G}(I_{\mathfrak{p}})_+}(\mathrm{G}(I_{\mathfrak{p}})) \cong \mathrm{H}^i_{\mathfrak{p}A_{\mathfrak{p}}}(A_{\mathfrak{p}})$ by [TI] and thus $\operatorname{depth} \mathrm{G}(I_{\mathfrak{p}}) = \operatorname{depth} A_{\mathfrak{p}}$. Since A satisfies Serre's condition (S_ℓ), we get $\operatorname{depth} A_{\mathfrak{p}} \geq \min\{\dim A_{\mathfrak{p}}, \ell\}$. Therefore $\operatorname{depth} S_{\mathfrak{p}} \geq \min\{\dim S_{\mathfrak{p}}, \ell\}$ because $S_{\mathfrak{p}} = \mathrm{G}(I_{\mathfrak{p}})$ and $\dim S_{\mathfrak{p}} = \dim A_{\mathfrak{p}}$. Then thanks to Proposition 2.1, we get $\mathrm{a}^*(S) = \max\{\mathrm{a}(S), r - \ell\}$. We obtain that $\mathrm{a}(S) = -2$ from $\mathrm{R}(I)$ is a Gorenstein ring (see [I]). And we have $r \leq \ell - 2$, so that $\mathrm{a}^*(S) = -2$ by our formula. If $i < \dim A$, then $\mathrm{a}_i(S) = -1$ or $-\infty$ by [TI] and thus we get $\mathrm{a}_i(S) = -\infty$, as $\mathrm{a}^*(S) = -2$. Therefore S is a Cohen-Macaulay ring, and then so is A. Since A is quasi-Gorenstein by [I], we get A is Gorenstein. Conversely, when A is Gorenstein, we have S is Gorenstein and $\mathrm{a}(S) = -2$ by [I], so that $r - \ell \leq -2$ (recall that the a-invariant formula in [U]). $\square$

3 Proof of Proposition 1.2.

The goal of this section is to prove Proposition 1.2. In what follows let t be an indeterminate over A. We define $\mathrm{R}'(I) := A[It, t^{-1}] \subseteq A[t, t^{-1}]$, which we call the extended Rees algebra. Then $\mathrm{G}(I) \cong \mathrm{R}'(I)/t^{-1}\mathrm{R}'(I)$ as graded rings. By our standard assumption, the ring $\mathrm{R}'(I)$ is a homomorphic image of a Gorenstein graded ring $S = \bigoplus_{i\in\mathbb{Z}} S_i$ over a local ring S_0. We may assume $\dim S = \dim R'(I)$. Let K_A, $\mathrm{K}_{\mathrm{R}(I)}$, $\mathrm{K}_{\mathrm{R}'(I)}$, and $\mathrm{K}_{\mathrm{G}(I)}$ denote the graded canonical modules of A, $\mathrm{R}(I)$, $\mathrm{R}'(I)$, and $\mathrm{G}(I)$, respectively. We denote $\mathrm{R}'(I)$ simply by R'. Let $\mathfrak{M} = t^{-1}R' + \mathfrak{m}R' + ItR'$. Let $\mathrm{H}^i_{\mathfrak{M}}(\ )$ $(i \in \mathbb{Z})$ denote the graded ith local cohomology functor of R' with respect to $\mathfrak{M}$. We put $R = \mathrm{R}(I)$, $G = \mathrm{G}(I)$, and $a = \mathrm{a}(G)$. We always assume that $\mathrm{K}_A = A$ and $\dim A \geq 2$. To begin with, we note

Lemma 3.1. *Let k be an integer and let $b = \mathrm{a}(\mathrm{G}(I^k))$. Assume that $\sqrt{I} = \mathfrak{m}$. Then $(b+1)k = a+1$ if $\mathrm{K}_{\mathrm{R}'(I^k)} \cong \mathrm{R}'(I^k)(b+1)$ as graded $\mathrm{R}'(I)$-modules.*

Proof. It is routine to check the inequality $(b+1)k \leq a+1$ (cf., e.g., [GI], Proof of 4.1). Since $\sqrt{I} = \mathfrak{m}$, we have the equality $\mathrm{a}(\mathrm{G}(I^k)) = [\mathrm{a}(\mathrm{G}(I))/k]$ where $[\ \ ]$ denotes the smallest integral part. Hence $(b+1)k \geq a+1$. $\square$

Lemma 3.2. *Assume that the ring A satisfies Serre's condition (S_3) and that the ring R is Cohen-Macaulay. Then $\mathrm{K}_G \cong \mathrm{K}_{R'}/t^{-1}\mathrm{K}_{R'}(-1)$ as graded G-modules, and hence $\mathrm{K}_G \cong G(a)$ if and only if $\mathrm{K}_{R'} \cong R'(a+1)$.*

Proof. Look at the canonical exact sequence

$$0 \to R'(1) \xrightarrow{t^{-1}} R' \to G \to 0 \tag{$\sharp$}$$

of graded R'-modules. We apply the graded local cohomology functors $\mathrm{H}^i_{\mathfrak{M}}(*)$ $(i \in \mathbb{Z})$ to the exact sequences $(\sharp)$. Then we have the resulting exact sequence

$$\mathrm{H}^{d-1}_{\mathfrak{M}}(G) \to \mathrm{H}^d_{\mathfrak{M}}(R')(1) \xrightarrow{t^{-1}} \mathrm{H}^d_{\mathfrak{M}}(R')$$

of graded R'-modules. By [TI], $\mathrm{H}^{d-1}_{\mathfrak{M}}(G) \cong \mathrm{H}^{d-1}_{\mathfrak{m}}(A)$. Since the quasi-Gorenstein ring A satisfies Serre's condition (S_3), we get $\mathrm{H}^{d-1}_{\mathfrak{m}}(A) = (0)$ by [S2], and hence $\mathrm{H}^{d-1}_{\mathfrak{M}}(G) = (0)$. Thus we have an injection $\mathrm{H}^d_{\mathfrak{M}}(R') \xrightarrow{t^{-1}} \mathrm{H}^d_{\mathfrak{M}}(R')$, so that $\mathrm{H}^d_{\mathfrak{M}}(R')$ must be the zero module. Taking the S-dual of the exact sequence $(\sharp)$ of graded R'-modules, we get the resulting exact sequence

$$0 \to \mathrm{K}_{R'} \xrightarrow{t^{-1}} \mathrm{K}_{R'}(-1) \to \mathrm{K}_G \to \mathrm{Ext}^1_S(R', S)$$

of graded R'-modules. Since $\mathrm{H}^d_{\mathfrak{M}}(R') = (0)$, $\mathrm{Ext}^1_S(R', S) = (0)$ by the local duality theorem. Therefore $\mathrm{K}_G \cong \mathrm{K}_{R'}/t^{-1}\mathrm{K}_{R'}(1)$ as graded G-modules.

We will show the last assertion. Assume $\mathrm{K}_G \cong G(a)$ as graded G-modules. Then $\mathrm{K}_{R'}/t^{-1}\mathrm{K}_{R'}(-1) \cong G(a)$. Let $\{\omega_i\}_{i \in \mathbb{Z}}$ stand for the canonical I-filtration of A (see [GI], 1.1 and notice that the canonical filtration exists if the base ring A satisfies Serre's condition (S_2)). Then we have an isomorphism $\bigoplus_{i \geq -a} \omega_{i-1}/\omega_i \cong G(a)$ as graded G-modules.

By induction on i, we will see that $\omega_{i-a-1} = I^i$ for all integers $i \geq 1$. Since the equality $\omega_{-a-1} = A$ always holds, an isomorphism $A/\omega_{-a} \cong A/I$ follows from the graded isomorphism above, and hence we get $\omega_{-a} = I$. Let $i > 1$ and assume $\omega_{i-a} = I^{i-1}$. We note that $\omega_{i-a-1} \supseteq I^i$. From the graded isomorphism above we obtain that $I^{i-1}/I^i \cong \omega_{i-a}/\omega_{i-a-1} = I^{i-1}/\omega_{i-a-1}$, and hence the natural surjective map $I^{i-1}/I^i \to I^{i-1}/\omega_{i-a-1}$ is bijective. Thus we get $\omega_{i-a-1} = I^i$ for all $i \geq 1$. This means that $\mathrm{K}_{R'} \cong R'(a+1)$ as graded R'-modules. $\square$

We now come to the proof of Proposition 1.2.

Proof of Proposition 1.2. Thanks to [O] 4.3, the implication (2) $\Rightarrow$ (1) holds true. Conversely, suppose $\mathrm{R}(Q^k)$ is a Gorenstein ring. Then $\mathrm{a}(\mathrm{G}(Q^k)) = -2$ and $\mathrm{K}_{\mathrm{G}(Q^k)} \cong \mathrm{G}(Q^k)(-2)$ as graded G-modules by [I]. From Lemma 3.2 we obtain the isomorphism $\mathrm{K}_{R'(Q^k)} \cong \mathrm{R}'(Q^k)(-1)$ as graded $\mathrm{R}'(Q^k)$-modules, so that the equality $(-2+1)k = \mathrm{a}(\mathrm{G}(Q)) + 1$ holds true by Lemma 3.1. We have $\mathrm{a}(\mathrm{G}(Q)) = -d$, and hence $k = d-1$.

Let $\{\omega_i\}_{i\in\mathbb{Z}}$ stand for the canonical Q-filtration of A (see [GI], 1.1 and notice that the canonical filtration exists if the base ring A satisfies Serre's condition (S_2)). We note that $\omega_i \supseteq Q^{i-d+1}$ for all $i \in \mathbb{Z}$, as $\omega_{d-1} = A$. Since $\mathrm{K}_{\mathrm{R}'(Q)} \cong \bigoplus_{i\in\mathbb{Z}} \omega_i$ as graded $\mathrm{R}'(Q)$-modules, $\mathrm{K}_{\mathrm{R}'(Q^{d-1})} \cong \bigoplus_{i\in\mathbb{Z}} \omega_{i(d-1)}$ as graded $\mathrm{R}'(Q^{d-1})$-modules, so that $\{\omega_{i(d-1)}\}_{i\in\mathbb{Z}}$ is the canonical Q^{d-1}-filtration of A. The uniqueness of the canonical filtration implies that equalities $\omega_{i(d-1)} = (Q^{d-1})^{i-1}$ for all $i \in \mathbb{Z}$ because $\mathrm{K}_{\mathrm{R}'(Q^{d-1})} \cong \mathrm{R}'(Q^{d-1})(-1)$ as graded $\mathrm{R}'(Q^{d-1})$-modules. Therefore we get equalities $\omega_i = Q^{i-d+1}$ for all integers $i >> 0$ because $\{\omega_i\}_{i\in\mathbb{Z}}$ is a stable Q-filtration (cf. [GI], 2.2 (1)).

Look at the canonical homomorphisms

$$\mathrm{G}(Q)(-d) \xrightarrow{f} \bigoplus_{i\in\mathbb{Z}} \omega_{i-1}/\omega_i \xrightarrow{\sim} \mathrm{K}_{\mathrm{R}'(Q)}/t^{-1}\mathrm{K}_{\mathrm{R}'(Q)}(-1) \hookrightarrow \mathrm{K}_{\mathrm{G}(Q)}$$

of graded $\mathrm{G}(Q)$-modules, where f is induced by the inclusions $Q^{i-d+1} \subseteq \omega_i$. Then $\ker f$ is finitely graded, as $\omega_i = Q^{i-d+1}$ for all integers $i >> 0$. Therefore $\ker f = (0)$ because $x_1 t$ is a $\mathrm{G}(Q)$-regular element, and hence we get an embedding $\mathrm{G}(Q)(-d) \hookrightarrow \mathrm{K}_{\mathrm{G}(Q)}$ of graded $\mathrm{G}(Q)$-modules. Consider it's homogeneous component of degree d that is an injection $A/Q \hookrightarrow [\mathrm{K}_{\mathrm{G}(Q)}]_d$. We have $[\mathrm{K}_{\mathrm{G}(Q)}]_d \cong \mathrm{Hom}_{A/Q}([\mathrm{H}^d_{\mathfrak{M}}(\mathrm{G}(Q))]_{-d}, \mathrm{E}_{A/Q}(A/\mathfrak{m}))$, and hence $\mathrm{length}_A(A/Q) \leq \mathrm{length}_A([\mathrm{H}^d_{\mathfrak{M}}(\mathrm{G}(Q))]_{-d})$. From [GY], 4.1, it follows that $[\mathrm{H}^d_{\mathfrak{M}}(\mathrm{G}(Q))]_{-d} \cong A/\mathrm{M}(Q)$ where $\mathrm{M}(Q) := \sum_{i=1}^d [(x_1, \dots x_{i-1}, x_{i+1}, \dots x_d) :_A x_i] + Q$, and hence $\mathrm{length}_A(A/Q) \leq \mathrm{length}_A(A/\mathrm{M}(Q))$. But we have $Q \subseteq \mathrm{M}(Q)$, and then the equality $Q = \mathrm{M}(Q)$ holds true. This implies that the ring A is Cohen-Macaulay by [GY], 3.15. $\square$

4 Certain Gorenstein Rees algebras.

In this section we consider the existence of an ideal I in A such that the Rees algebra $\mathrm{R}(I)$ is Gorenstein. In what follows we always assume that $d = \dim A \geq 3$ and that $A = \mathrm{K}_A$, which is the canonical module of A. To state our result, we set up some notation. Put $\mathfrak{a}(A) = \Pi_{p=0}^{d-1}(0) :_A \mathrm{H}^p_{\mathfrak{m}}(A)$ and $\mathrm{NCM}(A) = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid A_{\mathfrak{p}} \text{ is not a Cohen-Macaulay ring}\}$. Then $\mathrm{NCM}(A) = \mathrm{V}(\mathfrak{a}(A))$. Suppose that $\dim \mathrm{NCM}(A) \leq 1$. Then there is a system of parameters $x_1, x_2, \dots, x_d$ of A such that $x_2, x_3 \dots, x_d \in \mathfrak{a}(A)$. Set $J = (x_2, x_3, \dots, x_d)$. Taking enough large power of the element x_1, we may assume $J : x_1 = J : x_1^2$. Set $I = J : x_1$. Then the Rees algebra $\mathrm{R}(I^k)$ is Cohen-Macaulay for all integers $k \geq d-2$ by [K2]. With this notation the main result in this section can be stated as follows.

Theorem 4.1. *Assume that* $\dim \mathrm{NCM}(A) \leq 1$. *Let k be an integer with $k \geq d-2$. Consider the following two conditions.*

(1) $\mathrm{R}(I^k)$ *is a Gorenstein ring.*

(2) $\dim \mathrm{NCM}(A) \leq 0$ *and* $k = d-2$.

Then one has the implication (2) $\Rightarrow$ (1). *Furthermore if the ring A satisfies Serre's condition* (S_3), *then the above two conditions are equivalent to each other.*

Proof. (2) $\Rightarrow$ (1). $\mathrm{R}(I^{d-2})$ is a Cohen-Macaulay ring by [K2]. Thus it is enough to show that $\mathrm{K}_{\mathrm{R}(I^{d-2})} \cong \mathrm{R}(I^{d-2})(-1)$ as graded $\mathrm{R}(I^{d-2})$-modules. Put $\mathcal{A} = \mathrm{Ass}_A \mathrm{G}(I^{d-2})$. The ring $\mathrm{G}(I^{d-2})$ is quasi-unmixed because so is A (see [HIO], 18.24). And $\mathrm{G}(I^{d-2})$ fulfills Serre's condition (S_1) because the Rees algebra $\mathrm{R}(I^{d-2})$ is Cohen-Macaulay and the ring A fulfills Serre's condition (S_2) (see [V], 3.53). So $\mathrm{G}(I^{d-2})$ is unmixed (i.e. all associated prime ideals of $\mathrm{G}(I^{d-2})$ have same codimension), and hence $\mathcal{A} = \{\mathfrak{p} \in \mathrm{V}(I) \mid \dim A_\mathfrak{p} = \ell(I_\mathfrak{p})\}$ by [M], 4.1 (recall that $\ell(I) = \ell(I^{d-2})$). Since the ideal $J = (x_2, x_3, \ldots, x_d)$ is a reduction of I by [K1], (3.2.1), we have $\ell(I) = d-1$. Therefore $\mathfrak{m} \notin \mathcal{A}$ because $\ell(I_\mathfrak{p}) \leq \ell(I)$.

Take any $\mathfrak{p} \in \mathcal{A}$. Since $\mathfrak{p} \neq \mathfrak{m}$, we have $\mathrm{ht}_A \mathfrak{p} = d-1$, and hence $I_\mathfrak{p} = (J : x_1)_\mathfrak{p} = J_\mathfrak{p}$. Therefore $I_\mathfrak{p}$ is a parameter ideal of the ring $A_\mathfrak{p}$, which is Gorenstein because $\mathrm{K}_A = A$ and $\dim \mathrm{NCM}(A) \leq 0$. Thus the rings $\mathrm{R}(I_\mathfrak{p}^{d-2})$ and $\mathrm{G}(I_\mathfrak{p}^{d-2})$ are Gorenstein, and then $\mathrm{a}(\mathrm{G}(I_\mathfrak{p}^{d-2})) = -2$ (cf. [O], 4.3). By [HHK], we have the equality $\mathrm{a}(\mathrm{G}(I^{d-2})) = \max\{\mathrm{a}(\mathrm{G}(I_\mathfrak{p}^{d-2})) \mid \mathfrak{p} \in \mathcal{A}\}$. Therefore $\mathrm{a}(\mathrm{G}(I^{d-2})) = -2$.

Thanks to a theorem given by [TVZ], there exists an I^{d-2}-filtration $\{\kappa_i\}_{i\geq 0}$ of A such that $\mathrm{K}_{\mathrm{R}(I^{d-2})} \cong \bigoplus_{n\geq 1} \kappa_n$ and $\mathrm{K}_{\mathrm{G}(I^{d-2})} \cong \bigoplus_{n\geq 1} \kappa_{n-1}/\kappa_n$ as graded $\mathrm{R}(I^{d-2})$-modules because $\mathrm{R}(I^{d-2})$ is a Cohen-Macaulay ring. By the equality $\mathrm{a}(\mathrm{G}(I^{d-2})) = -2$, we have $\kappa_0/\kappa_1 \cong [\mathrm{K}_{\mathrm{G}(I^{d-2})}]_1 = (0)$ and hence $A = \kappa_0 = \kappa_1$. So it follows that $(I^{d-2})^{i-1} \subseteq \kappa_i$ for all integers i, which imply the natural graded embedding

$$\varphi : \mathrm{R}(I^{d-2})(-1) \hookrightarrow \mathrm{K}_{\mathrm{R}(I^{d-2})}$$

and then we have $\mathrm{coker}\varphi = \bigoplus_{i\geq 2} \dfrac{\kappa_i}{(I^{d-2})^{i-1}}$. We want to show that $\mathrm{coker}\varphi = (0)$. Take any $\mathfrak{p} \in \mathcal{A}$. It is enough to prove $(I^{d-2})_\mathfrak{p}^{i-1} = (\kappa_i)_\mathfrak{p}$ for all integers $i \geq 2$ because $\mathcal{A} = \bigcup_{i\geq 1} \mathrm{Ass}_A A/(I^{d-2})^i$. Since $\mathrm{K}_{\mathrm{R}(I_\mathfrak{p}^{d-2})} \cong \bigoplus_{n\geq 1} (\kappa_n)_\mathfrak{p}$ and $\mathrm{K}_{\mathrm{G}(I_\mathfrak{p}^{d-2})} \cong \bigoplus_{n\geq 1} (\kappa_{n-1})_\mathfrak{p}/(\kappa_n)_\mathfrak{p}$ as graded $\mathrm{R}(I_\mathfrak{p}^{d-2})$-modules, the filtration $\{(\kappa_i)_\mathfrak{p}\}_{i\in\mathbb{Z}}$ (where $(\kappa_i)_\mathfrak{p} = A_\mathfrak{p}$ for all integers $i < 0$) is the canonical $I_\mathfrak{p}^{d-2}$-filtration of $A_\mathfrak{p}$ (cf. [GI], Proof of 2.5). From $\mathrm{G}(I_\mathfrak{p}^{d-2})$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I_{\mathfrak{p}}^{d-2})) = -2$, we obtain the required equality $(\kappa_i)_{\mathfrak{p}} = (I_{\mathfrak{p}}^{d-2})^{i-1}$ because the canonical $I_{\mathfrak{p}}^{d-2}$-filtration of $A_{\mathfrak{p}}$ is uniquely determined by [GI], 1.1.

(1) $\Rightarrow$ (2). Assume the ring A satisfies Serre's condition (S_3). Take any $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathfrak{p} \neq \mathfrak{m}$. Then $A_{\mathfrak{p}}$ is a quasi-Gorenstein ring has finite local cohomology modules. Recall that $\mathrm{ht}_A I = d-1$. Let $\mathfrak{p} \in \mathrm{V}(I)$. Then $\mathrm{ht}_A \mathfrak{p} = d-1$. Since $\mathrm{R}(I_{\mathfrak{p}}^k)$ is a Gorenstein ring, it follows that the ring $A_{\mathfrak{p}}$ is Gorenstein and that $k = d-2$ from Proposition 1.2. Let $\mathfrak{p} \notin \mathrm{V}(I)$. Then the Gorenstein ring $\mathrm{R}(I_{\mathfrak{p}}^k)$ is isomorphic to the polynomial ring $A_{\mathfrak{p}}[t]$, and therefore $A_{\mathfrak{p}}$ is a Gorenstein ring. Thus $\dim \mathrm{NCM}(A) \leq 0$. $\square$

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Quasi-socle ideals in Gorenstein numerical semi-group rings

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この報告は明治大学の後藤四郎教授と松岡直之さんとの共同研究によるものです。

1　はじめに

以下、A は可換環, $I \subseteq A$ イデアルとする。記号 $\overline{I}$ によって, イデアル I の整閉包を表す事にする。Q を環 A のイデアルで $Q \subseteq I$ とする時, Q が I の節減 (reduction) であるとは, ある整数 $r \geq 0$ に対し等式 $I^{r+1} = QI^r$ が成り立つことをいう。もし環 A が Noether 環ならば, $I \subseteq \overline{Q}$ である事と Q が I の reduction である事は同値である。イデアル Q が I の reduction の時, $\mathrm{r}_Q(I) = \min\{r \in \mathbb{Z} \mid I^{r+1} = QI^r\}$ と置き, I の Q に関する reduction number と呼ぶ。イデアル I に対して

$$\mathrm{G(I)} = \bigoplus_{\mathrm{n}\geq 0} \mathrm{I^n/I^{n+1}}$$

と置き, イデアル I の随伴次数環と呼ぶ。環 A を極大イデアルを m を持つ局所環とした時

$$\mathrm{F(I)} = \bigoplus_{\mathrm{n}\geq 0} \mathrm{I^n/mI^n}$$

と置き, I の Fiber 環と呼ぶ。

本研究の目的は, 次の問いを解析することにある。

問題 1.1. A を極大イデアル $\mathfrak{m}$ を持つ Cohen-Macaulay 局所環とし, $Q \subseteq A$ を任意の A の巴系イデアルとして, $I = Q : \mathfrak{m}^q$ (但し, $q \geq 1$ は整数) とする。この時

(1) いつ Q は I の reducticn になるか？

(2) もし Q が I の reduction であるならば, その reduction number はいくつか？

(3) いつ $\mathfrak{m}^q I = \mathfrak{m}^q Q$ は成り立つか？

(4) いつ G(I), F(I) は Cohen-Macaulay 環であるのか？

この問いは, すでによく知られている次の結果を一般化することを目的としたものである。

定理 1.2 ([CP], [CHV], [CPV], [G]). A を極大イデアル $\mathfrak{m}$ を持つ Cohen-Macaulay 局所環とし, $Q \subseteq A$ を A の巴系イデアルとして, $I = Q : \mathfrak{m}$ とする。この時次の 3 条件は同値である。

(1) $I^2 \neq QI$ である。

(2) $Q = \overline{Q}$ が成り立つ。

(3) A は正則局所環で, $\mu_A(\mathfrak{m}/Q) \leq 1$ が成り立つ。但し, $\mu_A(*)$ により極小生成系の個数を表す。

特に, 局所環 A が Cohen-Macaulay であるがしかし正則でないとすると, 任意の A の巴系イデアル Q に対して $I^2 = QI$ が成り立ち, よって $\mathrm{G}(I)$ と $\mathrm{F}(I)$ は Cohen-Macaulay 環である。但し, $I = Q : \mathfrak{m}$ とする。

また, 次の最近の結果もある。

定理 1.3 ([GMT]). A を極大イデアル $\mathfrak{m}$ を持つ Gorenstein 局所環で $\mathrm{e}^0_{\mathfrak{m}}(A) \geq 3$ とする。但し, $\mathrm{e}^0_{\mathfrak{m}}(A)$ によって環 A の $\mathfrak{m}$ に関する重複度を表す。$Q \subseteq A$ を A の巴系イデアルとして, $I = Q : \mathfrak{m}^2$ とおく。この時次が成立する。

(1) $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$ が成り立つ。

(2) $I^3 = QI^2$ が成り立つ。

(3) $\mathrm{G}(I)$ は Cohen-Macaulay 環である。

(4) $\mathrm{F}(I)$ は Cohen-Macaulay 環である。

しかし, 環 A が Gorenstein でないときは解析が非常に困難である。

本稿では, 環 A が numerical semi-group ring の時に限定した計算を紹介したい。その主結果は, 定理 1.3 に完全に含まれているが, しかし手法が完全に違い, semi-group の言葉で証明を与えている。また, 環 A が Gorenstein でないときの解析の困難さにも触れたい。

ここで, 本稿の構成に触れておく。第2章は2つの節から成る。第1節では, $Q:\mathfrak{m}^2$ について考える。特に環 A が Gorenstein のときとそうでないときの違いを明らかにする。第2節では $q=3$ とした場合は環 A が Gorenstein であっても制御が難しいことを示す。第3章では, 環 A を hypersurface とした場合に, 問題 1.1 について考察する。以下, A-加群 M に対して $\ell_A(M)$ によって M の A-加群としての長さを表すこととする。

2 numerical semi-group ring 上での計算

2.1 $I=Q:\mathfrak{m}^2$ の場合

第2章を通して, 以下の記号を用いる。$\ell>0$ を整数とする。$0<a_1<a_2<\cdots<a_\ell$ を $\gcd(a_1,a_2,\cdots,a_\ell)=1$ なる整数を取り

$$H=\langle a_1,a_2,\cdots,a_\ell\rangle:=\{\sum_{i=1}^{\ell}c_ia_i\mid 0\le c_i\in\mathbb{Z}\}$$

と置き, $a_1,a_2,\cdots,a_\ell$ で生成される半群と呼ぶ。$\gcd(a_1,a_2,\cdots,a_\ell)=1$ なので,

$$c=c(H)=\min\{c\in\mathbb{Z}\mid \text{整数}\, n\, \text{に対して}, n\ge c\, \text{ならば}\, n\in H\}$$

が定まる。これを H における conductor と呼ぶ。ここで, k を体, $k[[t]]$ (t は k 上の不定元) を冪級数環とし,

$$A=k[[H]]:=k[[t^{a_1},t^{a_2},\cdots,t^{a_\ell}]]\subseteq k[[t]]$$

と置くと, A は $\mathfrak{m}=(t^{a_1},t^{a_2},\cdots,t^{a_\ell})$ を唯一の極大イデアルにもつ1次元の Cohen-Macaulay 局所環である。また, $\mathfrak{m}$ における重複度 $e^0_{\mathfrak{m}}(A)$ は a_1 と等しい。

半群 H が symmetric であるとは, 任意の整数 n について, $n\in H$ と $(c-1)-n\notin H$ が同値であることを言う。環 $A=k[[H]]$ の Gorenstein 性については次の事実がよく知られている。

Fact 2.1. 半群 H が symmetric である事と, A が Gorenstein 局所環である事は同値である。

半群 H が symmetric である事は, 次の補題によって判定される。

補題 2.2. $a=a_1\ge 3$ とする。整数 $\alpha\ge a$ で, 任意の整数 $0\le n\le\alpha$ に対して $n\in H$ と $\alpha-n\notin H$ が同値となるものが存在するとする。この時, $\alpha=c-1$ である。

証明. 任意の整数$1 \leq m < a \leq \alpha$を取れば$m \notin H$なので, 仮定より$\alpha - m \in H$である。従って, $\alpha-1, \alpha-2, \cdots, \alpha-(a-1) \in H$であり, よって$\alpha+(a-1), \alpha+(a-2), \cdots, \alpha+1 \in H$である。一方で, 仮定から$\alpha = \alpha - 0 \notin H$であるので, $\alpha + a \in H$を示せば良い。ここで, $\alpha - a + 1 \in H$に注目する。$\alpha - a + 1 = aq + r$(但し, $q \geq 0$, $0 \leq r < a$は整数) と表す。このとき$i = \min\{n \mid an + r \in H\}$と置く。もちろん$0 \leq i \leq q$である。もし$i > 0$ならば, iの最小性から$a(i-1) + r \notin H$である。また, $a > 1$より, $0 \leq a(i-1) + r < aq + r = \alpha - a + 1 < \alpha$であるから, $\alpha - (a(i-1) + r) \in H$となり, $\alpha + a = (\alpha - (a(i-1) + r)) + (ai + r) \in H$がわかる。次に, $i = 0$とすると$r \in H$となる。しかし, 今$a > r$であるから, $r = 0$となる。従って, $\alpha = aq + (a-1)$であり, $q \geq 1$がわかる。$\alpha - 1 = aq + (a-2), \alpha - 2 = aq + (a-3), \cdots, \alpha - (a-1) = aq \in H$であったから, $a \geq 3$に注意すると$aq + 1 \in H$である。ここで, $j = \min\{n \mid an + 1 \in H\}$と置く。$j$の最小性から, $a(j-1)+1 \notin H$である。また, $0 \leq a(j-1)+1 < aq+(a-1) = \alpha$であるから, $\alpha - (a(j-1)+1) = a(q-(j-1))+a-2 \in H$となり, $\alpha + a = aq + 2a - 1 = (aj+1) + (a(q-(j-1)) + a - 2) \in H$となる。 □

以下, 任意の元$0 < s \in H$を取り, $Q = (t^s)$と置き, $I = Q : \mathfrak{m}^2$として問題 1.1 について考える。次の補題は, いつ$I \subseteq \overline{Q}$, つまりQがIの reduction となるかという問いに対する答えである。

補題 2.3. 次が正しい。

(1) 任意の整数$n \geq c$に対して, $t^n \in \mathfrak{m}^2$が成り立つとする。この時$I \subseteq \overline{Q}$である。

(2) Aは Gorenstein 局所環とする。この時$a_1 \geq 3$である事と, 任意の整数$n \geq c$に対して, $t^n \in \mathfrak{m}^2$が成り立つ事が同値である。

特にAが Gorenstein 局所環で, $a_1 \geq 3$とすると$I \subseteq \overline{Q}$である。

証明. (1) $I = (t^n \in A \mid t^n \in I,\ n \geq s)$を示せば良い。最初に$I$は単項式で生成されている事に注意しておく。元$t^n \in I$(但し, $n \in H$) を取る。もし$n < s$ならば$c - 1 + (s - n) \geq c$であるので, 仮定より$t^{c-1+(s-n)} \in \mathfrak{m}^2$となる。従って, $t^{s+c-1} = t^n t^{c-1+(s-n)} \in Q$となり, $c - 1 \in H$となってしまうので矛盾である。

(2) Hは$a_1, a_2, \cdots, a_\ell$によって極小に生成されていると仮定してよい。全ての整数$n \geq c$に対して$t^n \in \mathfrak{m}^2$が成り立つ時, $a_1 \geq 3$である事は自明である。$a_1 \geq 3$とする。よって, $\ell \geq 2$である事に注意する。ある整数$n \geq c$で, $t^n \notin \mathfrak{m}^2$であるようなものが存在すると仮定する。すると, ある整数$1 \leq i \leq \ell$で, $n = a_i$と表せる。

$$K = \{c_1 a_1 + c_2 a_2 + \cdots + c_{i-1} a_{i-1} + c_{i+1} a_{i+1} + \cdots + c_\ell a_\ell \mid 0 \leq c_j \in \mathbb{Z}\}$$

と置く。H が $a_1, a_2, \cdots, a_\ell$ によって極小に生成されていることから $a_i \notin K$ である。まず, $\gcd(a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_\ell) = 1$ を示そう。整数 $1 \le m < a_1$ を取れば, $n \ge c$ より $n + m \in H$ である。従って, $n + m = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_\ell a_\ell$ $(\alpha_1, \alpha_2, \cdots, \alpha_\ell \ge 0)$ と表せる。もし, $\alpha_i > 0$ ならば, $m = n + m - a_i \in H$ である。しかしこれは, $1 \le m < a_1$ より $m \notin H$ である事に反する。従って, $\alpha_i = 0$ となり, $n + m \in K$ となる。この事から, $a_1 \ge 3$ である事に注意すると, $n + 1, n + 2 \in K$ となるので, $gcd(a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_\ell) = 1$ となる。次に補題 2.2 を使って, K の conductor が c と一致する事を示そう。任意に整数 $1 \le j \le c - 1$ を取る。$j \in K \subseteq H$ とする。H は symmetric より, $(c - 1) - j \notin H$ であるから, $(c - 1) - j \notin K$ である。逆に $(c - 1) - j \notin K$ の時, もし $(c - 1) - j \in H$ ならば, $(c - 1) - j < n = a_i$ より, $(c - 1) - j \in K$ となり矛盾。従って, $(c - 1) - j \notin H$ であるから, $j \in H$ であり, さらに $j \le c - 1 < n = a_i$ より, $j \in K$ となる。従って, 補題 2.2 から K の conductor と c は一致する。よって, $n \ge c$ であるから $n = a_i \in K$ となるが, これは矛盾。 □

本稿の主結果は次の通りである。

定理 2.4. 局所環 A は Gorenstein とし, $a_1 \ge 3$ とする。この時次が成立する。

(1) $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$ である。

(2) $I^2 \cap Q = QI$ と $I^3 = QI^2$ が成り立つ。従って, $\mathrm{r}_Q(I) \le 2$ であり, G(I) は Cohen-Macaulay 環である。

(3) 全ての整数 $n \ge 1$ に対して, $\mathfrak{m}I^{n+1} = \mathfrak{m}QI^n$ が成立する。従って, F(I) は Cohen-Macaulay 環である。

証明. 明らかに $t^{s+c-1} \in [Q : \mathfrak{m}] \setminus Q$ であり, 今 A は Gorenstein 局所環であるので, $Q : \mathfrak{m} = Q + (t^{s+c-1})$ である。また $a_1 \ge 3$ であるので, 補題 2.3 より $I \subseteq \overline{Q}$ である。このことから, $I = (t^n \in A \mid t^n \in I,\ n \ge s)$ である事が解る。

(1) I は単項式で生成されているので, $t^n \in I$, $t^{a_i} t^{a_j} \in \mathfrak{m}^2$ $(1 \le i, j \le \ell)$ に対して, $t^{a_i} t^{a_j} t^n \in \mathfrak{m}^2 Q$ を言えば十分である。今 $t^{a_i} t^n \in \mathfrak{m}I \subseteq Q : \mathfrak{m} = Q + (t^{s+c-1})$ であるので, $t^{a_i} t^n \in Q$ かまたは $t^{a_i} t^n \in (t^{s+c-1})$ である。もし, $t^{a_i} t^n \in Q$ なら, ある $u \in H$ で, $t^{a_i} t^n = t^s t^u$ と表せる。従って, $a_i + n = s + u$ である。また, $n - s \ge 0$ より $u = a_i + (n - s) > 0$ であるので, $t^u \in \mathfrak{m}$ である。よって $t^{a_i} t^{a_j} t^n = t^s t^u t^{a_j} \in Q\mathfrak{m}^2$ が成り立つ。次に, $t^{a_i} t^n \in (t^{s+c-1})$ なら, ある $u \in H$ で, $t^{a_i} t^n = t^{s+c-1} t^u$ と表せる。従って, $a_j + u + (c - 1) \ge c$ である事から, $t^{a_j + u + (c-1)} \in \mathfrak{m}^2$ であり, よって $t^{a_i} t^{a_j} t^n = t^{a_j + u + (c-1)} t^s \in \mathfrak{m}^2 Q$ である。よって, 等式 $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$ が成り立つ。

(2) まず, $I^2 \cap Q = QI$ を示す。まず, $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$ であることから, 任意の整数 $n \geq 1$ に対して等式 $\mathfrak{m}^n I = \mathfrak{m}^n Q$ が成り立つことに注意する。元 $x \in I^2 \cap Q$ を取る。従って, ある $y \in A$ で, $x = t^s y$ と表せる。ここで, 任意の $\alpha \in \mathfrak{m}^2$ に対して, $\alpha x \in \mathfrak{m}^2 I^2 = \mathfrak{m}^2 Q^2$ である。従って, $\alpha y \in Q$ である。ゆえに, $x \in QI$ であるので, $I^2 \cap Q = QI$ である事が解った。次に, $I^3 = QI^2$ を示す。$I^2 \cap Q = QI$ であったので, $I^2 \not\subseteq Q$ として良い。従って, $Q \subsetneq Q + I^2 \subseteq Q : \mathfrak{m} = Q + (t^{s+c-1})$ である。今, A は Gorenstein 局所環なので $\ell_A([Q : \mathfrak{m}]/Q) = 1$ であるが, $Q \neq Q + I^2$ であるので $\ell_A([I^2 + Q]/Q) = 1$ と $Q + I^2 = Q : \mathfrak{m} = Q + (t^{s+c-1})$ がわかる。$I^2 \cap Q = QI$ であるので, $1 = \ell_A([I^2 + Q]/Q) = \ell_A(I^2/QI)$ であり, $t^{s+c-1} \in I^2$ が成り立つ。従って, $I^2 = QI + (t^{s+c-1})$ となるので, $I^3 = QI^2 + (t^{s+c-1})I$ となる。従って, $It^{s+c-1} \subseteq QI^2$ を示せば良い。元 $t^n \in I$ を取る。$n > s$ として良い。ここで, $h = n + s + c - 1 - 2s = (n - s) + c - 1$ とする。もちろん $h \geq c$ であるので $h \in H$ である。$t^h \in I$ を示そう。任意に元 $\alpha \in \mathfrak{m}^2$ を取る。すると, $\alpha t^n t^{s+c-1} \in \mathfrak{m}^2 I^3 = \mathfrak{m}^2 Q^3$ であるので, $\alpha t^h \in Q$ である。従って $t^h \in I$ となる。ゆえに $t^n t^{s+c-1} = t^{2s} t^h \in Q^2 I \subseteq QI^2$t となる。よって $I^3 = QI^2$ である。従って, 任意の整数 n に対して, $I^{n+1} \cap Q = QI^n$ が成り立つから, G(I) は Cohen-Macaulay 環である。

(3) $I^3 = QI^2$ なので, $\mathfrak{m}I^2 = \mathfrak{m}QI$ を示せば十分である。$I^2 \not\subseteq Q$ として良い。従って $I^2 = QI + (t^{s+c-1})$ である。故に, $\mathfrak{m}I^2 = \mathfrak{m}QI + \mathfrak{m}(t^{s+c-1})$ であるので, $\mathfrak{m}t^{c-1} \subseteq \mathfrak{m}I$ を示せば良い。今, $t^{s+c-1} \in I^2$ なので, $s + c - 1 \geq 2s$ であり, よって $c > s$ がわかる。さて, $\mathfrak{m}^2 t^{s+c-1} \subseteq \mathfrak{m}^2 I^2 = \mathfrak{m}^2 Q^2$ より, $\mathfrak{m}^2 t^{c-1} \subseteq Q$ である。従って, $\mathfrak{m}t^{c-1} \subseteq Q : \mathfrak{m} = Q + (t^{s+c-1})$ である。元 $t^{a_i} t^{c-1} \in \mathfrak{m}t^{c-1} \subseteq Q + (t^{s+c-1})$ $(1 \leq i \leq \ell)$ を取る。もし $t^{a_i} t^{c-1} \in Q$ ならば, ある $u \in H$ で, $t^{a_i} t^{c-1} = t^s t^u$ と表せる。従って, $a_i + c - 1 = s + u$ である。ここでもし $u = 0$ ならば, $s = a_i + c - 1 \geq c$ となり矛盾である。従って, $u > 0$ である事が解り, $t^{a_i} t^{c-1} = t^u t^s \in \mathfrak{m}Q \subseteq \mathfrak{m}I$ となる。次に, $t^{a_i} t^{c-1} \in (t^{s+c-1})$ ならば, $(t^{s+c-1}) \subseteq I^2 \subseteq \mathfrak{m}I$ より, $t^{a_i} t^{c-1} \in \mathfrak{m}I$ が成り立つ。以上より, $\mathfrak{m}t^{c-1} \subseteq \mathfrak{m}I$ が示された。 □

次に, 環 A が Gorenstein 局所環でない時には定理 2.4 が成り立たないような例を挙げる。そのような例は [GMT] でも与えられているが, それとは別の例を紹介する。

例 2.5. $A = k[[t^5, t^6, t^{13}]]$ とする。この時, A は Gorenstein 局所環でない。任意の元 $0 < s \in H := \langle 5, 6, 13 \rangle$ に対して $Q = (t^s)$ と置き, $I = Q : \mathfrak{m}^2$ とすると次が確かめられる。

(1) $I \subseteq \overline{Q}$ である。

(2) $\mathfrak{m}^2 I = \mathfrak{m}^2 Q \iff s = 6, 13$ である。

(3) G(I) が Cohen-Macaulay 環 $\Longleftrightarrow s = 6, 13$ である。

(4) F(I) が Cohen-Macaulay 環 $\Longleftrightarrow s = 6, 13$ である。

(5) $\mathrm{r}_Q(I) = 4 \Longleftrightarrow s \neq 6, 13$ である。

2.2 $I = Q : \mathfrak{m}^q$ の場合

本節の目的は, たとえ半群環 A が Gorenstein であったとしても, 整数 $q \geq 3$ に対しては $I = Q : \mathfrak{m}^q$ を制御することは難しいということを示すことにある。本稿では特に $q = 3$ の場合の例を与える。まず, 次の例は $\mathfrak{m}^q I = \mathfrak{m}^q Q$ が一般には成り立たないことを示している。

例 2.6. $A = k[[t^7, t^{10}, t^{18}, t^{22}]]$ とする。 この時, 環 A は Gorenstein 局所環である。任意の元 $0 < s \in H := \langle 7, 10, 18, 22 \rangle$ に対して $Q = (t^s)$ と置き, $I = Q : \mathfrak{m}^3$ とすると次が確かめられる。

(1) $I \subseteq \overline{Q}$ が成り立つ。

(2) $\mathrm{r}_Q(I) \leq 3$ である。

(3) $\mathfrak{m}^3 I = \mathfrak{m}^3 Q \Longleftrightarrow s = 7$ である。

(4) G(I) が Cohen-Macaulay 環 $\Longleftrightarrow s = 7, 14, 21, 22, 29$ である。

それでは, いつ $\mathfrak{m}^3 I = \mathfrak{m}^3 Q$ が成り立つかを考え, 得られた結果が次のものである。

定理 2.7. 半群環 A は Gorenstein 局所環とし, 次の 2 条件を仮定する。

(i) 任意の整数 $n \geq c$ に対して $t^n \in \mathfrak{m}^q$ が成り立つ。

(ii) $n \in H$ に対して, $t^n \notin \mathfrak{m}^{q-1}$ ならば $n < a_1(q-1)$ である。

すると, 次が正しい。

(1) $I \subseteq \overline{Q}$ である。

(2) $\mathfrak{m}^q I = \mathfrak{m}^q Q$ である。

(3) $s \geq c$ ならば, 等式 $I^2 = QI$ が成り立つ。従って, G(I) は Cohen-Macaulay 環である。

証明. (1) 補題 2.3 (1) と同様に示される。

(2) 任意に整数 $1 \le i_1, i_2, \cdots, i_q \le \ell$ と, 任意の元 $t^n \in I$ をとる。$u := a_{i_1} + a_{i_2} + \cdots + a_{i_{q-1}}$ と置く。すると, $t^u t^n \in \mathfrak{m}^{q-1} I \subseteq Q : \mathfrak{m} = Q + (t^{s+c-1})$ であるから, $t^u t^n \in Q$ であるかまたは, $t^u t^n \in (t^{s+c-1})$ である。$t^u t^n \in Q$ の時, ある元 $h \in H$ で, $t^u t^n = t^s t^h$ と書ける。従って, $h = u + (n - s) \ge a_1(q-1)$ より, 仮定 (ii) から $t^h \in \mathfrak{m}^{q-1}$ であるから, $t^u t^n \in \mathfrak{m}^{q-1} Q$ である。よって, $t^{a_q} t^u t^n \in \mathfrak{m}^q Q$ が成り立つ。次に, $t^u t^n \in (t^{s+c-1})$ の時, ある元 $h \in H$ で, $t^u t^n = t^{s+c-1} t^h$ と書けるので, $(a_q + h) + c - 1 \ge c$ に注意すると, 仮定 (i) から $t^{(a_q+h)+c-1} \in \mathfrak{m}^q$ である。よって, $t^{a_q} t^u t^n = t^{(a_q+h)+c-1} t^s \in \mathfrak{m}^q Q$ となる。以上より, $\mathfrak{m}^q I = \mathfrak{m}^q Q$ が示された。

(3) $\mathfrak{m}^q I = \mathfrak{m}^q Q$ より, $I^2 \cap Q = QI$ が定理 2.4(2) と同様に確かめられる。元 $t^n t^m \in I^2$ を取ると, $n, m \ge s \ge c$ である。よって, $n + m - s = n + (m - s) \ge c$ であるから, $n + m - s \in H$ となり, $t^n t^m \in Q$ となる。従って, $I^2 \subseteq Q$ が示されたから, $I^2 = QI$ である。 □

これまでの例では, 全て $\mathfrak{m}^q I = \mathfrak{m}^q Q$ ならば $\mathrm{G}(I)$ は Cohen-Macaulay 環となっていた。しかし, これは一般には成り立たない。本節の最後に $\mathfrak{m}^q I = \mathfrak{m}^q Q$ であるが G(I) が Cohen-Macaulay 環にはならない例を $q = 3$ の場合に挙げる。

例 2.8. $A = k[[t^{10}, t^{13}, t^{16}, t^{17}, t^{19}]]$ とする。 この時 A は Gorenstein 局所環であって, 定理 2.7 の条件 (i)(ii) を満たす。任意の元 $0 < s \in H = \langle 10, 13, 16, 17, 19 \rangle$ に対して $Q = (t^s)$ と置き, $I = Q : \mathfrak{m}^3$ とすると定理 2.7 により $\mathfrak{m}^3 I = \mathfrak{m}^3 Q$ である。しかし, $s = 16$ の時に限り G(I) は Cohen-Macaulay 環でなく, この時 $\mathrm{r}_Q(I) = 5$ である。

3 hypersurface 上での計算

本章では, ある特殊な hypersurface を取り上げ, その環について問題 1.1 を考える。以下, $a > 1$ を整数として, $H = \langle a, a+1 \rangle := \{c_1 a + c_2(a+1) \mid 0 \le c_1, c_2 \in \mathbb{Z}\}$ と置き, その coductor を c とする。このとき, $c = a(a-1)$ である。$k[[t]]$ を体 k 上の冪級数環とし, $A = k[[H]] := k[[t^a, t^{a+1}]] \subseteq k[[t]]$ とする。任意に元 $0 < s \in H$ を取って $Q = (t^s)$ と置き, $I = Q : \mathfrak{m}^q$(但し, $q > 0$ は整数) とする。最初に次の補題を示す。

補題 3.1. 次が正しい。

(1) 任意の整数 $\ell \ge 0$ に対して,

$$\mathfrak{m}^\ell = (t^{a\ell+i} \mid 0 \le i \le \ell,\ i \in \mathbb{Z}) = (t^n \mid n \in H,\ n \ge a\ell)$$

が成り立つ。

(2) 整数 $0 \le q,\ i \le a-1$ に対して, $aq+i \in H$ である事と $i \le q$ である事は同値である。

証明. (1) 任意に整数 $\ell \ge 0$ を取る。任意の整数 $0 \le i \le \ell$ に対して, $a\ell + i = a(\ell - i) + (a+1)i \in H$ であるから, $\mathfrak{m}^\ell = (t^a, t^{a+1})^\ell = (t^{a\ell+i} \mid 0 \le i \le \ell,\ i \in \mathbb{Z})$ となる。次に, $\mathfrak{m}^\ell = (t^n \mid n \in H, n \ge a\ell)$ を示す。元 $n \in H$ で $n \ge a\ell$ なるものを取り, $n = aq+r$ $(q \ge 0,\ 0 \le r < a)$ と書く。すると, $q \ge \ell$ である事が解る。一方, ある整数 $\alpha \ge 0$, $\beta \ge 0$ で, $n = a\alpha + (a+1)\beta$, と書ける。従って, $\beta \equiv r \mod a$ なので, ある整数 m で $\beta = am + r$ と書ける。すると, $m \ge 0$ となる事も解る。以上から $n = a(\alpha+\beta+m)+r$ となるので, $q = \alpha+\beta+m = \alpha + (a+1)m + r \ge r$ である。よって,

$$t^n \in (t^{a(\alpha+\beta+m)+i} \mid 0 \le i \le (\alpha+\beta+m),\ i \in \mathbb{Z}) = \mathfrak{m}^{\alpha+\beta+m} = \mathfrak{m}^q \subseteq \mathfrak{m}^\ell$$

となる。

(2) $aq+i \in H$ を仮定する。故に, ある整数 $\alpha \ge 0$, $\beta \ge 0$ で, $aq+i = a\alpha+(a+1)\beta$ と書けるから, $\beta \equiv i \mod a$ である。従って, ある整数 m で $\beta = am+i$ と書けて, $m \ge 0$ である事が解る。故に, $aq+i = a(\alpha+\beta+m)+i$ より, $q = \alpha+\beta+m = \alpha+(a+1)m+i \ge i$ である。逆に, $i \le q$ を仮定すると, $aq+i = a(q-i)+(a+1)i \in H$ である。 $\square$

次の定理が本章の鍵となる結果であり, Q が I の reduction となることを特徴付けるものである。

定理 3.2. 次の 3 条件は互いに同値である。

(1) $I \subseteq \overline{Q}$ である。

(2) 等式 $\mathfrak{m}^q I = \mathfrak{m}^q Q$ が成り立つ。

(3) $q < a$ である。

証明. (1) ⇒ (2) 仮定より $I = (t^n \in A \mid t^n \in I,\ n \ge s)$ である。元 $t^n \in I$, $t^{aq+i} \in \mathfrak{m}^q$ に対して, $t^{aq+i}t^n \in Q$ より, $aq+i+(n-s) \in H$ である。また, $aq+i+(n-s) \ge aq$ より, 補題 3.1 (1) から $t^{aq+i+(n-s)} \in \mathfrak{m}^q$ となり, $t^{aq+i}t^n \in \mathfrak{m}^q Q$ が成り立つ。従って, $\mathfrak{m}^q I = \mathfrak{m}^q Q$ である。

(2) ⇒ (1) determinantal trick に依る。

(3) ⇒ (1) $I = (t^n \in A \mid t^n \in I,\ n \ge s)$ を示せば良い。元 $t^n \in I$ を取る。もし $n < s$ ならば, $c-1+(s-n) \ge c$ となるので, $c-1+(s-n) \in H$ であり, $t^{c-1+(s-n)} = t^{a(a-1)+((s-n)-1)} \in \mathfrak{m}^{a-1} \subseteq \mathfrak{m}^q$ となる。従って, $t^{s+c-1} = t^n t^{c-1+(s-n)} \in Q$ となり, $c-1 \in H$ となってしまうので矛盾である。

(1) $\Rightarrow$ (3) $q \geq a$ とする。今 $s \geq a$ である。もし $s-a \in H$ なら, $n \geq aq$ なる任意の元 $n \in H$ に対して, $(n+s-a)-s = n-a \geq aq-a \geq a^2-a = c$ より, $(n+s-a)-s \in H$ であり, $t^n t^{s-a} = t^{(n+s-a)-s} t^s \in Q$ である。従って, $t^{s-a} \in I \subseteq \overline{Q}$ となる。しかし $s-a < s$ より, これは矛盾である。よって, $s-a \notin H$ である。ここで, $s = a\ell + r$ $(\ell \geq 0, 0 \leq r < a)$ と書くと, もちろん $\ell > 0$ であり, $s-a = a(\ell-1)+r \notin H$ である。一方, $s-a \notin H$ より, $s-a < c = a(a-1)$ であるから, $\ell-1 \leq a-2 < a-1$ となり, 補題 3.1 (2) より, $r > \ell-1$ となるので, $r \geq \ell$ となる。また, $\ell \leq a-1$, $s = a\ell + r \in H$ だったので, 補題 3.1 (2) より, $r \leq \ell$ となるから, $r = \ell$ となる事が解り, $s = (a+1)\ell$ となる。ここで, $n \geq aq$ なる任意の元 $n \in H$ に対して, $a\ell + n - s \geq a(\ell+q) - (a+1)\ell = aq - \ell \geq a^2 - (a-1) = c+1$ であるので, $a\ell + n - s \in H$ であり, $t^n t^{a\ell} = t^{a\ell+n-s} t^s \in Q$ となるから, $t^{a\ell} \in I \subseteq \overline{Q}$ となる。従って, $a\ell \geq s = (a+1)\ell$ となるから, $\ell = 0$ となり, これは $\ell > 0$ に反する。 $\square$

以下, $q < a$ を仮定する。

命題 3.3. $s \geq aq$ ならば等式 $I^2 = QI$ が成立する。従って, G(I) は Cohen-Macaulay 環である。

証明. $q < a$ から, $\mathfrak{m}^q I = \mathfrak{m}^q Q$ であるので, $I^2 \cap Q = QI$ となる。ここで, 元 $t^n \in I$ を取ると, $n \geq s \geq aq$ であるので, $t^n \in \mathfrak{m}^q$ である。故に, $I \subseteq \mathfrak{m}^q$ である事が解る。従って, $I^2 \subseteq \mathfrak{m}^q I = \mathfrak{m}^q Q \subseteq Q$ となるから, $I^2 = QI$ である。 $\square$

そこで, さらに $s < aq$ を仮定する。以下, $s = a\ell + r$ $(\ell \geq 0,\ 0 \leq r < a)$ と書くと, $1 \leq \ell < q \leq a-1$ であり, 補題 3.1 (2) より $r \leq \ell$ である。ここで, $p = (a-1-q)+\ell = (a-1)+(\ell-q)$ とおくと, $1 \leq \ell \leq p < a-1$ であり, $0 \leq r \leq p-\ell+r \leq p$ である。これらの記号の元にイデアル I の形を決定する命題が次のものである。

命題 3.4. 次が正しい。

(1) 等式

$$\begin{aligned} I &= Q + \mathfrak{m}^{p+1} + (t^{ap+i} \mid p-\ell+r < i \leq p,\ i \in \mathbb{Z}) \\ &= Q + \mathfrak{m}^{p+1} + (t^{ap+i} \mid r \leq i \leq p,\ i \in \mathbb{Z}) \end{aligned}$$

が成り立つ。

(2) $r = 0$ ならば $I = Q + \mathfrak{m}^p$ である。

証明. (1) $\mathfrak{m}^q = (t^{aq+i} \mid 0 \le i \le q,\ i \in \mathbb{Z})$ であったので, 元 $n \in H$ について, $t^n \in I$ である事と, 任意の整数 $0 \le i \le q$ について $aq + i + (n - s) \in H$ である事は同値である。従って, $n \ge a(p+1)$ となる元 $n \in H$ について, $n - s + aq \ge a(p+1) - s + aq = c + (a - r) \ge c$ なので, $t^n \in I$ となる。また, 整数 $r \le i \le p$ について, $n := ap + i = c - aq + s - r + i \ge c - aq + s$ より, $aq + (n - s) \ge c$ なので, $t^n \in I$ となる。従って, $I \supseteq Q + \mathfrak{m}^{p+1} + (t^{ap+i} \mid r \le i \le p, i \in \mathbb{Z}) := K$ である。逆の包含を示そう。任意に元 $t^n \in I$ を取り, $n = aq_1 + r_1 (q_1 \ge 0, 0 \le r_1 < a)$ と書く。$n \ge ap + r$ の時, $q_1 \ge p$ であるが, もし $q_1 = p$ なら, $r \le r_1 < a$ である。また, $1 \le q_1 = p < a - 1$ であったので, 補題 3.1 (2) より, $r_1 \le q_1 = p$ となるから, $t^n \in (t^{ap+i} \mid r \le i \le p, i \in \mathbb{Z}) \subseteq K$ である。もし $q_1 > p$ なら, $n = aq_1 + r_1 \ge aq_1 \ge a(p+1)$ より, $t^n \in \mathfrak{m}^{p+1} \subseteq K$ である。次に, $n < ap + r$ の時, $t^n \in Q$ を示そう。$0 \le n - s < ap + r - s = a(p - \ell)$ より, $0 < \ell < p$ である。ここで, $n - s = aq_2 + r_2 (q_2 \ge 0,\ 0 \le r_2 < a)$ と書くと, $0 \le aq_2 + r_2 = n - s < ap + r - s = a(p - \ell)$ より, $0 < p - \ell - q_2 = a - 1 - q - q_2$ であるから, $q + q_2 < a - 1$ である。

Claim 3.5. $r_2 \le q_2$ である。

証明. もし, $r_2 + q \le a - 1$ ならば, 今 $t^n \in I$ より, $a(q+q_2) + (r_2+q) = aq + q + (n-s) \in H$ であったので, $0 \le q + q_2 < a - 1$ に注意すると, 補題 3.1 (2) から, $r_2 + q \le q + q_2$ となるから, $r_2 \le q_2$ となる。次に, $r_2 + q > a - 1$ の時を考える。$i = a - 1 - r_2$ と置くと, $0 \le i < q$ であり, 今 $t^n \in I$ であるから, $a(q + q_2) + (a - 1) = a(q + q_2) + ((a - 1 - r_2) + r_2) = a(q + q_2) + (r_2 + i) = aq + i + (n - s) \in H$ である。従って, 補題 3.1 (2) より, $a - 1 \le q + q_2 < a - 1$ となり矛盾である。従って, $r_2 + q \le a - 1$ 故, $r_2 \le q_2$ である。 □

従って, 補題 3.1 (2) より $n - s \in H$ となるので, $t^n \in Q$ となる。以上より, $I \subseteq K$ となるので, $I = K$ である。最後に, 整数 $r \le i \le p - \ell + r \le p$ について, $ap + i = s + a(p - \ell) + (i - r)$ であり, 補題 3.1 (2) より, $a(p - \ell) + (i - r) \in H$ であるから, $t^{ap+i} \in Q$ である。従って, $Q + \mathfrak{m}^{p+1} + (t^{ap+i} \mid p - \ell + r < i \le p, i \in \mathbb{Z}) = K$ である事が示された。

(2) (1) と補題 3.1 (1) に依る。 □

系 3.6. $q = a - 1$ のとき, $r = 0,\ \ell$ ならば G(I) は Cohen-Macaulay 環である。

証明. $r = 0$ とすると, $Q \subseteq \mathfrak{m}^\ell$ より, $I = Q + \mathfrak{m}^\ell = \mathfrak{m}^\ell$ となる。従って, 補題 3.1 (1) より, 全ての整数 n について $I^{n+1} \cap Q = QI^n$ が成り立つので, G(I) は Cohen-Macaulay 環である。

以下, $r=\ell$ とする。

$$\begin{aligned} I &= Q+\mathfrak{m}^{\ell+1} \\ &= (t^{(a+1)\ell}, t^{a(\ell+1)}, t^{a(\ell+1)+1}, \cdots, t^{a(\ell+1)+(\ell-1)}, t^{a(\ell+1)+\ell}, t^{(a+1)(\ell+1)}) \\ &= Q+(t^{a(\ell+1)}, t^{a(\ell+1)+1}, \cdots, t^{a(\ell+1)+(\ell-1)}) \end{aligned}$$

より, 任意に整数 $n\geq 1$ を取ると, $I^{n+1}=QI^n+(t^{a(\ell+1)(n+1)+i} \mid 0\leq i\leq (\ell-1)(n+1))$ である。

Claim 3.7. $I^{n+1}=QI^n+(t^{a(\ell+1)(n+1)+i} \mid 0\leq i\leq \ell-1)$ である。

証明. 任意の整数 $\ell\leq i\leq(\ell-1)(n+1)$ に対して, $0\leq i-\ell\leq(\ell-1)(n+1)-\ell=n(\ell-1)-1<n(\ell-1)$ より, $i-\ell+1\leq n(\ell-1)$ である。$(a(\ell+1)(n+1)+i)-s=a((\ell+1)(n+1)-i)+(a+1)(i-\ell)\in H$ であり, $a(\ell+1)(n+1)+i)-s=a(\ell+1)n+(i-\ell+1)$ であるから, $t^{(a(\ell+1)(n+1)+i)-s}\in I^n$ となる。従って, $t^{a(\ell+1)(n+1)+i}\in QI^n$ である。 □

任意に整数 $0\leq i\leq\ell-1$ を取り, $u=a(\ell+1)(n+1)+i$ とする。まず, $(\ell+1)(n+1)-i\geq a+1$ の時を考える。今, $i+a>\ell$ より, $(i+a)-j\ell\geq 0$, $(i+a)-(j+1)\ell<0$ を満たす整数 $j>0$ が存在する。よって, $u-js=a((\ell+1)(n+1)-(i+a+1))+(a+1)((i+a)-j\ell)\in H$ であり, $((\ell+1)(n+1)-(i+a+1))+((i+a)-j\ell)=n\ell+\ell-j\ell+n\geq n\ell+\ell-j\ell+n+(1-j)=(n+1-j)(\ell+1)$ である。故に, $j\leq n+1$ ならば, $t^{u-js}\in\mathfrak{m}^{((n+1)-j)(\ell+1)}\subseteq I^{(n+1)-j}$ であるから, $t^u\in Q^jI^{(n+1)-j}\subseteq QI^n$ である。$j>n+1$ ならば, $t^u\in Q^j\subseteq QI^n$ である。次に, $(n+1)(\ell+1)-i<a+1$ の時, $u-s=(\ell+1)(n+1)+i-(a\ell+\ell)<(i+a+1)+i-(a\ell+\ell)\leq(\ell-1+a+1)+\ell-1-(a\ell+\ell)=(a-1)(1-\ell)\leq 0$ 従って, $u=s$ かまたは, $u-s<0$ である。$u=s$ ならば $i=0$ であり, また $\ell=1$ となるので, $s=a+1$ である。しかし, $n\geq 1$ に注意すると, $t^{a+1}=t^u\in\mathfrak{m}^{n+1}\subseteq\mathfrak{m}^2$ となり矛盾である。従って, $u-s<0$ であるから, $u-s\notin H$ となるので, $t^u\notin Q$ となる。以上から, 任意の整数 n に対して, $I^{n+1}\cap Q=QI^n$ が成り立つ事が示されたので, G(I) は Cohen-Macaulay 環である。 □

本稿の最後に, 一般には $G(I)$ が Cohen-Macaulay 環となるとは限らないことに触れたい。実際, 次の定理が成り立つ。

定理 3.8. $\ell\geq 2$, $a\geq\ell+3$ とする。この時, $r=\ell-1$, $q=a-1$ であるならば, G(I) は Cohen-Macaulay 環でない。

証明. 命題3.4の (1) より,

$$\begin{aligned} I &= Q + \mathfrak{m}^{\ell+1} + (t^{a\ell+\ell}) \\ &= (t^{a\ell+\ell-1}, t^{a(\ell+1)}, t^{a(\ell+1)+1}, \cdots, t^{a(\ell+1)+\ell+1}, t^{a\ell+\ell}) \\ &= (t^{a\ell+\ell-1}, t^{a\ell+\ell}, t^{a\ell+a}, t^{(a\ell+a)+1}, \cdots, t^{(a\ell+a)+(\ell+1)}) \end{aligned}$$

である。$\alpha_1 = a\ell+\ell-1, \alpha_2 = a\ell+\ell, \alpha_3 = a\ell+a, \alpha_4 = (a\ell+a)+1, \cdots, \alpha_n = (a\ell+a)+(\ell+1)$ (但し, $n = \ell+4$) とおく。もちろん, $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \cdots < \alpha_n$ である。

Claim 3.9. $(t^{a\ell+\ell})^{a-\ell} \in I^{a-\ell} \cap Q =$ であり, $(t^{a\ell+\ell})^{a-\ell} \notin QI^{(a-\ell)-1}$ である。

証明. $\ell \geq 2$, $a \geq \ell+3$ に注意すると, $((a\ell+\ell)(a-\ell)-s)-c = (\ell-1)a^2 - (\ell^2-1)a - \ell^2 - \ell + 1 > 0$ であるので, $(a\ell+\ell)(a-\ell)-s \in H$ である。従って, $(t^{a\ell+\ell})^{a-\ell} \in I^{a-\ell} \cap Q$ である。ここで, $(t^{a\ell+\ell})^{a-\ell} \in QI^{(a-\ell)-1}$ とすると, $t^{(a\ell+\ell)(a-\ell)-s} \in I^{a-\ell-1} = (t^{\alpha_1\beta_1}t^{\alpha_2\beta_2}\cdots t^{\alpha_n\beta_n} \mid 0 \leq \beta_1, \beta_2, \cdots, \beta_n \in \mathbb{Z}, \beta_1+\beta_2+\cdots+\beta_n = a-\ell-1)$ となるので, $\beta_1+\beta_2+\cdots+\beta_n = a-\ell-1$ となる整数 $0 \leq \beta_1, \beta_2, \cdots, \beta_n$ と, ある元 $h \in H$ で $(a\ell+\ell)(a-\ell)-s = \alpha_1\beta_1+\alpha_2\beta_2+\cdots+\alpha_n\beta_n+h$ と書ける。従って, $u := (a\ell+\ell)(a-\ell)-s = a^2\ell - a\ell^2 - \ell^2 - \ell + 1 \geq \sum_{i=1}^n \alpha_i\beta_i$ である。一方, $\alpha_1(a-\ell-1) = (a\ell+\ell-1)(a-\ell-1) = a^2\ell - a\ell^2 - \ell^2 - a + 1 < u$ であり, $0 < a-\ell < a$ 故, $u - \alpha_1(a-\ell-1) = a-\ell \notin H$ であるから, $\beta_1 < a-\ell-1$ である。従って, ある整数 $1 < i \leq n$ で, $\beta_i \geq 1$ である。よって, $u \geq \sum_{j=1}^n \alpha_j\beta_j = (\sum_{i\neq j}\alpha_j\beta_j)+\alpha_i\beta_i \geq \sum_{i\neq j}\alpha_1\beta_j)+\alpha_i\beta_i = (\sum_{j=1}^n \alpha_1\beta_j)+(\alpha_i-\alpha_1)\beta_i = \alpha_1(a-\ell-1)+(\alpha_i-\alpha_1)\beta_i \geq \alpha_1(a-\ell-1)+(\alpha_i-\alpha_1)$ となる。$i=2$ である。実際, $i \geq 3$ とすると, $u \geq \alpha_1(a-\ell-1)+(\alpha_i-\alpha_1) \geq a^2\ell-a\ell^2-\ell^2-a+1+(\alpha_3-\alpha_1) = a^2\ell-a\ell^2-\ell^2-\ell+2 = u+1$ となり矛盾である。従って, $\beta_2 \geq 1$ であり, $\beta_3 = \beta_4 = \cdots = \beta_n = 0$ である。故に, $\alpha_1\beta_1+\alpha_2\beta_2 = a^2\ell - a\ell^2 - \ell^2 - \ell - \beta_1$ となり, $h = u - (\alpha_1\beta_1+\alpha_2\beta_2) = 1+\beta_1 \in H$ となる。しかし, $0 \leq \beta_1 < a-\ell-1$ であったので, $1 \leq 1+\beta_1 < a-\ell < a$ であるから, $1+\beta_1 \notin H$ であるので矛盾である。 □

従って, G(I) は Cohen-Macaulay 環でない。 □

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$\mathfrak{m}$-primary ideals with the small first Hilbert coefficients

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1 Introduction

以下，$(A, \mathfrak{m})$ は Cohen-Macaulay 局所環とし次元は $d = \dim A > 0$ とする。環 A の剰余体 $A/\mathfrak{m}$ は無限体であると仮定する。I を環 A の $\mathfrak{m}$-準素イデアルとし，Q をイデアル I の極小節元とする。剰余体 $A/\mathfrak{m}$ が無限体であるので，イデアル Q は環 A の巴系イデアルである。以下，

$$
\begin{aligned}
R &= \mathcal{R}(I) = \bigoplus_{n \geq 0} I^n, \\
T &= \mathcal{R}(Q) = \bigoplus_{n \geq 0} Q^n, \\
G &= \mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}
\end{aligned}
$$

とおき，R をイデアル I の Rees 代数，T をイデアル Q の Rees 代数，G をイデアル I の随伴次数環という。

$$B = T/\mathfrak{m}T \ (\cong (A/\mathfrak{m})[X_1, X_2, \cdots, X_d])$$

とおく。但し，$(A/\mathfrak{m})[X_1, X_2, \cdots, X_d]$ は体 $A/\mathfrak{m}$ 上の多項式環をなす。$N = \mathfrak{m}T + T_+$ を Rees 代数 T の極大次数付きイデアルとおく。ある整数 $\mathrm{e}_i(I)$ $(0 \leq i \leq d)$ が存在して，十分大きい整数 $n \gg 0$ に対して，

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

と表せることが知られており，これをイデアル I の Hilbert 多項式と呼ぶ。但し，A-加群 M に対し，$\ell_A(M)$ によって A-加群としての M の長さを表す。このとき，各 $\mathrm{e}_i(I)$ をイデアル I の第 i 番目の Hilbert 係数と呼ぶ。

本研究の目的は，Cohen-Macaulay 局所環内の $\mathfrak{m}$-準素イデアル I について，この Hilbert 係数 $\mathrm{e}_i(I)$ と随伴次数環 $\mathrm{G}(I)$ の環構造の関係を探ることにある。特に，本報告では第 1 番目の Hilbert 係数 $\mathrm{e}_1(I)$ について考えたい。その上で基本的な手がかりとなる結果は次のものである。

命題 1 (cf.[N], [H]). 次の不等式が正しい。

$$0 \leq \mu_A(I) - d \leq \mathrm{e}_0(I) - \ell_A(A/I) \leq \mathrm{e}_1(I).$$

さらに，次の条件 (1)，(2)，(3) が成り立つ。

(1) $\mu_A(I) - d = 0 \Longleftrightarrow I = Q$,

(2) $\mathrm{e}_0(I) - \ell_A(A/I) = \mu_A(I) - d \Longleftrightarrow \mathfrak{m}I \subseteq Q$,

(3) $\mathrm{e}_0(I) - \ell_A(A/I) = \mathrm{e}_1(I) \Longleftrightarrow I^2 = QI$.

但し，A-加群 M に対し $\mu_A(M)$ によって A-加群としての M の極小生成元の個数を表すこととする。

この命題により，$\mathrm{e}_1(I) = 0$ ならば $I = Q$ である。よって，イデアル I は巴系イデアルであり，その随伴次数環 $\mathrm{G}(I)$ の環構造は非常によくわかる。さらに，$\mathrm{e}_1(I) = 1$ ならば $I^2 = QI$ が成り立つことから，随伴次数環 $\mathrm{G}(I)$ は Cohen-Macaulay 環であることがわかる。すると，自然に次の問いを考えたくなるであろう。

問題 2. $\mathrm{e}_1(I)$ が十分小さい $\mathfrak{m}$-準素イデアル I について，その随伴次数環 $\mathrm{G}(I)$ の環構造を解析せよ。

そこで，本稿ではこの問いに対し，まず $\mathrm{e}_1(I) = 2$ のときを解析した結果を報告したい。但し，上でも述べた通り，等式 $I^2 = QI$ が成り立つならば，随伴次数環 $\mathrm{G}(I)$ が Cohen-Macaulay 環になるということは，良く知られた事実であるという事を注意しておきたい。従って，本稿では $I^2 \neq QI$ であるイデアル I を研究対象とした。

主結果を述べる前に，必要となる記号を準備したい。C を可換環，J を環 C のイデアル，M を C-加群としたとき，整数 $0 \leq i \leq d$ に対して，

$$\mathrm{H}^i_J(M) = \lim_{n\to\infty} \mathrm{Ext}^i_C(C/J^n, M)$$

によって C-加群 M のイデアル J に関する i 番目の局所コホモロジー加群を表す。また，

$$S = IR/IT$$

と定め，イデアル I の Q に関する Sally 加群と呼ぶ。この Sally 加群は本稿の議論において非常に重要な役割を果たす。Sally 加群 S は次の次数付けによって，次数付き T-加群の構造を持つ。

$$S_n = \begin{cases} I^{n+1}/Q^nI & (n \geq 1) \\ (0) & (n \leq 0). \end{cases}$$

Sally 加群については，次の事実がよく知られている。

事実 3. 次が正しい。

(1) $S=(0) \Longleftrightarrow I^2=QI$.

(2) $S=TS_1 \Longleftrightarrow I^3=QI^2$.

(3) $\mathfrak{m}S=(0)$ ならば $\mathfrak{m}I^2 \subseteq QI$ が成り立つ。

(4) 任意の整数 $n \geq 0$ に対して，次の等式が成り立つ。

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{d+n}{d} - (\mathrm{e}_0(I)-\ell_A(A/I))\binom{d+n-1}{d-1} - \ell_A(S_n)$$

(5) $\mathrm{Ass}_T\, S \subseteq \{\mathfrak{m}T\}$ が成り立つ。

(6) $\mathrm{e}_1(I)=\mathrm{e}_0(I)-\ell_A(A/I)+\ell_{T_\mathcal{P}}(S_\mathcal{P})$ が成り立つ。但し，$\mathcal{P}=\mathfrak{m}T$ である。

これらの記号の下に，本稿の主結果を述べれば下記の通りである。

定理 4. 次の条件はそれぞれ同値である。

(1) 等式 $\mathrm{e}_1(I)=2$ かつ不等式 $I^2 \neq QI$ が成り立つ。

(2) 次数付き T-加群としての同型 $S \cong B(-1)$ が成り立ち，かつ，等式 $\ell_A(I/Q)=1$ が成り立つ。

(3) 等式 $I^3=QI^2$，$\ell_A(I/Q)=1$ かつ $\operatorname{depth} \mathrm{G}(I)=d-1$ が成り立つ。

さらにこのとき，随伴次数環 $\mathrm{G}(I)$ の環構造，特に Buchusbaum 性について次の結果が得られる。

定理 5. $\mathrm{e}_1(I)=2$ とし，$I^2 \neq QI$ であると仮定する。このとき次の条件は同値である。

(1) $\mathrm{G}(I)$ は *Buchsbaum* 環である。

(2) $\mathrm{H}_N^{d-1}(\mathrm{G}(I))=[\mathrm{H}_N^{d-1}(\mathrm{G}(I))]_{1-d}$ である。

また，このとき $\ell_A([\mathrm{H}_N^{d-1}(\mathrm{G}(I))]_{1-d})=1$ が成り立つ。さらに，$d \geq 2$ のとき次の条件も同値である。

(3) $\mathcal{R}(I)$ は *Buchsbaum* 環である。

但し，次数付き加群 M に対し M_i によって M の i 番目の斉次部分を表す。

このように，$\mathrm{e}_1(I)=2$ の場合については，イデアル I の構造と随伴次数環 $\mathrm{G}(I)$ の環構造に関する興味深い結果を得ることができた。しかし，$\mathrm{e}_1(I) \geq 3$ の場合については，その構造を制御する方法が今のところ見つかっていない。

ここで本稿の構成を説明しておく。第 2 節において定理 4 の証明を与え，定理 5 の証明は第 3 節で与える。第 4 節では，定理 5 の条件を満たす例に触れたい。

2 定理4の証明

本節では，定理4の証明を与える。以下，$r_Q(I) = \min\{r \geq 0 | I^{r+1} = QI^r\}$ とおく。定理4の証明は次の定理が本質的である。

定理 6 (後藤-西田). $d = 2$ とする。このとき，次の条件は同値である。

(1) $\mathfrak{m}S = (0)$ かつ $\mathrm{rank}_B S = 1$.

(2) 次数付き T-加群としての同型 $S \cong B(-1)$ か又は $S \cong B_+$ が存在する。

この定理4の (2) の同型は Sally 加群 S の極小自由分解を表している。特に，次数付き T-加群としての同型 $S \cong B_+$ と完全列

$$0 \to B(-2) \to B(-1) \oplus B(-1) \to S \to 0$$

は同様の事を表している。

それでは定理4の証明を与えよう。

定理4の証明. (1) ⇒ (3) $\mathrm{e}_1(I) = 2$ であるから，等式

$$\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + \ell_{T_\mathcal{P}}(S_\mathcal{P}) = 2$$

を得る。ここで，$I^2 \neq QI$ より $S \neq (0)$ となる。このとき，$\mathrm{Ass}_T S = \{\mathcal{P}\}$ であるから，$S_\mathcal{P} \neq (0)$ となり，$\ell_{T_\mathcal{P}}(S_\mathcal{P}) \neq 0$ が得られる。さらに，$I \neq Q$ より $\mathrm{e}_0(I) - \ell_A(A/I) \neq 0$ が分かる。従って，

$$\ell_{T_\mathcal{P}}(S_\mathcal{P}) = 1,$$
$$\mathrm{e}_0(I) - \ell_A(A/I) = 1$$

が得られる。すると，A は Cohen-Macaulay 環で，Q は I の節元であるから，

$$\ell_A(I/Q) = \ell_A(A/Q) - \ell_A(A/I) = \mathrm{e}_0(Q) - \ell_A(A/I) = \mathrm{e}_0(I) - \ell_A(A/I) = 1$$

を得る。よって特に，$\mathfrak{m}I \subseteq Q$ である。また，S から $S_\mathcal{P}$ への単射が存在し，$\ell_{T_\mathcal{P}}(S_\mathcal{P}) = 1$ であるから，$\mathcal{P}S = (0)$ となる。よって $\mathfrak{m}S = (0)$ が得られる。従って，$\mathrm{rank}_B S = 1$ となることが分かる。$I^2 \subseteq Q$ かつ $I^2 \neq QI$ より，$Q \cap I^2 \neq QI$ が成立する。従って，随伴次数環 $\mathrm{G}(I)$ は Cohen-Macaulay 環ではない。

Claim 7. 等式 $\ell_A(I^2/QI) = 1$ が成り立つ。

Proof. $\ell_A(I/Q) = 1$ より，ある $x \in I \backslash Q$ が存在し，等式 $I = Q + (x)$ と表せる。このとき，$I^2 = QI + (x^2)$ とかける。ここで，$\mathfrak{m}I^2 \subseteq QI$ かつ $I^2 \neq QI$ より，$\ell_A(I^2/QI) = 1$ が得られる。 □

$d=1$ の場合，$r_Q(I) \leq \mathrm{e}_1(I)$ が成り立つ。よって $I^3 = QI^2$ が得られる。$d=2$ の場合，上の議論により，$\mathfrak{m}S=(0)$ かつ $\mathrm{rank}_B\, S = 1$ が成り立つことから，後藤-西田の定理により次数付き T-加群としての同型 $S \cong B(-1)$ か又は $S \cong B_+$ が得られる。しかし，次数付き T-加群としての同型 $S \cong B_+$ が成り立つとき，1 次の斉次部分 $S_1 \cong B_1$ を見るに，次数付き加群 B は剰余体 $A/\mathfrak{m}$ 上の 2 変数の多項式環と同型であることから，$\ell_A(S_1) = 2$ となってしまうことが分かる。従って，$\ell_B(S_1) = \ell_A(I^2/QI) = 1$ であるため，次数付き T-加群としての同型 $S \cong B(-1)$ としか取り得ない。これより，Sally 加群 S は S_1 によって生成されていることが分かり，$I^3 = QI^2$ が成り立つ。ここで，完全列

$$0 \to IT \to T \to T/IT \to 0$$

において，イデアル Q の Rees 代数 T は 3 次元の Cohen-Macaulay 環で，剰余加群 T/IT は剰余環 A/I 上の 2 変数の多項式環と同型であるため，加群 IT は 3 次元の Cohen-Macaulay 加群をなすことが分かる。さらに，Sally 加群 $S \cong B(-1)$ であることから，次の完全列

$$0 \to IT \to IR \to S \to 0$$

より，$\mathrm{depth}_T\, IR \geq 2$ が得られる。次数付き R-加群としての同型 $IR \cong R_+(1)$ であることに注意しながら完全列

$$0 \to R_+ \to R \to A \to 0$$

を見るに，$\mathrm{depth}\, R \geq 2$ が分かる。さらに，完全列

$$0 \to R_+(1) \to R \to \mathrm{G}(I) \to 0$$

により，$\mathrm{depth}\, \mathrm{G}(I) \geq 1$ が成り立つことが分かる。ここで，随伴次数環 $\mathrm{G}(I)$ は Cohen-Macaulay 環ではないため，$\mathrm{depth}\, \mathrm{G}(I) = 1$ である。

$d \geq 3$ の場合。$d-1$ まで正しいと仮定する。イデアル Q の生成元 $a_1 \in Q$ をイデアル I の上表元としてよい。このとき，$\overline{A} = A/(a_1)$，$\overline{I} = A/(a_1)$，そして $\overline{Q} = Q/(a_1)$ とおくと，$\mathrm{e}_1(\overline{I}) = \mathrm{e}_1(I) = 2$ である。もし，$\overline{I}^2 = \bar{Q}\bar{I}$ ならば，イデアル $\overline{I}$ の随伴次数環 $\mathrm{G}(\overline{I})$ は Cohen-Macaulay 環となる。従って，特に [S] により $f = a_1 t$ は随伴次数環 $\mathrm{G}(I)$ の正則元であり，同型 $\mathrm{G}(I)/f\,\mathrm{G}(I) \cong \mathrm{G}(\overline{I})$ となるから，イデアル I の随伴次数環 $\mathrm{G}(I)$ も Cohen-Macaulay 環となってしまい，これは矛盾である。故に，$\overline{I}^2 \neq \bar{Q}\bar{I}$ である。よって，帰納法の仮定により $\overline{I}^3 = \bar{Q}\bar{I}^2$ と $\mathrm{depth}\, \mathrm{G}(\overline{I}) = (d-1)-1 > 0$ が得られる。よって，[S] により $f = a_1 t$ は随伴次数環 $\mathrm{G}(I)$ の正則元となり，同型 $\mathrm{G}(I)/f\,\mathrm{G}(I) \cong \mathrm{G}(\overline{I})$ が成り立つ。従って，$\mathrm{depth}\, \mathrm{G}(I) = d-1$ であり，

$$I^3 = (a_1, a_2, \cdots, a_d)I^2 + (a_1) \cap I^3 = (a_1, a_2, \cdots, a_d)I^2 + a_1 I^2 = QI^2$$

が成り立つ。

(3) $\Rightarrow$ (2) 仮定より，$\ell_A(I/Q) = 1$ に従い $\mathfrak{m}I \subseteq Q$ で，ある元 $x \in I$ に対して，$I = Q + (x)$ とかける。故に，等式 $I^2 = QI + (x^2)$ と表せて，$\mathfrak{m}I^2 \subseteq QI$ より，$\ell_A(I^2/QI) \leq 1$ が得られる。ここで，随伴次数環 $\mathrm{G}(I)$ は Cohen-Macaulay 環ではないので，$\ell_A(I^2/QI) \neq 0$ を得る。従って，$\ell_A(I^2/QI) = 1$ である。故に，$B(-1)$ から S への全射が存在し，このとき，Sally 加群 S は B-torsion free であるため，同型 $S \cong B(-1)$ が得られる。

(2) $\Rightarrow$ (1) $\mathfrak{m}S = (0)$ で $\mathrm{rank}_B\, S = 1$ である。従って，等式

$$\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1 = \ell_A(I/Q) + 1 = 2$$

が成り立つ。また，$S \neq (0)$ より $I^2 \neq QI$ である。 □

3 定理5の証明

本節では定理5の証明を与える。以下，イデアル I に対し

$$\widetilde{I} = \bigcup_{n>0} [I^{n+1} : I^n]$$

とおき，イデアル I の Ratliff-Rush 閉包と呼ぶ。

定理5の証明の前に，次の命題を準備しておく。

命題 8. $\mathrm{e}_1(I) = 2$ とし，$I^2 \neq QI$ であると仮定する。このとき次が正しい。

(1) $\mathrm{a}(\mathrm{G}(I)) = 2 - d$ かつ $\ell_A([\mathrm{H}_N^d(\mathrm{G}(I))]_{2-d}) = 1$.

(2) $\mathrm{a}_{d-1}(\mathrm{G}(I)) = 1 - d$ かつ $\ell_A([\mathrm{H}_N^{d-1}(\mathrm{G}(I))]_{1-d}) = 1$ である。

但し，整数 $0 \leq i \leq d$ に対し $\mathrm{a}_i(\mathrm{G}(I)) = \sup\{n \in \mathbb{Z} \mid [\mathrm{H}_N^i(\mathrm{G}(I))]_n \neq (0)\}$ とおく。特に，$\mathrm{a}(\mathrm{G}(I)) = \mathrm{a}_d(\mathrm{G}(I))$ と表し，随伴次数環 $\mathrm{G}(I)$ の a-不変量という。

Proof. $d = \dim A$ に関する帰納法により示す。$d = 1$ とする。定理4より，特に $I^3 = QI^2$ であるから同型

$$\mathrm{H}_N^0(\mathrm{G}(I)) = [\mathrm{H}_N^0(\mathrm{G}(I))]_0 \cong \widetilde{I}/I$$

が得られる。従って，随伴次数環 $\mathrm{G}(I)$ は Cohen-Macaulay 環ではないため，$\ell_A(\widetilde{I}/I) > 0$ に従い，不等式

$$\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_A(A/I) + 1 = \mathrm{e}_0(I) - \ell_A(A/\widetilde{I}) - \ell_A(\widetilde{I}/I) + 1 = \mathrm{e}_0(\widetilde{I}) - \ell_A(A/\widetilde{I}) - \ell_A(\widetilde{I}/I) + 1$$
$$\leq \mathrm{e}_0(\widetilde{I}) - \ell_A(A/\widetilde{I}) \leq \mathrm{e}_1(\widetilde{I}) = \mathrm{e}_1(I)$$

により，$\ell_A(\widetilde{I}/I) = 1$ が得られる。よって，$\ell_A(\mathrm{H}_N^0(\mathrm{G}(I))) = 1$ であるから，$N\,\mathrm{H}_N^0(\mathrm{G}(I)) = (0)$ となる。従って，$d = 1$ のとき，随伴次数環 $\mathrm{G}(I)$ は Buchsbaum 環であることが分かる。

随伴次数環 $\mathrm{G}(I)$ の元 $f = a_1 t$ をとり，完全列

$$0 \to \mathrm{H}_N^0(\mathrm{G}(I))(-1) \to \mathrm{G}(I)(-1) \xrightarrow{\hat{f}} \mathrm{G}(I) \to \mathrm{G}(I)/f\,\mathrm{G}(I) \to 0$$

を見る。これより，完全列

$$0 \to \mathrm{H}_N^0(\mathrm{G}(I)) \to \mathrm{G}(I)/f\,\mathrm{G}(I) \to \mathrm{H}_N^1(\mathrm{G}(I))(-1) \to \mathrm{H}_N^1(\mathrm{G}(I)) \to 0$$

が得られる。ここで，$\mathrm{G}(I)/f\,\mathrm{G}(I) = A/I \oplus I/Q \oplus I^2/QI$ から $\mathrm{a}(\mathrm{G}(I)) \leq 1$ が分かる。任意の正の整数 $n > 0$ に対して，$[\mathrm{H}_N^0(\mathrm{G}(I))]_n = 0$ かつ $\mathrm{a}(\mathrm{G}(I)) \leq 1$ により同型

$$[\mathrm{H}_N^1(\mathrm{G}(I))]_1 \cong [\mathrm{G}(I)/f\,\mathrm{G}(I)]_2 \cong I^2/QI$$

が得られる。従って，$\ell_A(I^2/QI) = 1$ より，随伴次数環 $\mathrm{G}(I)$ の a-不変量 $\mathrm{a}(\mathrm{G}(I)) = 1$ となり，$\ell_A([\mathrm{H}_N^1(\mathrm{G}(I))]_1) = 1$ が得られる。

$d > 1$ と仮定し，$d-1$ まで正しいとする。定理 4 より，$\operatorname{depth} \mathrm{G}(I) = d-1 > 0$ であることから，$f = a_1 t$ は $\mathrm{G}(I)$ の正則元ととれる。このとき，$\overline{A} = A/(a_1)$，$\overline{I} = I/(a_1)$，そして $\overline{Q} = Q/(a_1)$ とおく。この条件の下で，$\mathrm{e}_1(\overline{I}) = 2$ と $\overline{I}^2 \neq \bar{Q}\overline{I}$ が成り立つ。(定理 4 の証明 (1) $\Rightarrow$ (3) を参照。) ここで，同型 $\mathrm{G}(I)/f\,\mathrm{G}(I) \cong \mathrm{G}(\overline{I})$ に注意すると，完全列

$$0 \to \mathrm{H}_N^0(\mathrm{G}(I))(-1) \to \mathrm{G}(I)(-1) \xrightarrow{\hat{f}} \mathrm{G}(I) \to \mathrm{G}(\overline{I}) \to 0$$

が存在する。この完全列から導かれる完全列

$$0 \to \mathrm{H}_N^{d-2}(\mathrm{G}(\overline{I})) \to \mathrm{H}_N^{d-1}(\mathrm{G}(I))(-1) \to \mathrm{H}_N^{d-1}(\mathrm{G}(I)) \to \mathrm{H}_N^{d-1}(\mathrm{G}(\overline{I}))$$
$$\to \mathrm{H}_N^{d}(\mathrm{G}(I))(-1) \to \mathrm{H}_N^{d}(\mathrm{G}(I)) \to 0$$

を見る。帰納法の仮定により，$a_{d-2}(\mathrm{G}(\overline{I})) = 2-d$ であるから，$a_{d-1}(\mathrm{G}(I)) \leq 1-d$ が分かる。このとき，$[\mathrm{H}_N^{d-2}(\mathrm{G}(\overline{I}))]_{2-d} \neq (0)$ であるから $a_{d-1}(\mathrm{G}(I)) = 1-d$ であって，同型

$$[\mathrm{H}_N^{d-2}(\mathrm{G}(\overline{I}))]_{2-d} \cong [\mathrm{H}_N^{d-1}(\mathrm{G}(I))]_{1-d}$$

が得られる。帰納法の仮定により，$\ell_A([\mathrm{H}_N^{d-2}(\mathrm{G}(\overline{I})]_{2-d}) = 1$ であることから，この同型を見るに，$\ell_A([\mathrm{H}_N^{d-1}(\mathrm{G}(I))]_{1-d}) = 1$ である。一方で，帰納法の仮定により，$\mathrm{a}(\mathrm{G}(\overline{I})) = 3-d$ なので，$\mathrm{a}(\mathrm{G}(I)) \leq 2-d$ を得る。任意の整数 $n > 1-d$ に対して，$[\mathrm{H}_N^{d-1}(\mathrm{G}(I))]_n = (0)$ である。よって $[\mathrm{H}_N^{d-1}(\mathrm{G}(\overline{I})]_{3-d}$ から $[\mathrm{H}_N^{d}(\mathrm{G}(I))]_{2-d}$ への単射が存在し，$\mathrm{a}(\mathrm{G}(I)) \leq 2-d$ からこれは同型

$$[\mathrm{H}_N^{d-1}(\mathrm{G}(\overline{I}))]_{3-d} \cong [\mathrm{H}_N^{d}(\mathrm{G}(I))]_{2-d}$$

である。従って，帰納法の仮定により，$\ell_A([\mathrm{H}_N^{d-1}(\mathrm{G}(I))]_{3-d}) = 1$ であるから，随伴次数環 $\mathrm{G}(I)$ の a-不変量 $\mathrm{a}(\mathrm{G}(I)) = 2-d$ となり，$\ell_A([\mathrm{H}_N^{d}(\mathrm{G}(I))]_{2-d}) = 1$ が得られる。 □

これより，定理5の証明を行う。

定理5の証明. (1) $\Leftrightarrow$ (2) $d = \dim A$ に関する帰納法により示す。$d = 1$ のときは，共にいつも成り立つ。よって，$d \geq 2$ と仮定する。

(1) $\Rightarrow$ (2) 随伴次数環 $\mathrm{G}(I)$ は Buchusbaum 環とする。$\overline{A} = A/(a_1)$，$\overline{I} = I/(a_1)$，$\overline{Q} = Q/(a_1)$ とおくと，$\mathrm{e}_1(\overline{I}) = 2$ かつ $\overline{I}^2 \neq \bar{Q}\overline{I}$ が成り立つ。さらに，$\operatorname{depth} \mathrm{G}(I) > 0$ だから，$f = a_1 t$ は随伴次数環 $\mathrm{G}(I)$ の正則元であって，同型 $\mathrm{G}(I)/f\,\mathrm{G}(I) \cong \mathrm{G}(\overline{I})$ より，イデアル $\overline{I}$ に関する随伴次数環 $\mathrm{G}(\overline{I})$ もまた Buchsbaum 環である。命題8及びその証明から，同型

$$\mathrm{H}_N^{d-2}(\mathrm{G}(\overline{I})) \cong \mathrm{H}_N^{d-1}(\mathrm{G}(I))(-1)$$

が得られる。従って，帰納法の仮定により $\mathrm{H}_N^{d-1}(\mathrm{G}(I)) = [\mathrm{H}_N^{d-1}(\mathrm{G}(I))]_{1-d}$ が得られる。

(1) $\Leftarrow$ (2) 命題8より，$\ell_A(\mathrm{H}_N^{d-1}(\mathrm{G}(I))) = 1$ であるから，$N\,\mathrm{H}_N^{d-1}(\mathrm{G}(I)) = (0)$ となる。従って，随伴次数環 $\mathrm{G}(I)$ は Buchsbaum 環である。以上より (1) $\Leftrightarrow$ (2) が示せた。このとき，命題8の (2) により，$\ell_A([\mathrm{H}_N^{d-1}(\mathrm{G}(I))]) = 1$ であることが分かる。

以下 $d \geq 2$ とし (1) $\Leftrightarrow$ (3) を示す。

次数付き R-加群 $L = R_+$ とおく。次数付き R-加群としての完全列,

$$0 \to L \to R \to A \to 0,$$

$$0 \to L(1) \to R \to \mathrm{G}(I) \to 0$$

及び，$\operatorname{depth} A = d$ であることと，$\operatorname{depth} \mathrm{G}(I) = d-1$ であることを見るに，$\operatorname{depth} R = d$ が得られる。このことから，次の2つの完全列が得られる。

$$0 \to \mathrm{H}_N^d(L) \to \mathrm{H}_N^d(R) \to \mathrm{H}_\mathfrak{m}^d(A) \qquad (*)$$

$$0 \to \mathrm{H}_N^{d-1}(\mathrm{G}(I)) \to \mathrm{H}_N^d(L)(1) \to \mathrm{H}_N^d(R) \to \mathrm{H}_N^d(\mathrm{G}(I)) \qquad (**)$$

任意の整数 $n > 2 - d = \mathrm{a}(\mathrm{G}(I))$ に対して，全射

$$[\mathrm{H}_N^d(L)]_{n+1} \to [\mathrm{H}_N^d(R)]_n \to 0$$

が成り立ち，単射

$$0 \to [\mathrm{H}_N^d(L)]_{n+1} \to [\mathrm{H}_N^d(R)]_{n+1}$$

が成り立つ。従って，任意の整数 $n > 2-d$ に対して $[\mathrm{H}_N^d(R)]_n = (0)$ と $[\mathrm{H}_N^d(L)]_n = (0)$ が得られる。

(1) $\Rightarrow$ (3) 随伴次数環 $\mathrm{G}(I)$ は Buchusbaum 環であるとする。すると，$\mathrm{H}_N^{d-1}(\mathrm{G}(I)) = [\mathrm{H}_N^{d-1}(\mathrm{G}(I))]_{1-d}$ となり，任意の整数 $n < 1 - d$ に対して，単射

$$0 \to [\mathrm{H}_N^d(L)]_{n+1} \to [\mathrm{H}_N^d(R)]_n$$

が成り立つ。さらに，$n+1 \leq 1-d<0$ より

$$[\mathrm{H}_N^d(L)]_{n+1} \cong [\mathrm{H}_N^d(R)]_{n+1}$$

が得られる。すると今，局所コホモロジー次数加群 $\mathrm{H}_N^d(R)$ は有限生成であるから (cf.[BS])，任意の整数 $n \leq 1-d$ に対して $[\mathrm{H}_N^d(R)]_n=(0)$ となる。故に，$\mathrm{H}_N^d(R)=[\mathrm{H}_N^d(R)]_{2-d}$ が成り立つ。さて，このとき同型

$$[\mathrm{H}_N^{d-1}(\mathrm{G}(I))]_{1-d} \cong [\mathrm{H}_N^d(L)]_{2-d}$$

が得られ，仮定によりイデアル I の随伴次数環 $\mathrm{G}(I)$ が Buchsbaum 環であることから，$\ell_A([\mathrm{H}_N^{d-1}(\mathrm{G}(I))]_{1-d})=1$ である。

$d \geq 3$ の場合，$[\mathrm{H}_\mathfrak{m}^d(A)]_{2-d}=(0)$ より，単射

$$0 \to [\mathrm{H}_N^d(L)]_{2-d} \to [\mathrm{H}_N^d(R)]_{2-d}$$

を得る。従って，$\ell_A([\mathrm{H}_N^d(R)]_{2-d})=1$ であるから，Rees 代数 R は Buchusbaum 環である。$d=2$ の場合。$[\mathrm{H}_N^2(L)]_1=(0)$ より同型

$$[\mathrm{H}_N^2(R)]_0 \cong [\mathrm{H}_N^2(\mathrm{G}(I))]_0$$

が得られる。一方で，完全列

$$0 \to \mathrm{H}_N^1(\mathrm{G}(I)) \to \mathrm{H}_N^1(\mathrm{G}(\overline{I})) \to \mathrm{H}_N^2(\mathrm{G}(I))(-1) \to \mathrm{H}_N^2(\mathrm{G}(I)) \to 0$$

を見るに，$\mathrm{H}_N^1(\mathrm{G}(I))=[\mathrm{H}_N^1(\mathrm{G}(I))]_{-1}$ かつ $\mathrm{a}(\mathrm{G}(I))=0$ であることから，同型

$$[\mathrm{H}_N^1(\mathrm{G}(\overline{I})]_1 \cong [\mathrm{H}_N^2(\mathrm{G}(I))]_0.$$

が得られる。命題8の (1) を1次元の場合に適応させることにより，$\ell_A([\mathrm{H}_N^1(\mathrm{G}(\overline{I}))]_1)=1$ が分かる。従って，$\ell_A([\mathrm{H}_N^2(\mathrm{G}(I))]_0)=1$ かつ $\ell_A([\mathrm{H}_N^2(R(I))]_0)=1$ である。よって，Rees 代数 $R(I)$ は Buchusbaum 環である。

(3) ⇒ (1) Rees 代数 R が Buchusbaum 環とする。このとき，完全列 (*)，(**) により，$N\,\mathrm{H}_N^{d-1}(\mathrm{G}(I))=(0)$ が分かり，随伴次数環 $\mathrm{G}(I)$ は Buchusbaum 環となる。 □

4 定理5を満たす例

最後に定理5の条件を満たすようなイデアル I の例を挙げる。

例 9. 整数 $m \geq d>0$ をとる。集合 $\{1,2,\cdots,m\}$ の部分集合 Λ を，$\Lambda \cap \{1,2,\cdots,d\}=\emptyset$ を満たすようにとる。$U=k[[X_1,X_2,\cdots,X_m,V,A_1,A_2,\cdots,A_d]]$ を体 k 上の冪級数環とし，

$$K=(X_1,\cdots,X_m)\cdot(X_1,\cdots,X_m,V)+(V^2-\Sigma_{i=1}^d A_iX_i)$$

とおく。ここで，$A=U/K$ と定める。このとき環 A は *Cohen-Macaulay* 局所環でその次元 $\dim A=d$ である。環 A のイデアル I と Q を

$$I=(a_1,a_2,\cdots,a_d)A+(x_\alpha \mid \alpha\in\Lambda)A+vA,\quad Q=(a_1,a_2,\cdots,a_d)A$$

ととる。但し，a_i, x_α, v はそれぞれ A_i, X_α, V の A 内での像であるとする。すると，イデアル I は環 A の $\mathfrak{m}$-準素イデアルであり，イデアル Q は I の節元をなす。このとき，次が正しい。

(1) $\mathrm{G}(I)$ は *Buchsbaum* 環であり，$\operatorname{depth}\mathrm{G}(I)=d-1$，そして $\ell_A(\mathrm{H}_N^{d-1}(\mathrm{G}(I)))=1$ である。

(2) $\mathrm{e}_0(I)=m+2$ かつ $\mathrm{e}_1(I)=\sharp\Lambda+2$.

(3) $d\geq 2$ ならば $\mathrm{e}_2(I)=1$ であり，$d\geq 3$ ならば $\mathrm{e}_3(I)=\mathrm{e}_4(I)=\cdots=\mathrm{e}_d(I)=0$ である。

故に，もし $\Lambda=\emptyset$ ならば，イデアル I は定理 5 の条件を満たすことが分かる。

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Uniform test exponent for ideal-adic tight closures of parameter ideals [1]

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This work is motivated by a joint work with Shunsuke TAKAGI.

Introduction

In 1980's, Hochster and Huneke [8] introduced the notion of tight closure in positive characteristic. The tight closure gives a powerful tool in the theory of Commutative rings. For example, this gives a short proof of Briançon–Skoda theorem, the Cohen–Macaulay property of a pure subring of a regular domain. However, many problems (localization, completion, etc.) remain still open. see e.g. [8, 11].

In 2003, Hara and the author [6] introduced the notion of *ideal-adic tight closure* and defined the *generalized test ideal*, which is an analogue of the multiplier ideals in Algebraic Geometry (in equi-characteristic zero). Roughly speaking, the modulo p reduction of a multiplier ideal ($p \gg 0$) coincides the generalized test ideal; see [6, Theorem 6.8].

In this talk, we consider some fundamental problems (localization etc.) with respect to ideal-adic tight closures and generalized test ideals.

1. Preliminaries

Throughout this talk, let R be a Noetherian ring containing $\mathbb{Z}/p\mathbb{Z}$ with the Frobenius map $F\colon R \to R$ ($a \mapsto a^p$). The ring R viewed as an R-module via the e-times iterated Frobenius map F^e is denoted by ${}^eR$. For an R-module M and $e \in \mathbb{N}$, we put $\mathbb{F}_R^e(M) = {}^eR \otimes_R M$ and regard it as an R-module by the action of $R = {}^eR$ from the left. The induced e-times iterated Frobenius map on M defined by $F^e\colon M \to \mathbb{F}_R^e(M)$ $\left(m \mapsto m^{p^e} := F^e(m) := 1 \otimes m\right)$. For an R-submodule N of M and $q = p^e$, we put $N_M^{[q]} = \mathrm{Ker}(\mathbb{F}_R^e(M) \to \mathbb{F}_R^e(M/N))$. For an ideal I of R, $I^{[q]} = I_R^{[q]}$ is the ideal generated by all elements a^q for $a \in I$.

Let R° denote the complement of the union of all minimal prime ideals of R. A collection of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ of R is called a *graded family of ideals* in R if the following conditions are satisfied:

$$(a)\ \mathfrak{a}_m\mathfrak{a}_n \subseteq \mathfrak{a}_{m+n} \text{ for all } m,\, n \in \mathbb{N}; \qquad (b)\ \mathfrak{a}_1 \cap R^\circ \neq \emptyset.$$

Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number. Then $\mathfrak{a}_n = R$, $\mathfrak{a}_n = \mathfrak{a}^{\lceil tn \rceil}$, or $\mathfrak{a}_n = \mathfrak{a}^{(n)}$ (the *symbolic power* of $\mathfrak{a}^n$) are those examples.

[1] The 28th symposium on Commutative Ring Theory; Nov.13–16, 2006; at Tama

For any graded family $\{\mathfrak{a}_\bullet\}$ of ideals in R, we define the notion of $\mathfrak{a}_\bullet$-tight closure.

Definition 1.1 (Ideal-adic tight closure [4, 6]**).** Let $\mathfrak{a}_\bullet$ be a graded family of ideals in R, and let $N \subseteq M$ be R-modules. For $z \in M$,

$$z \in N_M^{*\mathfrak{a}_\bullet} \quad \overset{def}{\Longleftrightarrow} \quad \exists c \in R^\circ \text{ such that } cz^q \mathfrak{a}_q \subseteq N_M^{[q]} \text{ for all } q = p^e,\ e \gg 0.$$

The R-module $N_M^{*\mathfrak{a}_\bullet}$ is called the *$\mathfrak{a}_\bullet$-tight closure* of N in M. For an ideal I of R, we define $I^{*\mathfrak{a}_\bullet} = I_R^{*\mathfrak{a}_\bullet}$.

Remark 1.2. In case of $\mathfrak{a}_\bullet = \{R\}$, $N_M^{*\mathfrak{a}_\bullet} = N_M^*$ is the *tight closure* introduced by Hochster and Huneke [8]. Moreover, in case of $\mathfrak{a}_\bullet = \{\mathfrak{a}^{\lceil tn \rceil}\}$, $N_M^{*\mathfrak{a}_\bullet} = N_M^{*\mathfrak{a}^t}$ is the *$\mathfrak{a}^t$-tight closure* introduced by Hara and the author [6].

The test ideal $\tau(R) = \bigcap_{I \subseteq R} I : I^*$ plays an important role in the theory of tight closure. In [6], Hara and the author generalized this notion. The test ideal is generated by "test elements", but the generalized test ideal $\tau(\mathfrak{a}_\bullet)$ is not necessarily generated by $\mathfrak{a}_\bullet$-test elements. An existence of $\mathfrak{a}_\bullet$-test element will be discussed in Section 2.

Definition 1.3 (Generalized test ideal [4, 6]**).** Put $E = \oplus_{\mathfrak{m} \in \mathrm{Max}(R)} E_R(R/\mathfrak{m})$. Let $\mathfrak{a}_\bullet$ be a graded family of ideals in R. Then we define

$$\tau(\mathfrak{a}_\bullet) = \bigcap_M \mathrm{Ann}_R(0)_M^{*\mathfrak{a}_\bullet} \left(= \mathrm{Ann}_R(0)_E^{*\mathfrak{a}_\bullet, \mathrm{fg}} \right),$$

where M runs through all finitely generated A-modules (in E). We call this ideal *the (generalized) test ideal with respect to* $\mathfrak{a}_\bullet$ or simply the $\mathfrak{a}_\bullet$-test ideal. We also define $\widetilde{\tau}(\mathfrak{a}_\bullet) = \mathrm{Ann}_R(0)_E^{*\mathfrak{a}_\bullet}$.

Definition 1.4 ($\mathfrak{a}_\bullet$-test element). An element $c \in R^\circ$ is called an *$\mathfrak{a}_\bullet$-test element* of R if $cz^q\mathfrak{a}_q \subseteq I^{[q]}$ holds whenever I is an ideal of R and $z \in I^{*\mathfrak{a}_\bullet}$ and $q = p^e$.

For example, if R is an excellent reduced local ring, then $\tau(\mathfrak{a}_\bullet) = \bigcap_I I : I^{*\mathfrak{a}_\bullet}$, where I runs through all ideals in R. Moreover, if R is Gorenstein and $x_1, \ldots, x_d$ forms a system of paramters, then

$$\tau(\mathfrak{a}_\bullet) = \bigcap_{\ell=1}^{\infty} (x_1^\ell, \ldots, x_d^\ell) : (x_1^\ell, \ldots, x_d^\ell)^{*\mathfrak{a}_\bullet}.$$

The main purpose of this talk is to prove the following theorem:

Theorem 1.5 (Uniform test exponent for s.s.o.p). *Let $(R, \mathfrak{m})$ be an excellent, equidimensional, reduced local ring of characteristic $p > 0$. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_n\}$ be a graded family of ideals. Then for any element $c \in R^\circ$, there exists $e_0 \in \mathbb{N}$ for which the following property holds: For any ideal I which is generated by subsystem of parameters,*

$$(\#) \quad cz^q\mathfrak{a}_q \subseteq I^{[q]} \text{ for some } q = p^e,\ e \geq e_0 \Longrightarrow z \in I^{*\mathfrak{a}}.$$

If c is an $\mathfrak{a}_\bullet$-test element, then the converse is also true.

For an ideal I, a positive integer e_0 which satisfies the condition (#) is called a *test exponent* of I (when $\mathfrak{a}_\bullet = \{R\}$); see [10]. Therefore the above theorem means that *there exists an uniform $\mathfrak{a}_\bullet$-test exponent for ideals which is generated by subsystem of parameters.* Note that e_0 depends on $\mathfrak{a}_\bullet$ and $c \in R^\circ$.

Roughly speaking, "the exsitence of ($\mathfrak{a}_\bullet$-)test exponent" if and only if "($\mathfrak{a}_\bullet$-)tight closure commutes with localization". In particular we have: Let $(R, \mathfrak{m})$ and $\mathfrak{a}_\bullet = \{\mathfrak{a}_n\}$ be as in Theorem 1.5. Let I be an ideal which is generated by subsystem of parameters in R. Then for any multiplicatively closed subset $W \subseteq R$, we have

$$I^{*\mathfrak{a}_\bullet} R_W = (IR_W)^{*\mathfrak{a}_{W,\bullet}},$$

where $\mathfrak{a}_{W,\bullet}$ denotes the graded family $\{\mathfrak{a}_n R_W\}$ of ideals in R_W. We can similarly show that an existence of an uniform test exponent for ideals in an excellent regular local ring; see Section 4.

Moreover, as its application, we can prove the following proposition without F-finiteness of the ring:

Proposition 1.6 (Localization of generalized test ideals). *Let $(R, \mathfrak{m})$ be a complete, Gorenstein, reduced local ring of characteristic $p > 0$. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_n\}$ be a graded family of ideals. Then for any multiplicatively closed subset $W \subseteq R$, we have*

$$\tau(\mathfrak{a}_\bullet) R_W = \tau(\mathfrak{a}_{W,\bullet}).$$

Remark 1.7. The inclusion $\subseteq$ follows from (a weak version of) Corollary 3.3. The proof of the converse requires an analogy of F-ideal which is introduced by Smith [15]. In the F-finite normal $\mathbb{Q}$-Gorenstein case, the result is already known; see Hara and Takagi [5].

2. Proof of existence of $\mathfrak{a}_\bullet$-test elements

In this section, we prove an existence theorem of $\mathfrak{a}_\bullet$-test elements for graded family of ideals $\mathfrak{a}_\bullet$, which plays a key role in the proof of our main theorem. Before stating the theorem, let us recall the following lemma.

Lemma 2.1 ([7, Remark 3.2]). *Let R be an F-finite reduced Noetherian ring of characteristic $p > 0$. If the localization R_c of R at an element $c \in R^\circ$ is strongly F-regular, then there exists an integer $n \geq 0$, depending only on R and c, satisfying the following property:*

For any $d \in R^\circ$, there exists a power q' of p and an R-linear map $\phi \colon R^{1/q'} \to R$ sending $d^{1/q'}$ to c^n.

The following theorem has been proved in the case $\mathfrak{a}_\bullet = \{\mathfrak{a}^{\lceil tn \rceil}\}$ in [6]. In fact, it is an analogy of the main theorems in [8].

Theorem 2.2 (cf. [6, Theorem 1.7]). *Let R be a Noetherian reduced ring of characteristic $p > 0$ and let $c \in R^\circ$. Assume that one of the following conditions holds:*

(1) *R is F-finite and the localized ring R_c is strongly F-regular.*

(2) *R is an algebra of finite type over an excellent local ring B, and the localized ring R_c is Gorenstein and F-regular.*

Then some power c^n of c is an $\mathfrak{a}_\bullet$-test element for all graded family of ideals $\mathfrak{a}_\bullet$ on R and for all integers $k \geq 1$.

Proof. **The case (1):** Take c^n which satisfies the condition of the above lemma. Let I be an ideal of R, $z \in I^{*\mathfrak{a}_\bullet}$ and q a power of p. Since $z \in I^{*\mathfrak{a}_\bullet}$, there exists $d \in R^\circ$ such that $dz^Q\mathfrak{a}_Q \subseteq I^{[Q]}$ for every power Q of p. By the above lemma, there exists a power q' of p and $\phi \in \mathrm{Hom}_R(R^{1/q'}, R)$ such that $\phi(d^{1/q'}) = c^n$. Since $dz^{qq'}(\mathfrak{a}_q)^{[q']} \subseteq dz^{qq'}\mathfrak{a}_{qq'} \subseteq I^{[qq']}$, one has $d^{1/q'}z^q\mathfrak{a}_qR^{1/q'} \subseteq I^{[q]}R^{1/q'}$. Applying ϕ to both sides gives $c^nz^q\mathfrak{a}_q \subseteq I^{[q]}$. Hence c^n is an $\mathfrak{a}_\bullet$-test element.

The case (2): Assume that R is of finite type over an excellent local ring B and R_c is Gorenstein F-regular. Put $R' = R \otimes_B \widehat{B}$. Since B is excellent, the canonical ring homomorphism $R \to R'$ is faithfully flat with regular fibers. In particular, R' is reduced and R'_c is Gorenstein, F-regular by [9, Theorem 7.3 (c)]. Now suppose that $d = c^n \in R^\circ \subseteq (R')^\circ$ is an $\mathfrak{a}_\bullet R'$-test element. Let I be an ideal of R, $z \in I^{*\mathfrak{a}_\bullet}$ and a power q of p. By assumption, $c^nz^q\mathfrak{a}_qR' \subseteq I^{[q]}R'$. Then $c^nz^q\mathfrak{a}_q \subseteq I^{[q]}R' \cap R = I^{[q]}$ because R' is faithfully flat over R. Hence c^n is an $\mathfrak{a}_\bullet$-test element. Thus we may assume that B is complete.

Let k be a coefficient field of R, and let Λ be a p-base for k. For any cofinite subset Γ of Λ, if we put $k_e^\Gamma = k[\lambda^{1/p^e} : \lambda \in \Gamma]$, then $R^\Gamma = \cup_{e\geq 0} k_e^\Gamma[[R]]$ is F-finite and is faithfully flat over R with Gorenstein fibers. Moreover, by [9, Lemmas 6.13, 6.19], we can take Γ for which R^Γ is reduced and $(R^\Gamma)_c$ is Gorenstein, F-regular, and thus strongly F-regular. See [9, Section 6] for more details.

As R^Γ is F-finite, there exists a power $c^n \in R^\circ \subseteq (R^\Gamma)^\circ$ which is an $\mathfrak{a}_\bullet R^\Gamma$-test element by (1). Let I be an ideal of R, $z \in I^{*\mathfrak{a}_\bullet}$ and a power q of p. Then $c^nz^q\mathfrak{a}_qR^\Gamma \subseteq I^{[q]}R^\Gamma \cap R = I^{[q]}$ by the choice of c^n. Therefore c^n is an $\mathfrak{a}_\bullet$-test element. □

3. Uniform test exponent for subsystem of parameters

In this section, we prove the main theorem in this talk, which states an existence of an uniform test exponent for subsystem of parameters. Recently, R. Y. Sharp proved a similar result for original tight closures. Moreover, as an application, we show that ideal-adic tight closures of ideals generated by subsystem of parameters commute with localization in the case of excellent Gorenstein reduced local rings.

Theorem 3.1 (Uniform test exponent for s.s.o.p). *Let $(R, \mathfrak{m})$ be an excellent, equidimensional, reduced local ring of characteristic $p > 0$. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_n\}$ be a graded family of ideals. Then for any element $c \in R^\circ$, there exists $e_0 \in \mathbb{N}$ for which the following property holds: For any ideal I which is generated by subsystem of parameters,*

$$(\#) \quad cz^q\mathfrak{a}_q \subseteq I^{[q]} \text{ for some } q = p^e,\ e \geq e_0 \Longrightarrow z \in I^{*\mathfrak{a}}.$$

If c is an $\mathfrak{a}_\bullet$-test element, then the converse is also true.

Before starting our proof, we need the following lemma.

Lemma 3.2 (cf. [10, Proposition 2.6]). *Assume that R has a test element. Let $c \in R^\circ$, and let $\mathfrak{a}_\bullet$ be a graded family of ideals on R. Let $N \subseteq M$ be R-modules such that*

(1) *M/N is Artinian.*
(2) *$(0)^*_{F^e(M)/N_M^{[q]}} = (0)^{*fg}_{F^e(M)/N_M^{[q]}}$ holds for every integer $e \geq 1$.*

Then there exists an integer e_0 such that for any $\xi \in M$, $\xi \in N_M^{*\mathfrak{a}_\bullet}$ holds whenever $c\xi^q \mathfrak{a}_q \subseteq N_M^{[q]}$ for some $q = p^e$, $e \geq e_0$.

Proof. This proof is essentially due to Hochster and Huneke; see [10]. Put

$$N_e = \{\xi \in M \,:\, c\xi^{p^e}\mathfrak{a}_{p^e} \subseteq (N_M^{[p^e]})^F\},$$

where $N_M^F = \{\xi \in M \,:\, \xi^{q'} \in N_M^{[q']}$ for some power q' of $p\}$ denotes the Frobenius closure of N in M.

First we show that $N_{e+1} \subseteq N_e$. Let $\xi \in N_{e+1}$. If we put $q = p^e$, then $c\xi^{pq}\mathfrak{a}_{pq} \subseteq (N_M^{[pq]})^F$. By definition, there exists $q' = p^{e'}$ such that $c^{q'}\xi^{pqq'}\mathfrak{a}_{pq}^{[q']} \subseteq (N_M^{[pq]})^{[q']} = N_M^{[pqq']}$. Hence $c^{pq'}\xi^{pqq'}\mathfrak{a}_q^{[pq']} \subseteq c^{q'}\xi^{pqq'}\mathfrak{a}_{pq}^{[q']} \subseteq (N_M^{[q]})^{[pq']}$. This implies that $c\xi^q\mathfrak{a}_q \subseteq (N_M^{[q]})^F$, that is, $\xi \in N_e$.

Since M/N is Artinian and each N_e contains N, a descending sequence $N_1 \supseteq N_2 \supseteq \cdots$ stabilizes. Hence there exists $e_0 \in \mathbb{N}$ such that $N_e = N_{e_0}$ for all $e \geq e_0$.

To see the lemma, it is enough to show the following claim.

Claim: Fix e_0 which satisfies the condition above. Then $N_{e_0} \subseteq N_M^{*\mathfrak{a}_\bullet}$.

For any $\xi \in N_{e_0}$ and for $q = p^e$, $e \geq e_0$, we have $c\xi^q\mathfrak{a}_q \subseteq (N_M^{[q]})^F \subseteq (N_M^{[q]})^*$. Take a test element $d \in R^\circ$ and fix it. Then $dc\xi^q\mathfrak{a}_q \subseteq N_M^{[q]}$ in $\mathbb{F}^e(M)$ because the tight closure of (0) coincides the finitistic tight closure of (0) in $\mathbb{F}^e(M)/N_M^{[q]}$ by the assumption (2). This yields that $\xi \in N_M^{*\mathfrak{a}_\bullet}$. $\square$

Proof of Theorem 3.1. Since R is excellent and reduced, R has an $\mathfrak{b}_\bullet$-test element d for any graded family $\mathfrak{b}_\bullet$ of ideals in R. Note that $H^d_\mathfrak{m}(R)$ is Artinian and $\mathbb{F}^e(H^d_\mathfrak{m}(R)) \cong H^d_\mathfrak{m}(R)$. As R is excellent equidimensional, we have $(0)^{*\mathrm{fg}}_{H^d_\mathfrak{m}(R)} = (0)^*_{H^d_\mathfrak{m}(R)}$. Applying the above lemma to $N = (0) \subseteq M = H^d_\mathfrak{m}(R)$, we can take an integer e_0 such that

$$(\#\#) \quad \xi \in H^d_\mathfrak{m}(R),\ c\xi^q\mathfrak{a}_q = 0 \quad \text{for some } q = p^e,\ e \geq e_0 \implies \xi \in (0)^{*\mathfrak{a}_\bullet}_{H^d_\mathfrak{m}(R)}.$$

Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters in R, and put $I = (x_1, \ldots, x_d)R$. Then we prove the following claim:

Claim 1: If $cz^q\mathfrak{a}_q \subseteq I^{[q]}$ for some $q = p^e$, $e \geq e_0$, then $z \in I^{*\mathfrak{a}_\bullet}$.

Put $\xi = z + (x_1, \ldots, x_d) \in H^d_\mathfrak{m}(R) = \varinjlim R/(x_1^\ell, \ldots, x_d^\ell)$. Then $c\xi^q\mathfrak{a}_q = 0$ by assumption. Hence $\xi \in (0)^{*\mathfrak{a}_\bullet}_{H^d_\mathfrak{m}(R)}$ from $(\#\#)$. Since d is an $\mathfrak{a}_\bullet$-test element, for all powers Q of p, $d\xi^Q\mathfrak{a}_Q = 0$. That is, there exists an integer $\ell = \ell(Q) \geq 0$ depending on Q such that $dz^Q\mathfrak{a}_Q(x_1\cdots x_d)^\ell \subseteq (x_1^{\ell+Q},\ldots,x_d^{\ell+Q})$. By the colon-capturing property for tight closure, we get

$$dz^Q\mathfrak{a}_Q \subseteq (x_1^{\ell+Q}\ldots x_d^{\ell+Q})\colon (x_1\cdots x_d)^\ell \subseteq (I^{[Q]})^*.$$

Hence $d^2z^Q\mathfrak{a}_Q \subseteq I^{[Q]}$ for all powers Q. This means that $z \in I^{*\mathfrak{a}_\bullet}$, as required.

More generally, we prove the following claim:

Claim 2: A similar result as Claim 1 holds for any subsystem of parameters.

Let I be an ideal generated by a subsystem of parameters $x_1,\ldots,x_h$, and suppose that $cz^q\mathfrak{a}_q \subseteq I^{[q]}$ for some $q = p^e$, $e \geq e_0$. We may assume that $x_1,\ldots,x_d$ is a system of parameters. Then for all integers $\ell \geq 1$, we have $cz^q\mathfrak{a}_q \subseteq (x_1,\ldots,x_h,x_{h+1}^\ell,\ldots x_d^\ell)^{[q]}$. Applying Claim 1, we obtain that $z \in (x_1,\ldots,x_h,x_{h+1}^\ell,\ldots,x_d^\ell)^{*\mathfrak{a}_\bullet}$. Since $d \in R^\circ$ is an $\mathfrak{a}_\bullet$-test element, $dz^Q\mathfrak{a}_Q \subseteq (x_1^Q,\ldots,x_h^Q,x_{h+1}^{\ell Q},\ldots,x_d^{\ell Q})$ for all powers Q of p. Hence

$$dz^Q\mathfrak{a}_Q \subseteq \bigcap_{\ell=1}^{\infty}(x_1^Q,\ldots,x_h^Q,x_{h+1}^{\ell Q},\ldots,x_d^{\ell Q}) = I^{[Q]}.$$

Therefore $z \in I^{*\mathfrak{a}_\bullet}$, as required. □

As an application, we prove that any $\mathfrak{a}_\bullet$-adic tight closure commutes with localization for ideals generated by subsystem of parameters.

Corollary 3.3 (Localization of ideal-adic tight closures). *Let $(R,\mathfrak{m})$ and $\mathfrak{a}_\bullet = \{\mathfrak{a}_n\}$ be as in* Theorem 3.1. *Let I be an ideal which is generated by subsystem of parameters in R. Then for multiplicatively closed subset $W \subseteq R$, we have*

$$I^{*\mathfrak{a}_\bullet}R_W = (IR_W)^{*\mathfrak{a}_{W,\bullet}}.$$

Proof. It is enough to show that $(IR_W)^{*\mathfrak{a}_{W,\bullet}} \subseteq I^{*\mathfrak{a}_\bullet}R_W$. Suppose that $\alpha \in (IR_W)^{*\mathfrak{a}_{W,\bullet}}$. By definition, there exist $c \in (R_W)^\circ \cap R$ and an integer e_1 such that $c\alpha^q\mathfrak{a}_qR_W \subseteq I^{[q]}R_W$ for all $q = p^e$, $e \geq e_1$. We may assume that $c \in R^\circ$ by prime avoidance. Take a positive integer e_2 such that $z \in I^{*\mathfrak{a}_\bullet}$ holds whenever $cz^q\mathfrak{a}_q \subseteq I^{[q]}$ for some $q = p^e$, $e \geq e_2$. Fix $e \geq e_0 := \max\{e_1,e_2\}$. Take $u \in W$ such that $uc\alpha^q\mathfrak{a}_q \subseteq I^{[q]}$. Then $c(u\alpha)^q\mathfrak{a}_q \subseteq I^{[q]}$ and thus $u\alpha \in I^{*\mathfrak{a}_\bullet}$. That is, $\alpha \in I^{*\mathfrak{a}_\bullet}R_W$. We have finished the proof of the proposition. □

In the proof of Theorem 3.1, we use the following theorem. So we need to assume that "R is equidimensional".

Theorem 3.4 (Colon–Capturing for TC, [8]). *Let $(R,\mathfrak{m})$ be an excellent, equidimensional local ring of characteristic $p > 0$. Then for any subsystem of parameters $x_1,\ldots,x_h$,*

$$(x_1,\ldots,x_{i-1}) : x_i \subseteq (x_1,\ldots,x_{i-1})^*$$

holds for each $i = 1,\ldots,h$.

One cannot relax the assumption that "R is equdimensional".

Example 3.5. Let $R = k[[x,y,z]]/(x) \cap (y,z)$. Put $a_1 = x^n + y$, $a_2 = x^n + z$. Then $(a_1) : a_2 = (x,y)$ and $(a_1)^* = (x^n, y)$. Thus the Colon–Capturing does *not* hold in general.

4. Localization problem for test ideals

In this section, we show that any $\mathfrak{a}_\bullet$-test ideal commutes with localization for complete Gorenstein reduced local rings. One implication follows from Theorem 3.1. The other implication follows from Smith' argument; see [15, 16].

We emphasize that this problem was settled for $\widetilde{\tau}(\mathfrak{a}^t)$ in F-finite local rings; see [5] for more details. Our attempt is to remove the assumption that "R is F-finite".

Proposition 4.1 (cf. [5, Proposition 3.1]). *Let $(R, \mathfrak{m})$ be a complete Gorenstein reduced local ring of characteristic $p > 0$, and let $\mathfrak{a}_\bullet$ be a graded family of ideals on R. Let W denote a multiplicatively closed subset of R. Then*

$$\tau(\mathfrak{a}_{W,\bullet}) = \tau(\mathfrak{a}_\bullet) R_W.$$

We first show that $\tau(\mathfrak{a}_{W,\bullet}) \subseteq \tau(\mathfrak{a}_\bullet) R_W$, which is easy to prove.

Lemma 4.2. *Assume that $(R, \mathfrak{m})$ is an excellent Gorenstein reduced local ring of characteristic $p > 0$. Let $\mathfrak{a}_\bullet$ be a graded family of ideals on R. Let W denote a multiplicatively closed subset of R. Then*

$$\tau(\mathfrak{a}_\bullet) R_W \subseteq \tau(\mathfrak{a}_{W,\bullet}).$$

Proof. Let $c \in \tau(\mathfrak{a}_\bullet)$ and $z \in (IR_W)^{*\mathfrak{a}_{W,\bullet}} \cap R$, where I is any given ideal of R such that $IR_W \neq R_W$. Let $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathfrak{p} \cap W = \emptyset$. If we prove $\tau(\mathfrak{a}_\bullet) R_\mathfrak{p} \subseteq \tau(\mathfrak{a}_{\mathfrak{p},\bullet})$ for such any prime $\mathfrak{p}$, then $\frac{cz}{1} = \frac{c}{1} \cdot \frac{z}{1} \in IR_\mathfrak{p}$ since $z \in (IR_\mathfrak{p})^{*\mathfrak{a}_{\mathfrak{p},\bullet}} \cap R$. It follows that $\frac{cz}{1} \in IR_W$. Namely, we obtain that $\frac{c}{1} \in \tau(\mathfrak{a}_{W,\bullet})$, as required.

Let $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathfrak{p} \cap W = \emptyset$. Then it suffices to show $\tau(\mathfrak{a}_\bullet) R_\mathfrak{p} \subseteq \tau(\mathfrak{a}_{\mathfrak{p},\bullet})$. Take an R-sequence $x_1, \dots, x_h$ in $\mathfrak{p}$ whose images in $R_\mathfrak{p}$ forms a system of parameters, and put $I^{[\ell]} = (x_1^\ell, \dots, x_h^\ell)$. Then $(I^{[\ell]})^{*\mathfrak{a}_\bullet} R_\mathfrak{p} = (I^{[\ell]} R_\mathfrak{p})^{*\mathfrak{a}_{\mathfrak{p},\bullet}}$ by the proposition. Since R is Gorenstein, we get

$$\begin{aligned}
\tau(\mathfrak{a}_{\mathfrak{p},\bullet}) &= \bigcap_{\ell=1}^{\infty} I^{[\ell]} R_\mathfrak{p} \colon (I^{[\ell]} R_\mathfrak{p})^{*\mathfrak{a}_{\mathfrak{p},\bullet}} \\
&= \bigcap_{\ell=1}^{\infty} I^{[\ell]} R_\mathfrak{p} \colon (I^{[\ell]})^{*\mathfrak{a}_\bullet} R_\mathfrak{p} \\
&= \bigcap_{\ell=1}^{\infty} \left(I^{[\ell]} \colon (I^{[\ell]})^{*\mathfrak{a}_\bullet}\right) R_\mathfrak{p} \\
&\supseteq \left(\bigcap_{\ell=1}^{\infty} I^{[\ell]} \colon (I^{[\ell]})^{*\mathfrak{a}_\bullet}\right) R_\mathfrak{p} \supseteq \tau(\mathfrak{a}_\bullet) R_\mathfrak{p},
\end{aligned}$$

as required. □

Proof of Proposition 4.1. It suffices to show that $\tau(\mathfrak{a}_{W,\bullet}) \subseteq \tau(\mathfrak{a}_\bullet)R_W$. We first consider the case $W = R \setminus \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} R$. Put $h = \operatorname{height}(\mathfrak{p})$, $N = (0)^{*\mathfrak{a}_\bullet}_{H^d_\mathfrak{m}(R)}$ and $U = \operatorname{Ann}_R N$. Then

$$\tau(\mathfrak{a}_{\mathfrak{p},\bullet}) = \operatorname{Ann}_{R_\mathfrak{p}} (0)^{*\mathfrak{a}_{\mathfrak{p},\bullet}}_{H^h_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})}$$

by definition. On the other hand, since R is complete and N is Artinian,

$$\tau(\mathfrak{a}_\bullet)R_\mathfrak{p} = (\operatorname{Ann}_R N)R_\mathfrak{p} = \operatorname{Ann}_{R_\mathfrak{p}} N^{\vee_\mathfrak{m}\vee_\mathfrak{p}},$$

where $N^{\vee_\mathfrak{m}\vee_\mathfrak{p}} = \operatorname{Hom}_R(\operatorname{Hom}_R(N, E), E_R(R/\mathfrak{p})) \subseteq E_R(R/\mathfrak{p})$ by [15, Lemma 3.1(ii)]. So it is enough to prove the following claim:

Claim: $N^{\vee_\mathfrak{m}\vee_\mathfrak{p}} \subseteq (0)^{*\mathfrak{a}_{\mathfrak{p},\bullet}}_{H^h_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})}$ in $E_R(R/\mathfrak{p}) \cong H^h_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})$.

To see this claim, we will show that $\mathfrak{a}_q F^e(N^{\vee_\mathfrak{m}\vee_\mathfrak{p}}) \subseteq N^{\vee_\mathfrak{m}\vee_\mathfrak{p}}$ for all $q = p^e$. Take a system of parameters $x_1, \ldots, x_d$ for which $\frac{x_1}{1}, \ldots \frac{x_h}{1}$ forms a system of parameters in $R_\mathfrak{p}$. Assume that

$$\eta = \left[\frac{z}{1} + \left(\frac{x_1^t}{1}, \cdots, \frac{x_h^t}{1}\right)\right] \in N^{\vee_\mathfrak{m}\vee_\mathfrak{p}}.$$

To see that $\mathfrak{a}_q\eta^q \subseteq N^{\vee_\mathfrak{m}\vee_\mathfrak{p}}$ for all $q = p^e$, $e \geq 1$, we may assume that $t = 1$ without loss of generality. Since $N^{\vee_\mathfrak{m}\vee_\mathfrak{p}} = \operatorname{Ann}_{H^h_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})} U_\mathfrak{p}$ by [16, Lemma 2.1 (iv)], we have

$$\frac{z}{1}U_\mathfrak{p} \subseteq \left(\frac{x_1}{1}, \cdots, \frac{x_h}{1}\right) R_\mathfrak{p}.$$

Take an element $a \in R \setminus \mathfrak{p}$ such that $azU \subseteq (x_1, \ldots, x_h)$ and put

$$\eta_\ell = \left[az + (x_1, \ldots, x_h, x_{h+1}^\ell, \ldots, x_d^\ell)\right] \in H^d_\mathfrak{m}(R),$$

for every integer $\ell \geq 1$. Then $\eta_\ell \in \operatorname{Ann}_{H^d_\mathfrak{m}(R)} U = (0)^{*\mathfrak{a}_\bullet}_{H^d_\mathfrak{m}(R)}$. It follows that $\mathfrak{a}_q\eta_\ell^q \subseteq \operatorname{Ann}_{H^d_\mathfrak{m}(R)} U$ by a similar argument as in the proof of [6, Proposition 1.15]. That is,

$$(az)^q\mathfrak{a}_q U \subseteq (x_1^q, \ldots, x_h^q, x_{h+1}^{\ell q}, \ldots, x_d^{\ell q})$$

for every $\ell \geq 1$. In particular,

$$\left(\frac{z}{1}\right)^q \mathfrak{a}_q U_\mathfrak{p} \subseteq \left(\frac{x_1^q}{1}, \ldots, \frac{x_h^q}{1}\right) R_\mathfrak{p}$$

and thus $\mathfrak{a}_q F^e(\eta) \subseteq N^{\vee_\mathfrak{m}\vee_\mathfrak{p}}$.

Since there exists $\frac{c}{1} \in \tau(\mathfrak{a}_\bullet)R_\mathfrak{p} \cap (R_\mathfrak{p})^\circ \subseteq \tau(\mathfrak{a}_{\mathfrak{p},\bullet}) \cap (R_\mathfrak{p})^\circ$ by the previous lemma, we obtain that $N^{\vee_\mathfrak{m}\vee_\mathfrak{p}} \subseteq (0)^{*\mathfrak{a}_{\mathfrak{p},\bullet}}_{H^h_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p})}$.

Next, we consider the general case. Suppose that $c \in \tau(\mathfrak{a}_{W,\bullet}) \cap R$. To see $c \in \tau(\mathfrak{a}_\bullet)R_W \cap R$, it suffices to show that $c \in \tau(\mathfrak{a}_\bullet)R_\mathfrak{p} \cap R$ for all $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathfrak{p} \cap W = \emptyset$. Take a system of parameters $x_1, \ldots, x_d$ such that $\frac{x_1}{1}, \ldots, \frac{x_h}{1}$ forms a system of parameters in $R_\mathfrak{p}$. Then

$$\frac{c}{1}(I^{[\ell]}R_\mathfrak{p})^{*\mathfrak{a}_{\mathfrak{p},\bullet}} = \frac{c}{1}(I^{[\ell]})^{*\mathfrak{a}_\bullet}R_\mathfrak{p} \subseteq I^{[\ell]}R_\mathfrak{p}$$

for every $\ell \geq 1$. Hence $\frac{c}{1} \in \tau(\mathfrak{a}_{\mathfrak{p},\bullet}) = \tau(\mathfrak{a}_\bullet)R_\mathfrak{p}$. $\square$

5. Uniform test exponents for ideals in an RLR

This section is devoted to prove the following theorem, which implies an existence of an uniform test exponent for ideals in an regular local ring.

Theorem 5.1 (Uniform test exponent for ideals in an RLR). *Let $(R, \mathfrak{m})$ be an excellent regular local ring of characteristic $p > 0$. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_n\}$ be a graded family of ideals. Then for any element $c \in R^\circ$, there exists $e_0 \in \mathbb{N}$ for which the following property holds: For any ideal I,*

$$cz^q\mathfrak{a}_q \subseteq I^{[q]} \text{ for some } q = p^e,\ e \geq e_0 \Longrightarrow z \in I^{*\mathfrak{a}}.$$

If c is an $\mathfrak{a}_\bullet$-test element, then the converse is also true.

Proof. Applying Lemma 3.2 to the case $N = (0) \subseteq M = H^d_{\mathfrak{m}}(R)$, we can find an integer $e_0 = e_0(c, \mathfrak{a}_\bullet) \geq 1$ such that

$$(\natural)\quad \xi \in E,\ c\xi^q\mathfrak{a}_q = 0 \text{ in } \mathbb{F}^e(E) \text{ for some } q = p^e,\ e \geq e_0 \quad \Longrightarrow \xi \in (0)^{*\mathfrak{a}_\bullet}_E.$$

For the above integer e_0 and for any ideal I of R, it suffices to prove the following claim:

Claim: $z \in E$, $cz^q\mathfrak{a}_q \subseteq I^{[q]}$ for some $q = p^e$, $e \geq e_0 \Longrightarrow z \in I^{*\mathfrak{a}_\bullet}$.

First suppose that I is an $\mathfrak{m}$-primary ideal of R. Then $E_R(R/I)$ is isomorphic to a direct sum of finite copies of $E = E_R(R/\mathfrak{m})$. Let $\varphi\colon R/I \hookrightarrow E_R(R/I) = E^\mu$. Put $\varphi(z + I) = (\xi_1, \dots, \xi_\mu)$. The assumption $cz^q\mathfrak{a}_q \subseteq I^{[q]}$ implies that $c\xi_j^q\mathfrak{a}_q = 0$ for each $j = 1, \dots, \mu$. By $(\natural)$, $\xi_j \in (0)^{*\mathfrak{a}_\bullet}_E$. That is, $\varphi(z + I) \in (0)^{*\mathfrak{a}_\bullet}_{E_R(R/I)}$. On the other hand, since Frobenius map is flat, the induced map $\mathbb{F}^e(R/I) \to \mathbb{F}^e(E_R(R/I))$ is injective. Therefore

$$z + I \in (0)^{*\mathfrak{a}_\bullet}_{E_R(R/I)} \cap R/I = (0)^{*\mathfrak{a}_\bullet}_{R/I} = I^{*\mathfrak{a}_\bullet}/I.$$

Thus $z \in I^{*\mathfrak{a}_\bullet}$.

Next we consider the general case. Suppose that $cz^q\mathfrak{a}_q \subseteq I^{[q]}$ for some $q = p^e$, $e \geq e_0$. Then $cz^q\mathfrak{a}_q \subseteq (I + \mathfrak{m}^n)^{[q]}$ for every integer $n \geq 1$. By the above argument, we have $z \in (I + \mathfrak{m}^n)^{*\mathfrak{a}_\bullet}$. Since 1 is an $\mathfrak{a}_\bullet$-test element, we get

$$z^{q'}\mathfrak{a}_{q'} \subseteq \bigcap_{n=1}^{\infty}(I + \mathfrak{m}^n)^{[q']} = \bigcap_{n=1}^{\infty}\left(I^{[q']} + (\mathfrak{m}^{[q']})^n\right) = I^{[q']}$$

for all powers q' of p. Thus $z \in I^{*\mathfrak{a}_\bullet}$. □

This theorem implies that any $\mathfrak{a}_\bullet$-tight closure commutes with localization for any ideal in an excellent regular local ring. In fact, this is valid without excellence. See [17].

6. Completion

In order to replace "complete" with "excellent" in Proposition 1.6, we must prove that generalized test ideal commutes with regular base change. But we do *not* have any satisfactory answer to this problem. In fact, we can show that the following conjecture is true for ideal-adic filtrations $\mathfrak{a}_\bullet = \{\mathfrak{a}^{tn}\}$ only.

Conjecture 6.1. *Let R be an excellent reduced local ring, and I an $\mathfrak{m}$-primary ideal. Then for any graded family $\mathfrak{a}_\bullet$ of ideals, we have $I^{*\mathfrak{a}_\bullet}\widehat{R} = (I\widehat{R})^{*\mathfrak{a}_\bullet}$.*

Remark 6.2. There exists a non-excellent local ring for which the above proposition does *not* hold; see Loepp and Rotthaus [14].

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GENERALIZED COMPLETE INTERSECTIONS

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Introduction

Let $S = K[X_1, \dots, X_n]$ be a polynomial ring over the field K and consider a monomial ideal $I \subset S$. Then we consider the graded local cohomology modules $H^i_{\mathfrak{m}}(S/I)$ with regard to the graded maximal ideal $\mathfrak{m} = (X_1, \dots, X_n)$. We say that S/I has *FLC (finite local cohomology)*, if $H^i_{\mathfrak{m}}(S/I)$ has finite length for all $i < \dim S/I$. We have many examples of the residue class rings S/I with FLC property. Namely, the projective coordinate rings of any Cohen-Macaulay projective schemes have FLC property. For Stanley-Reisner rings, i.e., the case that I is generated by square free monomial ideals, FLC property for S/I is equivalent to Buchsbaumness of S/I, and Buchsbaum Stanley-Reisner rings are well understood through topological characterization of the corresponding sinmplicial compliexes. However, for monomial ideals that are not necessarily generated by square free monomials, FCL property is not well understood. In [7], the second author gave combinatorial characterizations of FLC monomial ideals for $d \leq 3$. But the problem to find fairly large classes of FLC monimial ideals has been open.

The aim of this paper is to give a partial answer to this problem. We introduce the notion of generalized complete intersection (gCI), which is a Stanley-Reisner ideal $I \subset S$ whose powers I^n $(n = 1, 2, \dots)$ all have FLC property. We give a ring theoretic (Theorem 1.2) and combinatorial (Theorem 2.1) characterizations of gCI. Moreover, as a special class of gCI, we consider the gCI $I \subset S$ whose powers I^n $(n = 1, 2, \dots)$ all have linear resolutions. We give the complete classification of the simplicial complexes corresponding to such ideals (Theorem 3.1).

1. Generalized complete intersection

Let Δ be a simplicial complex over the vertex set $[n] = \{1, \dots, n\}$. We will always assume that $\{i\} \in \Delta$ for all $i \in [n]$. For a face $F \in \Delta$, we define the *link* by $\mathrm{lk}_\Delta(F) = \{G \mid G \cup F \in \Delta, G \cap F = \emptyset\}$. We will denote by $K[\Delta]$ the Stanley-Reisner ring corresponding to the simplicial complex Δ.

Definition 1.1. A Stanley-Reisner ring $K[\Delta]$ is called a *generalized complete intersection* (gCI) if Δ is pure and $K[\mathrm{lk}_\Delta(\{i\})]$ is complete intersection for all $i \in [n]$. We also call Δ to be gCI if $K[\Delta]$ is gCI.

If $K[\Delta]$ is a complete intersection, then it is also a gCI.

Now we give a ring theoretic characterization of *gCI*.

Theorem 1.2 (cf. Theorem 2.5 [4]). *Let $n \geq 1$ be an integer and let Δ be a simplicial complex over the vertex set $[n]$. Then the following conditions are equivalent.*

(1) *$K[\Delta]$ is a gCI.*
(2) *$S/{I_\Delta}^{\ell+1}$ has FLC for arbitrary integer $\ell \geq 0$.*
(3) *The set $\{\ell \geq 0 \mid$ the ring $S/{I_\Delta}^{\ell+1}$ has FLC$\}$ is infinite.*

If one of these conditions holds, $K[\Delta]$ is Buchsbaum.

Proof. If I is a generically complete intersection of a Cohen-Macaulay local ring S, the follwing conditions are equivalent ([2, 1, 5, 10]):

1. I is generated by an S-regular sequence,
2. $S/I^{\ell+1}$ is Cohen-Macaulay for arbitrary integer $\ell \geq 0$,
3. $\Lambda = \{\ell \geq 0 \mid S/I^{\ell+1}$ is Cohen-Macaulay$\}$ is an infinite set.

Since Stanley-Reisner ideals are generically complete intersection, by considering a localized version of this, we obtain the required result. □

2. Combinatorial characterization of gCI

For $F \in \Delta$, we define $\mathrm{star}_\Delta(F) = \{G \mid G \cup F \in \Delta\}$. We also define $\mathrm{core}[n] = \{i \in [n] \mid \mathrm{star}_\Delta(i) \neq \Delta\}$. Then the core of Δ is defined by $\mathrm{core}\,\Delta = \{F \cap \mathrm{core}[n] \mid F \in \Delta\}$.

We characterize a generalized complete intersection ideal $I_\Delta \subset S$ in terms of combinatorics of simplicial complex $\mathcal{F}_\Delta$ generated by the supports of the minimal set of monomial generators.

Theorem 2.1 (cf. Theorem 3.16 [4]). *Let $K[\Delta]$ be a Stanley-Reisner ring with $\Delta = \mathrm{core}\,\Delta$. Let $G(I_\Delta) = \{u_1, \dots, u_\ell\}$ and $\mathcal{F}_\Delta = \{\mathrm{supp}(u_j) \mid j = 1, \dots, \ell\}$. Assume that $K[\Delta]$ is not a complete intersection. Then $K[\Delta]$ is a gCI if and only if the following conditions hold:*

1. *for every $S \in \mathcal{F}_\Delta$ with $\sharp S \geq 3$, there exists a non-empty set $\mathcal{C}(S)(\subset [n])$ such that*
   (a) *$\mathcal{C}(S) \cap S = \emptyset$,*
   (b) *for every $i \in \mathcal{C}(S)$, we have $E_{ij} := \{i, j\} \in \mathcal{F}_\Delta$ for all $j \in S$. Moreover if $S \cap T \neq \emptyset$ for $T \in \mathcal{F}_\Delta$, then $T = E_{ij}$ for some i, j.*
   (c) *for every $k \notin \mathcal{C}(S) \cup S$, we have $\{i, k\} \in \mathcal{F}_\Delta$ for all $i \in \mathcal{C}(S)$.*

2. *Any two elements $i, j \in [n]$ are linked with a path $P = \{\{i_k, i_{k+1}\} \mid k = 1, \dots, r\}$, with edges $\{i_k, i_{k+1}\} \in \mathcal{F}_\Delta$ for $k = 1, \dots, r$ such that $i = i_1$ and $j = i_{r+1}$.*
3. *If there exists a length 4 path $P = \{\{i_p, i_{p+1}\} \in \mathcal{F}_\Delta \mid p = 1, 2, 3, 4\}$, then there must be an edge $\{i_1, i_q\} \in \mathcal{F}_\Delta$ with $q = 3, 4$ or 5.*
4. *Δ is pure.*

Moreover, if $\Delta \neq \operatorname{core}\Delta$, $K[\Delta]$ is a gCI if and only if it is a complete intersection.

Proof. The proof is carried out by a purely combinatorial discussion on finite sets. □

We show a few examples of gCI Stanley-Reisner ideals.

Example 2.2. Examples of non Cohen-Macaulay edge ideals:
1. $I_\Delta = (X_1X_3, X_1X_4, X_2X_3, X_2X_4) = (X_1, X_2) \cap (X_3, X_4) \subset k[X_1, \dots, X_4]$ and $\Delta = \langle\{1,2\}, \{3,4\}\rangle$ (two disjoint edges).
2. $I_\Delta = (X_1X_2, X_2X_3, X_1X_3, X_3X_4, X_4X_5, X_1X_5) = (X_2, X_3, X_5) \cap (X_1, X_3, X_5) \cap (X_1, X_3, X_4) \cap (X_1, X_2, X_4) \subset k[X_1, \dots, X_5]$ and $\Delta = \langle\{1,4\}, \{4,2\}, \{2,5\}, \{5,3\}\rangle$ (a path of length 4).
3. $I_\Delta = (X_1X_2, X_2X_3, X_1X_3, X_3X_4, X_4X_5, X_1X_5, X_2X_5) = (X_2, X_3, X_5) \cap (X_1, X_3, X_5) \cap (X_1, X_2, X_4) \subset k[X_1, \dots, X_5]$ and $\Delta = \langle\{1,4\}, \{4,2\}, \{5,3\}\rangle$ (disjoint union a path of length 2 and an edge).
4. $I_\Delta = (X_1X_2, X_1X_5, X_2X_3, X_2X_5, X_3X_4) = (X_2, X_4, X_5) \cap (X_2, X_3, X_5) \cap (X_1, X_2, X_4) \cap (X_1, X_2, X_3) \cap (X_1, X_3, X_5) \subset k[X_1, \dots, X_5]$ and $\Delta = \langle\{1,3\}, \{3,5\}, \{5,4\}, \{4,1\}\{4,2\}\rangle$ (an edge attached to a circle).
5. $I_\Delta = (X_1, \dots, X_n) \cap (X_{n+1}, \dots, X_{2n}) \subset k[X_1, \dots, X_{2n}]$. Notice that $G(I_\Delta)$ is a bipartite graph.

Example 2.3. Examples of Cohen-Macaulay edge ideals:
1. $I_\Delta = (X_1X_2, X_2X_3, X_3X_4) = (X_2, X_4) \cap (X_2, X_3) \cap (X_1, X_3) \subset k[X_1, \dots, X_4]$ and $\Delta = \langle\{1,3\}, \{1,4\}, \{4,2\}\rangle$ (a path of length 3)
2. $I_\Delta = (X_1X_2, X_2X_3, X_3X_4, X_4X_5, X_5X_1) = (X_2, X_4, X_5) \cap (X_1, X_2, X_4) \cap (X_1, X_3, X_4) \cap (X_1, X_3, X_5) \cap (X_2, X_3, X_5) \subset k[X_1, \dots, X_5]$ and $\Delta = \langle\{1,3\}, \{3,5\}, \{5,2\}, \{2,4\}, \{4,1\}\rangle$ (a circle)

For Cohen-Macaulay edge ideals, see [9].

Example 2.4. Ideals whose generators contain degree ≥ 3 monomials:
1. $I_\Delta = (X_1X_2X_3) + (X_1, X_2, X_3) * (X_4, X_5, X_6) + (X_4X_7, X_5X_7, X_6X_7) \subset k[X_1, \dots, X_7]$ and

$$\Delta = \langle\{1,2,7\}, \{1,3,7\}, \{2,3,7\}, \{4,5,6\}\rangle,$$

which is a not Cohen-Macaulay complex since it is not connected.

2. $I_\Delta = (X_1X_2X_3X_4) + (X_1, X_2, X_3, X_4) * (X_5, X_6, X_7)$ and

$$\Delta = \langle \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{5,6,7\} \rangle,$$

which is a not Cohen-Macaulay complex since it is not connected.

3. gCI WITH LINEAR RESOLUTIONS

By Theorem 2.1, we know that if $I_\Delta \subset S$ is a generalized complete intersection with a linear resolution, then it must be generated in degree 2. In this case, $\mathcal{F}_\Delta$ is a chordal graph ([3]). Also by Theorem 3.2 in [6], I^ℓ have also linear resolutions (and FLC) for all $\ell \geq 1$. The following result characterizes the simplicial complexes Δ corresponding to generalized complete intersections I_Δ with linear resolutions.

Theorem 3.1 (cf. Theorem 2.3 [8]). *Let* $K[\Delta]$ *be a gCI with* $\dim K[\Delta] = d+1$ *and* $\mathrm{core}\Delta = \Delta$. *Then* I_Δ^ℓ *has a linear resolution for all* $\ell \geq 1$ *if and only if* Δ *is a finite set of points,i.e.* $d = 0$, *or otherwise* Δ *is as follows:*

case ($\dim \Delta = 1$): Δ *is the disjoint union of paths* $\Gamma_1, \dots, \Gamma_s$ *of arbitrary lengths.*

case ($\dim \Delta \geq 2$): Δ *is the disjoint union of the following types of subcomplexes:*

- **type 1:** $\langle F, G \rangle$, *where* F *and* G *are d-simplexes such that* $F \cap G \neq \emptyset$ *and* $|F \setminus G| = |G \setminus F| = 1$,
- **type 2:** $\langle H \rangle$, *where* H *is a d-simplex.*

Notice that, since $\mathrm{core}\Delta = \Delta$, *we exclude the case that* Δ *itself is of type 1 or type 2.*

Proof. The proof is carried out by a purely combinatorial discussion to determine the pure simplicial complexes Δ satisfying the following conditions:

1. The 1-skeleton Δ_1 is chordal,
2. The conditions 2. and 3. in Theorem 2.1.

□

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ON VANISHING AND NON-VANISHING DEGREES OF LOCAL COHOMOLOGIES OF GRADED ISOLATED NON F-RATIONAL SINGULARITIES

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Introduction

Let K be a field of characteristic $\operatorname{char} K = p > 0$. Let $(R, \mathfrak{m})$ be a finitely generated $\mathbb{N}$-graded K-algebra with $d = \dim R$, $R = \bigoplus_{n \geq 0} R_n$, $\mathfrak{m} = \bigoplus_{n > 0} R_n$ and $R_0 = K$. We assume that R is reduced, equidimensional and R_P is F-rational for all primes $P(\neq \mathfrak{m})$, namely R is an $\mathbb{N}$-graded isolated non F-rational singularity. Since R is Cohen-Macaulay on the punctured spectrum, it is a generalized Cohen-Macaulay ring, i.e., $\ell_R(H^i_{\mathfrak{m}}(R)) < \infty$ for all $i < d$, where $H^i_{\mathfrak{m}}(R)$ is the ith local cohomology module and $\ell_R(-)$ denotes the length as a R-module. In this paper, we are interested in the structure of local cohomologies $H^i_{\mathfrak{m}}(R)$, in particular vanishing/non-vanishing degrees of them.

A distinguished property of $\mathbb{N}$-graded isolated non F-rational singularity is Kodaira type vanishing of cohomologies, which, in terms of local cohomologies, can be stated that lower local cohomologies vanish for negative degrees [10, 11]: $[H^i_{\mathfrak{m}}(R)]_n = 0$ for all $i < d$ and all $n < 0$. If R is a generic mod p reduction of an isolated non-rational singularity (over the field of characteristic 0), Kodaira vanishing holds. This result has been established by Huneke, Smith, Hara and Watanabe [6, 11, 7]. This is in fact the case of *liftable to the second Witt vectors* in the sence of Deligne-Illusie [1]. Notice that an isolated non F-rational singularity is not necessary a mod p reduction of an isolated non-rational singularity and Kodaira vanishing for positive characteristic is in general false, eg. [16]. This means that vanishing/non-vanishing of local cohomologies for isolated non F-rational singularities in general is widely open.

In this papaer, we first show that lower local cohomologies $H^i_{\mathfrak{m}}(R)$, $i < d$, of an $\mathbb{N}$-graded isolated non F-rational singularity R can be described as quotient modules of the tight closures by the limit closures of suitable parameter ideals (see Theorem 17). This is a refinement of the results by Schenzel [12] and Smith [14]. With this representation, Huneke-Smiths interpretation of Kodaira vanishing in terms of tight closure [11] can be recovered immediately in a slightly general form.

A Noetherian local ring $(R, \mathfrak{m})$ is an isolated non F-rational singularity if and only if R is generalized Cohen-Macaulay and the tight closure of zero in the highest local cohomology $(0)^*_{H^d_{\mathfrak{m}}(R)}$, $d = \dim R$, has finite length (cf. [4]). This suggests that vanishing/non-vanishing of lower local cohomologies may be controlled by $(0)^*_{H^d_{\mathfrak{m}}(R)}$ to some extent. In fact, the above mensioned Kodaira vanishing theorem by Huneke-Smith-Hara-Watanabe has been proved by investigating the structure of $(0)^*_{H^d_{\mathfrak{m}}(R)}$. In this paper, we will consider isolated singularities and show that $[(0)^*_{H^d_{\mathfrak{m}}(R)}]_n \neq 0$ for some $n \in \mathbb{Z}$ implies non-vanishing of $H^{d-1}_{\mathfrak{m}}(R)$ for certain degree, under some subtle conditions (see Theorem 24).

In section 1 we will summerize the known results on isolated non F-rational singularities that are necessary in this paper. In section 2, after preparing some definitions and facts on standard sequence, limit closure and germ closure, we give a representation of lower local cohomologies in terms of tight closure and limit closure. We also give some direct consequences from this representation. $\mathbb{N}$-graded isolated singularities are considered in section 3. After giving a graded version of Goto-Nakamura's representation of tight closure of zero $(0)^*_{H^d_{\mathfrak{m}}(R)}$, we investigate non-vanishing of $(0)^*_{H^d_{\mathfrak{m}}(R)}$ and a generic hypersurface intersection $\overline{R}$ of R. Here we use the characteristic p version of Flenner's Bertini theorem. Then finally we consider the non-vanishing of $H^{d-1}_{\mathfrak{m}}(R)$ in terms of non-vanishing of $(0)^*_{H^d_{\mathfrak{m}}(R)}$.

1. Graded Isolated non F-rational Singularities

This section summerizes the basic definitions and results concerning isolated non F-rational singularities. See, for example, [10, 9, 14] for the detail.

Definition 1. *A sequence $x_1, \ldots, x_i$ in a ring R is called* parameters *if for every $P \in \operatorname{Spec} R$ containing the sequnece, the image of the sequence in R_P forms a part of sop. In other words,* $\operatorname{ht}((x_1, \ldots, x_i)) = i$. *An ideal generated by a set of parameters is called a* parameter ideal.

Definition 2. *For an ideal I of a ring R, the* tight closure *I^* of I is defined by $I^* = \{r \in R \mid$ there exists $c \in R^o$ such that $cz^q \in I^{[q]}$ for all $q = p^e \gg 0\}$, where R^o is the set of elements outside the union of mininal primes of R and $I^{[q]} = (r^q \mid r \in I)$.*

Definition 3. *A Noetherian ring R of prime characteristic is said to be F-*rational *if every parameter ideal of R is tightly closed.*

We recall the basic properties of F-rational rings needed in this paper.

Proposition 1 (cf. [9]). *Let R be a Noetherian ring that is the homomorphic image of a Cohen-Macaulay ring. We have the following:*

1. *If R is F-rational, then R is Cohen-Macaulay.*
2. *A localization of an F-rational ring is F-rational.*
3. *If R is regular, then R is F-rational.*

Definition 4. *Let K be a field. Then a finitely generated $\mathbb{N}$-graded K-algebra $(R, \mathfrak{m})$ is said to be a* generalized Cohen-Macaulay *if one of the following equivalent conditions holds:*

1. $\operatorname{Proj}(R)$ *is an equidimensional Cohen-Macaulay projective scheme,*
2. *R_P is an equidimensional Cohen-Macaulay local ring for every $P \in \operatorname{Spec} R - \{\mathfrak{m}\}$,*
3. *$H^i_{\mathfrak{m}}(R)$ is a finite length module for all $i < \dim R$. In particular, there exists $N_i \in \mathbb{Z}$ such that $[H^i_{\mathfrak{m}}(R)]_n = 0$ for all $n < N_i$.*

Definition 5. *Let $(R, \mathfrak{m})$ be a finitely generated $\mathbb{N}$-graded K-algebra or a Noetherian local ring with the maximal ideal $\mathfrak{m}$. Then we say that R is an* isolated non F-rational singularity *if R_P is F-rational for all primes $P(\neq \mathfrak{m})$.*

In this paper, when we consider an isolated non F-rational singularity, we always assume that it is equidimensional.

Definition 6. *An element $c \in R^o$ is called a* parameter test element *if, for arbitrary ideal I generated by an sop and arbitrary $x \in I^*$, we have $cx^q \in I^{[q]}$ for all $q = p^e$.*

In this definition, we can replace the ideal I by any parameter ideal, including $I = (0)$, because of the following result, which is a special case of Exercise 2.12 [9].

Proposition 2. *Let $(R, \mathfrak{m})$ be a local ring and assume that $c \in R^o$ is a test element for parameter ideals generated by sops. Then c is a test element for arbitrary parameter ideal $I = (x_1, \ldots, x_i)$, $i < d = \dim R$, including the case of $i = 0$, i.e., $I = 0$.*

Every isolated non F-rational singularity has an $\mathfrak{m}$-primary parameter test ideal. Namely,

Proposition 3. *Let $(R, \mathfrak{m})$ be a reduced isolated non-F-rational singularity of* $\operatorname{char} R = p > 0$*. Then there exists an $\mathfrak{m}$-primary parameter* *test ideal $J \subset R$, i.e., every element $a \in J$ is a test element for any parameter ideal in R.*

Proof. This follows immediately from (3.9) [17]. See also Exer. 2.12 [9]. □

By Proposition 1, we know that an isolated non F-rational singularity is generalized Cohen-Macaulay. Also, in view of Proposition 3, we will always consider reduced rings.

2. Lower local cohomologies of isolated non F-rational singularities

This section gives a representation of lower local cohomologies of isolated non F-rational singularities in terms of tight closure and limit closure of unconditioned strong d-sequences (USD-sequences), which is a refinement of the representation given by P. Schenzel and K. E. Smith.

2.1. equidimensional hull, limit closure and germ closure. In this subsection, we summerize some of the definitions and results needed to give our representation of lower local cohomologies.

Definition 7. *A sequence of elements $x_1, \dots, x_n$ in a commutative ring is said to be a d-*sequence *if for every $0 \leq i \leq n-1$ and $k > i$ we have*

$$(x_1, \dots, x_i) : x_{i+1}x_k = (x_1, \dots, x_i) : x_k.$$

A sequence $x_1, \dots, x_n$ is said to be a strong d-sequence *if $x_1^{m_1}, \dots, x_n^{m_n}$ is a d-sequence for every arbitrary $m_i \geq 1$, $i = 1, \dots, n$. Finally, a sequence $x_1, \dots, x_n$ is said to be a* USD-sequence *(unconditioned strong d-sequence) if every permutation of it is a strong d-sequence.*

Definition 8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $d = \dim R$. A system of parameters $x_1, \dots, x_d$ is called* standard *if*

$$(x_1, \dots, x_d)H^j_{\mathfrak{m}}(R/(x_1, \dots, x_{i-1})) = 0 \qquad \text{for all } 0 \leq i + j \leq d,\ i \geq 1$$

where we set $(x_1, \dots, x_{i-1}) = 0$ if $i = 1$.

Notice that for a standard sop we have

$$(x_1, \dots, x_d)H^i_{\mathfrak{m}}(R) = 0 \quad \text{for all } i < d.$$

We also note that the empty sequence is trivially a USD and standard sequence. The notions of USD-sequence and standard sequence are actually equivalent:

Proposition 4 (cf. (3.8) [12]). *For an sop $x_1, \dots, x_d$ in a Noetherian local ring $(R, \mathfrak{m})$, the following are equivalent:*

(i) *$x_1, \dots, x_d$ is a standard system of parameters;*
(ii) *$x_1, \dots, x_d$ is an unconditioned strong d-sequence (USD-sequence).*

The USD or standad property of the sequences is preserved by hypersurface intersection. Namely,

Proposition 5 (cf. (3.2) [12]). *Let $x_1, \dots, x_i$ be a standard sequence in a Noetherian local ring $(R, \mathfrak{m})$. Then the image of $x_1, \dots, x_{i-1}$ in $\overline{R} := R/x_iR$ is again a standard sequence of $\overline{R}$.*

For the existence of USD-sequence, we have

Proposition 6 ((6.19) [5]). *Let $(R, \mathfrak{m})$ be a generalized Cohen-Macaulay local ring. Then there exists $t \in \mathbb{N}$ such that any system of parameters in $\mathfrak{m}^t$ is a USD-sequence.*

This implies that we can always obtain a USD-sequence by taking hight enough power of an sop:

Corollary 7. *Let $(R, \mathfrak{m})$ be a generalized Cohen-Macaulay local ring and let $x_1, \dots, x_d$, $d = \dim R$, be any system of parameters. Then $x_1^n, \dots, x_d^n$ is a USD-sequence for $n \gg 0$.*

Proposition 8 (cf. (5.11) [10]). *Let $x_1, \dots, x_i \in R$ be parameters which are also parameter test elements. Then $x_1, \dots, x_i$ are a USD-sequence.*

Definition 9. *Let R be a Noetherian ring. For an ideal $I \subset R$, consider the minimal primary decomposition $I = \bigcap_i Q_i$. Then we define $I^{umn} = \bigcap Q_i$, where Q_i runs over the primary components such that $\dim R/Q_i = \dim R/\sqrt{Q_i} = \dim R/I$. We call I^{umn} the* unmixed hull *of I.*

Definition 10. *For a parameter ideal $I = (x_1, \dots, x_i)$ of a ring R, we define the* limit closure *of I as follows. If $i \geq 0$, we define*

$$\begin{aligned} I^{\lim} &= \{z \in R \mid \mathbf{x}^{s-1}z \in (x_1^s, x_2^s, \dots, x_i^s) \text{ for some } s \in \mathbb{N}\} \\ &= \{z \in R \mid \mathbf{x}^{s-1}z \in (x_1^s, x_2^s, \dots, x_i^s) \text{ for all } s \gg 0\} \\ &= \bigcup_{s=1}^{\infty} (x_1^s, \dots, x_i^s) : \mathbf{x}^{s-1} \end{aligned}$$

where $\mathbf{x} = x_1 \cdots x_i$. For $i = 0$, we define $(0)^{\lim} = 0$.

Proposition 9 (cf. (2.5) [11]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I = (x_1, \dots, x_i)$ $(i \geq 0)$ be a parameter ideal. Then the following are equivalent:*

(i) $z \in I^{\lim}$

(ii) *an element* $\eta = \left[\dfrac{z}{x_1 \cdots x_i}\right] \in H_I^i(R)$ *is* 0.

Proof. Clear from the Čech complex representation of local cohomology. □

Proposition 10 ((5.4) [10]). *Let R be equidimensional and the homomorphic image of a Cohen-Macaulay ring. If $I \subset R$ is a parameter ideal, then we have $I^{\lim} \subseteq I^*$.*

Proposition 11 ((5.8) [10]). *Let $(R, \mathfrak{m})$ be an equidimensional graded Noetherian ring over a field with an $\mathfrak{m}$-primary parameter test ideal. Then for a parameter ideal $I \subset R$ such that $\operatorname{ht} I < \dim R$, we have $I^{umn} = I^*$.*

Corollary 12. *Let $(R, \mathfrak{m})$ be an $\mathbb{N}$-graded isolated non F-rational singularity and let $I = (x_1, \ldots, x_i)$ be a parameter ideal such that $\operatorname{ht}(I) < \dim R$. Then we have $I^{unm} = I^*$.*

Proof. Immediate from Proposition 3 and Proposition 11. □

Definition 11. *Let $I = (x_1, \ldots, x_i)$ be a parameter ideal. Then we define the* germ closure *of I as follows. For $i \geq 1$, we define*

$$I^{germ} = \sum_{j=1}^{i} (x_1, \ldots, \hat{x}_j, , \ldots, x_i)^*.$$

Also, we define $(0)^{germ} = 0$.

In the above definition, we note that $I^{germ} = (0)^*$ if $i = 1$ and we have $I \subset I^{germ}$ for $i \neq 1$. Also we always have $I^{germ} \subset I^*$.

The following proposition is a slight modification of Theorem 5.12 [10]. The only difference is that, for $I = (x_1, \ldots, x_i)$ for $i \leq d$, only the case of $i = d$ is considered in [10]. But with almost the same proof we have the following extension.

Proposition 13 (cf. (5.12) [10]). *Let $(R, \mathfrak{m})$ be an equidimensional local ring that is the homomorphic image of a Cohen-Macaulay local ring. Let $I = (x_1, \ldots, x_i)$, where $0 \leq i \leq d$ and $x_1, \ldots, x_d$ is an sop which are parameter test elements Then we have $I^{germ} + I = I^{\lim}$, in particular if $i \neq 1$ we have $I^{germ} = I^{\lim}$.*

Proof. See (5.12) [10] in the case of $i \geq 2$. The case of $i = 0, 1$ can also be proved easily. □

2.2. representation of lower local cohomologies.

Proposition 14 (cf. (6.8) and (6.9) [14]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring that is equidimensional, the homomorphic image of a Cohen-Macaulay local ring and R_P is Cohen-Macaulay for all primes $P \neq \mathfrak{m}$. Then there exists an sop $x_1, \ldots, x_d$, $(d = \dim R)$ such that for $0 \leq i < d$ we have*

$$H^i_{\mathfrak{m}}(R) \cong \frac{(x_1, \ldots, x_i)^{unm}}{\sum_{j=1}^{i}(x_1, \ldots, \hat{x}_j, \ldots, x_i)^{unm} + (x_1, \ldots, x_i)}$$

Moreover if $i \neq 1$ we have

$$H^i_{\mathfrak{m}}(R) \cong \frac{(x_1, \ldots, x_i)^{unm}}{\sum_{j=1}^{i}(x_1, \ldots, \hat{x}_j, \ldots, x_i)^{unm}}$$

In fact, any sop $x_1, \ldots, x_d$ in $\mathfrak{m}^N$ for $N \gg 0$ has such a property.

Proof. By Schenzel's formula (3.3) [12] together with Lemma 15 below. □

Lemma 15 (6.9 [14]). *In ths situation of Proposition 14, we have $(x_1, \ldots, x_i) : x_{i+1} = (x_1, \ldots, x_i)^{unm}$ for $0 \leq i < d$, by replacing the sequence $x_1, \ldots, x_d$ by high powers, if necessary.*

Notice that the equidimensional hull of a graded ideal is again graded. Then, by considering a graded version of Proposition 3.3 in [12], which Proposition 14 bases on, we easily deduce the following.

Corollary 16. *Assume that $(R, \mathfrak{m})$ is generalized Cohen-Macaulay. Then there exists a homogeneous sop $x_1, \ldots, x_d$, $(d = \dim R)$ such that for $0 \leq i < d$ we have*

$$H^i_{\mathfrak{m}}(R) \cong \frac{(x_1, \ldots, x_i)^{unm}}{\sum_{j=1}^{i}(x_1, \ldots, \hat{x}_j, \ldots, x_i)^{unm} + (x_1, \ldots, x_i)}(\delta_i)$$

where $\delta_i = \sum_{j=1}^{i} \deg(x_j)$. Moreover if $i \neq 1$ we have

$$H^i_{\mathfrak{m}}(R) \cong \frac{(x_1, \ldots, x_i)^{unm}}{\sum_{j=1}^{i}(x_1, \ldots, \hat{x}_j, \ldots, x_i)^{unm}}(\delta_i).$$

Theorem 17. *Let $(R, \mathfrak{m})$ be an $\mathbb{N}$-graded Noetherian K-algebra of dimension $d = \dim R$ and $\operatorname{char} K = p > 0$. Assume that R is an equidimensional isolated non F-rational singularity. Then for any homogeneous sop $x_1, \ldots, x_d \in \mathfrak{m}^N$ with $N \gg 0$, we have*

$$H^i_{\mathfrak{m}}(R) \cong \frac{(x_1, \ldots, x_i)^*}{(x_1, \ldots, x_i)^{\lim}}(\delta_i) \cong \frac{(x_1^n, \ldots, x_i^n)^*}{(x_1^n, \ldots, x_i^n)^{\lim}}(n\delta_i)$$

for $i = 0, \ldots, d-1$ and $n \in \mathbb{N}$, where we define $\delta_i = \sum_{j=1}^{i} \deg x_j$.

Proof. By a straightforward calculation using Corollaries. 7, 16, 12 and Proposition 13. $\square$

As an immediate application of this representation, we recover some of the known results in a slightly general way.

Corollary 18 (cf. (2.7) [11]). *Let $(R, \mathfrak{m})$ be an equidimensional isolated non F-rational singularity and let $I = (x_1, \ldots, x_i)$, $i < d = \dim R$, be a homogeneous parameter ideal such that with $I \subset \mathfrak{m}^\ell$ for $\ell \gg 0$. Then for an integer N_i the following are equivalent*

(i) $[H^i_{\mathfrak{m}}(R)]_n = 0$ *for all* $n < N_i$.

(ii) $I^* \subseteq I^{\lim} + R_{\geq \delta_i + N_i}$ *where* $\delta_i = \sum_{j=1}^{i} \deg(x_j)$.

In a special case, we can give a lower bound of the vanishing degree N_i. For an $\mathbb{N}$-graded module M, we set $b(M) = \min\{i \mid M_i \neq 0\}$. If $(R, \mathfrak{m})$ is a generalized Cohen-Macaulay local ring, we have

$$[H^i_{\mathfrak{m}}(R)]_n = 0 \qquad \text{for} \quad n < -it, \quad i < \dim R,$$

if there exists a homogeneous sop $x_1, \ldots, x_d$ as in Theorem 17 with $t = \min\{\deg x_1, \ldots, \deg x_d\}$ ((2.4)[8]). The following result gives a refinement for isolated non F-rational singularities.

Corollary 19. *Let K be an algebraically closed field of* $\operatorname{char} K = p > 0$ *and let $(R, \mathfrak{m})$ be a Noetherian standard $\mathbb{N}$-graded domain with $R_0 = K$ and $\mathfrak{m} = \bigoplus_{n>0} R_n$. Assume that R is an isolated non F-rational singularity and let*

$$t = \min\{N \in \mathbb{N} \mid \mathbf{x} = x_1, \ldots, x_d \in \mathfrak{m}^N \text{ where } \mathbf{x} \text{ is as in Theorem 17}\}.$$

Then for all $i < d = \dim R$ and for all $n < -(i-1)t$ we have $[H^i_{\mathfrak{m}}(R)]_n = 0$.

For another application, we can consider, under the condition of Theorem 17, the natural map $\rho_i : H^i_{\mathfrak{m}}(R) \longrightarrow H^{i+1}_{\mathfrak{m}}(R)$ for $i = 0, \ldots, d-2$ defined by $\rho_i(r + (x_1, \ldots, x_i)^{\lim}) = r + (x_1, \ldots, x_{i+1})^{\lim}$ with $r \in (x_1, \ldots, x_i)^*$. For the degree preserving homomorphism, we can also consider $\psi_i(r + (x_1, \ldots, x_i)^{\lim}) = r x_{i+1} + (x_1, \ldots, x_{i+1})^{\lim}$. Unfortunately, they are all trivial.

Corollary 20. *The maps ρ_i and ψ_i are trivial.*

3. Tight Closure of Zeros and Isolated Singularities

In this section, we consider the question of how the tight closure of zero in the highest local cohomology $H^d_{\mathfrak{m}}(R)$ controls the vanishing of the lower local cohomologies of isolated singularities.

Definition 12. *For a R-module M, we define*

$$(0)^*_M = \left\{ x \in M \mid \begin{array}{l} \text{there exists } c \in R^o \text{ such that } cx^q = 0 \text{ in } M \\ \text{for } q = p^e \gg 0 \end{array} \right\}$$

and call it tight closure of 0 *in M.*

We are particularly interested in the tight closure of zero in the highest local cohomology $H^d_{\mathfrak{m}}(R)$, $d = \dim R$. We give a graded version of the result by Goto and Nakamura.

Proposition 21. *Let $(R, \mathfrak{m})$ be an $\mathbb{N}$-graded isolated non F-rational singularity. Suppose that a homogeneous sop $\mathbf{x} = x_1, \dots, x_d$ forms a USD-sequence. Then, by replacing $\mathbf{x}$ by a higher power $\mathbf{x}^N$ $(N > 0)$ if necessary, we have*

$$(0)^*_{H^d_{\mathfrak{m}}(R)} = \bigcup_{n>0} Z_n(n\delta) \qquad \text{where} \quad Z_n = \frac{(x_1^n, \dots, x_d^n)^*}{(x_1^n, \dots, x_d^n)^{\lim}},$$

where $\delta = \sum_{j=1}^d \deg(x_j)$.

Proof. Consider a graded version of Proposition 2.1 [4], and apply Corollary. 12, Proposition. 13 and Lemma 15. $\square$

Notice that, for $i < d = \dim R$, we have a natural map

$$H^i_{\mathfrak{m}}(R) \cong \frac{(x_1, \dots, x_i)^*}{(x_1, \dots, x_i)^{\lim}}(\delta_i) \longrightarrow \frac{(x_1, \dots, x_d)^*}{(x_1, \dots, x_d)^{\lim}}(\delta_d) \subset (0)^* \subset H^d_{\mathfrak{m}}(R)$$

by sending $r + (x_1, \dots, x_i)^{\lim}$ to $r + (x_1, \dots, x_d)^{\lim}$ or as a degree preserving map $rx_{i+1} \cdots x_d + (x_1, \dots, x_d)^{\lim}$. But they are actually 0-maps, which can be proved similarly to Cor. 20.

Now we consider relation between the tight closures of zero of $H^d_{\mathfrak{m}}(R)$ and $H^{d-1}_{\mathfrak{m}}(\overline{R})$, where $\overline{R}$ is a hypersurface intersetion of R. It is well known that Flenner's Bertini theorem also holds for positive characteristics, which is, as far as the author is concerned, not stated in the literature. We give the statement for the readers convenience.

Proposition 22 (cf. Satz (4.1) [3]). *Let k be an infinite field and $(R, \mathfrak{m})$ be a Noetherian local k-algebra, whose residue class ring $K := R/\mathfrak{m}$ is separable over k. Let $x_1, \dots, x_d \in \mathfrak{m}$ be such that $I = (x_1, \dots, x_d)$ is an $\mathfrak{m}$-primary ideal. Assume that R_P is regular for all $P \in \operatorname{Spec} R - \{\mathfrak{m}\}$. Then for a general linear form $x_\alpha = \sum_{i=1}^d \alpha_i x_i$ with $\alpha \in k^d$ and any $P \in \operatorname{Spec} R/x_\alpha - \{\mathfrak{m}'\}$ where $\mathfrak{m}' = \mathfrak{m}/x_\alpha R$, $(R/x_\alpha R)_P$ is again regular.*

Let $(R, \mathfrak{m})$ be an isolated singularity with $d = \dim R$. Then, by Proposition 5, 22 and 21 together with Cor. 7, we can choose a homogeneous sop $x_1, \dots, x_d$, which is a USD-sequence, such that $\overline{R} = R/x_dR$ is also an isolated singularity and

$$(0)^*_{H^d_{\mathfrak{m}}(R)} = \bigcup_{n>0} \frac{(x_1^n, \dots, x_d^n)^*}{(x_1^n, \dots, x_d^n)^{\lim}}(n\delta_d)$$

and

$$(0)^*_{H^{d-1}_{\mathfrak{m}}(\overline{R})} = \bigcup_{n>0} \frac{(\overline{x}_1^n, \dots, \overline{x}_{d-1}^n)^*}{(\overline{x}_1^n, \dots, \overline{x}_{d-1}^n)^{\lim}}(n\delta_{d-1})$$

where $\delta_j = \sum_{k=1}^{j} \deg(x_k)$ and $\overline{x}_j$ is the image of x_j in $\overline{R}$. Now we can prove

Proposition 23. *Let $(R, \mathfrak{m})$ be an $\mathbb{N}$-graded isolated non F-rational singularity over the field K of* $\operatorname{char} K = p > 0$ *with $R_0 = K$. Let $d = \dim R$. Assume that $x_1, \dots, x_d$ is a homogeneous sop and consider the natural surjection*

$$\varphi : R \longrightarrow \overline{R} = R/x_dR,$$

whose image $\varphi(r)$ will be denoted by $\overline{r}$. Then,

(*i*) *For any $n \in \mathbb{Z}$ we have*

$$\text{if} \quad \left[\frac{(x_1, \dots, x_d)^*}{(x_1, \dots, x_d)^{\lim}}\right]_n \neq 0, \quad \text{then} \quad \left[\frac{(\overline{x}_1, \dots, \overline{x}_{d-1})^*}{(\overline{x}_1, \dots, \overline{x}_{d-1})^{\lim}}\right]_n \neq 0,$$

(*ii*) *Assume also that $x_1, \dots, x_d$ is a USD-sequence and that $R/x_d^{\ell}R$ is a non F-rational singularity for all $\ell \in \mathbb{N}$. If $[(0)^*_{H^d_{\mathfrak{m}}(R)}]_n \neq 0$ for $n \in \mathbb{Z}$, then there exists $\ell \in \mathbb{N}$ such that $[(0)^*_{H^{d-1}_{\mathfrak{m}}(R/x_d^{\ell}R)}]_{n+a} \neq 0$, where $a = \ell \cdot \deg(x_d)$.*

Proof. We first show (*i*). By persistence of tight closure, φ induces the homomorphism

$$\overline{\varphi} : (x_1, \dots, x_d)^* \longrightarrow \frac{(\overline{x}_1, \dots, \overline{x}_{d-1})^*}{(\overline{x}_1, \dots, \overline{x}_{d-1})^{\lim}}.$$

Since $\operatorname{Ker} \overline{\varphi} \subset (x_1, \dots, x_d)^{\lim}$, we have the following diagram:

$$(1) \qquad \frac{(x_1, \dots, x_d)^*}{(x_1, \dots, x_d)^{\lim}} \xleftarrow{nat} \frac{(x_1, \dots, x_d)^*}{\operatorname{Ker} \overline{\varphi}} \overset{\hat{\varphi}}{\hookrightarrow} \frac{(\overline{x}_1, \dots, \overline{x}_{d-1})^*}{(\overline{x}_1, \dots, \overline{x}_{d-1})^{\lim}}$$

where *nat* is the natural surjection and $\hat{\varphi}$ is the induced embedding. Taking the degree n fraction of this diagram, we immediately know that (*i*) holds.

Now we show (ii). Assume that $[(0)^*_{H^d_{\mathfrak{m}}(R)}] \neq 0$. Then by Proposition 21 there exists $\ell \in \mathbb{N}$ such that

$$0 \neq \left[\frac{(x_1^\ell, \ldots, x_d^\ell)^*}{(x_1^\ell, \ldots, x_d^\ell)^{\lim}}(\ell \cdot \delta_d)\right]_n = \left[\frac{(x_1^\ell, \ldots, x_d^ell l)^*}{(x_1^\ell, \ldots, x_d^\ell)^{\lim}}\right]_{n+\ell\cdot\delta_d}.$$

Thus by (i) with $x_1, \ldots, x_d$ replaced by $x_1^\ell, \ldots, x_d^\ell$, we have

$$0 \neq \left[\frac{(\overline{x}_1^\ell, \ldots, \overline{x}_{d-1}^\ell)^*}{(\overline{x}_1^\ell, \ldots, \overline{x}_{d-1}^\ell)^{\lim}}\right]_{n+\ell\cdot\delta_d} = \left[\frac{(\overline{x}_1^\ell, \ldots, \overline{x}_{d-1}^\ell)^*}{(\overline{x}_1^\ell, \ldots, \overline{x}_{d-1}^\ell)^{\lim}}(\ell \cdot \delta_{d-1})\right]_{n+\ell\cdot\deg(x_d)}$$

Then we obtain the required result by Proposition 21. □

Proposition 23 assumes that hypersurface intersections of isolated non F-rational singularities are again isolated non F-rational singularities. However, F-rationality behaves rather badly with hypersurface intersections, since an F-rational local ring has negative a-invariant (see [2]) but hypersurface intersection increases a-invariant. But for isolated singularity, which is a special case of isolated non F-rational singularity, hypersurface intersections behave well thanks to Bertini type theorems.

Now we come to the main result of this section.

Theorem 24. *Let $(R, \mathfrak{m})$ be a reduced $\mathbb{N}$-graded equidimensional isolated singularity over the field K of $\operatorname{char} K = p > 0$ with $R_0 = K$, $d = \dim R$ and $\sharp K = \infty$. Assume that $[(0)^*_{H^d_{\mathfrak{m}}(R)}]_n \neq 0$ for some $n \in \mathbb{Z}$ and consider a homogeneous regular element x, $a = \deg(x)$, that is also a part of a USD-sequence. Then*

- (i) *If the multiplication $[H^d_{\mathfrak{m}}(R)]_n \xrightarrow{x} [H^d_{\mathfrak{m}}(R)]_{n+a}$ is injective, then we have*
  $[H^{d-1}_{\mathfrak{m}}(R)]_{n+\ell\cdot a} \neq 0$ for some $\ell \in \mathbb{N}$.
- (ii) *Otherwise, if there is a USD sequence $x_1, \ldots, x_d$ with $x = x_d$ such that the degree $n+\delta_d$ fragment of $(x_1, \ldots, x_d)^{\lim} - \varphi^{-1}((\overline{x}_1, \ldots, \overline{x}_{d-1})^{\lim})$ is nonempty, where $\varphi : R \longrightarrow R/x_dR$ is the natural surjection whose image of $r \in R$ is denoted by $\overline{r}$. Then we have $[H^{d-1}_{\mathfrak{m}}(R)]_{n+a} \neq 0$.*

Proof. Let $x_1, \ldots, x_d \in \mathfrak{m}$ be a USD-sequence with x_d general enough. Consider the short exact sequence:

$$0 \longrightarrow R(-a) \xrightarrow{x_d} R \xrightarrow{\varphi} \overline{R} \longrightarrow 0, \qquad a := \deg(x_d)$$

where we set $\overline{R} := R/(x_d)R$ and $\varphi : R \longrightarrow R/(x_d)R$ is the natural map. $\overline{R}$ is again an isolated singularity by Proposition 22.

Since x_d is standard, we have $x_d H_{\mathfrak{m}}^{d-1}(R) = 0$. Thus, from the above short exact sequence, we obtain the degree $n+a$ fragment long exact sequence:

$$0 \to [H_{\mathfrak{m}}^{d-1}(R)]_{n+a} \to [H_{\mathfrak{m}}^{d-1}(\overline{R})]_{n+a} \xrightarrow{\psi} [H_{\mathfrak{m}}^{d}(R)]_n \xrightarrow{x_d} [H_{\mathfrak{m}}^{d}(R)]_{n+a} \to 0.$$

Now we show (i). Assume that the multiplcation by x_d is injective, we have an isomorphism $[H_{\mathfrak{m}}^{d-1}(R)]_{n+a} \cong [H_{\mathfrak{m}}^{d-1}(\overline{R})]_{n+a}$. By Proposition 21 we have the inclusion

$$\frac{(\overline{x}_1, \dots, \overline{x}_{d-1})^*}{(\overline{x}_1, \dots, \overline{x}_{d-1})^{\lim}}(\delta_{d-1}) \subset (0)^*_{H_{\mathfrak{m}}^{d-1}(\overline{R})} \subset H_{\mathfrak{m}}^{d-1}(\overline{R})$$

so that, if

$$\left[\frac{(\overline{x}_1, \dots, \overline{x}_{d-1})^*}{(\overline{x}_1, \dots, \overline{x}_{d-1})^{\lim}}(\delta_{d-1})\right]_{n+a} \neq 0,$$

then we have $[H_{\mathfrak{m}}^{d-1}(\overline{R})]_{n+a} \cong [H_{\mathfrak{m}}^{d-1}(R)]_{n+a} \neq 0$. Then (i) follows by Proposition 23(ii) by replacing, if necessary, x_d by x_d^{ℓ} for some $\ell \in \mathbb{N}$.

Next we show (ii). If $x_d \in \mathfrak{m}^N$ for $N \gg 0$, x_d is a parameter test element by Proposition 3. By Prop 4.4 [13] we know $J \subset \operatorname{Ann}_R((0)^*_{H_{\mathfrak{m}}^d(R)})$ where J is the parameter test ideal. Thus $x_d : [H_{\mathfrak{m}}^{d}(R)]_n \xrightarrow{x_d} [H_{\mathfrak{m}}^{d}(R)]_{n+a}$ is not injective if $[(0)^*_{H_{\mathfrak{m}}^d(R)}]_n \neq 0$. Thus we have the exact sequence

$$0 \longrightarrow [H_{\mathfrak{m}}^{d-1}(R)]_{n+a} \longrightarrow [H_{\mathfrak{m}}^{d-1}(R)]_{n+a} \xrightarrow{\psi} \operatorname{Im}\psi \longrightarrow 0$$

where $0 \neq [(0)^*_{H_{\mathfrak{m}}^d(R)}]_n \subset \operatorname{Im}\psi$. By considering the Čech complex representation of local cohomology, we know that, for

$$\eta = \left[\frac{\overline{r}}{\overline{x}_1 \cdots \overline{x}_{d-1}}\right] \in H_{\mathfrak{m}}^{d-1}(\overline{R})$$

we have

$$\psi(\eta) = \left[\frac{r}{x_1 \cdots x_d}\right] \in H_{\mathfrak{m}}^{d}(R).$$

Also we know that

$$\begin{aligned} \operatorname{Ker}\psi &= \left\{\eta = \left[\frac{\overline{r}}{\overline{x}_1 \cdots \overline{x}_{d-1}}\right] \mid \left[\frac{r}{x_1 \cdots x_d}\right] = 0 \text{ in } H_{\mathfrak{m}}^{d}(R)\right\} \\ &= \left\{\eta = \left[\frac{\overline{r}}{\overline{x}_1 \cdots \overline{x}_{d-1}}\right] \mid r \in (x_1, \dots, x_d)^{\lim}\right\} \end{aligned}$$

by Proposition 9. Now we can show that $\operatorname{Ker}\psi \neq 0$. Thus we must have $[H_{\mathfrak{m}}^{d-1}(R)]_{n+a} \neq 0$ as required. □

Remark 1. As we saw in the proof, the condition (i) of Theorem 24 is satisfied only when $\deg(x)$ is small large enough. On the other hand, Proposition 6 tells that, generally speaking, in order to obtain a part of USD-sequence we need to choose elements of high degree. Thus (i) could be satisfied in relatively few cases.

Also, since $[H^{d-1}R]_k = 0$ for $k \gg 0$, Theorem 24 means that $(x_1, \dots, x_d)^{\lim}$ is the preimage of $(\overline{x}_1, \dots, \overline{x}_{d-1})^{\lim}$ via the natural map φ when $\deg(x_d)$ is large enough.

The following result describes the injectivity of the morphism *nat* in the diagram (1).

Corollary 25. *Let $(R, \mathfrak{m})$ be a reduced $\mathbb{N}$-graded equidimensional isolated singularity over the field K of $\operatorname{char} K = p > 0$ with $R_0 = K$, $d = \dim R$ and $\sharp K = \infty$. For an sop $x_1, \dots, x_d \in \mathfrak{m}^N$ with $N \gg 0$ and for every $n \in \mathbb{Z}$ with $[(0)^*_{H^d_{\mathfrak{m}}(R)}]_n \neq 0$, we have*

$$\left[\frac{(x_1, \dots, x_d)^*}{(x_1, \dots, x_d)^{\lim}}\right]_{n+\delta_d} \subset \left[\frac{(\overline{x}_1, \dots, \overline{x}_{d-1})^*}{(\overline{x}_1, \dots, \overline{x}_{d-1})^{\lim}}\right]_{n+\delta_d}$$

where $\overline{x}_j$ is the image of x_j in $\overline{R} := R/x_dR$.

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TAYLOR RESOLUTIONS OF MONOMIAL IDEALS WITH LINEAR QUOTIENTS

MUNETAKA OKUDAIRA AND YUKIHIDE TAKAYAMA

INTRODUCTION

Let $R = K[X_1, \dots, X_n]$ be a polynomial ring over a field K and consider a monomial ideal $I \subset S$. It is well known that the Taylor complex $(T_\bullet, d_\bullet)$ gives an R-free resolution of I, but it is in general far from minimal. The aim of this paper is to determine some of the cases in which $T_\bullet$ is minimal.

In particular, we consider the case that I is an ideal with linear quotients. It is known that stable ideal, squarefree stable ideals and (poly)matroidal ideals are all ideals with linear quotients and they have Eliahou-Kervaire type minimal resolutions [2].

We first characterize when a monomial ideal I with linear quotients has the minimal Taylor resolution (Theorem 2.1). Then, as a special case, we give a complete description of stable ideals with minimal Taylor resolutions (Theorem 2.3). Finally, we determine monomial ideals with linear resolutions whose Taylor resolutions are minimal (Theorem 3.1).

1. PREPARATION

In this section, we give definitions and known results that are needed to describe out results. For a monomial ideal I, we denote a minimal set of monomial generators of I by $\mathrm{G}(I)$.

Definition 1. Let I be a monomial ideal of R. Then :

(i) I is *stable* if, for an arbitrary monomial $w \in I$, we have $X_i w / X_m \in I$ for all $i < m := \max(w)$. Here, we set $\max(w) = \max\{i \mid a_i > 0\}$ with $w = X_1^{a_1} \cdot \dots \cdot X_n^{a_n}$.

(ii) I is said to have *linear quotients* if, for some order $u_1, \dots, u_r$ of the elements of $\mathrm{G}(I)$ and all $j = 1, \dots, r$, the colon ideals $(u_1, \dots, u_{j-1}) : u_j$ are generated by a subset of $\{X_1, \dots, X_n\}$, namely $(u_1, \dots, u_{j-1}) : u_j = (X_{i_1}, \dots, X_{i_s})$ for some $\{i_1, \dots, i_s\} \subset \{1, 2, \dots, n\}$. For an ideal I with linear quotients, we consider the order of generators $u_1, \dots, u_r$ when we denote $G(I) = \{u_1, \dots, u_r\}$.

We also set $\operatorname{set}(u_j) = \{i \mid X_i \in \mathrm{G}((u_1, \ldots, u_{j-1}) : u_j)\} = \{i_1, \ldots, i_s\}$ for $j = 1, \ldots, r$.

Remark . Let $I \subset R$ be a stable ideal, then I has linear quotients in the following order: For $G(I) = \{u_1, \ldots, u_r\}$, we have $\deg u_1 \leq \deg u_2 \leq \cdots \deg u_r$ and if $\deg u_i = \deg u_{i+1}$ we have $u_i > u_{i+1}$ with reverse lexicographic order with regard to $X_1 > X_2 > \cdots > X_n$. In the following, we always assume that $G(I) = \{u_1, \ldots, u_r\}$ is arranged by this order when I is stable.

Theorem 1.1 (cf. lemma 1.5 [2]). *Let I be a monomial ideal with linear quotients, and $\mathrm{G}(I) = \{u_1, \ldots, u_r\}$. If $\deg u_1 \leq \cdots \leq \deg u_r$ then we have*

$$\beta_q(I) = \sum_{j=1}^{r} \binom{\sharp \operatorname{set}(u_j)}{q}. \tag{1}$$

Remark . Let I be a stable ideal, $\mathrm{G}(I) = \{u_1, \ldots, u_r\}$ whose order of elements is as in the above remark, and β_q the q-th Betti number of I. Then theorem 1.1 and the fact $\sharp\operatorname{set}(u) = \max(u) - 1$ (see [2]) implies,

$$\beta_q(I) = \sum_{u \in \mathrm{G}(I)} \binom{\max(u) - 1}{q} = \sum_{i=1}^{n} \mathrm{m}_i(\mathrm{G}(I)) \binom{i-1}{q} \tag{2}$$

where $\mathrm{m}_i(\mathrm{G}(I)) \overset{\text{def.}}{=} \sharp\{u \in \mathrm{G}(I) \mid \max(u) = i\}$. This is the same resolution as in [1].

We recall the definition of Taylor resolution.

Definition 2. Let I be a monomial ideal of R with $\mathrm{G}(I) = \{u_1, \ldots, u_r\}$. We define the complex

$$T_\bullet : 0 \longrightarrow T_{r-1} \xrightarrow{d_{r-1}} T_{r-2} \xrightarrow{d_{r-2}} \cdots \longrightarrow T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} I \longrightarrow 0 \tag{3}$$

called the *Taylor resolution* of I as follows: Let L be the R-free module with free basis $\{e_1, \cdots, e_r\}$ and set $T_q = \bigwedge^{q+1} L$, the $(q+1)$-th exterior product. We define $d_q \colon T_q \to T_{q-1}$ and $\varepsilon \colon T_0 \to I$ by

$$\begin{aligned} d_q(e_{i_0} \wedge \cdots \wedge e_{i_q}) & \\ = \sum_{s=0}^{q} (-1)^s \cdot \frac{\operatorname{lcm}(u_{i_0}, \ldots, u_{i_q})}{\operatorname{lcm}(u_{i_0}, \ldots, \hat{u_{i_s}}, \ldots, u_{i_q})} \cdot e_{i_0} \wedge \cdots \wedge \hat{e_{i_s}} \wedge \cdots \wedge e_{i_q} & \\ \text{with } 1 \leq i_0 < \cdots < i_q \leq r, \quad q \geq 1 & \\ \varepsilon(e_i) = u_i \qquad (i = 1, \ldots, r) & \end{aligned}$$

Remark . As is well known, a Taylor resolution is not generally minimal. Notice that $\operatorname{rank}(T_q) = \binom{r}{q+1}$.

2. Monomial Ideal with linear quotients

The goal of this section is to give the complete description of stable ideals with minimal Taylor resolutions. Before that we give a general result for monomial ideals with linear quotients.

We can show

Lemma 1. *Let $I = (u_1, \ldots, u_q)$ be a monomial ideal with linear quotients in this order. Then,* $\sharp\mathrm{set}(u_i) \leq i - 1 \quad$ for $i = 1, \ldots, q$.

Theorem 2.1. *Let I be a monomial ideal with linear quotients with $\mathrm{G}(I) = \{u_1, \ldots, u_r\}$. If $\deg u_1 \leq \cdots \leq \deg u_r$ with $r \leq n$, then the Taylor resolution $T_\bullet$ of I is minimal if and only if $\sharp\mathrm{set}(u_i) = i - 1$ for $i = 1, \ldots, r$.*

Proof. By Theorem 1.1 and Lemma 1 together with the remark after Definition 2. □

Now we consider the special case of stable ideals.

Proposition 2.2. *Let I be a stable ideal of R. Then the following conditions are equivalent :*

(i) *The Taylor resolution $T_\bullet$ of I is minimal;*
(ii) $\sharp\mathrm{G}(I) = \max\{\max(u) | u \in \mathrm{G}(I)\}$;
(iii) $\mathrm{m}_i(\mathrm{G}(I)) = \begin{cases} 1 & \text{for } 1 \leq i \leq \sharp\mathrm{G}(I) \\ 0 & \text{for } \sharp\mathrm{G}(I) < i \leq n. \end{cases}$

Proof. By the remarks after Theorem 1.1 and Definition 2. □

Using the above proposition we can determine the form of the stable ideals whose Taylor resolutions are minmal.

Theorem 2.3. *Let I be a stable monomial ideal of R and $T(I)_\bullet$ the Taylor resolution of I. Then the following conditions are equivalent :*

(i) *The Taylor resolution $T_\bullet$ of I is minimal;*
(ii) *$I = (u_1, \ldots, u_r)$ with $r \leq n$ and there exist positive integers $s_1, \ldots, s_r$ such that $u_1 = X_1^{s_1}$, $u_{j+1} = X_1^{s_1 - 1} \cdots X_j^{s_j - 1} X_{j+1}^{s_{j+1}}$ for $j = 1, \ldots, r - 1$.*

3. Equigenerated case

As a special case of Theorem 2.3, we can consider a stable monomial ideal $I \subset R$ generated by monomials with the same degree. In this case, I has a linear resolution. But it turns out that this case is very special even if we do not assume the stableness. Namely,

Theorem 3.1. *Let $I \subset R$ be a monomial ideal with a linear resolution. Then the following conditions are equivalent:*

(i) *The Taylor resolution $T_\bullet$ of I is minimal;*

(ii) *$I = u \cdot (X_{i_1}, \dots, X_{i_r})$ for some $1 \leq i_1 < \cdots < i_r \leq n$ and u is a monomial.*

In this case, I is an ideal with linear quotients.

Proof. We use the induction on $\sharp \mathrm{G}(I)$ and the fact that the ideal which is generated by the subset of $\mathrm{G}(I)$ also has the minimal Taylor resolution and a linear resolution when I has a linear resolution and its Taylor resolution is minimal. $\square$

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LINEARITY DEFECTS OF FACE RINGS

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1. Introduction

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K, and $E := \bigwedge \langle y_1, \dots, y_n \rangle$ an exterior algebra. We regard them as graded rings with $\deg x_i = \deg y_i = 1$. As usual, for a graded S (or E)-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ and $j \in \mathbb{Z}$, $M(j)$ denotes the shifted module of M with $M(j)_i = M_{i+j}$.

Let $A = \bigoplus_{i \geq 0} A_i$, $A_0 = K$, be a Koszul algebra (e.g. $A = S$ or E), and $^*\mathrm{mod}\, A$ the category of finitely generated graded left A-modules. For a minimal free resolution $P_\bullet$ of $M \in {}^*\mathrm{mod}\, A$, Eisenbud et al. [3] defined the *linear part* $\mathrm{lin}(P_\bullet)$ of $P_\bullet$, which is the chain complex obtained by erasing all terms of degree ≥ 2 from the matrices representing the differential maps of $P_\bullet$ (hence $\mathrm{lin}(P_\bullet)_i = P_i$ for all i). Following Herzog and Iyengar [7], we call

$$\mathrm{ld}_A(M) := \sup\{\, i \mid H_i(\mathrm{lin}(P_\bullet)) \neq 0 \,\}$$

the *linearity defect* of M.

For $M \in {}^*\mathrm{mod}\, A$ and $i \in \mathbb{Z}$, $M_{\langle i \rangle}$ denotes the submodule of M generated by its degree i part M_i. We say M is *componetwise linear* ([6]), if each $M_{\langle i \rangle}$ has a linear free resolution, in other words, the minimal free resolution is of the form

$$\cdots \to A^{\oplus \beta_j}(-i-j) \to \cdots \to A^{\oplus \beta_2}(-i-2) \to A^{\oplus \beta_1}(-i-1) \to A^{\oplus \beta_0}(-i) \to M_{\langle i \rangle} \to 0.$$

If M is componentwise linear, so is its i^{th} syzygy $\Omega_i(M)$ for all $i \geq 0$. Moreover, we have

$$\mathrm{ld}_A(M) = \inf\{\, i \mid \Omega_i(M) \text{ is componentwise linear} \,\}.$$

Clearly, $\mathrm{ld}_A(M) \leq \mathrm{proj.dim}_A(M)$. Hence $\mathrm{ld}_S(M) < \infty$ for all $M \in {}^*\mathrm{mod}\, S$. But even if A is a Koszul commutative algebra, it might occur that $\mathrm{ld}_A(M) = \infty$ (c.f. [7]). But we have the following.

Theorem 1.1. *We have the following.*

(1) (Eisenbud et. al. [3]) *We have* $\mathrm{ld}_E(M) < \infty$ *for all* $M \in {}^*\mathrm{mod}\, E$.

(2) (Y. [16]) *If* $n \geq 2$, *then* $\sup\{\, \mathrm{ld}_E(M) \mid M \in {}^*\mathrm{mod}\, E \,\} = \infty$. *But we have*

$$\mathrm{ld}_E(M) \leq w^{n!} 2^{(n-1)!},$$

where $w := \max\{\, \dim_K M_i \mid i \in \mathbb{Z} \,\}$.

(3) (Herzog-Römer [10]) *If J is a monomial ideal of* $E = K\langle y_1, \dots, y_n \rangle$, *then*

$$\mathrm{ld}_E(E/J) \leq n - 1.$$

The parts (1) and (2) of the theorem are closely related to *Bernstein-Gel'fand-Gel'fand correspondence* (*BGG correspondence*), which gives a derived equivalence $D^b({}^*\mathrm{mod}\, S) \cong D^b({}^*\mathrm{mod}\, E)$. The main results of this paper improve/refine the part (3).

This note is based on our paper [9] which has been submitted to a journal.

Set $[n] := \{1, \dots, n\}$, and let $\Delta \subset 2^{[n]}$ be a simplicial complex (i.e., $F \subset G \in \Delta$ implies $F \in \Delta$). To Δ, we assign monomial ideals of S and E as follows:

$$I_\Delta := (\prod_{i \in F} x_i \mid F \subset [n], F \notin \Delta) \subset S, \qquad J_\Delta := (\prod_{i \in F} y_i \mid F \subset [n], F \notin \Delta) \subset E.$$

Any monomial ideal of E is of the form J_Δ for some Δ. We call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ, and $K\langle\Delta\rangle := E/J_\Delta$ the *exterior face ring* of Δ. Both are important in combinatorial commutative algebra. The following theorem follows from BGG correspondence for $K[\Delta]$ and $K\langle\Delta\rangle$.

Theorem 1.2. *Let $\Delta \neq \emptyset$ be a simplicial complex on $[n]$. Then;*

(1) $\mathrm{ld}_S(K[\Delta]) = \mathrm{ld}_E(K\langle\Delta\rangle)$. *(So we denote this value by* $\mathrm{ld}(\Delta)$.*)*
(2) *If $\Delta \neq 2^T$ for any $T \subset [n]$, then $\mathrm{ld}(\Delta)$ is a topological invariant of the geometric realization $|\Delta^\vee|$ of the dual complex $\Delta^\vee := \{ F \subset [n] \mid [n] - F \notin \Delta \}$ of Δ.*
(3) *For $n \geq 4$, $\mathrm{ld}(\Delta) = n - 2$ (this is the largest possible value) if and only if Δ is an n-gon.*

2. BGG correspondence and Linearity Defect

We use the same notation as the previous section.

Let $P_\bullet : \cdots \to P_1 \to P_0 \to 0$ be a graded minimal S-free resolution of $M \in {}^*\mathrm{mod}\, S$. For $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, we have natural numbers $\beta_{i,j}(M)$ such that $P_i \cong \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}(M)}$. The *(Castelnuovo-Mumford) regularity*

$$\mathrm{reg}_S(M) := \max\{ j - i \mid \beta_{i,j}(M) \neq 0 \}$$

of M is an important invariant introduced in [4]. For the conveniense, we set the regularity of the 0 module to be $-\infty$. For $N \in {}^*\mathrm{mod}\, E$, we can define $\beta_{i,j}(N)$ by the same way.

For a cochain complex $(N^\bullet, \partial^\bullet)$ in ${}^*\mathrm{mod}\, E$, we can make $L(N^\bullet) := S \otimes_K N^\bullet$ a cochain complex of graded free S-modules as follows. Set $L(N^\bullet)^m = \bigoplus_{i+j=m} S \otimes_K N^i_j$, and the degree of an element in $S_l \otimes_K N^i_j$ to be $l - j$. Finally, the differential defined by

$$L(N^\bullet)^m \supset S \otimes_K N^i_j \ni 1 \otimes z \mapsto \sum_{1 \leq l \leq n} x_l \otimes y_l z + (-1)^m (1 \otimes \partial^i(z)) \in L(N^\bullet)^{m+1}.$$

This operation gives the functor $\mathcal{L} : D^b({}^*\mathrm{mod}\, E) \to D^b({}^*\mathrm{mod}\, S)$.

Similarly, for a cochain complex $M^\bullet$ in ${}^*\mathrm{mod}\, S$, we can make $R(M^\bullet) = \mathrm{Hom}_K(E, M^\bullet)$ a cochain complex of graded free E-modules. If $M^\bullet$ is a module M, then $R^m(M) = \mathrm{Hom}_K(E, M_m) \cong E^{\oplus \dim_k M_m}(n + m)$, and the differential is given by

$$R^m(M) = \mathrm{Hom}_K(E, M_m) \ni f \mapsto [e \mapsto \sum_{1 \leq i \leq d} x_i f(y_i e)] \in \mathrm{Hom}_K(E, M_{m+1}) = R^{m+1}(M).$$

This operation induces the functor $\mathcal{R} : D^b({}^*\mathrm{mod}\, S) \to D^b({}^*\mathrm{mod}\, E)$. See [3] for details.

Theorem 2.1 (BGG correspondence, c.f.[3]). *The functors $\mathcal{L}$ and $\mathcal{R}$ give a category equivalence $D^b({}^*\mathrm{mod}\, S) \cong D^b({}^*\mathrm{mod}\, E)$.*

BGG correspondence is a special case of *Koszul duality.* That is, the fact that S and E are Koszul algebras and they are the Kosuzl dual of each other (i.e., $S \cong E^! := \mathrm{Ext}^\bullet_E(K, K)$ and $E \cong S^! := \mathrm{Ext}^\bullet_S(K, K)$) is essential.

Since $\beta_{i,j}(M) = \dim_K[\operatorname{Ext}^i_S(M,K)]_{-j}$ and $\mathcal{R}(K) = E$, we have the equality

$$\beta_{i,j}(M) = \dim_k H^{j-i}(R(M))_{-j}.$$

Hence we have the following.

Theorem 2.2 (Eisenbud et al. [3]). *For $M \in {}^*\mathrm{mod}\, S$, we have*

$$\mathrm{reg}_S(M) = \max\{\, i \mid H^i(R(M)) \neq 0 \,\}.$$

The "linear part" of the free resolution of a graded module (over S or E) has been introduced by Eisenbud and his coworkers. In the previous section, we mentioned this notion. But we give a precise description here. Let $P_\bullet$ be a *minimal* E-free resolution of $N \in {}^*\mathrm{mod}\, E$. Consider the decomposition $P_i := \bigoplus_{j \in \mathbb{Z}} P_{i,j}$ such that $P_{i,j} \cong E^{\oplus m}(-j)$ for some m. For an integer l, we define the *l-linear strand* $\mathrm{lin}_l(P_\bullet)$ of $P_\bullet$ is the complex defined as follows: The term $\mathrm{lin}_l(P_\bullet)_i$ of homological degree i is $P_{i,l+i}$ and the differential $P_{i,l+i} \to P_{i-1,l+i-1}$ is the corresponding component of the differential $P_i \to P_{i-1}$ of $P_\bullet$. So the differential of $\mathrm{lin}_l(P_\bullet)$ is represented by matrices whose entries are in $E_1 = \langle y_1, \dots, y_n \rangle$. We call $\mathrm{lin}(P_\bullet) := \bigoplus_{l \in \mathbb{Z}} \mathrm{lin}_l(P_\bullet)$ the *linear part* of the minimal free resolution $P_\bullet$ of N. $\mathrm{lin}(P_\bullet)$ is not acyclic in general. It is acyclic if and only if N is componentwise linear.

To state the next result, we have to introduce a few operations. For a complex $M^\bullet$, set $\mathcal{H}(M^\bullet)$ to be the complex such that $\mathcal{H}(M^\bullet)^i = H^i(M)$ for all i and the differential maps are zero. It is easy to see that $(-)^\vee := \mathrm{Hom}_K(-,K)$ gives an exact duality functor from ${}^*\mathrm{mod}\, E$ to itself. (One might think $N^\vee$ is a *right* E-module. But, for graded E-modules, we do not have to distinguish left modules form right ones.) Clearly, $(-)^\vee$ can be extended to a operation on complexes.

Proposition 2.3 ([15]). *Let $P_\bullet$ be a minimal free resolution of $N \in {}^*\mathrm{mod}\, E$. Then we have*

$$\mathrm{lin}(P_\bullet) = (R \circ \mathcal{H} \circ L(N^\vee))^\vee.$$

For a minimal free resolution $P_\bullet$ of $M \in {}^*\mathrm{mod}\, S$, we can also define $\mathrm{lin}(P_\bullet)$ by the same way. In this case, $\mathrm{lin}(P_\bullet) = L \circ \mathcal{H} \circ R(M)$.

Definition 2.4 (Herzog-Iyengar [7]). Let $N \in {}^*\mathrm{mod}\, E$, and $P_\bullet$ its minimal free resolution. We call $\mathrm{ld}_E(N) := \sup\{\, i \mid H_i(\mathrm{lin}(P_\bullet)) \neq 0 \,\}$ the *linearity defect* of N. We can also define $\mathrm{ld}_S(M)$ for $M \in {}^*\mathrm{mod}\, S$.

Clearly, $\mathrm{ld}_S(M) \leq \mathrm{proj.dim}_S(M) \leq n$. While $\mathrm{proj.dim}_E(N) = \infty$ unless N is free, Eisenbud et al. [3] showed that $\mathrm{ld}_E(N) < \infty$ for all $N \in {}^*\mathrm{mod}\, E$. (The second author showed that this result can be generalized as follows: If A is a Koszul complete intersection commutative algebra, then $\mathrm{ld}_{A^!}(N) < \infty$ for all $N \in {}^*\mathrm{mod}\, A^!$.)

The next result follows form Theorem 2.2 and Proposition 2.3.

Theorem 2.5. *If $N \in {}^*\mathrm{mod}\, E$, then we have*

$$\mathrm{ld}_E(N) = \max\{\, \mathrm{reg}_S(H^i(L(N^\vee))) + i \mid i \in \mathbb{Z} \,\}.$$

Remark 2.6. Eisenbud et al [3] gives a similar description of $\mathrm{ld}_E(N)$. But we believe that the above one is more practical. For example, Theorem 1.1 (2) follows from the above description. Moreover, it is not so difficult to write a Macaulay2 script which construct $L(N^\vee)$ for $N \in {}^*\mathrm{mod}\, E$ (c.f [2]). So $\mathrm{ld}_E(N)$ can be computed actually.

3. Linearity Defect of Face Rings

For a simplicial complex $\Delta \subset 2^{[n]}$, let $I_\Delta := (\prod_{i \in F} x_i \mid F \subset [n], F \not\in \Delta)$ be a monomial ideal of S, and $J_\Delta := (\prod_{i \in F} y_i \mid F \subset [n], F \not\in \Delta)$ a monomial ideal of E. Set $K[\Delta] := S/I_\Delta$ and $E := E/J_\Delta$.

Theorem 3.1. *For a simplicial complex $\Delta \subset 2^{[n]}$, we have*

$$\mathrm{ld}_E(K\langle\Delta\rangle) = \mathrm{ld}_S(K[\Delta]) \qquad \textit{and} \qquad \mathrm{ld}_E(J_\Delta) = \mathrm{ld}_S(I_\Delta).$$

Remark 3.2. If $\mathrm{ld}_S(K[\Delta]) \geq 1$, then we have $\mathrm{ld}_S(K[\Delta]) = \mathrm{ld}_S(I_\Delta) + 1$. On the other hand, we have $\mathrm{ld}_S(K[\Delta]) = 0 \iff \Delta = 2^T$ for some $T \subset [n] \implies \mathrm{ld}_S(I_\Delta) = 0$.

Idea of the proof. There might exists a direct proof. But, in [9], we use the fact that BGG correspondence has special meaning for $K[\Delta]$ and $K\langle\Delta\rangle$ (c.f. [14]). From this, we can show that both $\mathrm{ld}_S(K(\Delta))$ and $\mathrm{ld}_E(K\langle\Delta\rangle)$ equal

$$\max\{\, i - \mathrm{depth}_S(\, \mathrm{Ext}_S^{n-i}(I_{\Delta^\vee}, S)\,) \mid 0 \leq i \leq n\,\}. \tag{3.1}$$

Here

$$\Delta^\vee := \{\, F \subset [n] \mid [n] \setminus F \not\in \Delta \,\}$$

is the *Alexander dual* of Δ.

We remark that the number in (3.1) is closely related to the notion of *sequentially Cohen-Macaulay modules* (c.f. [11, Theorem 2.11]).

Theorem 3.1 suggests that we may set

$$\mathrm{ld}(\Delta) := \mathrm{ld}_S(K[\Delta]) = \mathrm{ld}_E(K(\langle\Delta\rangle)).$$

A simplicial complex Δ gives the topological space $|\Delta|$ which is called the *geometric realization* of Δ. In other words, Δ is a "triangulation" of $|\Delta|$. It is well-known that many homological/ring theoretical invariants of $K[\Delta]$ only depend on the topological space $|\Delta|$ (and $\mathrm{char}(K)$). But, for $\mathrm{ld}(\Delta)$, the Alexander dual $\Delta^\vee$ is essential.

Theorem 3.3 ([9]). *If $\Delta \neq 2^{[n]}$, then $\mathrm{ld}_S(I_\Delta)\,(= \mathrm{ld}_E(J_\Delta))$ is a topological invariant of the geometric realization $|\Delta^\vee|$ of the Alexander dual $\Delta^\vee$. If $\Delta \neq 2^T$ for any $T \subset [n]$, $\mathrm{ld}(\Delta)$ is a topological invariant of $|\Delta^\vee|$.*

The above result follows from the fact that $\mathrm{ld}(\Delta)$ equals the number given in (3.1) and "sheaf method in the Stanley-Reisner ring theory" introduced in [13].

Remark 3.4. (1) For the first statement of Theorem 3.3, the assumption that $\Delta \neq 2^{[n]}$ is necessary. In fact, if $\Delta = 2^{[n]}$, then $I_\Delta = 0$ and $\Delta^\vee = \emptyset$. On the other hand, if we set $\Gamma := 2^{[n]} \setminus [n]$, then $\Gamma^\vee = \{\emptyset\}$ and $|\Gamma^\vee| = \emptyset = |\Delta^\vee|$. In view of (3.1), it might be natural to set $\mathrm{ld}_S(I_\Delta) = \mathrm{ld}_S(0) = -\infty$. But, $I_\Gamma \cong S(-n)$ and hence $\mathrm{ld}_S(I_\Gamma) = 0$. One might think it is better to set $\mathrm{ld}_S(0) = 0$ to avoid the problem. But this convention does not help so much, if we consider $K[\Delta]$ and $K[\Gamma]$. In fact, $\mathrm{ld}_S(K[\Delta]) = 0$ and $\mathrm{ld}_S(K[\Gamma]) = 1$.

(2) $\mathrm{ld}(\Delta)$ depends on the characteristic $\mathrm{char}(K)$. In fact, when $|\Delta^\vee|$ is homeomorphic to a real projective plane $\mathbb{P}^2\mathbb{R}$, we have

$$\mathrm{ld}(\Delta) = \begin{cases} 3 & \text{if } \mathrm{char}(K) = 2 \\ 1 & \text{otherwise.} \end{cases}$$

4. An upper bound of linearity defects.

In the previous section, we have seen that $\mathrm{ld}_E(K\langle\Delta\rangle) = \mathrm{ld}_S(K[\Delta])\,(=: \mathrm{ld}(\Delta))$ for a simplicial complex Δ. In this section, we will give an upper bound of them, and see that the bound is sharp.

Let $N \in {}^*\mathrm{mod}\, E$. We call $\mathrm{indeg}_E(N) := \min\{\, i \mid N_i \neq 0 \,\}$ minimal the *initial degree* of N, and $\mathrm{indeg}_S(M)$ is similarly defined as $\mathrm{indeg}_S(M) := \min\{\, i \mid M_i \neq 0 \,\}$ for $M \in {}^*\mathrm{mod}\, S$. Note that for a simplicial complex Δ on $[n]$ we have $\mathrm{indeg}_S(I_\Delta) = \mathrm{indeg}_E(J_\Delta) = \min\{\, \sharp F \mid F \subset [n]\,, F \notin \Delta \,\}$, where $\sharp F$ denotes the cardinal number of F. (Recall that $\dim F = \sharp F - 1$.) So we set

$$\mathrm{indeg}(\Delta) := \mathrm{indeg}_S(I_\Delta) = \mathrm{indeg}_E(J_\Delta).$$

Let $P_\bullet$ be a minimal free resolution of $M \in {}^*\mathrm{mod}\, S$. Recall that, for $l \in \mathbb{Z}$, the l-linear strand $\mathrm{lin}_l(P_\bullet)$ is a direct summand of $\mathrm{lin}(P_\bullet)$ with $\mathrm{lin}_l(P_\bullet)_i = S(-l-i)^{\beta_{i,l+i}(M)}$. We often make use of the following facts:

Lemma 4.1. *Let $M \in {}^*\mathrm{mod}\, S$ and let $P_\bullet$ be a minimal free resolution of M. Then*

(1) $\mathrm{lin}_i(P_\bullet) = 0$ *for all $i < \mathrm{indeg}_S(M)$;*
(2) $\mathrm{lin}_{\mathrm{indeg}_S(M)}(P_\bullet)$ *is a subcomplex of $P_\bullet$;*
(3) *if $M = K[\Delta]$, then $\mathrm{lin}(P_\bullet) = \bigoplus_{0 \le l \le n} \mathrm{lin}_l(P_\bullet)$, and $\mathrm{lin}_l(P_\bullet)_i = 0$ for all $i > n - l$ and all $0 \le l \le n$, where the subscript i is a homological degree.*

Proposition 4.2 ([9]). *For a simplicial complex Δ on $[n]$, we have*

$$\mathrm{ld}(\Delta) \le \max\{1, n - \mathrm{indeg}(\Delta)\}.$$

Proof. By Lemma 4.1, we have

$$\mathrm{ld}(\Delta) = \mathrm{ld}_S(K[\Delta]) \le \mathrm{proj.\,dim}_S(K[\Delta]) \le n - \mathrm{indeg}(\Delta) + 1.$$

As a bound for $\mathrm{ld}(\Delta)$, this is weaker than the required one. But, checking the "tail" of the minimal free resolution $P_\bullet$ of $K[\Delta]$ carefully, we can improve it. See [9] for detail. □

Let Δ, Γ be simplicial complexes on $[n]$. We denote $\Delta * \Gamma$ for the join $\{\, F \cup G \mid F \in \Delta, G \in \Gamma \,\}$ of Δ and Γ, and for our convenience, set $\mathrm{ver}(\Delta) := \{\ v \in [n] \mid \{v\} \in \Delta\ \}$.

Lemma 4.3. *Let Δ be a simplicial complex on $[n]$. Assume that $\mathrm{indeg}(\Delta) = 1$, or equivalently $\mathrm{ver}(\Delta) \neq [n]$. Then we have $\mathrm{ld}(\Delta) = \mathrm{ld}(\Delta * \{v\})$ for $v \in [n] \setminus \mathrm{ver}(\Delta)$.*

Proof. We may assume that $v = 1$. Let $P_\bullet$ be a minimal free resolution of $K[\Delta * \{1\}]$ and $\mathcal{K}(x_1)$ the Koszul complex with respect to x_1. Then $P_\bullet \otimes_S \mathcal{K}(x_1)$ is a minimal free resolution of $K[\Delta]$, and moreover, we can easily verify that $\mathrm{lin}(P_\bullet \otimes_S \mathcal{K}(x_1)) = \mathrm{lin}(P_\bullet) \otimes_S \mathcal{K}(x_1)$. This yields the required assertion. □

We can infer the following from the above lemma:

Proposition 4.4 ([9]. See also. [15, Proposition 4.15]). *Let Δ be a simplicial complex on $[n]$. If $\mathrm{indeg}\,\Delta = 1$, then we have $\mathrm{ld}(\Delta) \le \max\{1, n-3\}$. Hence, for any Δ, we have $\mathrm{ld}(\Delta) \le \max\{1, n-2\}$.*

Given a simplicial complex Δ on $[n]$, we denote $\Delta^{(i)}$ for the i^{th} skeleton of Δ, which is defined as $\Delta^{(i)} := \{\, F \in \Delta \mid \dim F \le i \,\}$. We set $\Delta_F := \{\, G \in \Delta \mid G \subset F \,\}$.

We sometimes use the following well-known result due to Hochster; there is an isomorphism

$$[\mathrm{Tor}^i_S(K[\Delta], K)]_F \cong \tilde{H}^{\sharp F - i - 1}(\Delta_F; K) \tag{4.1}$$

as K-vector spaces.

Example 4.5 ([9]). Here we show that the inequality of Proposition 4.2 is optimal, if $\mathrm{indeg}(\Delta) \geq 2$.

Set $\Sigma := 2^{[n]}$, and let Γ be a simplicial complex on $[n]$ whose geometric realization $|\Gamma|$ is homeomorphic to the $(d-1)$-dimensional sphere S^{d-1} with $2 \leq d < n-1$. (For $m > d$, there exists a triangulation of S^{d-1} with m vertices. See, for example, [1, Proposition 5.2.10].) Set $\Delta := \Gamma \cup \Sigma^{(d-2)}$.

Since $\Sigma^{(d-2)}$ has no faces of dimension $\geq d-1$, we have $\tilde{C}_{d-1}(\Delta_F; K) = \tilde{C}_{d-1}(\Gamma_F; K)$ and hence $\tilde{H}_{d-1}(\Delta_F; K) = \tilde{H}_{d-1}(\Gamma_F; K)$. On the other hand, our assumption that $|\Gamma| \approx S^{d-1}$ implies that

$$\begin{aligned} \beta_{n-d,n}(K[\Delta]) &= \dim_K \tilde{H}_{d-1}(\Gamma; K) = 1 \neq 0; \\ \beta_{n-d-1,n-1}(K[\Delta]) &= \sum_{F \subset [n], \sharp F = n-1} \dim_K \tilde{H}_{d-1}(\Gamma_F; K) = 0, \end{aligned}$$

from which it follows that $\mathrm{ld}(\Delta) \geq n - d$. On the other hand, we have $\mathrm{indeg}(\Delta) \geq d$ by the construction. Hence $\mathrm{ld}(\Delta) = n - \mathrm{indeg}(\Delta)$ by Proposition 4.2, and Δ attains the inequality of Proposition 4.2.

5. A simplicial complex Δ with $\mathrm{ld}(\Delta) = n-2$ is an n-gon

We say that a simplicial complex on $[n]$ is an *n-gon* if its facets are $\{1,2\}, \cdots, \{n-1,n\}$, and $\{n,1\}$ after a suitable permutation of vertices. Consider the simplicial complex Δ on $[n]$ given in Example 4.5. If we set $d = 2$, then Δ is an n-gon. Thus if a simplicial complex Δ on $[n]$ is an n-gon, we have $\mathrm{ld}(\Delta) = n-2$. Actually, the inverse holds:

Theorem 5.1. *Let Δ be a simplicial complex on $[n]$ with $n \geq 4$. Then $\mathrm{ld}(\Delta) = n - 2$ if and only if Δ is an n-gon.*

Lemma 5.2. *Let Δ be a simplicial complex on $[n]$ with $\mathrm{indeg}(\Delta) \geq 2$, and $P_\bullet$ a minimal free resolution of $K[\Delta]$. We denote $Q_\bullet$ for the subcomplex of $P_\bullet$ such that $Q_i := \bigoplus_{j \leq i+1} S(-j)^{\beta_{i,j}} \subset \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}} = P_i$. Assume $n \geq 4$. Then the following are equivalent.*

(1) $\mathrm{ld}(\Delta) = n - 2$;
(2) $H_{n-2}(\mathrm{lin}_2(P_\bullet)) \neq 0$;
(3) $H_{n-3}(Q_\bullet) \neq 0$.

It is noteworthy that in the case $n \geq 5$ the condition (3) is equivalent to say that $H_{n-3}(\mathrm{lin}_1(P_\bullet)) \neq 0$.

Proof. The assertion follows from the exact sequence of complexes:

$$0 \longrightarrow Q_\bullet \longrightarrow P_\bullet \longrightarrow \tilde{P}_\bullet := P_\bullet / Q_\bullet \longrightarrow 0,$$

See [9] for details. □

The sketch of the proof of Theorem 5.1. First, we need some observation. Set $V := S_1 = \langle x_1, \dots, x_n \rangle$, and let $P_\bullet$ be a minimal free resolution of $K[\Delta]$. We can define two differential map ϑ, ∂ on the chain complex $K[\Delta] \otimes_K \bigwedge V \otimes_K S$ as follows:

$$\vartheta(f \otimes \wedge^G \mathbf{x} \otimes g) = \sum_{i \in G} (-1)^{\alpha(i,G)} (x_i f \otimes \wedge^{G \setminus \{i\}} \mathbf{x} \otimes g);$$
$$\partial(f \otimes \wedge^G \mathbf{x} \otimes g) = \sum_{i \in G} (-1)^{\alpha(i,G)} (f \otimes \wedge^{G \setminus \{i\}} \mathbf{x} \otimes x_i g).$$

By a routine, it is easy to see that $\partial\vartheta + \vartheta\partial = 0$, and that $P_i \cong H_i((K[\Delta] \otimes_K \bigwedge V \otimes_K S, \vartheta))$ for all i. Since, moreover, the differential maps of $\operatorname{lin}(P_\bullet)$ is induced by ∂ due to Eisenbud-Goto [4] and Herzog-Simis-Vasconcelos [8], $\operatorname{lin}_l(P_\bullet)_i \longrightarrow \operatorname{lin}_l(P_\bullet)_{i-1}$ can be identified with

$$\bigoplus_{F \subset [n], \sharp F = i+l} [\operatorname{Tor}_i^S(K[\Delta], K)]_F \otimes_K S \xrightarrow{\bar{\partial}} \bigoplus_{F \subset [n], \sharp F = i-1+l} [\operatorname{Tor}_{i-1}^S(K[\Delta], K)]_F \otimes_K S,$$

where $\bar{\partial}$ is induced by ∂, and hence by (4.1) with

$$\bigoplus_{F \subset [n], \sharp F = i+l} \tilde{H}^{l-1}(\Delta_F; K) \otimes_K S \longrightarrow \bigoplus_{F \subset [n], \sharp F = i-1+l} \tilde{H}^{l-1}(\Delta_F; K) \otimes_K S.$$

Let Δ be a 1-dimensional simplicial complex on $[n]$ (i.e., Δ is essentially a simple graph). A *cycle* C in Δ of length t (≥ 3) is a sequence of edges of Δ of the form (v_1, v_2), (v_2, v_3), ..., (v_t, v_1) joining distinct vertices $v_1, \dots v_t$. We say C has a *chord* if there exists an edge (v_i, v_j) of G such that $j \not\equiv i+1 \pmod t$, and C is said to be *minimal* if it has no chord. It is easy to see that the 1st homology of Δ is generated by those of minimal cycles contained in Δ, that is, we have the surjective map:

$$\bigoplus_{\substack{C \subset \Delta \\ C:\text{minimal cycle}}} \tilde{H}_1(C; K) \longrightarrow \tilde{H}_1(\Delta; K). \tag{5.1}$$

Now we are ready to prove. The implication "$\Leftarrow$" has been already done in the beginning of this section. So we shall show the inverse. By Proposition 4.4, we may assume that $\operatorname{indeg}(\Delta) \geq 2$. Let $P_\bullet$ be a minimal free resolution of $K[\Delta]$ and $Q_\bullet$ as in Lemma 5.2. Note that $Q_\bullet$ is determined only by $[I_\Delta]_2$ and that it follows $[I_\Delta]_2 = [I_{\Delta^{(1)}}]_2$. If the 1-skeleton $\Delta^{(1)}$ of Δ is an n-gon, then so is Δ itself. Thus by Lemma 5.2, we may assume that $\dim \Delta = 1$. Since $\operatorname{ld}(\Delta) = n-2$, by Lemma 5.2 we have

$$\tilde{H}_1(\Delta; K) \cong \tilde{H}^1(\Delta; K) \cong [\operatorname{Tor}_{n-2}^S(K[\Delta], K)]_{[n]} \neq 0,$$

and hence Δ contains at least one cycle as a subcomplex. So it suffices to show that Δ has no cycles of length $\leq n-1$. Suppose not, i.e., Δ has some cycles of length $\leq n-1$. To give a contradiction, we shall show $0 \to \operatorname{lin}_2(P_\bullet)_{n-2} \to \operatorname{lin}_2(P_\bullet)_{n-3}$ is exact; in fact then it follows $H_{n-2}(\operatorname{lin}_2(P_\bullet)) = 0$, which contradicts to Lemma 5.2. By the above observation, this is equivalent to say that

$$0 \longrightarrow \tilde{H}^1(\Delta; K) \otimes_K S \longrightarrow \bigoplus_{i \in [n]} \tilde{H}^1(\Delta_{-\{i\}}; K) \otimes_K S. \tag{5.2}$$

is exact. Here $-\{i\}$ denotes the subset $[n] \setminus \{i\}$ of $[n]$.

Now by our assumption that Δ contains a cycle of length $\leq n-1$ (that is, Δ itself is not a minimal cycle), and by (5.1), we have the surjective map $\bar{\eta} : \bigoplus_{i \in [n]} \tilde{H}_1(\Delta_{-\{i\}}; K) \to$

$\tilde{H}_1(\Delta; K)$ where $\bar{\eta}$ is induced by the chain map $\eta : \bigoplus \tilde{\mathcal{C}}_\bullet(\Delta_{-\{i\}}; K) \longrightarrow \tilde{\mathcal{C}}_\bullet(\Delta; K)$, and η is the sum of $\eta_i : \tilde{\mathcal{C}}_\bullet(\Delta_{-\{i\}}; K) \ni e_G \mapsto (-1)^{\alpha(i,G)} e_G \in \tilde{\mathcal{C}}_\bullet(\Delta; K)$. Here e_G is the basis element of $\tilde{\mathcal{C}}_\bullet(\Delta_{-\{i\}}; K)$ (and $\tilde{\mathcal{C}}_\bullet(\Delta; K)$) corresponding to $G \in \Delta_{-\{i\}}$. Taking the K-dual of $\bar{\eta}$, we have the injective map $\bar{\eta}^* : \tilde{H}^1(\Delta; K) \longrightarrow \bigoplus_{i\in[n]} \tilde{H}^1(\Delta_{-\{i\}}; K)$, which is composed by the K-dual $\bar{\eta}_i^* : \tilde{H}^1(\Delta; K) \to \tilde{H}^1(\Delta_{-\{i\}}; K)$ of $\bar{\eta}_i$. Then we see that the second morphism in (5.2) is given by

$$z \otimes y \mapsto \sum_{i=1}^{n} \bar{\eta}_i^*(z) \otimes x_i y$$

for $z \in \tilde{H}^1(\Delta; K)$ and $y \in S$, and hence (5.2) is injective. □

Remark 5.3. (1) If Δ is an n-gon, then $\Delta^\vee$ is an $(n-3)$-dimensional Buchsbaum complex with $\tilde{H}_{n-4}(\Delta^\vee; K) = K$. If $n = 5$, then $\Delta^\vee$ is a triangulation of the Möbius band. But, for $n \geq 6$, $\Delta^\vee$ is not a homology manifold. In fact, let $\{1,2\}, \{2,3\}, \cdots, \{n-1,n\}, \{n,1\}$ be the facets of Δ, for $F := [n] \setminus \{1,3,5\}$, easy computation shows that $\mathrm{lk}_{\Delta^\vee} F$ is a 0-dimensional complex with 3 vertices, and hence $\tilde{H}_0(\mathrm{lk}_{\Delta^\vee} F; K) = K^2$.

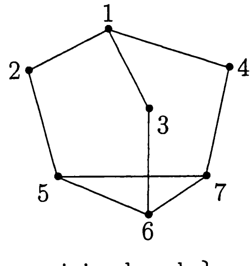
(2) If $\operatorname{indeg} \Delta \geq 3$, then the simplicial complexes given in Example 4.5 are not the only examples which attain the equality $\mathrm{ld}(\Delta) = n - \operatorname{indeg}(\Delta)$. We shall give two examples of such complexes.

Let Δ be the triangulation of the real projective plane $\mathbb{P}^2\mathbb{R}$ with 6 vertices which is given in [1, figure 5.8, p.236]. Since $\mathbb{P}^2\mathbb{R}$ is a manifold, $K[\Delta]$ is Buchsbaum. Hence we have $H^2_{\mathfrak{m}}(K[\Delta]) = [H^2_{\mathfrak{m}}(K[\Delta])]_0 \cong \tilde{H}_1(\Delta; K)$. So, if $\mathrm{char}(K) = 2$, then we have $\operatorname{depth}_S(\operatorname{Ext}^4_S(K[\Delta], \omega_S)) = 0$. Note that we have $\Delta = \Delta^\vee$ in this case. Therefore, easy computation shows that $\mathrm{ld}(\Delta^\vee) = \mathrm{ld}(\Delta) = 3 = 6 - 3 = 6 - \operatorname{indeg}(\Delta)$.

Next let Δ be a triangulation of a cylinder with 6 vertices. A cylinder is a manifold, and so as above we have $H^2_{\mathfrak{m}}(K[\Delta]) = [H^2_{\mathfrak{m}}(K[\Delta])]_0 \cong \tilde{H}_1(\Delta; K) = 1 \neq 0$, since it is homotopic to a circle. Thus we deduce that $\mathrm{ld}(\Delta^\vee) = 3 = 6 - \operatorname{indeg}(\Delta^\vee)$. Here we use the fact that in general $\operatorname{indeg}(\Delta^\vee) = n - \dim \Delta - 1$ holds for a simplicial complex Δ on $[n]$. However $\Delta^\vee$ is not a simplicial complex in Example 4.5; otherwise the dimension of $\Delta^\vee$ must be $\operatorname{indeg}(\Delta^\vee) - 1 = 2$, but $\dim \Delta^\vee = 6 - \operatorname{indeg} \Delta - 1 = 3$.

In our talk, we said that for a 1-dimensional simplicial complex Δ, $\mathrm{ld}(\Delta) = \max\{\sharp F - 2 \mid \Delta_F$ itself is a minimal cycle$\}$ holds for $\mathrm{ld}(\Delta) \geq 2$, but to our regret we found this assertion false. So we correct and apologies. The assertion was "proved" after submission of our paper [9], and so this error make no influence on it. We give a counterexample as follows:

Example 5.4. Let Δ be the simplicial complex as follows:



Then $\max\{\sharp F - 2 \mid \Delta_F$ itself is a minimal cycle$\} = 3$, while we see $\mathrm{ld}(\Delta) = 4$, by computation with the software system Macaulay 2 ([5]).

Since an inequality of one side holds (in more general condition), we shall introduce it.

Proposition 5.5. *For a simplicial complex* Δ, *we have*

$$\operatorname{ld}(\Delta) \geq \max\{\, \sharp F - \dim \Delta_F - 1 \mid F \subset [n], \Delta_F \textit{ is Gorenstein}\}.$$

In particular, if $\dim \Delta = 1$,

$$\operatorname{ld}(\Delta) \geq \max\{\, \sharp F - 2 \mid \Delta_F \textit{ itself is a minimal cycle}\}.$$

Proof. Here we use the same notation as the sketch of the proof of Theorem 5.1. Since $\sharp F - \dim \Delta_F - 1 = \sharp \operatorname{core} F - \dim \Delta_{\operatorname{core} F} - 1$, we may assume that Δ_F is a *Gorenstein* complex* (see [11]) on F. Set $d := \dim \Delta_F + 1$. Then we have $\tilde{H}^{d-1}(\Delta_F; K) \neq 0$, whence $\operatorname{lin}_d(P_\bullet)_{\sharp F - d} \supset \tilde{H}^{d-1}(\Delta_F; K) \otimes_K S \neq 0$. Now take a base z of the free S-module $\tilde{H}^{d-1}(\Delta_F; K) \otimes_K S$. Then by minimality of $P_\bullet$, z is not in $\partial^{\langle d \rangle}(\operatorname{lin}_d(P_\bullet)_{\sharp F - d + 1})$. On the other hand, though $\partial^{\langle d \rangle}(z) \in \bigoplus_{G \subset F, \sharp G = \sharp F - 1} \tilde{H}^{d-1}(\Delta_G; K) \otimes_K S$ holds, we have $\tilde{H}^{d-1}(\Delta_G; K) = 0$ for all such G, for Δ_F is Gorenstein*, and so $\partial^{\langle d \rangle}(z) = 0$. Therefore we conclude $H_{\sharp F - d}(\operatorname{lin}_d(P_\bullet)) \neq 0$. $\square$

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Acyclicity of flat complexes over a noetherian ring

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Dedicated to Professor Masayoshi Nagata on his eightieth birthday

Abstract

Let R be a noetherian commutative ring, and

$$\mathbb{F} : \cdots \to F_2 \to F_1 \to F_0 \to 0$$

a flat R-complex. We prove that if $\kappa(\mathfrak{p}) \otimes_R \mathbb{F}$ is acyclic for any $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathbb{F}$ is acyclic, and $H_0(\mathbb{F})$ is R-flat. It follows that if $\mathbb{F}$ is a (possibly unbounded) complex of flat R-modules and $\kappa(\mathfrak{p}) \otimes_R \mathbb{F}$ is exact for any $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathbb{G} \otimes_R \mathbb{F}$ is exact for any R-complex $\mathbb{G}$. If, moreover, $\mathbb{F}$ is a complex of projective R-modules, then it is null-homotopic (follows from Neeman's theorem).

1. Introduction

Throughout this paper, R denotes a noetherian commutative ring. The symbol $\otimes$ without any subscript means $\otimes_R$. For $\mathfrak{p} \in \operatorname{Spec} R$, let $?(\mathfrak{p})$ denote the functor $\kappa(\mathfrak{p})\otimes ?$, where $\kappa(\mathfrak{p})$ is the field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. An R-complex of the form

$$\mathbb{F} : \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

is said to be acyclic if $H_i(\mathbb{F}) = 0$ for any $i > 0$.

It has been known that, for an R-linear map of R-flat modules $\varphi : F_1 \to F_0$, if $\varphi(\mathfrak{p})$ is injective for any $\mathfrak{p} \in \operatorname{Spec} R$, then φ is injective and $\operatorname{Coker} \varphi$ is R-flat (see [1, Lemma I.2.1.4] and Corollary 6).

In this paper, we prove:

Theorem 1. *Let*

$$\mathbb{F} : \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

be an R-flat complex. If $\mathbb{F}(\mathfrak{p})$ is acyclic for any $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathbb{F}$ is acyclic, and $H_0(\mathbb{F})$ is R-flat. In particular, $M \otimes \mathbb{F}$ is acyclic for any R-module M.

The case of a map $\varphi : F_1 \to F_0$ is a special case of the theorem such that $F_i = 0$ for any $i > 1$.

By the theorem, it follows immediately that if $\mathbb{F}$ is an (unbounded) R-flat complex and $\mathbb{F}(\mathfrak{p})$ is exact for any $\mathfrak{p}$, then $\mathbb{F}$ is K-flat and exact. Combining this and Neeman's result, we can also prove that an (unbounded) projective R-complex $\mathbb{P}$ is null-homotopic if $\mathbb{P}(\mathfrak{p})$ is exact for any $\mathfrak{p} \in \operatorname{Spec} R$.

The author is grateful to H. Brenner for a valuable discussion. Special thanks are also due to A. Neeman for sending his preprint [2] to the author.

2. Acyclicity of flat complexes

We give a proof of Theorem 1.

Proof of Theorem 1. It suffices to prove that $R/I \otimes \mathbb{F}$ is acyclic for any ideal I of R. Indeed, if so, then considering the case that $I = 0$, we have that $\mathbb{F}$ is acyclic so that it is a flat resolution of $H_0(\mathbb{F})$. Since $R/I \otimes \mathbb{F}$ is acyclic for any ideal I, we have that $\operatorname{Tor}_i^R(R/I, H_0(\mathbb{F})) = 0$ for any $i > 0$. Thus $H_0(\mathbb{F})$ is R-flat. So $\operatorname{Tor}_i^R(M, H_0(\mathbb{F})) = 0$ for any $i > 0$, and the last assertion follows.

Assume the contrary, and let I be maximal among the ideals J such that $R/J \otimes \mathbb{F}$ is not acyclic. Then replacing R by R/I and $\mathbb{F}$ by $R/I \otimes \mathbb{F}$, we may assume that $R/I \otimes \mathbb{F}$ is acyclic for any nonzero ideal I of R, but $\mathbb{F}$ itself is not acyclic.

Assume that R is not a domain. There exists some filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = R$$

such that for each i, $M_i/M_{i-1} \cong R/P_i$ for some $P_i \in \operatorname{Spec} R$. Since each P_i is a nonzero ideal, $R/P_i \otimes \mathbb{F}$ is acyclic. So $M_i \otimes \mathbb{F}$ is acyclic for any i. In particular, $\mathbb{F} \cong M_r \otimes \mathbb{F}$ is acyclic, and this is a contradiction. So R must be a domain.

For each $x \in R \setminus 0$, there is an exact sequence

$$0 \longrightarrow \mathbb{F} \xrightarrow{x} \mathbb{F} \longrightarrow R/Rx \otimes \mathbb{F} \longrightarrow 0.$$

Since $R/Rx \otimes \mathbb{F}$ is acyclic, we have that $x : H_i(\mathbb{F}) \longrightarrow H_i(\mathbb{F})$ is an isomorphism for any $i > 0$. In particular, $H_i(\mathbb{F})$ is a K-vector space, where $K = \kappa(0)$ is the field of fractions of R. So

$$H_i(\mathbb{F}) \cong K \otimes H_i(\mathbb{F}) \cong H_i(K \otimes \mathbb{F}) = H_i(\mathbb{F}(0)) = 0 \qquad (i > 0),$$

and this is a contradiction. $\square$

Let A be a ring. A complex $\mathbb{F}$ of left A-modules is said to be *K-flat* if the tensor product $\mathbb{G} \otimes^\bullet_A \mathbb{F}$ is exact for any exact sequence $\mathbb{G}$ of right A-modules, see [3].

For a chain complex

$$\mathbb{H} : \cdots \longrightarrow H_{i+1} \xrightarrow{d_{i+1}} H_i \xrightarrow{d_i} H_{i-1} \longrightarrow \cdots$$

of left or right A-modules, we denote the complex

$$\cdots \longrightarrow H_{i+1} \longrightarrow \operatorname{Ker} d_i \longrightarrow 0$$

by $\tau_{\geq i}\mathbb{H}$ or $\tau^{\leq -i}\mathbb{H}$. Since $\mathbb{G} \cong \varinjlim \tau^{\leq n}\mathbb{G}$, $\mathbb{F}$ is K-flat if and only if $\mathbb{G} \otimes^\bullet_A \mathbb{F}$ is exact for any exact sequence $\mathbb{G}$ of right A-modules bounded above (i.e., $\mathbb{G}_{-i} = \mathbb{G}^i = 0$ for $i \gg 0$). A flat complex $\mathbb{F}$ of left A-modules is K-flat if it is bounded above, as can be seen easily from the spectral sequence argument. A null-homotopic complex $\mathbb{F}$ is K-flat, since $\mathbb{G} \otimes_A \mathbb{F}$ is null-homotopic for any complex $\mathbb{G}$.

Lemma 2. *Let A be a ring, and*

$$\mathbb{F} : \cdots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots$$

a complex of flat left A-modules. Then the following are equivalent.

1 *$M \otimes_A \mathbb{F}$ is exact for any right A-module M.*

2 *$\mathbb{F}$ is exact, and* $\operatorname{Im} d_i$ *is flat for any i.*

3 $\mathbb{F}$ *is K-flat and exact.*

Proof. **1⇒2** Obviously, $\mathbb{F} \cong A \otimes_A \mathbb{F}$ is exact. Thus

$$\mathbb{F}' : \cdots \longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow 0$$

is a flat resolution of $\operatorname{Im} d_i$, where F_{n+i} has the homological degree n in $\mathbb{F}'$. For any $i \in \mathbb{Z}$,

$$\operatorname{Tor}_1^A(M, \operatorname{Im} d_i) \cong H_1(M \otimes_A \mathbb{F}') \cong H_{i+1}(M \otimes_A \mathbb{F}) = 0$$

for any right A-module M. Thus $\operatorname{Im} d_i$ is A-flat.

2⇒1 For any $i \in \mathbb{Z}$,

$$H_{i+1}(M \otimes_A \mathbb{F}) \cong H_1(M \otimes_A \mathbb{F}') \cong \operatorname{Tor}_1^A(M, \operatorname{Im} d_i) = 0,$$

where $\mathbb{F}'$ is as above.

1,2⇒ 3 We prove that $\mathbb{G} \otimes_A \mathbb{F}$ is exact for any bounded above complex $\mathbb{G}$ of right A-modules. Since $\mathbb{F} \cong \varinjlim \tau^{\leq n}\mathbb{F}$ and $\tau^{\leq n}\mathbb{F}$ satisfies **2** (and hence **1**), we may assume that $\mathbb{F}$ is also bounded above. Then by an easy spectral sequence argument, $\mathbb{G} \otimes_A \mathbb{F}$ is exact.

3⇒1 Let $\mathbb{P}$ be a projective resolution of M. Since $\mathbb{P}$ is a bounded above flat complex of left A^{op}-modules and $\mathbb{F}$ is an exact complex of right A^{op}-modules, $\mathbb{P} \otimes_A \mathbb{F}$ is exact. Let $\mathbb{Q}$ be the mapping cone of $\mathbb{P} \to M$. Then $\mathbb{Q} \otimes_A \mathbb{F}$ is also exact, since $\mathbb{Q}$ is exact and $\mathbb{F}$ is K-flat. By the exact sequence of homology groups

$$H_i(\mathbb{P} \otimes_A \mathbb{F}) \longrightarrow H_i(M \otimes_A \mathbb{F}) \longrightarrow H_i(\mathbb{Q} \otimes_A \mathbb{F}),$$

we have that $M \otimes_A \mathbb{F}$ is also exact. □

Corollary 3. *Let*

$$\mathbb{F} : \cdots \longrightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots$$

be a (possibly unbounded) flat complex of R-modules. If $\mathbb{F}(\mathfrak{p})$ is exact for any $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathbb{F}$ is K-flat and exact.

Proof. By Lemma 2, it suffices to show that for any $n \in \mathbb{Z}$ and any R-module M, $H_n(M \otimes \mathbb{F}) = 0$. But this is trivial by Theorem 1 applied to the complex

$$\cdots \longrightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \longrightarrow 0.$$

□

The following was proved by A. Neeman [2, Corollary 6.10].

Theorem 4. *Let A be a ring, and $\mathbb{P}$ a complex of projective left A-modules. If $\mathbb{P}$ is K-flat and exact, then $\mathbb{P}$ is null-homotopic.*

By Corollary 3 and Theorem 4, we have

Corollary 5. *Let $\mathbb{P}$ be a complex of R-projective modules. If $\mathbb{P}(\mathfrak{p})$ is exact for any $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathbb{P}$ is null-homotopic.*

The following also follows.

Corollary 6 (cf. [1, Lemma I.2.1.4]). *Let $\varphi : F_1 \to F_0$ be an R-linear map between R-flat modules. Then the following are equivalent.*

1 *φ is injective and $\operatorname{Coker} \varphi$ is R-flat.*

2 *φ is pure.*

3 *$\varphi(\mathfrak{p})$ is injective for any $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. **1**$\Rightarrow$**2**$\Rightarrow$**3** is obvious. **3**$\Rightarrow$**1** is a special case of Theorem 1. $\square$

Corollary 7 ([1, Corollary I.2.1.6]). *Let F be a flat R-module. If $F(\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \operatorname{Spec} R$, then $F = 0$.*

Proof. Consider the zero map $F \to 0$, and apply Corollary 6. We have that this map is injective, and hence $F = 0$. $\square$

Corollary 8. *Let $\varphi : F_1 \to F_0$ be an R-linear map between R-flat modules. If $\varphi(\mathfrak{p})$ is an isomorphism for any $\mathfrak{p} \in \operatorname{Spec} R$, then φ is an isomorphism.*

Proof. By Corollary 6, φ is injective and $C := \operatorname{Coker} \varphi$ is R-flat. Since $C(\mathfrak{p}) \cong \operatorname{Coker}(\varphi(\mathfrak{p})) = 0$ for any $\mathfrak{p} \in \operatorname{Spec} R$, we have that $C = 0$ by Corollary 7. $\square$

Corollary 9. *Let M be an R-module. If $\operatorname{Tor}_i^R(\kappa(\mathfrak{p}), M) = 0$ for any $i > 0$, then M is R-flat. If $\operatorname{Tor}_i^R(\kappa(\mathfrak{p}), M) = 0$ for any $i \geq 0$, then $M = 0$.*

Proof. For the first assertion, Let $\mathbb{F}$ be a projective resolution of M, and apply Theorem 1. The second assertion follows from the first assertion and Corollary 7. $\square$

Corollary 10. *Let M be an R-module. If $\operatorname{Ext}_R^i(M, \kappa(\mathfrak{p})) = 0$ for any $i > 0$ (resp. $i \geq 0$), then M is R-flat (resp. $M = 0$).*

Proof. This is trivial by Corollary 9 and the fact

$$\operatorname{Ext}_R^i(M, \kappa(\mathfrak{p})) \cong \operatorname{Hom}_{\kappa(\mathfrak{p})}(\operatorname{Tor}_i^R(\kappa(\mathfrak{p}), M), \kappa(\mathfrak{p})). \quad \square$$

3. Some examples

Example 11. There is an acyclic projective complex

$$\mathbb{P} : \cdots \to P_1 \to P_0 \to 0$$

over a noetherian commutative ring R such that $H_0(\mathbb{P})$ is R-flat and $h_0(\mathfrak{p}) := \dim_{\kappa(\mathfrak{p})} H_0(\mathbb{P}(\mathfrak{p}))$ is finite and constant, but $H_0(\mathbb{P})$ is neither R-finite nor R-projective.

Proof. Set $R = \mathbb{Z}$, $M = \sum_p (1/p)\mathbb{Z}$, and $\mathbb{P}$ to be a projective resolution of M. Then M is R-torsion free, and is R-flat. Since $M_{(p)} = (1/p)\mathbb{Z}_{(p)}$, $h_0(\mathfrak{p}) = 1$ for any $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}$. A finitely generated nonzero torsion-free $\mathbb{Z}$-submodule of $\mathbb{Q}$ must be rank-one free, but M is not a cyclic module, and is not rank-one free. This shows that M is not R-finite. As R is a principal ideal domain, any R-projective module is free. If M is projective, then it is free of rank $h_0((0)) = 1$. But M is not finitely generated, so M is not projective. $\square$

Remark 12. Let $(R, \mathfrak{m})$ be a noetherian *local* ring, F a flat R-module, and c a non-negative integer. If $\dim_{\kappa(\mathfrak{p})} F(\mathfrak{p}) = c$ for any $\mathfrak{p} \in \operatorname{Spec} R$, then $F \cong R^c$, see [1, Corollary III.2.1.10].

Remark 13. Let

$$\mathbb{P} : 0 \to P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \xrightarrow{d^2} \cdots$$

be an R-flat complex such that P^0 is R-projective. Assume that $\mathbb{P}(\mathfrak{p})$ is acyclic (i.e., $H^i(\mathbb{P}(\mathfrak{p})) = 0$ for any $i > 0$) and $h^0_{\mathbb{P}}(\mathfrak{p}) := \dim_{\kappa(\mathfrak{p})} H^0(\mathbb{P}(\mathfrak{p}))$ is finite for any $\mathfrak{p} \in \operatorname{Spec} R$. If $h^0_{\mathbb{P}}$ is a locally constant function on $\operatorname{Spec} R$, then $H^0(\mathbb{P})$ is R-finite R-projective, and $M \otimes \mathbb{P}$ is acyclic and the canonical map $M \otimes H^0(\mathbb{P}) \to H^0(M \otimes \mathbb{P})$ is an isomorphism for any R-module M, see [1, Proposition III.2.1.14]. If, moreover, $\mathbb{P}$ is an R-projective complex, then $\operatorname{Im} d^i$ is R-projective for any $i \geq 0$, as can be seen easily from Theorem 4.

Example 14. Let M be an R-module. Even if $M(\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \operatorname{Spec} R$, M may not be zero. Even if $\operatorname{Tor}_1^R(\kappa(\mathfrak{p}), M) = 0$ for any $\mathfrak{p} \in \operatorname{Spec} R$, M may not be R-flat.

Indeed, let $(R, \mathfrak{m}, k)$ be a d-dimensional regular local ring, and E the injective hull of k. Then

$$\operatorname{Tor}_i^R(\kappa(\mathfrak{p}), E) \cong \begin{cases} k & (i = d \text{ and } \mathfrak{p} = \mathfrak{m}) \\ 0 & (\text{otherwise}) \end{cases}$$

E is not R-flat unless $d = 0$.

Proof. Since $\operatorname{supp} E = \{\mathfrak{m}\}$, $\operatorname{Tor}_i^R(\kappa(\mathfrak{p}), E) = 0$ unless $\mathfrak{p} = \mathfrak{m}$.

Let $\boldsymbol{x} = (x_1, \ldots, x_d)$ be a regular system of parameters of R, and $\mathbb{K}$ the Koszul complex $K(\boldsymbol{x}; R)$, which is a minimal free resolution of k. Note that $\mathbb{K}$ is self-dual. That is, $\mathbb{K}^* \cong \mathbb{K}[-d]$, where $\mathbb{K}^* = \operatorname{Hom}_R^\bullet(\mathbb{K}, R)$, and $\mathbb{K}[-d]^n = \mathbb{K}^{n-d}$. So

$$\begin{aligned}\operatorname{Tor}_i^R(k, E) &\cong H^{-i}(\mathbb{K} \otimes E) \cong H^{-i}(\mathbb{K}^{**} \otimes E) \cong H^{-i}(\operatorname{Hom}_R^\bullet(\mathbb{K}[-d], E)) \\ &\cong H^{-i}(\operatorname{Hom}_R^\bullet(k[-d], E)) \cong \begin{cases} k & (i = d) \\ 0 & (i \neq d) \end{cases} .\end{aligned}$$

$\square$

Example 15. There is a projective complex $\mathbb{P}$ over a noetherian commutative ring R such that for each $\mathfrak{m} \in \operatorname{Max}(R)$, $\mathbb{P}(\mathfrak{m})$ is exact, but $\mathbb{P}$ is not exact, where $\operatorname{Max}(R)$ denotes the set of maximal ideals of R.

Proof. Let R be a DVR with its field of fractions K, and $\mathbb{P}$ a projective resolution of K. $\square$

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On squarefree monomial ideals whose projective dimension is close to the number of generators

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This is a joint work with K. Kimura and K. Yoshida in Nagoya University.

§1. Hypergraphs and the main theorem

In [Te2] and [Te3] we gave a classification of almost complete intersection monomial ideals I ($\mu(I) = \mathrm{height} I + 1$) and equidimensional squarefree monomial ideals I with deviation 2 ($\mu(I) = \mathrm{height} I + 2$) to determine their arithmetical rank. In this short article we give a combinatorial characterization for the condition $\mu(I) = \mathrm{projdim} S/I + 1$ for squarefree monomial ideals I using hypergraphs.

By a *hypergraph H* on a vertex set V, we mean H is a family of subsets of V such that

$$\cup_{F \in H} F = V.$$

We define the dimension of F by $\dim F = \sharp(F) - 1$. For a hypergraph H, $V(H)$ stands for its vertex set.

Let I be a squarefree monomial ideal in the polynomial ring $S = k[x_1, x_2, \dots, x_n]$ over a field k. Put $I = (m_1, m_2, \dots, m_\mu)$, where $\{m_1, m_2, \dots, m_\mu\}$ is the minimal monomial generators. We define the hypergraph $H(I)$ on the vertex set $V = [\mu] := \{1, 2, \dots, \mu\}$ by the following way:

$$\begin{aligned} F \in H(I) \Leftrightarrow\ & \text{there exists } i\ (1 \le i \le n) \text{ such that for all } j \in V, \\ & m_j \text{ is divisible by } x_i \text{ if } j \in F \\ & \text{and } m_j \text{ is not divisible by } x_i \text{ if } j \in V \setminus F. \end{aligned}$$

Since $\{m_1, m_2, \dots, m_\mu\}$ is a minimal set of generators, the hypergraph $H = H(I)$ satisfies the following condition:

For all $i, j \in V (i \neq j)$, there exist $F, G \in H$ such that $i \in F \cap (V \setminus G)$ and $j \in G \cap (V \setminus F)$.

Conversely, a hypergraph H with the above condition can be written as $H = H(I)$ for a squarefree monomial ideal I in a polynomial ring with enough variables.

For a hypergraph H on a vertex set V, we define the i-subhypergraph of H by $H^i = \{F \in H; \dim F = i\}$. We can consider that $H^0 \subset V$, and put $B(H) = H^0$ and $W(H) = V \setminus B(H)$.

For a squarefree monomial ideal I, by Taylor resolution we have $\mathrm{bight} I \leq \mathrm{projdim} S/I \leq \mu(I)$, where $\mathrm{bight}\, I = \max\{\mathrm{height} P;\ P \in \mathrm{Min}(S/I)\}$. And the next proposition is easy and well known:

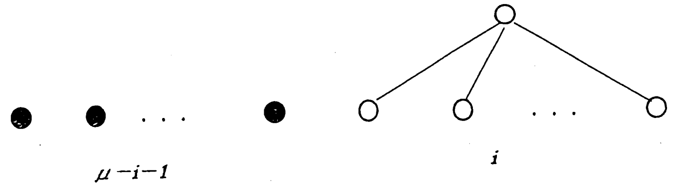
Proposition. *The following conditions are equivalent for a squarefree monomial ideal I:*
(1)$\mathrm{bight}\, I = \mu(I)$.
(2)$\mathrm{projdim}\, S/I = \mu(I)$.
(3)For the hypergraph $H := H(I)$ we have $B(H) = V(H)$.

Now we consider the condition $\mathrm{bight}\, I = \mu(I) - 1$.

A subset C of a hypergraph H on a vertex set V is called a *cover* of H if $\cup_{F \in C} F = V$. A cover C of H is called *minimal* if no proper subset of C is a cover of H.

Proposition. *The following conditions are equivalent for a squarefree monomial ideal I:*
(1)The ideal I has a prime component of height h.
(2)The hypergraph $H(I)$ has a minimal cover of cardinality h.

Corollary. *The following conditions are equivalent for a squarefree monomial ideal I:*
(1)$\mathrm{bight}\, I = \mu(I) - 1$.
(2)The hypergraph $H := H(I)$ satisfies $B(H) \neq V(H)$ and contains the following hypergraph $H_{\mu-i-1,i}$ for some $1 \leq i \leq \mu - 1$ as a spanning subhypergraph:

$H_{\mu-i-1,\ i} =$ 

where a "black vertex" belongs to $B(H)$ and a "white vertex" belongs to $W(H)$.

Next we give a combinatorial characterization for the condition $\mathrm{projdim}\, S/I = \mu(I) - 1$.
For $U \subset V(H)$, we define the restriction of a hypergraph H to U by $H_U = \{F \in H; F \subset U\}$.

Theorem. *The following conditions are equivalent for a squarefree monomial ideal I:*
(1)$\operatorname{projdim} S/I = \mu(I) - 1$.
(2)The hypergraph $H := H(I)$ satisfies $B(H) \neq V(H)$ and either one of the following conditions:
(i)The graph $(W(H), H^1_{W(H)})$ contains a complete bipartite graph as a spanning subgraph.
(ii)There exists $i \in B(H)$ such that $\{i, j\} \in H$ for all $j \in W(H)$.

Corollary. *Let $I = (m_1, m_2, \ldots, m_\mu)$ be a squarefree monomial ideal in S, where $\mu = \mu(I) \geq 3$. Let $3 \leq j \leq \mu$ be an integer. Put $S' = S[x_{n+1}]$, where x_{n+1} is a new indeterminate and let J be the monomial ideal $(m_1x_{n+1}, m_2x_{n+1}, \ldots, m_jx_{n+1}, m_{j+1}, \ldots, m_{\mu+1})$ in S'. Then* $\operatorname{projdim}_S S/I = \mu(I) - 1$ *if and only if* $\operatorname{projdim}_{S'} S'/J = \mu(J) - 1$.

Corollary. *The condition* $\operatorname{projdim} S/I = \mu(I) - 1$ *is independent of the base field k for a monomial ideal I.*

Remark. The condition $\operatorname{projdim} S/I = \mu(I)-2$ depends on the base field k for a monomial ideal I as the following examples shows:

Example. Let I be the monomial ideal defined by

$$I = (x_1x_2x_8x_9x_{10}, x_2x_3x_4x_5x_{10}, x_5x_6x_7x_8x_{10}, x_1x_4x_5x_6x_9, x_1x_2x_3x_6x_7, x_3x_4x_7x_8x_9).$$

Here $H(I)$ is the set of facets of the six-vertex triangulation of $\mathbf{P}^2(\mathbf{R})$. Then $\operatorname{projdim} S/I = 4$ if char $k = 2$, while $\operatorname{projdim} S/I = 3$ if char $k \neq 2$.

§2. Proof of the main theorem

Let $S = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n-variables over a field k. We fix a squarefree monomial ideal $I = (m_1, m_2, \ldots, m_\mu)$, where $\{m_1, m_2, \ldots, m_\mu\}$ is the minimal generating set of monomials for I.

By Lyubeznik[Ly] we have $\operatorname{projdim} S/I = \operatorname{cd} I := \max\{i; H^i_I(S) \neq 0\}$. Assuming that $\operatorname{projdim} S/I \leq \mu(I) - 1$, we have $\operatorname{projdim} S/I = \mu(I) - 1$ if and only if $H^{\mu-1}_I(S) \neq 0$. We give a combinatorial interpretation for the condition $H^{\mu-1}_I(S) \neq 0$. (See [Te1] for the cohomology with monomial ideal support.)

Consider the following Čech complex;

$$
\begin{aligned}
C^\bullet &= \bigotimes_{i=1}^{\mu}(0 \longrightarrow S \longrightarrow S_{m_i} \longrightarrow 0) \\
&= 0 \longrightarrow S \xrightarrow{\delta^1} \bigoplus_{1\le i\le \mu} S_{m_i} \xrightarrow{\delta^2} \bigoplus_{1\le i<j\le\mu} A_{m_i m_j} \xrightarrow{\delta^3} \cdots \xrightarrow{\delta^\mu} S_{m_1 m_2 \cdots m_\mu} \longrightarrow 0.
\end{aligned}
$$

We describe δ^{r+1} as follows; Put $R := S_{m_{i_1} m_{i_2} \cdots m_{i_r}}$ and $\{j_1, j_2, \ldots, j_s\} = \{1, 2, \cdots, \mu\} \setminus \{i_1, i_2, \ldots, i_r\}$, where $j_1 < j_2 < \ldots < \ldots j_s$ and $r + s = \mu$. Let $\psi_{j_p} : R \longrightarrow R_{m_{j_p}}$ be a natural map. For $u \in R$, we have

$$
\delta^{r+1}(u) = \sum_{p=1}^{s} (-1)^{|\{q; i_q < j_p\}|} \psi_{j_p}(u) = \sum_{p=1}^{s} (-1)^{j_p - p} \psi_{j_p}(u).
$$

For $F \in \mathbf{2}^{[n]}$, we define $x^F := \prod_{i\in F} x_i$. We define a simplicial complex $\Delta(F)$ on the vertex set $[\mu]$ by

$$
\Delta(F) = \{\{i_1, i_2, \ldots, i_r\};\ x^F \mid \prod_{j\in\{1,2,\cdots\mu\}\setminus\{i_1,i_2,\ldots,i_r\}} m_j\}.
$$

For $\alpha \in \mathbf{Z}^n$, there is a unique decomposition $\alpha = \alpha_+ - \alpha_-$ such that $\alpha_+, \alpha_- \in \mathbf{N}^n$ and $\operatorname{supp}\alpha_+ \cap \operatorname{supp}\alpha_- = \emptyset$. Then we have $\operatorname{supp}\alpha_- = \{i; \alpha_i < 0\}$.

Lemma(cf. [St, Lemma 2.5]). *For $\alpha \in \mathbf{Z}^n$ give an orientation for $\Delta(\operatorname{supp}\alpha_-)$ by $1 < 2 < \cdots < \mu$. Then we have the following isomorphism of complexes: $C^\bullet_\alpha \cong \tilde{C}_\bullet(\Delta(\operatorname{supp}\alpha_-))$ such that $C^r_\alpha \cong \tilde{C}_{\mu-r-1}(\Delta(\operatorname{supp}\alpha_-))$.*

Now we have the following isomorphisms:

$$
\begin{aligned}
H_I^{\mu-1}(S)_\alpha &= H^{\mu-1}(C^\bullet)_\alpha \\
&\cong \tilde{H}_0(\Delta(\operatorname{supp}\alpha_-); k) \\
&\cong \tilde{H}_0(\Delta(\operatorname{supp}\alpha_-)^{(1)}; k),
\end{aligned}
$$

where $\Delta(\operatorname{supp}\alpha_-)^{(1)} := \{F \in \Delta(\operatorname{supp}\alpha_-); \dim F \le 1\}$ is the 1-skeleton of $\Delta(\operatorname{supp}\alpha_-)$. Hence $H_I^{\mu-1}(S)_\alpha = 0$ if and only if $\Delta(\operatorname{supp}\alpha_-)^{(1)}$ is connected.

We claim that $\Delta(\operatorname{supp}\alpha_-)^{(1)}$ is connected for all $\alpha \in \mathbf{Z}^n$ if and only if the graph $(U, \binom{U}{2} \setminus H_U^1)$ is connected for all $W(H) \subset U \subset V(H)$, where $\binom{U}{2} := \{\{i, j\} \subset U;\ i \ne j\}$. Let U be the vertex set of $\Delta(\operatorname{supp}\alpha_-)^{(1)}$ for $\alpha \in \mathbf{Z}^n$. Then $U \supset W(H)$ and $\binom{U}{2} \setminus H_U^1 \subset \Delta(\operatorname{supp}\alpha_-)^{(1)}$. Hence if $(U, \binom{U}{2} \setminus H_U^1)$ is connected, then so is $\Delta(\operatorname{supp}\alpha_-)^{(1)}$. On the other hand, fix U

such that $W(H) \subset U \subset V(H)$. Put $U = W(H) \cup B'$, where $B' \subset B(H)$. By a suitable change of variables, we may assume that $B' = \{1, 2, \ldots, p\}$. For $1 \leq j \leq p$, set $\{x_i;\ x_i | m_j, x_i \nmid m_\ell \text{ for } \ell \neq j\} = \{x_{i_{j1}}, \ldots, x_{i_{js_j}}\}$. Take $\alpha \in \mathbf{Z}^n$ such that $\operatorname{supp} \alpha_- = [n] \setminus \{i_{11}, \ldots, i_{1s_1}, i_{21}, \ldots, i_{2s_2}, \ldots, i_{p1}, \ldots, i_{ps_p}\}$. Then we have $(U, \binom{U}{2} \setminus H_U^1) = \Delta(\operatorname{supp} \alpha_-)^{(1)}$. Hence we have the claim.

The graph $(U, \binom{U}{2} \setminus H_U^1)$ is connected for all U such that $W(H) \subset U \subset V(H)$ if and only if the following conditions (I) and (II) are satisfied:

(I)The graph $(W(H), \binom{W(H)}{2} \setminus H_{W(H)}^1)$ is connected.

(II)For $i \in B(H)$ set $U_i = W(H) \cup \{i\}$. The graph $(U_i, \binom{U_i}{2} \setminus H_{U_i}^1)$ is connected for all $i \in B(H)$.

Hence the condition $\operatorname{projdim} S/I = \mu(I) - 1$ holds if and only if one of the following conditions (i)' or (ii)' is satisfied:

(i)'The graph $(W(H), \binom{W(H)}{2} \setminus H_{W(H)}^1)$ is disconnected.

(ii)'The graph $(U_i, \binom{U_i}{2} \setminus H_{U_i}^1)$ is disconnected for some $i \in B(H)$.

The condition (i)' ((ii)', respectively) is equivalent to the condition (i) ((ii), respectively). Hence we are done. QED

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AB modules and weakly AB rings

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1 AB-rings

Let (R, m, k) be a commutative Noetherian local ring. We denote by mod$R$ the category of finitely generated R-modules.

Definition 1.1 (1) For non-zero R-modules M and N, we define $\mathrm{P}_R(M, N)$ as follows:

$$\mathrm{P}_R(M, N) = \sup\{ n \mid \mathrm{Ext}_R^n(M, N) \neq 0 \}$$

(2) We say R an *AB-ring* if the following condition holds:

$$\sup\{ \mathrm{P}_R(M, N) \mid \mathrm{P}_R(M, N) < \infty \ (M, N \in \mathrm{mod}R) \} < \infty$$

An AB-ring was introduced by C.Huneke and D.A.Jorgensen in [2]. They consider the following question.

Question 1.2 Are Gorenstein rings AB-rings?

The answer is No. D.A.Jorgensen and L.M.Şega showed that there exist Gorenstein rings which are not AB-rings. On the other hand, we can see that there exist AB-rings which are not Gorenstein rings.

Propositon 1.3 *Let R be a Cohen-Macaulay ring with a minimal multiplicity. Suppose $\mathrm{P}_R(M, N) < \infty$, then we have $\mathrm{pd}\, M < \infty$ or $\mathrm{id}\, N < \infty$. In particular, we have $\mathrm{P}_R(M, N) = \mathrm{depth}\, R - \mathrm{depth}\, M$ and R is an AB-ring.*

PROOF. Since projective dimension of M is finite if and only if projective dimension of ΩM is finite, we may assume that M is a maximal Cohen-Macaulay module. Let $0 \to Y \to X \to N \to 0$ be a Cohen-Macaulay approximation of N. (Here, Cohen-Macaulay approximation of N is an exact sequence such that injective dimension of N

is finite and X is a maximal Cohen-Macaulay module.) We can easily see that injective dimension of N is finite if and only if injective dimension of X is finite. Thus we may also assume that N is a maximal Cohen-Macaulay module. Let $\underline{x}$ be a maximal M-, N-, and R-regular sequence. Take $- \otimes R/(\underline{x})$, we may assume that R is artinian and $m^2 = 0$. We suppose that $\operatorname{pd} M = \infty$. In this case, we can check that $\mathrm{P}_R(\Omega M, N) < \infty$. On the other hand, since $m^2 = 0$, we have ΩM is a k-vector space. Thus we have $\mathrm{P}_R(k, N) < \infty$ and this implies $\operatorname{id} N < \infty$. □

2 Main results

In this section, we introduce the definition of AB-modules and weakly AB-rings, and give some properties about them. We consider the full subcategory $\mathcal{A}$ of mod$R$ as follows:

$$\mathcal{A} = \{\ M \in \operatorname{mod}R \mid \exists N \in \operatorname{mod}R \ s.t.\ \mathrm{P}_R(M, N) < \infty\}$$

Lemma 2.1 *The following conditions are equivalent.*

(1) $\mathcal{A} = \operatorname{mod}R$

(2) *R is Cohen-Macaulay.*

PROOF. We can check that the following implications hold:

$$\begin{aligned} \mathcal{A} = \operatorname{mod}R &\Rightarrow k \in \mathcal{A} \\ &\Rightarrow \exists N \in \operatorname{mod}R \ s.t.\ \mathrm{P}_R(k, N) < \infty \\ &\Rightarrow \exists N \in \operatorname{mod}R \ s.t.\ \operatorname{id} N < \infty \\ &\Rightarrow \exists N \in \operatorname{mod}R \ s.t.\ \mathrm{P}_R(M, N) < \infty\ (\forall M \in \operatorname{mod}R) \\ &\Rightarrow \mathcal{A} = \operatorname{mod}R \end{aligned}$$

On the other hand, It is known that R admits a finitely generated module of finite injective dimension if and only if R is Cohen-Macaulay (c.f. [4]). Thus we get $\mathcal{A} = \operatorname{mod}R$ if and only if R is Cohen-Macaulay. □

Definition 2.2 (1) For $M \in \mathcal{A}$,

(i) We define $\mathrm{P}_R(M)$ as follows:

$$\mathrm{P}_R(M) = \sup\{\ \mathrm{P}_R(M, N) \mid N \in \operatorname{mod}R \text{ with } \mathrm{P}_R(M, N) < \infty\}$$

(ii) We say M an *AB-module* if $\mathrm{P}_R(M) < \infty$.

(2) We say R a *weakly AB-ring* if every module in $\mathcal{A}$ is an AB-module.

Now we give a main theorem in this lecture.

Theorem 2.3 *Let M and N be R-modules. We assume that M is an AB-module and* $\operatorname{G-dim} M < \infty$. *Suppose* $\mathrm{P}_R(M,N) < \infty$, *then we have* $\mathrm{P}_R(M,N) = \operatorname{depth} R - \operatorname{depth} M$. *In particular, we have* $\mathrm{P}_R(M) = \operatorname{depth} R - \operatorname{depth} M$.

PROOF. **Case 1.** We assume $\operatorname{G-dim} M (= \operatorname{depth} R - \operatorname{depth} M) = 0$. In this case, note that $\mathrm{P}_R(M,R) = 0$ by the definition of G-dimension 0. To show $\mathrm{P}_R(M,N) = 0$, we assume $0 < \mathrm{P}_R(M,N) =: p < \infty$. We take the first syzygy of $N : 0 \to \Omega N \to F \to N \to 0$. Take $\operatorname{Hom}(M,-)$, we get a long exact sequence

$$\begin{array}{ccccccc} \cdots & \to & \operatorname{Ext}_R^p(M,\Omega N) & \to & \operatorname{Ext}_R^p(M,F) & \to & \operatorname{Ext}_R^p(M,N) & \to \\ & \to & \operatorname{Ext}_R^{p+1}(M,\Omega N) & \to & \operatorname{Ext}_R^{p+1}(M,F) & \to & \operatorname{Ext}_R^{p+1}(M,N) & \to \\ & \to & \operatorname{Ext}_R^{p+2}(M,\Omega N) & \to & \operatorname{Ext}_R^{p+2}(M,F) & \to & \operatorname{Ext}_R^{p+2}(M,N) & \to & \cdots . \end{array}$$

It comes from $\mathrm{P}_R(M,F) = 0$ and $\mathrm{P}_R(M,N) = p > 0$, we have $\mathrm{P}_R(M,\Omega N) = p+1$. By repeating this operation, we have $\mathrm{P}_R(M,\Omega^n N) = p+n$ $(\forall n \geq 0)$. But this contradicts that M is an AB-module. Therefore we have $\mathrm{P}_R(M,N) = 0$.

Case 2. We assume $\operatorname{G-dim} M (= \operatorname{depth} R - \operatorname{depth} M) =: p > 0$. We take a finite projective hull of $M : 0 \to M \to Y \to X \to 0$ (here, finite projective hull is an exact sequence such that $\operatorname{pd} Y < \infty$ and X is a maximal Cohen-Macaulay module). Since $\operatorname{depth} R > \operatorname{depth} M$ and X is maximal Cohen-Macaulay, we have $\operatorname{depth} Y = \operatorname{depth} M$. Thus we have $\operatorname{pd} Y = \operatorname{depth} R - \operatorname{depth} Y = \operatorname{depth} R - \operatorname{depth} M = p$ and $\mathrm{P}_R(Y,N) = p > 0$. It comes from case 1, we have $\mathrm{P}_R(X,N) = 0$. Take $\operatorname{Hom}(-,N)$ to a finite projective hull of M, we get a long exact sequence

$$\begin{array}{ccccccc} \cdots & \to & \operatorname{Ext}_R^p(X,N) & \to & \operatorname{Ext}_R^p(Y,N) & \to & \operatorname{Ext}_R^p(M,N) & \to \\ & \to & \operatorname{Ext}_R^{p+1}(X,N) & \to & \operatorname{Ext}_R^{p+1}(Y,N) & \to & \operatorname{Ext}_R^{p+1}(M,N) & \to & \cdots . \end{array}$$

Therefore we have $\mathrm{P}_R(M,N) = p = \operatorname{depth} R - \operatorname{depth} M$. □

It comes from Theorem 2.3, we can get following corollary.

Corollary 2.4 *Suppose R be Gorenstein, then the following conditions are equivalent.*

(1) *R is an AB-ring.*

(2) *R is a weakly AB-ring.*

PROOF. (1) $\Rightarrow$ (2): We can check that the following implications hold:

$$\begin{array}{rcl} R \text{ is an AB-ring.} & \Rightarrow & \sup\{ \mathrm{P}_R(M) \mid M \in \mathcal{A} \} < \infty \\ & \Rightarrow & \mathrm{P}_R(M) < \infty \ (\forall M \in \mathcal{A}) \\ & \Rightarrow & R \text{ is a weakly AB-ring.} \end{array}$$

(2) $\Rightarrow$ (1): Since R is Gorenstein, we have $\mathcal{A} = \mathrm{mod}R$ by Lemma 2.1 and every module has finite G-dimension. For any pair M, $N \in \mathrm{mod}R$ with $\mathrm{P}_R(M,N) < \infty$, we have $\mathrm{P}_R(M,N) = \operatorname{depth} R - \operatorname{depth} M \leq \operatorname{depth} R$ by Theorem 2.3. Therefore R is an AB-ring. $\square$

We can see that the following implications hold: "R is a Gorenstein AB ring." $\overset{(1)}{\Rightarrow}$ "$\mathcal{A} = \mathrm{mod}R$ and $\mathrm{P}_R(M) = \operatorname{depth} R - \operatorname{depth} M \quad (\forall M \in \mathrm{mod}R)$" $\overset{(2)}{\Rightarrow}$ "R is a Cohen-Macaulay AB ring." In general, the opposite of (1) is not hold (c.f. Proposition 1.3). We do not know if the opposite of (2) is hold. But I conjecture that the opposite of (2) is hold.

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Module categories and ring spectra

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Throughout this note, let R be a commutative Noetherian ring. We denote by $\operatorname{Mod} R$ the category of R-modules, and by $\operatorname{mod} R$ the full subcategory of $\operatorname{Mod} R$ consisting of finitely generated R-modules. A perfect R-complex is defined as a finite complex of finitely generated projective R-modules. Let $\mathcal{D}(R)$ denote the derived category of $\operatorname{Mod} R$, and $\mathcal{D}_{\mathrm{perf}}(R)$ the full subcategory of $\mathcal{D}(R)$ consisting of R-complexes isomorphic to perfect R-complexes. A thick subcategory of a triangulated category is defined as a triangulated full subcategory which is closed under direct summands. For an R-complex X, we denote by $H(X)$ the homology module of X, that is, $H(X) = \bigoplus_{i \in \mathbb{Z}} H_i(X)$. Around 1990, Hopkins [1] and Neeman [3] proved the following classification theorem.

Theorem 1 (Hopkins-Neeman). *One has a one-to-one correspondence*

$$\left\{ \begin{array}{l} \textit{thick subcategories} \\ \textit{of } \mathcal{D}_{\mathrm{perf}}(R) \end{array} \right\} \underset{g_1}{\overset{f_1}{\rightleftarrows}} \left\{ \begin{array}{l} \textit{subsets of } \operatorname{Spec} R \\ \textit{closed under specialization} \end{array} \right\}$$

which are given by $f_1(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} \operatorname{Supp} H(X)$ *and* $g_1(S) = \{ X \in \mathcal{D}_{\mathrm{perf}}(R) \mid \operatorname{Supp} H(X) \subseteq S \}$.

A Serre subcategory of $\operatorname{mod} R$ is defined as a full subcategory which is closed under submodules, quotient modules and extensions. In other words, a Serre subcategory is defined to be a full subcategory $\mathcal{M}$ of $\operatorname{mod} R$ such that for any exact sequence $0 \to A \to B \to C \to 0$ in $\operatorname{mod} R$, $B \in \mathcal{M}$ if and only if $A, C \in \mathcal{M}$. A coherent subcategory of $\operatorname{mod} R$ is defined as a full subcategory which is closed under kernels, cokernels and extensions. In other words, a coherent subcategory is defined to be a full subcategory $\mathcal{M}$ of $\operatorname{mod} R$ such that for any exact sequence $A \to B \to C \to D \to E$ in $\operatorname{mod} R$, if $A, B, D, E \in \mathcal{M}$ then $C \in \mathcal{M}$. Note that every Serre subcategory is coherent. The following classification of Serre subcategories is well-known to experts.

Proposition 2. *One has a one-to-one correspondence*

$$\left\{\begin{array}{l}\textit{Serre subcategories}\\ \textit{of } \operatorname{mod} R\end{array}\right\} \underset{g_2}{\overset{f_2}{\rightleftarrows}} \left\{\begin{array}{l}\textit{subsets of } \operatorname{Spec} R\\ \textit{closed under specialization}\end{array}\right\}$$

which are given by $f_2(\mathcal{M}) = \bigcup_{M\in\mathcal{M}} \operatorname{Supp} M$ *and* $g_2(S) = \{ M \in \operatorname{mod} R \mid \operatorname{Supp} M \subseteq S \}$.

This proposition especially says that one has the following relationships between two modules whose supports have inclusion relation.

Corollary 3. *Let* M, N *be finitely generated* R*-modules with* $\operatorname{Supp} M \subseteq \operatorname{Supp} N$*. Then* M *is in the Serre subcategory of* $\operatorname{mod} R$ *generated by* N*, i.e. the smallest Serre subcategory containing* N*.*

Taking advantage of the Hopkins-Neeman theorem and the above proposition, Hovey [2] proved the following.

Theorem 4 (Hovey)**.** *Let* R *be a quotient of a regular Noetherian ring. Then the following hold.*

(1) *One has a one-to-one correspondence*

$$\left\{\begin{array}{l}\textit{coherent subcategories}\\ \textit{of } \operatorname{mod} R\end{array}\right\} \underset{g_3}{\overset{f_3}{\rightleftarrows}} \left\{\begin{array}{l}\textit{thick subcategories}\\ \textit{of } \mathcal{D}_{\mathrm{perf}}(R)\end{array}\right\}$$

which are given by $f_3(\mathcal{M}) = \{ X \in \mathcal{D}_{\mathrm{perf}}(R) \mid H(X) \in \mathcal{M} \}$ *and* $g_3(\mathcal{X}) =$ *(the coherent subcategory generated by* $\{ H(X) \mid X \in \mathcal{X} \}$*).*

(2) *Every coherent subcategory of* $\operatorname{mod} R$ *is a Serre subcategory.*

The first main result of this note is the following, which says that the assumption on R in Hovey's theorem can be removed.

Theorem 5. *Let* R *be a commutative Noetherian ring. Then every coherent subcategory of* $\operatorname{mod} R$ *is a Serre subcategory. Consequently, one has the following commutative diagram of bijections:*

$$\begin{tikzcd}
\left\{\begin{array}{l}\textit{thick subcategories}\\ \textit{of } \mathcal{D}_{\mathrm{perf}}(R)\end{array}\right\} \arrow[r, shift left, "f_1"] \arrow[d, shift left, "g_3"] \arrow[dr, phantom, "\square"] & \left\{\begin{array}{l}\textit{subsets of } \operatorname{Spec} R\\ \textit{closed under specialization}\end{array}\right\} \arrow[l, shift left, "g_1"] \arrow[d, shift left, "g_2"] \\
\left\{\begin{array}{l}\textit{coherent subcategories}\\ \textit{of } \operatorname{mod} R\end{array}\right\} \arrow[u, shift left, "f_3"] \arrow[r, equal] & \left\{\begin{array}{l}\textit{Serre subcategories}\\ \textit{of } \operatorname{mod} R\end{array}\right\} \arrow[u, shift left, "f_2"]
\end{tikzcd}$$

Proof. Let $\mathcal{M}$ be a coherent subcategory of $\operatorname{mod} R$. We want to show that $\mathcal{M}$ is a Serre subcategory. For this, it is enough to check that $\mathcal{M}$ is closed under submodules. Assume that $\mathcal{M}$ is not closed under submodules. Then there exist an R-module M in $\mathcal{M}$ and an R-submodule N of M such that N does not belong to $\mathcal{M}$. Since R is Noetherian and M is a finitely generated R-module, M is a Noetherian R-module. Hence we can choose N to be a maximal element, with respect to inclusion relation, of the set of R-submodules N' of M such that N' does not belong to $\mathcal{M}$. Since N does not coincide with M, there is an element $x \in M - N$. Set $L = N + Rx$. Note that L is an R-submodule of M strictly containing N. By the maximality of N, the module L is in $\mathcal{M}$. Put $I = \{ a \in R \,|\, ax \in N \}$. This is an ideal of R, and we easily see that the quotient R-module L/N is isomorphic to R/I. There is an exact sequence

$$0 \to N \to L \xrightarrow{\pi} R/I \to 0$$

of R-modules. Since $N \notin \mathcal{M}$ and $L \in \mathcal{M}$ and $\mathcal{M}$ is closed under kernels, we see from this exact sequence that R/I must not be in $\mathcal{M}$.

On the other hand, the map π in the exact sequence induces a surjective homomorphism

$$\overline{\pi} : L/IL \to R/I$$

of R/I-modules, which sends the residue class of $y \in L$ in L/IL to $\pi(y)$. Of course R/I is a projective R/I-module, so $\overline{\pi}$ is a split epimorphism. Therefore R/I is isomorphic to a direct summand of L/IL. The Noetherian property of R implies that the ideal I is finitely generated; write $I = (a_1, a_2, \dots, a_n)R$ for some elements $a_1, a_2, \dots, a_n \in R$. There is an exact sequence

$$R^{\oplus n} \xrightarrow{(a_1,\dots,a_n)} R \longrightarrow R/I \longrightarrow 0$$

of R-modules. Tensoring the R-module L with this exact sequence yields another exact sequence of R-modules:

$$L^{\oplus n} \xrightarrow{(a_1,\dots,a_n)} L \longrightarrow L/IL \longrightarrow 0.$$

Note that $\mathcal{M}$ is closed under finite direct sums, cokernels, and direct summands. Hence the direct sum $L^{\oplus n}$ belongs to $\mathcal{M}$, and so does the module L/IL, and therfore so does R/I. This is a contradiction, which says that $\mathcal{M}$ is closed under submodules. Thus the proof of the theorem is completed. $\square$

A localizing subcategory of $\mathcal{D}(R)$ is defined as a triangulated full subcategory which is closed under arbitrary direct sums, and a smashing subcategory of $\mathcal{D}(R)$ is defined as a localizing subcategory such that Bousfield localization commutes with arbitrary direct sums. Neeman [3] showed the following theorem.

Theorem 6 (Neeman). *One has two one-to-one correspondences*

$$\begin{tikzcd}
\left\{\begin{array}{l}\textit{localizing subcategories}\\ \textit{of } \mathcal{D}(R)\end{array}\right\} \arrow[r, shift left, "f_4"] & \arrow[l, shift left, "g_4"] \left\{\textit{subsets of } \operatorname{Spec} R\right\} \\
\left\{\begin{array}{l}\textit{smashing subcategories}\\ \textit{of } \mathcal{D}_{\mathrm{perf}}(R)\end{array}\right\} \arrow[u, "\subseteq"] \arrow[r, shift left, "f_5"] \arrow[ur, phantom, "\square"] & \arrow[l, shift left, "g_5"] \left\{\begin{array}{l}\textit{subsets of } \operatorname{Spec} R\\ \textit{closed under specialization}\end{array}\right\} \arrow[u, "\subseteq"']
\end{tikzcd}$$

where f_4, g_4 *are given by* $f_4(\mathcal{X}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R X \not\cong 0 \text{ for some } X \in \mathcal{X}\}$, $g_4(S) =$ *(the localizing subcategory generated by* $\{\kappa(\mathfrak{p}) \mid \mathfrak{p} \in S\}$*), and* f_5, g_5 *are the restrictions of* f_4, g_4 *respectively.*

The second main result of this note is the following, which is a module version of Neeman's theorem.

Theorem 7. *Let R be a commutative Noetherian ring. Then there is a commutative diagram:*

$$\begin{tikzcd}
\left\{\begin{array}{l}\textit{full subcategories of } \operatorname{mod} R\\ \textit{closed under submodules}\\ \textit{and extensions}\end{array}\right\} \arrow[r, shift left, "f_6"] & \arrow[l, shift left, "g_6"] \left\{\textit{subsets of } \operatorname{Spec} R\right\} \\
\left\{\begin{array}{l}\textit{Serre subcategories}\\ \textit{of } \operatorname{mod} R\end{array}\right\} \arrow[u, "\subseteq"] \arrow[r, shift left, "f_2"] \arrow[ur, phantom, "\square"] & \arrow[l, shift left, "g_2"] \left\{\begin{array}{l}\textit{subsets of } \operatorname{Spec} R\\ \textit{closed under specialization}\end{array}\right\} \arrow[u, "\subseteq"']
\end{tikzcd}$$

The maps f_6 and g_6 make a one-to-one correspondence, which are given by $f_6(\mathcal{M}) = \bigcup_{M \in \mathcal{M}} \operatorname{Ass} M$ *and* $g_6(S) = \{M \in \operatorname{mod} R \mid \operatorname{Ass} M \subseteq S\}$*. The maps f_6 and g_6 induce f_2 and g_2, respectively.*

To prove this theorem, we prepare two lemmas.

Lemma 8. *Let $\mathcal{M}$ be a full subcategory of* $\operatorname{mod} R$ *which is closed under submodules and extensions, and let M be a finitely generated R-module. Suppose that M has a unique associated prime $\mathfrak{p}$. If $R/\mathfrak{p}$ is in $\mathcal{M}$, then so is M.*

Proof. Assume that M is not in $\mathcal{M}$. Set $M_0 = M$, and let $h_{0,1}, \dots, h_{0,s_0}$ be a system of generators of the R-module $\mathrm{Hom}_R(M_0, R/\mathfrak{p})$. There is an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_0 \xrightarrow{\begin{pmatrix} h_{0,1} \\ \vdots \\ h_{0,s_0} \end{pmatrix}} (R/\mathfrak{p})^{\oplus s_0}$$

of R-modules. Since $M_0 = M$ is not in $\mathcal{M}$ and $(R/\mathfrak{p})^{\oplus s_0}$ is in $\mathcal{M}$ and $\mathcal{M}$ is closed under submodules and extensions, it is easily seen that M_1 must not be in $\mathcal{M}$. In particular, $M_1 \neq 0$ and hence $\mathfrak{p}$ is the unique associated prime of M_1. Letting $h_{1,1}, \dots, h_{1,s_1}$ be a system of generators of the R-module $\mathrm{Hom}_R(M_1, R/\mathfrak{p})$, we have an exact sequence

$$0 \longrightarrow M_2 \longrightarrow M_1 \xrightarrow{\begin{pmatrix} h_{1,1} \\ \vdots \\ h_{1,s_1} \end{pmatrix}} (R/\mathfrak{p})^{\oplus s_1}.$$

Since M_1 is not in $\mathcal{M}$ and $(R/\mathfrak{p})^{\oplus s_1}$ is in $\mathcal{M}$, we see that M_2 is not in $\mathcal{M}$, and that $\mathfrak{p}$ is the unique associated prime of M_2. Iterating this procedure, for each integer $i \geq 0$ we obtain an exact sequence

$$0 \longrightarrow M_{i+1} \longrightarrow M_i \xrightarrow{\begin{pmatrix} h_{i,1} \\ \vdots \\ h_{i,s_i} \end{pmatrix}} (R/\mathfrak{p})^{\oplus s_i},$$

where $h_{i,1}, \dots, h_{i,s_i}$ is a system of generators of the R-module $\mathrm{Hom}_R(M_i, R/\mathfrak{p})$ and $\mathfrak{p}$ is the unique associated prime of M_i. Localizing the descending chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ at $\mathfrak{p}$ yields a descending chain

$$M_\mathfrak{p} = (M_0)_\mathfrak{p} \supseteq (M_1)_\mathfrak{p} \supseteq (M_2)_\mathfrak{p} \supseteq \cdots$$

of $R_\mathfrak{p}$-modules. Since the $R_\mathfrak{p}$-module $(M_i)_\mathfrak{p}$ has finite length for every i, there exists an integer t such that $(M_t)_\mathfrak{p} = (M_{t+1})_\mathfrak{p} = (M_{t+2})_\mathfrak{p} = \cdots$. The exact sequence

$$0 \longrightarrow (M_{t+1})_\mathfrak{p} \xrightarrow{=} (M_t)_\mathfrak{p} \xrightarrow{\begin{pmatrix} (h_{t,1})_\mathfrak{p} \\ \vdots \\ (h_{t,s_t})_\mathfrak{p} \end{pmatrix}} \kappa(\mathfrak{p})^{\oplus s_t},$$

shows that $\mathrm{Hom}_{R_\mathfrak{p}}((M_t)_\mathfrak{p}, \kappa(\mathfrak{p})) = R_\mathfrak{p}(h_{t,1})_\mathfrak{p} + \cdots + R_\mathfrak{p}(h_{t,s_t})_\mathfrak{p} = 0$. Therefore $(M_t)_\mathfrak{p} = 0$. This is a contradiction since $\mathfrak{p} \in \mathrm{Ass}\, M_t \subseteq \mathrm{Supp}\, M_t$. Thus we conclude that M is in $\mathcal{M}$. $\square$

Lemma 9. *Let $\mathcal{M}$ be a full subcategory of* $\operatorname{mod} R$ *which is closed under submodules and extensions. Let M be a finitely generated R-module. Suppose that $R/\mathfrak{p}$ belongs to $\mathcal{M}$ for every $\mathfrak{p} \in \operatorname{Ass} M$. Then M also belongs to $\mathcal{M}$.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the associated primes of M, and let

$$0 = N_1 \cap \cdots \cap N_s$$

be an irredundant primary decomposition of the zero submodule 0 of M, where N_i is a $\mathfrak{p}_i$-primary submodule of M for $1 \leq i \leq s$. Then the natural homomorphism

$$M = M/N_1 \cap \cdots \cap N_s \to M/N_1 \oplus \cdots \oplus M/N_s$$

is injective. Since $\mathfrak{p}_i$ is the unique associated prime of the R-module M/N_i, Lemma 8 implies that M/N_i belongs to $\mathcal{M}$ for $1 \leq i \leq s$. Hence $M/N_1 \oplus \cdots \oplus M/N_s$ belongs to $\mathcal{M}$, and so does M. $\square$

Now we can prove our theorem.

Proof of Theorem 7. Let S be a subset of $\operatorname{Spec} R$. The set $f_6 g_6(S)$ is the union of $\operatorname{Ass} M$ where M runs through finitely generated R-modules all of whose associated primes are in S. It is trivial that this set is contained in S. For a prime ideal $\mathfrak{p} \in S$, we have $\operatorname{Ass}_R R/\mathfrak{p} = \{\mathfrak{p}\} \subseteq S$. Hence $\mathfrak{p}$ belongs to $f_6 g_6(S)$, and therefore $f_6 g_6(S) = S$. Let $\mathcal{M}$ be a subcategory of $\operatorname{mod} R$ which is closed under submodules and extensions. We have that $g_6 f_6(\mathcal{M})$ is the subcategory of $\operatorname{mod} R$ consisting of all finitely generated R-modules N with $\operatorname{Ass} N \subseteq \bigcup_{M \in \mathcal{M}} \operatorname{Ass} M$, and it is obvious that $g_6 f_6(\mathcal{M})$ contains $\mathcal{M}$. Let N be a finitely generated R-module with $\operatorname{Ass} N \subseteq \bigcup_{M \in \mathcal{M}} \operatorname{Ass} M$. Fix a prime ideal $\mathfrak{p} \in \operatorname{Ass} N$. Then there exists an R-module $M \in \mathcal{M}$ with $\mathfrak{p} \in \operatorname{Ass} M$. There is an injective homomorphism $R/\mathfrak{p} \to M$, and $R/\mathfrak{p}$ belongs to $\mathcal{M}$ since $\mathcal{M}$ is closed under submodules. It follows from Lemma 9 that N is in $\mathcal{M}$. Hence $g_6 f_6(\mathcal{M}) = \mathcal{M}$. Thus we conclude that f_6 is a bijection whose inverse map is g_6.

On the other hand, let S be a subset of $\operatorname{Spec} R$ which is closed under specialization. Let M be a finitely generated R-module such that $\operatorname{Ass} M$ is contained in S, and take $\mathfrak{p} \in \operatorname{Supp} M$. Then there is a prime ideal $\mathfrak{q} \in \operatorname{Min} M \subseteq \operatorname{Ass} M$ that is contained in $\mathfrak{p}$. Since $\mathfrak{q}$ is in S and S is closed under specialization, $\mathfrak{p}$ is also in S. Thus $g_6(S) = \{ M \in \operatorname{mod} R \mid \operatorname{Ass} M \subseteq S \}$ coincides with $g_2(S) = \{ M \in \operatorname{mod} R \mid \operatorname{Supp} M \subseteq S \}$. Let $\mathcal{M}$ be a Serre subcategory of $\operatorname{mod} R$. Let $N \in \mathcal{M}$

and $\mathfrak{p} \in \operatorname{Supp} N$. Choose a prime ideal $\mathfrak{q} \in \operatorname{Min} N$ which is contained in $\mathfrak{p}$. Then $\mathfrak{q}$ is an associated prime of N, so there is an injective homomorphism $R/\mathfrak{q} \to N$. Since $\mathcal{M}$ is closed under submodules, the module $R/\mathfrak{q}$ is in $\mathcal{M}$. Noting that there is a surjective homomorphism $R/\mathfrak{q} \to R/\mathfrak{p}$ and that $\mathcal{M}$ is closed under quotient modules, $R/\mathfrak{p}$ is also in $\mathcal{M}$. Hence we get $\mathfrak{p} \in \operatorname{Ass} R/\mathfrak{p} \subseteq \bigcup_{M \in \mathcal{M}} \operatorname{Ass} M$. Therefore the set $f_2(\mathcal{M}) = \bigcup_{M \in \mathcal{M}} \operatorname{Supp} M$ is contained in $f_6(\mathcal{M}) = \bigcup_{M \in \mathcal{M}} \operatorname{Ass} M$, and we see that $f_6(\mathcal{M}) = f_2(\mathcal{M})$. It follows that f_2 and g_2 are induced from f_6 and g_6, respectively. □

Here, let us check that an analogous result to Corollary 3 holds. This actually follows from Lemma 9.

Corollary 10. *Let M and N be finitely generated R-modules with $\operatorname{Ass} M \subseteq \operatorname{Ass} N$. Then M is in the full subcategory of* $\operatorname{mod} R$ *closed under submodules and extensions which is generated by N.*

Proof. Let $\mathcal{E}$ be the full subcategory of $\operatorname{mod} R$ closed under submodules and extensions which is generated by N. According to Lemma 9, we have only to show that the R-module $R/\mathfrak{p}$ is in $\mathcal{E}$ for every $\mathfrak{p} \in \operatorname{Ass} M$. Let $\mathfrak{p}$ be a prime ideal in $\operatorname{Ass} M$. The assumption says that $\mathfrak{p}$ is in $\operatorname{Ass} N$. Hence there exists an injective homomorphism $R/\mathfrak{p} \to N$ of R-modules. Since N is in $\mathcal{E}$ and $\mathcal{E}$ is closed under submodules, $R/\mathfrak{p}$ is also in $\mathcal{E}$, as required. □

In the following example, we will give several correspondences between full subcategories of $\operatorname{mod} R$ which are closed under submodules and extensions and subsets of $\operatorname{Spec} R$, which are made by the isomorphisms f_6 and g_6. Before that, we need to prepare some notation. Let I be an ideal of R, and let M, N be R-modules. We denote by $\Gamma_I(M)$ the *I-torsion submodule* of M, namely, the set of elements of M which are annihilated by some power of I. Recall that M is called *I-torsion* if $\Gamma_I(M) = M$, and that M is called *I-torsionfree* if $\Gamma_I(M) = 0$. It is well-known and easy to see that M is I-torsion if and only if $\operatorname{Ass} M \subseteq V(I)$, and that M is I-torsionfree if and only if $\operatorname{Ass} M \cap V(I) = \emptyset$. We set $\operatorname{grade}(N, M) = \inf\{\, i \mid \operatorname{Ext}_R^i(N, M) \neq 0 \,\}$, $\operatorname{grade}(I, M) = \operatorname{grade}(R/I, M)$, $\operatorname{grade} I = \operatorname{grade}(I, R)$ and $\operatorname{grade} M = \operatorname{grade}(\operatorname{Ann} M, R)$.

Example 11. The bijections f_6 and g_6 make the following correspondences. Let n be a nonnegative integer, I an ideal of R and X a finitely generated R-module.

(1) $\{ M \in \operatorname{mod} R \mid M \text{ is } I\text{-torsion} \} \leftrightarrow V(I)$.
(2) $\{ M \in \operatorname{mod} R \mid \operatorname{grade}(X, M) > 0 \} \leftrightarrow \operatorname{Spec} R \setminus \operatorname{Supp} X$.
(3) $\{ M \in \operatorname{mod} R \mid M \text{ is } I\text{-torsionfree} \}$
$= \{ M \in \operatorname{mod} R \mid \operatorname{grade}(I, M) > 0 \} \leftrightarrow D(I)$.
(4) $\{ M \in \operatorname{mod} R \mid \operatorname{grade}(M, X) \geq n \} \leftrightarrow \{ \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{grade}(\mathfrak{p}, X) \geq n \}$.
(5) $\{ M \in \operatorname{mod} R \mid \operatorname{rank} M = 0 \} = \{ M \in \operatorname{mod} R \mid \operatorname{grade} M > 0 \}$
$\leftrightarrow \{ \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{grade} \mathfrak{p} > 0 \}$.
(6) $\{ M \in \operatorname{mod} R \mid \text{every } X\text{-regular element is } M\text{-regular} \}$
$\leftrightarrow \{ \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{grade}(\mathfrak{p}, X) = 0 \}$.
(7) $\{ M \in \operatorname{mod} R \mid M \text{ is torsionfree} \} \leftrightarrow \{ \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{grade} \mathfrak{p} = 0 \}$.
(8) $\{ M \in \operatorname{mod} R \mid \operatorname{ht} \operatorname{Ann} M \geq n \} \leftrightarrow \{ \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{ht} \mathfrak{p} \geq n \}$.
(9) $\{ M \in \operatorname{mod} R \mid \dim M \leq n \} \leftrightarrow \{ \mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} \leq n \}$.
(10) $\{ M \in \operatorname{mod} R \mid \ell(M) < \infty \} \leftrightarrow \operatorname{Max} R$.

Note that in the correspondences (1), (4), (5), (8), (9) and (10) in the above example, the left-hand subcategories of $\operatorname{mod} R$ are Serre subcategories and the right-hand subsets of $\operatorname{Spec} R$ are closed under specializations, hence those correspondences are in fact obtained by the bijections f_2 and g_2.

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On the universal family of deformations of modules

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Let k be an algebraically closed field and let R be an associative k-algebra. Fixing a left R-module M, we assume that $\mathrm{Ext}^1_R(M, M)$ is of finite dimension over k. Under such circumstances, it is known by an ancient theorem of Schlessinger that there exist the universal family of deformations of M and its parameter space. To be more precise, let $\mathcal{C}$ be the category of commutative artinian local k-algebra A with $A/\mathfrak{m}_A = k$ and k-algebra maps as morphisms. Now we consider a covariant functor $F : \mathcal{C} \to (Sets)$ defined as

$$\begin{aligned} F(A) := \ & \{\text{the isomorphism class of } (R, A)\text{-bimodule } X \\ & \text{which is flat as right } A\text{-module and } X \otimes_A k \cong M \text{ as left } R\text{-module}\}, \end{aligned}$$

for each $A \in \mathcal{C}$. Then the pro-representability theorem of Schlessinger says that there exists a noetherian complete local k-algebra Q with $Q/\mathfrak{m}_Q = k$ with the isomorphism of functors

$$F \cong \mathrm{Hom}_{k\text{-alg}}(Q, \quad).$$

This isomorphism is realized by the universal (or miniversal) family U which is an (R, Q)-bimodule and Q-flat. More precisely, for $A \in \mathcal{C}$, the isomorphism

$$\mathrm{Hom}_{k\text{-alg}}(Q, A) \xrightarrow{\phi(A)} F(A)$$

is given by $\phi(A)(f) = U \otimes_Q {}_fA$ where ${}_fA$ denotes the Q-algebra A through the k-algebra map $f : Q \to A$.

We call Q the parameter space of universal family of deformations of M. From the view point of representation theory, it seems to be important to know the properties of the parameter spaces. It is known by the obstruction theory that the embedding dimension of Q equals $\dim_k \mathrm{Ext}^1_R(M, M)$ and that the obstruction for Q being nonsingular lies in $\mathrm{Ext}^2_R(M, M)$.

EXAMPLE 0.1 (DEFORMATIONS OF JORDAN CANONICAL FORMS) Let $R = k[x]$ be a polynomial ring with one variable and let M be an R-module $R/(x^n)$ for some integer n. In this case, we have that $Q = k[[t_0, \dots, t_{n-1}]]$ and $U = Q[x]/(x^n + t_{n-1}x^{n-1} + \cdots + t_0)$.

EXAMPLE 0.2 Let R be a complete local k-algebra such that $R/\mathfrak{m}_R \cong k$ and let M be an R-module $R/\mathfrak{m}_R$. In this case, we have that $Q = R$ and $U = R$ (as an (R, R)-bimodule). Hence, every complete local ring can be a parameter space of a module.

Besides the obstruction theory, we consider it more ring-theoretically. The idea is to consider the functor between the derived categories

$$U \otimes_Q - \ : \ D(Q) \to D(R),$$

which induces the morphism between Yoneda algebras

$$\varphi^{\cdot} : \ \mathrm{Ext}^{\cdot}_Q(k, k) \to \mathrm{Ext}^{\cdot}_R(M, M).$$

$\varphi^0 : \ k \to \mathrm{End}_R(M)$ is always an injection. On the other hand, $\varphi^1 : \ \mathrm{Ext}^1_Q(k, k) \to \mathrm{Ext}^1_R(M, M)$ is an isomorphism by the construction.

Of the greatest interest is the map

$$\varphi^2 : \ \mathrm{Ext}^2_Q(k, k) \to \mathrm{Ext}^2_R(M, M).$$

Our main result is the following inequality.

Theorem 0.3

$$\dim_k \ \mathrm{Ker}(\varphi^2) \leqq \binom{\dim_k \mathrm{Ext}^1_R(M, M)}{2}.$$

This theorem contains all the results obtained by the obstruction theory. In fact, let us denote Q by S/I, where $S = k[[t_1, \dots, t_r]]$ $(r = \dim_k \mathrm{Ext}^1_R(M, M))$ and I is an ideal of S minimally generated by $\{f_1, \dots, f_\ell\}$ contained in $(t_1, \dots, t_r)^2 S$. Write

$$f_i = \sum_{j=1}^{r} a_{ij} t_j \quad (a_{ij} \in (t_1, \dots, t_r)S).$$

Considering the second syzygy of the Q-module k ;

$$0 \longrightarrow \Omega^2_Q k \longrightarrow \bigoplus_{i=1}^{r} Qe_i \longrightarrow Q \longrightarrow k \longrightarrow 0,$$

it is well-known that $\Omega^2_Q k$ is generated by the set

$$\{v_i = \sum_{j=1}^{r} a_{ij} e_j \mid i = 1, 2, \dots, \ell\} \cup \{t_i e_j - t_j e_i \mid 1 \le i < j \le r\}.$$

Therefore, $\mathrm{Ext}^2_Q(k, k)$ is generated as a k-vector space by the set

$$\{v_i^* \mid i = 1, 2, \dots, \ell\} \cup \{(t_i e_j - t_j e_i)^* \mid 1 \le i < j \le r\}.$$

Note that $\{v_i^* \mid i = 1, 2, \dots, \ell\}$ gives a k-base of $(I/\mathfrak{m}_S I)^*$. Hence we have the following result as a corollary of the theorem.

Corollary 0.4

$$\ell \le \dim_k \mathrm{Ext}^2_R(M, M).$$

It follows that $\mathrm{Ext}^2_R(M,M)=0$ implies the regularity of Q.

As to the proof of the theorem, we need enlarging the base category $\mathcal{C}$ to the category of associative (not necessarily commutative) local artinian k-algebras. Instead of writing $Q=S/I$, we need to describe Q as the residue ring of the noncommutative complete local ring $T=k\ll t_1,\dots,t_r\gg$ with the Jacobson radical $J=(t_1,\dots,t_r)T$;

$$Q=T/\overline{T\{f_1,\dots,f_\ell,[t_i,t_j]\ (1\le i<j\le r)\}T},$$

where $\overline{}$ means the closure in J-adic topology.

A nontrivial (noncommutative) small extension P of Q is a (not necessarily commutative) local k-algebra with a socle element $\sigma\in J_P^2$ such that the two-sided ideal generated by σ is one-dimensional as a k-vector space and that $Q=P/(\sigma)$. Let $\mathfrak{a}=\overline{T\{f_1,\dots,f_\ell,[t_i,t_j]\ (1\le i<j\le r)\}T}$. Then it is easy to see that the set of nontrivial small extensions bijectively corresponds to the set of subspaces of codimension one in $\mathfrak{a}/J_T\mathfrak{a}+\mathfrak{a}J_T$ that is spanned by $\{\overline{f_1},\dots,\overline{f_\ell},\overline{[t_i,t_j]}\ (1\le i<j\le r)\}$. Identifying $\overline{f_i}$ with v_i^* and $\overline{[t_i,t_j]}$ with $(t_ie_j-t_je_i)^*$, we can regard the set of pairs (P,σ) of nontrivial small extensions of Q and its socle element as $\mathrm{Ext}^2_Q(k,k)\setminus\{0\}$.

[Outline of the proof of the theorem.]

Let $\tau\neq 0\in\mathrm{Ext}^2_Q(k,k)$ be a nontrivial element. Take a corresponding nontrivial small extension P of Q. Suppose $\varphi^2(\tau)=0$ in $\mathrm{Ext}^2_R(M,M)$. Then we can show that there is an (R,P)-bimodule V which is flat over P and $V\otimes_P Q\cong U$ as left R-module. It then follows from the universality of U that P must be a noncommutative ring. Let π be the projection from $\mathrm{Ext}^2_Q(k,k)$ onto the subspace generated by $\{(t_ie_j-t_je_i)^*\mid 1\le i<j\le r\}$, and we have that $\pi(\tau)\neq 0$ because of the noncommutativity of P. This shows that the restriction of π to $\mathrm{Ker}(\varphi^2)$ is an injection, and thus $\dim_k\mathrm{Ker}(\varphi^2)\le\binom{r}{2}$. $\square$

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THE CORE OF AN IDEAL

CLAUDIA POLINI

To Professor Shiro Goto on the occasion of his sixtieth birthday

1. Introduction

This is a report on joint work with Alberto Corso, and Bernd Ulrich on the core of an ideal that has appeared in [1, 2, 11].

Let I be an ideal in a Noetherian local ring $(R,\mathfrak{m})$. Among the closure operations on I, the integral closure plays a central role. An ideal I is said to be *integral* over an ideal $J \subset I$ if the inclusion of Rees algebras $R[Jt] \hookrightarrow R[It]$ is module finite. The *integral closure* $\overline{I}$ of I is then defined to be the largest ideal integral over I and the ideal I is *integrally closed* if $\overline{I} = I$. A *reduction* of I is a subideal of I with the same integral closure as I, i.e. a subideal over which I is integral. One can think of reductions as simplifications of the ideal, which carry most of the information about I itself but, in general, with fewer generators. This notion, introduced by Northcott and Rees [10], has played a crucial role in the study of Rees algebras. *Minimal reductions*, reductions minimal with respect to inclusion, are the counterpart of the integral closure. However, unlike the integral closure, minimal reductions are not unique. For this reason one considers their intersection, called the *core* of I and denoted $\operatorname{core}(I)$. The core was introduced by Rees and Sally in the eighties to study mixed multiplicities and Briançon-Skoda type theorems. Indeed this object appears naturally in the context of the Briançon–Skoda theorem [8]. As shown first by Huneke and Swanson the core is related to adjoint and multiplier ideals [3, 7]. Furthermore a better understanding of the core could lead to a solution of Kawamata's conjecture on the non-vanishing of sections of certain line bundles [5, 6].

In this report we will describe algebraic properties of the core (see [1] for more details), and we will give explicit closed formulas (see [2, 11] for more details).

2. Algebraic properties of the core

Let $(R,\mathfrak{m})$ be a local Cohen-Macaulay ring with infinite residue field and $I = (f_1, \ldots, f_n)$ an R-ideal with grade g. The core of I is difficult to compute since a priori it is the intersection of an

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infinite number of ideals. Thus we first address the following issues: Is the core a *finite* intersection of minimal reductions of I? Can these minimal reductions taken to be *general*?

Recall that if the residue field of the ring R is infinite all minimal reductions have the same number of generators, namely the *analytic spread* $\ell = \ell(I)$ of I. By a *general* minimal reduction we mean an ideal generated by ℓ general elements in I. We give an affirmative answer to the above questions for the class of *weakly* $(\ell-1)$*-residually* S_2 ideals satisfying G_ℓ.

This class is fairly large: it includes $\mathfrak{m}$-primary ideals and more generally equimultiple ideals (ideals with $\ell = g$) as trivial cases, one-dimensional generic complete intersections and more generally generic complete intersection Cohen-Macaulay ideals with $\ell \leq g+1$, and strongly Cohen-Macaulay ideals satisfying G_ℓ. Examples of strongly Cohen-Macaulay ideals are quite common in the literature: Cohen-Macaulay almost complete intersections, Cohen-Macaulay ideals of Gorenstein rings that can be generated by $g+2$ elements, perfect ideals of codimension two as well as perfect Gorenstein ideals of codimension three are all strongly Cohen-Macaulay ideals. In general licci ideals (i.e. ideals in the linkage class of a complete intersection) are automatically strongly Cohen-Macaulay.

Theorem 2.1. *Let R be a local Cohen–Macaulay ring with infinite residue field and I an R-ideal of analytic spread ℓ. Assume that I is G_ℓ and weakly $(\ell-1)$-residually S_2, then* $\mathrm{core}(I)$ *is a finite intersection of general minimal reductions of I.*

The difficult part to prove is the fact that $\mathrm{core}(I)$ can be obtained by intersecting general minimal reductions of I. To show this we compare the multiplicities of modules defined by intersecting reduction ideals, generic ideals, and general ideals, respectively. In particular, we prove that under suitable assumptions, the multiplicity of I/J is independent of the choice of a minimal reduction J of I.

Notice that Theorem 2.1 gives an algorithm to compute the core of an ideal because if $J_1, \ldots J_{t+1}$ are general minimal reductions with $J_1 \cap \ldots \cap J_{t+1} = J_1 \cap \ldots \cap J_t$ then $\mathrm{core}(I) = J_1 \cap \ldots \cap J_t$.

Writing the core as intersection of a finite number of general minimal reductions allows us to prove that the core behaves well under flat extensions provided the map is local.

Theorem 2.2. *Let $R \hookrightarrow R'$ be a flat local extension of local rings with infinite residue fields. Assume R' is Cohen–Macaulay. Let I be an R-ideal of analytic spread ℓ such that IR' is G_ℓ and weakly $(\ell-1)$-residually S_2. Then* $\mathrm{core}(IR') = (\mathrm{core}(I))R'$.

Let x_{ij} be $\ell \times n$ variables and let S be the localized polynomial ring $R(\{x_{ij}\})$, $1 \leq i \leq \ell$, $1 \leq j \leq m$. The S-ideal $\mathcal{A}$ generated by the ℓ generic linear combinations $\sum x_{ij} f_j$ is called a *universal ℓ–generated ideal* in IS. Rees and Sally [12] have shown that if I is $\mathfrak{m}$-primary, then $\mathcal{A} \cap R \subset \mathrm{core(I)}$. We can prove that this containment is actually an equality and that the equality holds for the much broader class of universally weakly $(\ell-1)$-residually S_2 ideals satisfying G_ℓ.

Theorem 2.3. *Let R be a local Cohen–Macaulay ring with infinite residue field and I an R-ideal of analytic spread ℓ. Assume that I is G_ℓ and universally weakly $(\ell-1)$-residually S_2. Let $\mathcal{A}$ be a universal ℓ-generated ideal in IS. Then* $\operatorname{core}(I) = \mathcal{A} \cap R$.

Theorem 2.3 expresses the core in terms of a single minimal reduction (of the ideal IS)!! From the above equality we are able to write the core as a colon ideal in a polynomial ring over R, which provides a method for computing the core of a broad class of ideals generated by homogeneous elements not necessarily of the same degree. Furthermore, Theorem 2.3 leads to proving in general that the core behaves well under flat extensions.

Theorem 2.4. *Let $R \longrightarrow R'$ be a flat map of local Cohen–Macaulay rings with infinite residue fields. Let I be an R-ideal of analytic spread ℓ such that I and IR' are G_ℓ and universally weakly $(\ell-1)$-residually S_2. Then* $\operatorname{core}(IR') = (\operatorname{core}(I))R'$.

3. Formulas for the core

Huneke and Swanson [3] provided the first work in the literature with a 'closed formula' for the core of an ideal. More precisely, they showed that the core of integrally closed ideals in two dimensional regular local rings is still integrally closed and it is given by a formula that involves an ideal of minors of any presentation matrix of the ideal. They also relate the core to the adjoint of an ideal, denoted $\operatorname{adj}(I)$, introduced by Lipman in [7].

Theorem 3.1. [Huneke and Swanson, [3]] *Let R be a two dimensional regular local ring and I be an integrally closed R-ideal. Then* $\operatorname{core}(I) = I \cdot \operatorname{Fitt}_2(I) = I \cdot \operatorname{adj}(I) = \operatorname{adj}(I^2)$. *In particular,* $\operatorname{core}(I)$ *is integrally closed.*

Our goal is to generalize this results to a broader class of ideals and rings. To arrive at our more general formula we observe that residual intersections are the correct objects to replace the Fitting ideals occurring in Theorem 3.1. More precisely, in any Cohen-Macaulay ring we describe the core of ideals I that are *balanced* and in any Gorenstein ring we describe the core of ideals that have the *expected reduction number* $\leq \ell - g + 1$.

An ideal I is called *balanced* if $J : I$ does not depend on the minimal reduction J of I. The notion of balancedness was introduced in [13] as a tool for understanding reduction numbers. The reduction number is a key invariant in the study of blowup algebras and it measures how closely the ideal and its reductions are related. It is defined as $r(I) = \min\{r_J(I) \,|\, J \text{ a minimal reduction of } I\}$, where $r_J(I)$ is the least integer r such that $I^{r+1} = JI^r$.

For any universally weakly $(\ell-1)$-residually S_2 ideal satisfying G_ℓ, we prove that if I is balanced then $\operatorname{core}(I) = (J:I)J = (J:I)I$ with J any minimal reduction of I. Not only the converse holds as well, in fact balancedness is implied by the containment $(J:I)I \subset \operatorname{core}(I)$ for some minimal reduction J of I.

Theorem 3.2. *Let R be a local Cohen–Macaulay ring with infinite residue field. Let I be an R-ideal of height g and analytic spread ℓ. Suppose that I satisfies G_ℓ and is universally weakly $(\ell-1)$-residually S_2. Then the following conditions are equivalent:*

(*a*) $(J\colon I)I \subset \operatorname{core}(I)$ *for some minimal reduction J of I;*
(*b*) $(J\colon I)J = (J\colon I)I = \operatorname{core}(I)$ *for every minimal reduction J of I;*
(*c*) $J\colon I$ *does not depend on the minimal reduction J of I.*

If in addition R is Gorenstein and depth $R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$, we can relate the core to the the reduction number. Indeed, in [13] Ulrich proved that balancedness is equivalent to I having the *expected reduction number* $r(I) \leq \ell - g + 1$.

Theorem 3.3. *Let R be a local Gorenstein ring with infinite residue field. Let I be an R-ideal of height g and analytic spread ℓ. Suppose that I satisfies G_ℓ and depth $R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$. Then the following conditions are equivalent:*

(*a*) $(J\colon I)I \subset \operatorname{core}(I)$ *for some minimal reduction J of I;*
(*b*) $(J\colon I)J = (J\colon I)I = \operatorname{core}(I)$ *for every minimal reduction J of I;*
(*c*) $J\colon I$ *does not depend on the minimal reduction J of I;*
(*d*) *the reduction number of I is at most $\ell - g + 1$.*

In the case of perfect ideals of height two or perfect Gorenstein ideals of height three one can compute the core explicitly from a matrix presenting I.

Corollary 3.4. *Let R be a local Gorenstein ring with infinite residue field. Let I be a perfect R-ideal of height two and analytic spread ℓ, satisfying G_ℓ. Let φ be a matrix with n rows presenting I. Then the following conditions are equivalent:*

(*a*) *the reduction number of I is at most $\ell - 1$;*
(*b*) $\operatorname{core}(I) = I_{n-\ell}(\varphi) \cdot I$.

If any of these conditions hold and if R and I are normal, then $I_{n-\ell}(\varphi)$ and $\operatorname{core}(I)$ are integrally closed.

In the assumption of Theorem 3.1, the ideal I has reduction number at most one by [9] and is normal according to [14]. Thus Corollary 3.4 recovers Theorem 3.1.

Corollary 3.5. *Let R be a local Gorenstein ring with infinite residue field. Let I be a perfect Gorenstein R-ideal of height three, analytic spread ℓ and minimal number of generators n, satisfying G_ℓ. Let φ be a matrix with n rows presenting I. Then the following conditions are equivalent:*

(*a*) *the reduction number of I is $\ell - 2$;*
(*b*) $\operatorname{core}(I) = I_1(\varphi) \cdot I$.

If any of these conditions hold and if R and I are normal, then $I_1(\varphi)$ and $\operatorname{core}(I)$ are integrally closed.

4. THE CONJECTURE

We are now facing the following issue: How does the core look like if the ideal I is not balanced? In [2] we conjectured the formula below for the core of ideals of arbitrary reduction number.

Conjecture 4.1. *Let R be a local Cohen–Macaulay ring with infinite residue field. Let I be an R-ideal of analytic spread ℓ that satisfies G_ℓ and is weakly $(\ell-1)$-residually S_2. Let J be a minimal reduction of I and let r denote the reduction number of I with respect to J. Then*

$$\mathrm{core}(I) = J^{r+1} : I^r.$$

Hyry and Smith in [5] verified the above conjecture for equimultiple ideals in a Cohen-Macaulay ring of characteristic zero having Cohen-Macaulay Rees algebras. The role of the characteristic zero assumption on the residue field was a big surprise to us.

Subsequently (at the same time as in [11]), Huneke and Trung verified the conjecture without assuming the Cohen-Macaulayness of the Rees algebra [4]. In [11] we establish the conjecture for a broader class of ideals, which includes equimultiple ideals as first special case.

Theorem 4.2. *Let R be a local Gorenstein ring with infinite residue field k. Let I be an R-ideal of height $g > 0$ and analytic spread ℓ, and let J be a minimal reduction of I with $r = r_J(I)$. Assume I satisfies G_ℓ, $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$, and either $\operatorname{char} k = 0$ or $\operatorname{char} k > r - \ell + g$. Then*

$$\mathrm{core}(I) = J^{n+1} : I^n$$

for every $n \geq \max\{r - \ell + g, 0\}$.

To prove the Theorem 4.2 we identify the core with a graded component of a canonical module of the extended Rees algebra $R[It, t^{-1}] \subset R[t, t^{-1}]$ of I.

Theorem 4.3. *Let R be a local Gorenstein ring with infinite residue field k. Let I be an R-ideal of height $g > 0$ and analytic spread ℓ. Assume I satisfies G_ℓ, $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$. Then $[\omega_{R[It,t^{-1}]}]_g \subset \mathrm{core}(I)$. If in addition, $\operatorname{char} k = 0$ or $\operatorname{char} k > r - \ell + g$ then $\mathrm{core}(I) = [\omega_{R[It,t^{-1}]}]_g$.*

Theorem 4.3 is very useful in determining instances when the core is the adjoint or multiplier ideal of a power of I, or more generally, when the core is integrally closed. Indeed, Huneke and Swanson showed in [3] that the core of integrally closed ideals in a two dimensional regular local ring is $\mathrm{adj}(I^2)$, hence integrally closed. Unfortunately, this is false in general. However viewing the core as a component of a canonical module of $R[It, t^{-1}]$, we show that if $\mathrm{Proj}\,(R[It])$ satisfies Serre's condition R_1, then $\mathrm{core}(I)$ is integrally closed. We also identify the core with the adjoint of I^g provided R is a regular local ring essentially of finite type over a field of characteristic zero and $\mathrm{Proj}\,(R[It])$ has only rational singularities.

Corollary 4.4. *If in addition to the assumptions of Theorem 4.2, $\mathrm{Proj}\,(R[It])$ satisfies Serre's condition R_1, then $\mathrm{core}(I)$ is integrally closed.*

Corollary 4.5. *If in addition to the assumptions of Theorem 4.2, R is a regular local ring essentially of finite type over a field of characteristic zero and* $\operatorname{Proj}(R[It])$ *has only rational singularities, then* $\operatorname{core}(I) = \operatorname{adj}(I^g)$.

Without the assumption on the characteristic of the residue field in Theorem 4.2, we can still show that:

Theorem 4.6. *Let R be a local Gorenstein ring with infinite residue field, let I be an R-ideal with $g = \operatorname{ht} I$ and $\ell = \ell(I)$, and let J be a minimal reduction of I. Assume I satisfies G_ℓ and* $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ *for $1 \leq j \leq \ell - g$. Then*

$$J^{n+1} : I^n \subset \operatorname{core}(I) \subset J^{n+1} : \sum_{y \in I} (J, y)^n$$

for $n \geq \max\{r - \ell + g, 0\}$. In particular if $\mu(I) \leq \ell + 1$ then

$$\operatorname{core}(I) = J^{n+1} : I^n.$$

The second inclusion follows by lifting the formula for the core of any ideal of analytic spread one and positive height in a Cohen-Macaulay local ring:

Theorem 4.7. *Let R be a local Cohen-Macaulay ring with infinite residue field k, let I be an R-ideal with $\ell(I) = \operatorname{ht} I = 1$ and $r = r(I)$, and let J be a minimal reduction of I. Let $(y_1), \ldots, (y_t)$ be minimal reductions of I so that* $\operatorname{core}(I) = (y_1) \cap \cdots \cap (y_t)$ *and write* $s = \max\{r((J, y_i)) \mid 1 \leq i \leq t\}$.

(*a*)
$$\begin{aligned} \operatorname{core}(I) &= J^{n+1} : \sum_{y \in I} (J, y)^n = J(J^n : \sum_{y \in I} (J, y)^n) \\ &= J^{n+1} : \sum_{i=1}^{t} (J, y_i)^n = J(J^n : \sum_{i=1}^{t} (J, y_i)^n) \end{aligned}$$

for every $n \geq s$.

(*b*) *If* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > r$, *then*

$$\operatorname{core}(I) = J^{n+1} : I^n = J(J^n : I^n)$$

for every $n \geq r$.

We end this report by showing the failure of the formula of Theorem 4.2 if any of our assumptions is dropped.

Example 4.8. Let k be an infinite field of characteristic $p > 0$, let $q > p$ be an integer not divisible by p, consider the numerical semigroup ring $R = k[\![t^{p^2}, t^{pq}, t^{pq+q}]\!] \subset k[\![t]\!]$, and let $I = \mathfrak{m}$ be the maximal ideal of R. Now R is a one-dimensional local Gorenstein domain, and one has the *proper* containment $\operatorname{core}(I) \supsetneq J^{n+1} : I^n$ for any minimal reduction J of I and any $n \geq r(I)$. In fact $\operatorname{core}(I) = (t^{p^3}, \mathfrak{m}^{2p-1})$, whereas $J^{n+1} : I^n = \mathfrak{m}^{2p-1}$.

In the next example we show that the G_ℓ condition cannot be removed from Theorem 4.2.

Example 4.9. Let k be an infinite field, write $R = k[\![X,Y,Z,W]\!]/(X^2+Y^2+Z^2,ZW)$, let x,y,z,w denote the images of X,Y,Z,W in R, and consider the R-ideal $I=(x,y,z)$. Notice that R is a local Gorenstein ring, $\operatorname{ht} I = 1$, $\ell(I)=2$, R/I is Cohen-Macaulay, but I does not satisfy G_2. Let $J=(x,y)$. The ideal J is a minimal reduction of I with $r_J(I)=1$. One has $\operatorname{core}(I) = I^2 \subsetneq J^2 \colon I$. The same holds if one replaces J by a general minimal reduction of I.

Indeed, the special fiber ring $\operatorname{gr}_I(R)\otimes_R k$ is defined by a single quadric. Hence $\ell(I)=2$, and $r_K(I)=1$ for every minimal reduction K of I, which gives $I^2\subset \operatorname{core}(I)$. On the other hand, (x,y), (x,z) and (y,z) are minimal reductions of I, thus $\operatorname{core}(I)\subset (x,y)\cap(x,z)\cap(y,z) = I^2$. Therefore $\operatorname{core}(I)=I^2$. To conclude notice that $I^2 \subsetneq (I^2,xw,yw) = J^2\colon I$.

Finally, the formula of Theorem 4.2 does not hold for $g=0$ even if $\ell>0$:

Example 4.10. Let k be an infinite field, let $\Delta_i \in k[\![X,Y,Z]\!]$ be the maximal minor of the matrix

$$\begin{pmatrix} X & Y & 0 & Z \\ Y & 0 & Z & X \\ 0 & Z & X & Y \end{pmatrix}$$

obtained by deleting the i^{th} column, set $R=k[\![X,Y,Z]\!]/(\Delta_1,\Delta_2)$ and define $J=\Delta_3 R$, $I=(\Delta_3,\Delta_4)R$. Then R is a local Gorenstein ring, $\operatorname{ht} I=0$, $\ell(I)=1$, I satisfies G_1, R/I is Cohen-Macaulay, and J is a minimal reduction of I with $r=r_J(I)=2$. However, $\operatorname{core}(I)\subsetneq J^{n+1}\colon I^n$ for every $n\geq 1 = \max\{r-\ell+g,0\}$.

Indeed, [13] show that J is a minimal reduction of I with $r=2$. Writing $\mathfrak{m}=(X,Y,Z)R$ one has $J : I = \mathfrak{m}$. As $r=2=\ell-g+1$, Theorem 3.3 then gives $\operatorname{core}(I)=\mathfrak{m}J=\mathfrak{m}I$. On the other hand, a computation shows that $\mathfrak{m}J\subsetneq J^2\colon I = J^3\colon I^2$. The assertion now follows since $J^{n+1}\colon I^n$ form an increasing sequence of ideals for $n\geq r=2$.

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RECENT RESULTS ON THE CORE OF AN IDEAL

CLAUDIA POLINI

To Professor Shiro Goto on the occasion of his sixtieth birthday

1. Introduction

This is a report on joint work with Bernd Ulrich and Marie Vitulli, that will be published in [12], and on joint work with Louiza Fouli and Bernd Ulrich that will appear on [3].

The core of an ideal I, introduced and first studied by Rees and Sally [13], is a mysterious subideal of I that encodes information about the possible reductions of I. A reduction of I is a subideal of I over which I is integral. One reason to study the core is the fact that this object encodes information about reductions and reduction numbers. Another motivation is that a better understanding of cores would lead to improved versions of the celebrated Briancon-Skoda theorem and solve a conjecture of Kawamata on the non-vanishing of sections of line bundles.

Being the intersection of an a priori infinite number of ideals, the core is difficult to compute and the problem of finding algorithms and formulas to determine it was addressed in the work of Corso, Huneke, Hyry, Smith, Swanson, Trung, Ulrich, and myself [1, 2, 5, 6, 7, 8, 11]. These formulas unfortunately require the assumption that the characteristic of the residue field is zero. In the first section of this report (which describes work that will appear in [3]) we study cores of ideals in arbitrary characteristic.

Let $R = k[x_1,\dots,x_d]$ be a polynomial ring over an infinite field k, write $\mathfrak{m} = (x_1,\dots,x_d)$, and let I be a monomial ideal, that is, an R-ideal generated by monomials. Even though there may not exist any proper reduction of I which is monomial (or even homogeneous), the intersection of all reductions, the core, is again a monomial ideal. Lipman and Huneke-Swanson related the core to the adjoint ideal [5, 9]. The integral closure and the adjoint of a monomial ideal are again monomial ideals and can be described in terms of the Newton polyhedron of I [4]. Such a description cannot exist for the core, since the Newton polyhedron only recovers the integral closure of the ideal, whereas the core may change when passing from I to $\overline{I}$. When attempting to derive any kind of combinatorial description for the core of a monomial ideal from the known colon formulas (see Theorem 2.1 below), one faces the problem that the colon formula involves non-monomial

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ideals, unless I has a reduction J generated by a monomial regular sequence. Instead, we exploit the existence of such non-monomial reductions to devise an interpretation of the core in terms of monomial operations.

2. THE CORE IN ARBITRARY CHARACTERISTIC

Let I be an ideal in a Noetherian local ring $(R,\mathfrak{m})$. Recall that a subideal J of I is a *reduction* of I, or equivalently, I is *integral* over J, if $I^{r+1} = JI^r$ for some non-negative integer r. If J is a reduction of an R-ideal I, then the *reduction number* $r_J(I)$ of I with respect to J is the smallest nonnegative integer r with $I^{r+1} = JI^r$. *Minimal reductions* are reductions which are minimal with respect to inclusion. If the residue field of R is infinite, all minimal reductions have the same number of generators, called the *analytic spread* ℓ of I.

In [11] we prove a formula, previously conjectured in [3], for the core of ideals satisfying G_ℓ and $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$. Recall that an ideal I satisfies property G_ℓ if the minimal number of generators $\mu(I_\mathfrak{p})$ of $I_\mathfrak{p}$ is at most $\dim R_\mathfrak{p}$ for every prime ideal $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} \leq \ell - 1$. Since $\ell \leq \dim R$, the G_ℓ property is a weak requirement, not imposing any restriction on the global number of generators of I.

Theorem 2.1. *Let R be a local Gorenstein ring with infinite residue field k. Let I be an R-ideal of height $g > 0$ and analytic spread ℓ, and let J be a minimal reduction of I with $r = r_J(I)$. Assume I satisfies G_ℓ, $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$, and either $\operatorname{char} k = 0$ or $\operatorname{char} k > r - \ell + g$. Then*

$$\operatorname{core}(I) = J^{n+1} : I^n$$

for every $n \geq \max\{r - \ell + g, 0\}$.

Unfortunately our formula requires the assumption that the characteristic of the residue field is zero or large enough. The following example, which we will explain in Section 4, shows that the formula of Theorem 2.1 fails to hold in arbitrary characteristic even for 0-dimensional monomial ideals:

Example 2.2. Let $R = k[x,y]$ be a polynomial ring over an infinite field k, consider the ideal $I = (x^6, x^5y^3, x^4y^4, x^2y^8, y^9)$, and write $J = (x^6, y^9)$. One has $r_J(I) = 2$. If $\operatorname{char} k \neq 2$ then the formula of Theorem 2.1 gives $\operatorname{core}(I) = J^3 : I^2 = J(x^4, x^3y, x^2y^2, xy^5, y^6)$. On the other hand, if $\operatorname{char} k = 2$ then $\operatorname{core}(I) = (x^{10}, x^8y, x^7y^5, x^6y^6, x^4y^9, x^3y^{10}, x^2y^{11}, xy^{14}, y^{15}) \supsetneq J^3 : I^2$.

The next theorem describes conditions on the ideal I for when the above formula is valid in arbitrary characteristic.

Theorem 2.3. [Fouli-Polini-Ulrich] *Let R be a local Gorenstein ring with infinite perfect residue field. Let I be an R-ideal of height $g > 0$ and analytic spread ℓ, and let J be a minimal reduction of I with $r = r_J(I)$. Assume I satisfies G_ℓ, $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$, and that the* *special fiber ring $\mathcal{F}(I)$ of I has embedding dimension at most 1 locally at every minimal prime of dimension ℓ. Then*

$$\operatorname{core}(I) = J^{n+1} : I^n$$

for every $n \geq \max\{r-\ell+g, 0\}$.

If I is equimultiple (in particular, if I is 0-dimensional) we do not need to assume that the ambient ring is Gorenstein:

Theorem 2.4. [Fouli-Polini-Ulrich] *Let R be a local Cohen-Macaulay ring with infinite perfect residue field. Let I be an R-ideal of height $g > 0$ and analytic spread g, and let J be a minimal reduction of I with $r = r_J(I)$. Assume that the special fiber ring $\mathcal{F}(I)$ of I has embedding dimension at most 1 locally at every minimal prime of dimension g. Then*

$$\operatorname{core}(I) = J^{n+1} : I^n$$

for every $n \geq r$.

The assumption on the fiber ring is automatically satisfied if the fiber ring of I is reduced, in particular if I is an ideal generated by forms of the same degree in a positively graded reduced algebra over a perfect field. This result generalizes work by Hyry and Smith, who had treated the case of the maximal ideal in a standard graded ring, but with completely different methods.

We end this section by demonstrate the failure of the formula of Theorem 2.4 if the fiber ring $\mathcal{F}(I)$ of I has embedding dimension two locally at every minimal prime of dimension g.

Example 2.5. Let $R = k[x,y]_{(x,y)}$ be a localized polynomial ring over a field of characteristic 2 and $I = (x^6, y^9, x^5y^3, x^4y^4, x^2y^8)$ be an R-ideal. Notice that

a. $\mathcal{F}(I) \simeq k[a,b,c,d,e]/(d^2-ae, c^2, e^2, de, ce)$
b. $\operatorname{Min}(\mathcal{F}(I)) = \{\mathfrak{p} = (c,d,e)\}$
c. $\operatorname{emdim}(\mathcal{F}(I)_{\mathfrak{p}}) = 2$

Let $J = (x^6, y^9)$. As we have seen in Example 2.2, we have that $\operatorname{core}(I) \supsetneq J^3 : I^2 = J^{n+1} : I^n$ for every $n \geq r_J(I) = 2$.

3. Cores and adjoints for zero-dimensional monomial ideals

In this section we consider the relationship between cores and adjoints as defined in [9]. Howald has shown that if I is a monomial ideal then its adjoint (or multiplier ideal) $\operatorname{adj}(I)$ is the monomial ideal with exponent set $\{\alpha \in \mathbb{Z}^d_{\geq 0} \mid \alpha + \mathbf{1} \in \operatorname{NP}^\circ(I)\}$, where $\mathbf{1} = (1,1,\ldots,1) \in \mathbb{Z}^d_{\geq 0}$ and $\operatorname{NP}^\circ(I)$ denotes the interior of the Newton polyhedron of I [4]. Thus whenever the core of a monomial ideal is an adjoint one has a combinatorial description of the former in terms of a Newton polyhedron.

In [11] Ulrich and myself had shown that:

Theorem 3.1. *If in addition to the assumptions of Theorem 2.1, R is a regular local ring essentially of finite type over a field of characteristic zero and* $\operatorname{Proj}(R[It])$ *has only rational singularities, then* $\operatorname{core}(I) = \operatorname{adj}(I^g)$.

Now we can apply this result to asymptotically normal 0-dimensional monomial ideals:

Corollary 3.2. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k of characteristic 0. Let I be a 0-dimensional monomial ideal and let $\underline{\alpha}$ be a regular sequence generating a reduction of I. Then*

$$\operatorname{adj}(I^d) = (\underline{\alpha})^{t+1} : \overline{I^t} \subset (\underline{\alpha})^{t+1} : I^t = \operatorname{core}(I)$$

for every $t \geq \max\{r_{(\underline{\alpha})}(I), d-1\}$, and equality holds if $\overline{I^{dt}} = I^{dt}$ for some $t \geq \max\{r_{(\underline{\alpha})}(I), d-1\}$.

Even if neither I or any power of I is integrally closed the core may still be an adjoint as shown by Corollary 3.4. In Theorem 3.3 we compute the core of any ideal generated by monomials of the same degree in a polynomial ring in two variables.

Theorem 3.3. *Let $R = k[x,y]$ be a polynomial ring over an infinite field k and write $\mathfrak{m}$ for the homogeneous maximal ideal of R. Let I be an R-ideal generated by monomials of the same degree. Write $I = \mu(x^n, y^n, x^{n-k_1}y^{k_1}, \ldots, x^{n-k_s}y^{k_s})$ with μ a monomial and $0 < k_1 < \cdots < k_s < n$, and set $\delta = \gcd(k_1, \ldots, k_s, n)$. Then*

$$\operatorname{core}(I) = \mu\,(x^\delta, y^\delta)^{2\frac{n}{\delta}-1}.$$

Corollary 3.4. *In addition to the assumptions of Theorem3.3 suppose that $\mu = 1$, and $\delta = 1$. Then* $\operatorname{core}(I) = \operatorname{adj}(I^2) = \operatorname{adj}(\mathfrak{m}^{2n}) = \mathfrak{m}^{2n-1}$.

4. AN ALGORITHM FOR THE CORE OF ZERO-DIMENSIONAL MONOMIAL IDEALS

In this section using linkage theory we give an algorithm to compute the core of zero-dimensional monomial ideals. This algorithm provides a new interpretation of the core as the largest monomial ideal contained in a general locally minimal reduction of I. Furthermore this algorithm is more efficient in general than the formula of Theorem 2.1 and does not require any restriction on the characteristic.

Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over an infinite field k. Write $\mathfrak{m} = (x_1, \ldots, x_d)$ for the homogeneous maximal ideal of R. For an R-ideal L we let $\operatorname{mono}(L)$ denote the largest monomial ideal contained in L and $\operatorname{Mono}(L)$ the smallest monomial ideal containing L. Note that $\operatorname{Mono}(L)$ is easy to compute, being the ideal generated by the monomial supports of generators of L. The computation of $\operatorname{mono}(L)$ is also accessible; the algorithm provided in [14, 4.4.2] computes $\operatorname{mono}(L)$ by multi-homogenizing L with respect to d new variables and then contracting back to the ring R. The ideal $\operatorname{mono}(L)$ can be computed in CoCoA with the built-in command *MonsInIdeal*.

We first use linkage theory to express $\operatorname{mono}(L)$ in terms of $\operatorname{Mono}(L)$ for a class of ideals including $\mathfrak{m}$-primary ideals.

Theorem 4.1. *Let L be an unmixed R-ideal of height g and $\underline{\beta} \subset L$ a regular sequence consisting of g monomials. Then*

$$\operatorname{mono}(L) = (\underline{\beta}) : \operatorname{Mono}((\underline{\beta}) : L).$$

Now let I denote an $\mathfrak{m}$-primary monomial ideal. For each i let n_i be a power of x_i in I; such n_i exists since I is $\mathfrak{m}$-primary. Write $\underline{\alpha} = x_1^{dn_1}, \ldots, x_d^{dn_d}$ and let J be an ideal generated by d general k-linear combinations of minimal monomial generators of I. If the ideal I is generated by forms of the same degree, J is a general minimal reduction of I [10]. In general however, I and J may not even have the same radical. Nevertheless, $J_{\mathfrak{m}}$ is a general minimal reduction of $I_{\mathfrak{m}}$ by [10]. Consider the ideal $K = (J, \underline{\alpha})$. Observe that the $\mathfrak{m}$-primary ideal K is a reduction of I. Thus $\operatorname{core}(I) \subset \operatorname{mono}(K)$ since the core is a monomial ideal. The Briançon-Skoda theorem implies $(\underline{\alpha})_{\mathfrak{m}} \subset \operatorname{core}(I_{\mathfrak{m}})$. Hence $K_{\mathfrak{m}} = J_{\mathfrak{m}}$, and whenever I is generated by forms of the same degree then $K = J$. We call K a *general locally minimal reduction* of I.

In order to prove the other inclusion $\operatorname{core}(I) \supset \operatorname{mono}(K)$ we need to show that $\operatorname{mono}(K)$ does not depend on the general locally minimal reduction K. We do this by proving that the ideal $\operatorname{Mono}((\underline{\alpha}) : K)$ does not depend on the general locally minimal reduction K and then using Theorem 4.1. Now by [1] the core is a finite intersection of general minimal reductions $\operatorname{core}(I) = K_1 \cap \cdots \cap K_t \supset \operatorname{mono}(K_1) \cap \cdots \cap \operatorname{mono}(K_t) = \operatorname{mono}(K)$, where the last equality follows since $\operatorname{mono}(K)$ does not depend on the general locally minimal reduction K. Thus we obtain:

Theorem 4.2. *Let I denote an $\mathfrak{m}$-primary monomial ideal, then*

$$\operatorname{core}(I) = \operatorname{mono}(K)$$

for any general locally minimal reduction K of I.

Now we can finally give a proof for Example 2.2:

Remark 4.3. The formula of Theorem 2.1 does not hold in arbitrary characteristic. However, if J and I are monomial ideals, $J^{n+1} : I^n$ is obviously independent of the characteristic. On the other hand, the algorithm based on Theorem 4.2 works in any characteristic, but its output, $\operatorname{mono}(K)$, is characteristic dependent. In fact Example 2.2 is a zero-dimensional monomial ideal I for which $\operatorname{core}(I) = \operatorname{mono}(K)$ varies with the characteristic. As I has a reduction J generated by a monomial regular sequence this shows that the formula of Theorem 2.1 fails to hold in arbitrary characteristic even for 0-dimensional monomial ideals.

Example 4.4. Let $R = k[x,y]$ be a polynomial ring over an infinite field k, consider the ideal $I = (x^6, x^5y^3, x^4y^4, x^2y^8, y^9)$, and write $J = (x^6, y^9)$. One has $r_J(I) = 2$. If $\operatorname{char} k \neq 2$ then the formula of Theorem 2.1 as well as the algorithm of Theorem 4.2 give $\operatorname{core}(I) = J^3 : I^2 = J(x^4, x^3y, x^2y^2, xy^5, y^6) = (x^{10}, x^9y, x^8y^2, x^7y^5, x^6y^6, x^4y^9, x^3y^{10}, x^2y^{11}, xy^{14}, y^{15})$. On the other hand, if $\operatorname{char} k = 2$ then Theorem 4.2 shows that $\operatorname{core}(I) = (x^{10}, x^8y, x^7y^5, x^6y^6, x^4y^9, x^3y^{10}, x^2y^{11}, xy^{14}, y^{15}) \supsetneq J^3 : I^2$.

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HOW (PART OF) THE INTEGRAL CLOSURE OF IDEALS ARISES

ALBERTO CORSO

To Professor Shiro Goto on the occasion of his sixtieth birthday

1. Introduction

This article is an expanded version of a talk I gave during the 28-th Symposium on Commutative Ring Theory held in November 2006, in Japan. The Symposium was marked by the celebration of the sixtieth birthday of Professor Shiro Goto as well as his lasting contributions to Commutative Algebra. The topic of this paper relates to the computation (of part) of the integral closure of an ideal, an area where Professor Goto has been very influential. The format of this paper is highly colloquial as it reflects the nature of the original talk. It draws its material from the papers listed in the bibliography, to which we refer the interested reader for motivations, more accurate statements and additional references.

Throughout the paper, let R be a Noetherian ring and I an R-ideal. The *integral closure* of I is the ideal $\overline{I}$ of all elements of R that satisfy an equation of the form

$$X^m + p_1 X^{m-1} + \cdots + p_{m-1}X + p_m = 0, \quad p_i \in I^i.$$

In particular, one has the containments $I \subset \overline{I} \subset \sqrt{I}$, where $\sqrt{I}$ is the radical of I. An ideal $J \subset I$ is a *reduction* of I if $\overline{J} = \overline{I}$ or, equivalently, if $I^{r+1} = JI^r$, for some non-negative integer r. Finally, I is *integrally closed* (resp. *normal*) if $I = \overline{I}$ (resp. $I^m = \overline{I^m}$ for all m).

Given an R-ideal I, we would like to address the following issues:

- Design 'efficient' (and 'global') criteria do detect integrally closed ideals. The criteria should involve natural objects associated with I: eg, the powers of I, the radical of I, modules of syzygies, etc.
- If I fails the 'above' tests, find (relatively 'cheap') methods to compute (part of) the integral closure of I.

Example 1.1. It is not an easy task to compute the integral closure. For example, if R is a polynomial ring over a field and I is a monomial ideal, $\overline{I}$ is then the monomial ideal defined by the integral convex hull of the exponent vectors of I.

Example 1.2. However, if I is a binomial ideal then, unlike its radical $\sqrt{I}$, we have that $\overline{I}$ need not be a binomial ideal. Let $R = k[x, y, z, w]$ with $\mathrm{char}(k) = 0$. The ideal

$$I = (x^2 - xy, -xy + y^2, z^2 - zw, -zw + w^2)$$

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has integral closure (see [**5**])

$$\overline{I} = (x^2 - xy, -xy + y^2, z^2 - zw, -zw + w^2, \underline{xz - yz - xw + yw}).$$

The two issues outlined earlier are rather difficult and not much is known. A simple consequence of the *determinant trick* is that for every finitely generated R-module M then $I \subseteq IM : M \subseteq \overline{I}$. In particular, if I is integrally closed, for any such M, $IM : M = I$. Thus, we need to find appropriate 'test modules' for any ideal. An alternate approach is through the *Rees Algebra* $\mathcal{R}(I)$ of the ideal I, namely

$$\mathcal{R}(I) = R + It + I^2t^2 + \cdots + I^nt^n + \cdots.$$

One then looks for its integral closure inside $R[t]$:

$$\overline{\mathcal{R}(I)} = R + \overline{I}t + \overline{I^2}t^2 + \cdots + \overline{I^n}t^n + \cdots \subset R[t].$$

This is obviously wasteful of resources since the integral closure of all powers of I will be computed. An 'expensive' algorithm due to Vasconcelos works as follows. If $I = (a_1, \ldots, a_n)$ one then can represent its Rees algebra as $\mathcal{R}(I) = R[T_1, \ldots, T_n]/Q$, where Q is the kernel of the map $\varphi : R[T_1, \ldots, T_n] \longrightarrow \mathcal{R}(I)$, $T_i \mapsto a_i t$. If, in addition, $\mathcal{R}(I)$ is an affine domain over a field of characteristic zero and Jac denotes its Jacobian ideal, then Vasconcelos guarantees that the ring

$$\mathrm{Hom}_{\mathcal{R}(I)}(\mathrm{Jac}^{-1}, \mathrm{Jac}^{-1}) = \ldots = (\mathrm{Jac}\,\mathrm{Jac}^{-1})^{-1},$$

is larger than and integral over $\mathcal{R}(I)$, if the ring is not already normal. This process can be iterated until $\overline{\mathcal{R}(I)}$ has been obtained.

Example 1.3. Let k be a field of characteristic zero and let $I \subset R = [x, y]_{(x,y)}$ be the codimension two complete intersection ideal

$$I = (x^3 + y^6, xy^3 - y^5)$$

Iterating three times the method outlined before one can compute $\overline{I}$. To be precise

$$\begin{aligned} I_1 &= (x^3 + y^6, xy^3 - y^5, y^8) \\ I_2 &= (x^3 + y^6, xy^3 - y^5, x^2y^2 - y^6, y^7) \\ I_3 = \overline{I} &= (xy^3 - y^5, y^6, x^3, x^2y^2). \end{aligned}$$

For the records, we used 18 additional variables for our calculations (see [**5**]).

The above example brings up the issue of using methods and techniques inside the original ring in order to find integral elements over the given ideal. This is the line of investigation that we will follow in the rest of the paper. We conclude the introduction by recalling two older results, which are very inspirational. One is a 'local' criterion by Goto, whereas the second one by Burch shows that socle elements often provide integral elements.

Theorem 1.4 (Goto, 1987). *Let I be an ideal in a Noetherian ring R and assume that $\mu_R(I) = \mathrm{height}_R(I) = g$. Then the following conditions are equivalent:*

- $\overline{I} = I$, *i.e. I is integrally closed;*
- $\overline{I^n} = I^n$ *for all n, i.e. I is normal.*

- *for each* $\mathfrak{p} \in Ass_R(R/I)$, *the local ring* $R_\mathfrak{p}$ *is regular and*

$$\lambda_\mathfrak{p}\left(\frac{IR_\mathfrak{p} + \mathfrak{p}^2 R_\mathfrak{p}}{\mathfrak{p}^2 R_\mathfrak{p}}\right) \geq g - 1.$$

When this is the case, $Ass_R(R/I) = Min_R(R/I)$ *and* I *is generated by an* R*-regular sequence.*

Theorem 1.5 (Burch, 1968). *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring that is not regular and let* I *be an ideal of finite projective dimension. Then:*

$$\mathfrak{m}(I\colon \mathfrak{m}) = \mathfrak{m}I.$$

In particular, $I\colon \mathfrak{m} \subset \overline{I}$.

2. Linkage and reduction number of ideals

A global criterion and direct links. The only known criterion for integral closedness is stated in the following result. It involves the determination of the radical of an ideal, which is made possible by the seminal work of Eisenbud, Huneke and Vasconcelos [**7**].

Theorem 2.1 (Corso-Huneke-Vasconcelos, 1998). *Let* I *be an height unmixed ideal in a Cohen-Macaulay ring* R. *Suppose that* I *is generically a complete intersection. Then the following conditions are equivalent:*

- I *is integrally closed;*
- $I = IL\colon L$, *where* $L = I\colon \sqrt{I}$.

An earlier version of the above criterion had the condition $I = IL\colon L$ replaced by $\sqrt{I} = IL\colon L^2$. The proof is based on the previous local result of Goto and on a result of myself, Polini and Vasconcelos on direct links of ideals. It greatly generalizes the result of Burch as it includes the case of regular rings.

Theorem 2.2 (Corso-Polini-Vasconcelos, 1994; Corso-Polini, 1995). *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring and let* $J = (z_1, \ldots, z_g)$ *be an ideal generated by a regular sequence inside a prime ideal* $\mathfrak{p}$ *of height* g. *If we set* $I = J\colon \mathfrak{p}$ *then* $I^2 = JI$ *if one of the following two conditions holds:*

(L_1) $R_\mathfrak{p}$ *is not a regular local ring;*

(L_2) $R_\mathfrak{p}$ *is a regular local ring with dimension at least* 2 *and two of the* z_i*'s are in* $\mathfrak{p}^{(2)}$.

For simplicity, let us give an idea of the proof in the case of [**2**]. This proof brings up another player that we will consider later on: The Koszul homology module of an ideal. In the case of [**2**], the ring R is assumed to be Gorenstein; the general result for a Cohen-Macaulay ring is dealt with in [**3**] with completely different (ideal theoretic versus homological) methods. The first step in the proof is to use localization and reduce to the case $I = J\colon \mathfrak{m}$ and $J = (z_1, \ldots, z_d)$. We then analyze the first Koszul homology module of I. In particular, we observe that if R is Gorenstein and $\mu(I) = d + 1$ then

$$\lambda(I/J) = \lambda(I^2/JI) + \lambda(\delta(I)),$$

where $\delta(I)$ is the kernel of the map from the second symmetric power of I onto I^2 and $\lambda(\cdot)$ denotes length. Finally, in our specific case, namely $I = J\colon \mathfrak{m}$, we have that $\lambda(I/J) = 1$. Thus, in order to conclude that $I^2 = JI$ we need to argue that $\delta(I) \neq 0$.

The work of Goto-Sakurai. Goto and Sakurai have greatly generalized the previous result, by relaxing the Cohen-Macaulay assumption on the ring R. Two of my favorite results of theirs are listed below. One of the surprising issues is that the multiplicity of the ring R plays a role.

Theorem 2.3 (Goto-Sakurai). *Let $(R, \mathfrak{m})$ be a Buchsbaum local ring and assume that either* $\dim(R) \geq 2$ *or* $\dim(R) = 1$ *but* $e(R) \geq 2$. *Then there exists an integer $n > 0$ such that for every parameter ideal J of R which is contained in $\mathfrak{m}^n$, one has the equality $I^2 = JI$, where $I = J \colon \mathfrak{m}$.*

Theorem 2.4 (Goto-Sakurai). *Let $(R, \mathfrak{m})$ be a Buchsbaum local ring with* $\dim(R) \geq 1$ *and* $e(R) > 1$. *Let $J = (x_1, \ldots, x_d)$ be a parameter ideal in R and assume that $x_d \equiv ab$ mod $0 \colon \mathfrak{m}$ for some $a, b \in \mathfrak{m}$. Then $I^2 = JI$, where $I = J \colon \mathfrak{m}$.*

Links of powers. The case in which the prime ideal $\mathfrak{p}$ is replaced by a power $\mathfrak{p}^s$, for some $s \geq 1$, was first studied in [4] and, subsequently, in [**13**]. A simplified version of those results is stated below in the case of a power of the maximal ideal.

Theorem 2.5 (Corso-Polini, 1997; Polini-Ulrich, 1998). *If J is an $\mathfrak{m}$-primary complete intersection of a Gorenstein local ring $(R, \mathfrak{m})$ and $J \subset \mathfrak{m}^s$ but $J \not\subset \mathfrak{m}^{s+1}$ then one has an increasing sequence of ideals*

$$I_k = J \colon \mathfrak{m}^k$$

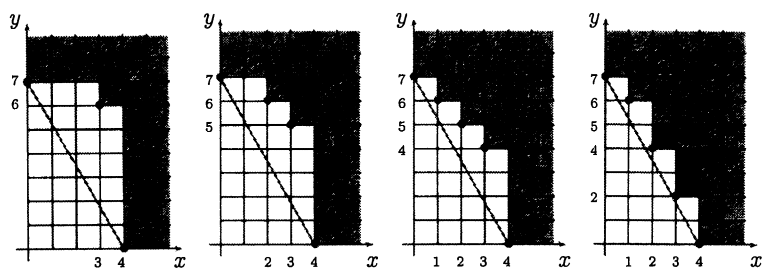
satisfying $I_k^2 = JI_k$ for $k = 1, \ldots, s$ if $\dim(R) \geq 3$ or for $k = 1, \ldots s-1$ if R is a regular local ring and $\dim(R) = 2$.

This says that the ideals I_k's are contained in the integral closure of I. Hence, instead of computing the integral closure of $R[Jt]$ one may start directly from $R[I_s t]$ (or $R[I_{s-1}t]$ if R is a regular local ring and $\dim(R) = 2$).

Example 2.6. Let us consider the localized polynomial ring $R = k[x, y]_{(x,y)}$ and set $\mathfrak{m} = (x, y)$ and $J = (x^4, y^7)$. Then the ideals I_k, with $k = 1, 2, 3$, listed below are all integral over J:

$$I_1 = J \colon \mathfrak{m} = (x^4, y^7, \underline{x^3y^6}) \quad \text{and} \quad I_1^2 = JI_1$$
$$I_2 = J \colon \mathfrak{m}^2 = (x^4, y^7, \underline{x^3y^5, x^2y^6}) \quad \text{and} \quad I_2^2 = JI_2$$
$$I_3 = J \colon \mathfrak{m}^3 = (x^4, y^7, \underline{x^3y^4, x^2y^5, xy^6}) \quad \text{and} \quad I_3^2 = JI_3.$$

However $\overline{I} = (x^4, y^7, \underline{x^3y^2, x^2y^4, xy^6})$ and $\overline{I}^2 = J\overline{I}$. The pictures below illustrate graphically the situation.



I_1 $\quad$ I_2 $\quad$ I_3 $\quad$ $\overline{I}$

A conjecture of Polini-Ulrich. The most general version of the linkage result that was proved by Polini and Ulrich essentially followed from their investigation of when an ideal is the unique maximal element of its linkage class, in the sense that it contains every ideal of the class:

Conjecture 2.7 (Polini-Ulrich, 1998). *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring of dimension $d \geq 2$, with $d \geq 3$ if R is regular. Let $s \geq 2$ be a positive integer. If $\{z_1, \ldots, z_d\}$ is a regular sequence contained in $\mathfrak{m}^s$, then*

$$(z_1, \ldots, z_d) \colon \mathfrak{m}^s \subset \mathfrak{m}^s$$

The conjecture has been recently proved in full generality by H.-J. Wang [**14**]. From the proof of the previous conjecture, Wang could also settle in the affirmative the following conjecture of myself and Polini:

Theorem 2.8. *Let R be a Noetherian ring. Let $\mathfrak{p}$ be a prime ideal of height $g \geq 2$, $J = (z_1, \ldots, z_g)$ an ideal generated by a regular sequence* <u>*inside*</u> *$\mathfrak{p}^{(k)}$, where $k \geq 2$ is a positive integer. Set $I_k = J \colon \mathfrak{p}^{(k)}$. Then*

$$I_k^2 = JI_k$$

if one of the following holds:

- *$R_\mathfrak{p}$ is not a regular local ring;*
- *$R_\mathfrak{p}$ is a regular local ring and $g \geq 3$;*
- *$R_\mathfrak{p}$ is a regular local ring, $g = 2$, and $J \subset \mathfrak{p}^{(k+1)}$.*

In a very recent work, Goto, Matsuoka and Takahashi [**12**] analyzed the case of ideals of the form $J \colon \mathfrak{m}^2$, where J is not necessarily contained in $\mathfrak{m}^2$. Their results are quite surprising as they show that the ideals are still integral over the complete intersection but the reduction number might go up! More precisely they show:

Theorem 2.9 (Goto-Matsuoka-Takahashi, 2006). *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim(R) > 0$ and assume that $e(R) \geq 3$. Then for every parameter ideal J in R one has that $\mathfrak{m}^2 I = \mathfrak{m}^2 J$ and $I^3 = JI^2$, where $I = J \colon \mathfrak{m}^2$.*

Linkage and Gorenstein linkage. The original result by Burch that we quoted in the introduction dealt with the integrality of elements of more general links. Thus, it is natural to inquire about the integrality of quotients where the complete intersection ideal J is replaced by a more general ideal. Two ideals I_1 and I_2 of height g are *linked* (*G-linked*, resp.) if there exists a complete intersection (perfect Gorenstein, resp.) ideal I of height g and with $I \subset I_1 \cap I_2$ such that

$$I_1 = I \colon I_2 \qquad I_2 = I \colon I_1.$$

The notion of linkage goes back to M. Noether, Apery and Gaeta. It was formulated in precise algebraic terms by Peskine and Szpiro. The notion of G-linkage was introduced by Schenzel and recently has been further developed by Kleppe, Migliore, Miro-Roig, Nagel, and Peterson.

Question 2.10. Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 3$. Let $I \subset \mathfrak{m}^t$, where $t \geq 2$, be an $\mathfrak{m}$-primary Gorenstein ideal. Set $L = I \colon \mathfrak{m}^{t-1}$. Is it true that

$$L^2 = IL?$$

In particular, L would be inside $\overline{I}$.

Thus far, the only known case is the following one.

Theorem 2.11 (Corso-Huneke-Vasconcelos, 1998). *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$. Suppose that $I \subset \mathfrak{m}^2$ is an $\mathfrak{m}$-primary ideal such that R/I is Gorenstein. Letting $L = I \colon \mathfrak{m}$ then*

$$L^2 = IL.$$

Idea of the Proof: As $L = (I, \Delta)$ one only needs to show that $\Delta^2 \in IL$. Now the key point is to observe that if we let $\mathfrak{m} = (x_1, \ldots, x_d)$, we can find $y_1, \ldots, y_d$ so that

$$x_i y_j \equiv \delta_{ij}\Delta \qquad \mod I.$$

Thus, for $i = 1, \ldots, d$ we have that $\Delta = x_i y_i + a_i$ for some $a_i \in I$. As $d \geq 2$, we can write:

$$\begin{aligned} \Delta^2 &= (x_1y_1 + a_1)(x_2y_2 + a_2) = x_1y_1x_2y_2 + x_1y_1a_2 + a_1x_2y_2 + a_1a_2 = \\ &= (x_1y_2)(x_2y_1) + (x_1y_1)a_2 + (x_2y_2)a_1 + a_1a_2 \in I(I, \Delta) = IL. \end{aligned}$$

as claimed. □

3. Annihilators of Koszul homology modules

Let $H_i = H_i(I)$ denote the homology modules of a *Koszul complex* $\mathbb{K}_*$ built on a minimal generating set $a_1, \ldots, a_n$ of an ideal I of height g. It is well known that all the Koszul homology modules H_i are annihilated by I, but in general their annihilators tend to be larger.

Question 3.1. Let R be a Cohen-Macaulay local ring and let I be an unmixed R-ideal:

$$\operatorname{Ann}(H_i) \overset{?}{\subset} \overline{I}?$$

In particular, one may want to look at the question "$\operatorname{Ann}(H_1) \overset{?}{\subset} \overline{I}$" as a non-traditional way to find part of $\overline{I}$. Notice also the unmixedness hypothesis on I.

Example 3.2. Let $R = k[x, y, z, w]_{(x,y,z,w)}$ with $\operatorname{char} k = 0$. The ideal

$$I = (x^2 - xy, -xy + y^2, z^2 - zw, -zw + w^2)$$

is a height two mixed ideal with

$$\operatorname{Ann}(H_1) = \overline{I} = (I, xz - yz - xw + yw)$$

$$\operatorname{Ann}(H_2) = \sqrt{I} = (x - y, z - w).$$

A first validation of the question is provided by the following result about integrally closed ideals.

Theorem 3.3 (Corso-Huneke-Katz-Vasconcelos, 2006). *Let I be an $\mathfrak{m}$-primary integrally closed ideal that is not a complete intersection. Then* $\operatorname{Ann}(H_1) = I$.

The proof of the above result is based on two facts:

- Let J_1 and J_2 be $\mathfrak{m}$-primary ideals with $J_1 \subseteq J_2$. Then

$$J_1 = J_2 \quad \Leftrightarrow \quad J_1 = J_2 \cap (J_1 : \mathfrak{m}).$$

- Let I be an $\mathfrak{m}$-primary ideal. If $c \in R$ is such that $cH_1 = 0$ and $c \in I\colon \mathfrak{m}$, then $c \in \overline{I}$.

A sketch of the proof of the second fact goes as follows. Suppose that $I = (a_1, \ldots, a_n)$. Notice that the condition $cH_1 = 0$ implies that

$$(a_1, \ldots, a_{n-1})\colon a_n \subseteq (a_1, \ldots, a_{n-1})\colon c. \qquad (*)$$

For if $r_n \in (a_1, \ldots, a_{n-1})\colon a_n$ then we can find $r_1, \ldots, r_{n-1} \in R$ so that $\sum_{i=0}^{n} r_i a_i = 0$. Thus $c(r_1, \ldots, r_n)$ is a Koszul relation and therefore

$$c(r_1, \ldots, r_n) = \sum_{i<j} s_{ij}(0, \ldots, a_j, \ldots, -a_i, \ldots, 0),$$

with a_j and $-a_i$ in the i-th and j-th places respectively. Thus, $cr_n \in (a_1, \ldots, a_{n-1})$.

On the other hand, $c \in I\colon \mathfrak{m}$. If $c\mathfrak{m} \subset I\mathfrak{m}$ then we have that $c \in \overline{I}$, by the determinant trick. Otherwise, $c\mathfrak{m} \not\subset I\mathfrak{m}$ so that we can choose a minimal generator for I of the form $a_n = cx$ for some $x \in \mathfrak{m}$. So we have $I = (a_1, \ldots, a_{n-1}, cx)$ with $(a_1, \ldots, a_{n-1})$ an $\mathfrak{m}$-primary ideal. Since we clearly have that

$$(a_1, \ldots, a_{n-1})\colon c \subseteq (a_1, \ldots, a_{n-1})\colon cx = (a_1, \ldots, a_{n-1})\colon a_n$$

we conclude that

$$(a_1, \ldots, a_{n-1})\colon c = (a_1, \ldots, a_{n-1})\colon cx = ((a_1, \ldots, a_{n-1})\colon c)\colon x,$$

as the reverse inclusion is given by equation $(*)$. As $R/(a_1, \ldots, a_{n-1})$ is an Artinian ring and $x \in \mathfrak{m}$, the above equation forces $c \in (a_1, \ldots, a_{n-1})$, which is impossible. $\square$

H_1 and the conormal module. The close relationship between H_1 and the conormal module I/I^2 is encoded in the following exact sequence of Simis and Vasconcelos

$$0 \to \delta(I) \longrightarrow H_1 \longrightarrow (R/I)^n \longrightarrow I/I^2 \to 0,$$

where $\delta(I)$ denotes the kernel of the natural map from the second symmetric power $\mathrm{Sym}_2(I)$ of I onto I^2.

The trouble would resolve if $\mathrm{Ann}(H_1) \subset \mathrm{Ann}(I/I^2)$. But this is not the case, in general. For example, let $R = k[x,y,z]_{(x,y,z)}$. The ideal $I = (x^7, y^7, z^7, x^3yz^2, x^2y^3z)$ is such that $I^2\colon I = I$ while $\mathrm{Ann}(H_1) = (I, x^6y^5, y^5z^6)$. Notice that $(\mathrm{Ann}(H_1))^2 = I \cdot \mathrm{Ann}(H_1)$, which implies that $\mathrm{Ann}(H_1) \subset \overline{I}$.

Ideals with a structure. The general case of Question 3.1 is rather complicated, thus in the sequel we will only deal with ideals that possess some specific structures. We first start presenting a 'loose' bound.

Proposition 3.4. *For any R-ideal I minimally presented by a matrix φ then* $\mathrm{Ann}(H_1) \subset I : I_1(\varphi)$. *If, in addition, I is syzygetic (that is $\delta(I) = 0$) then* $\mathrm{Ann}(H_1) = I\colon I_1(\varphi)$.

Proof: Indeed, if $x \in \mathrm{Ann}(H_1)$ one has that for $z \in Z_1$ the condition $xz \in B_1$ means that each coordinate of z is conducted into I by x. Thus $x \in I : I_1(\varphi)$. The converse holds if I is syzygetic. In fact, in this situation one actually has that $H_1 \hookrightarrow (I_1(\varphi)/I)^n$. Thus $I\colon I_1(\varphi) \subset \mathrm{Ann}(H_1)$. $\square$

Remark 3.5. In general, the ideal $I : I_1(\varphi)$ may be larger than $\overline{I}$. For example, the integrally closed R-ideal $I = (x,y)^2$, where $R = k[x,y]_{(x,y)}$, is such that $I\colon I_1(\varphi) = (x,y)$.

Next, in the case of height two perfect ideals in local Cohen-Macaulay rings, however, the Cohen-Macaulayness of the H_i's gets into the way.

Theorem 3.6. *Let I be a height two perfect R-ideal. Then for all i (with $H_i \neq 0$) one has $\operatorname{Ann}(H_i) = I$.*

Finally, for a height three perfect Gorenstein ideal I we still expect that $(\operatorname{Ann}(H_1))^2 = I \cdot \operatorname{Ann}(H_1)$, which would imply that $I \subsetneq \operatorname{Ann}(H_1) \subset \overline{I}$. Thus far, we can prove a weaker result.

Theorem 3.7. *Suppose that $\operatorname{char}(R) \neq 2$ and let I be a height three perfect Gorenstein ideal minimally generated by $n \geq 5$ elements. Then $(\operatorname{Ann}(H_1))^2 \subset \overline{I}$.*

Proof: Let V be any of the valuation overrings of R with valuation v. Let Z_1 and B_1 be the modules of cycles and boundaries, respectively, and let $c \in R$ such that $cZ_1 \subset B_1$. Then

$$c^{n-1} \det(Z_1 V) = \det(cZ_1 V) = \det(B_1 V).$$

On the other hand, we also have that

$$\det(Z_1 V) = I^2 V \qquad \text{and} \qquad \det(B_1 V) = I^{n-1} V.$$

So that $c^{n-1} I^2 V = I^{n-1} V$ implies that $c^{n-1} \in I^{n-3} V$. Now, we have that $(n-1)v(c) = v(c^{n-1}) \geq v(I^{n-3} V) = (n-3)v(IV)$ which yields

$$v(c^2) \geq 2\frac{n-3}{n-1} v(IV) \geq v(IV).$$

Therefore, $c^2 \in \overline{I}$. □

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MONOMIAL AND TORIC IDEALS ASSOCIATED TO FERRERS GRAPHS

ALBERTO CORSO

To Professor Shiro Goto on the occasion of his sixtieth birthday

This article is an expanded version of a talk I gave during the 28-th Symposium on Commutative Ring Theory held in November 2006, in Japan. The Symposium was marked by the celebration of the sixtieth birthday of Professor Shiro Goto as well as his lasting contributions to Commutative Algebra. The format of this paper is highly colloquial as it reflects the nature of original talk. It draws its material from a joint work with Uwe Nagel [**13**], to which I refer the interested reader for motivations, more accurate statements and additional references.

A *Ferrers graph* is a bipartite graph on two distinct vertex sets $\mathbf{X} = \{x_1, \ldots, x_n\}$ and $\mathbf{Y} = \{y_1, \ldots, y_m\}$ such that if (x_i, y_j) is an edge of G, then so is (x_p, y_q) for $1 \leq p \leq i$ and $1 \leq q \leq j$. In addition, (x_1, y_m) and (x_n, y_1) are required to be edges of G. For any Ferrers graph G there is an associated sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where λ_i is the degree of the vertex x_i. Notice that the defining properties of a Ferrers graph imply that $\lambda_1 = m \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1$; thus λ is a *partition*. Alternatively, we can associate to a Ferrers graph a diagram $\mathbf{T}_\lambda$, dubbed *Ferrers tableau*, consisting of an array of n rows of cells with λ_i adjacent cells, left justified, in the i-th row.

Ferrers graphs/tableaux have a prominent place in the literature as they have been studied in relation to chromatic polynomials [**2**, **20**], Schubert varieties [**18**, **17**], hypergeometric series [**31**], permutation statistics [**9**, **20**], quantum mechanical operators [**51**], inverse rook problems [**25**, **18**, **17**, **44**]. More generally, algebraic and combinatorial aspects of bipartite graphs have been studied in depth (see, e.g., [**47**, **32**] and the comprehensive monograph [**52**]). In this paper, which is the first of a series [**14**, **15**], we are interested in the algebraic properties of the *edge ideal* $I = I(G)$ and the *toric ring* $K[G]$ associated to a Ferrers graph G. The edge ideal is the monomial ideal of the polynomial ring $R = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ over the field K that is generated by the monomials of the form $x_i y_j$, whenever the pair (x_i, y_j) is an edge of G. $K[G]$ is instead the monomial subalgebra generated by the elements $x_i y_j$. An example is illustrated in Figure 1:

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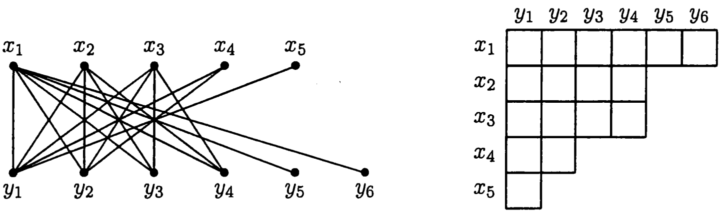

Ferrers graph *Ferrers tableau* with *partition* $\lambda = (6,4,4,2,1)$

$$I = (x_1y_1, x_1y_2, x_1y_3, x_1y_4, x_1y_5, x_1y_6, x_2y_1, x_2y_2, x_2y_3, x_2y_4, x_3y_1, x_3y_2, x_3y_3, x_3y_4, x_4y_1, x_4y_2, x_5y_1)$$

Figure 1: Ferrers graph, tableau and ideal

Throughout this article, $\lambda = (\lambda_1, \dots, \lambda_n)$ will always denote a fixed partition associated to a Ferrers graph G_λ with corresponding Ferrers ideal I_λ. In Section 2 of [**13**] we describe several fine numerical invariants attached to the ideal I_λ. In [**13**, Theorem 2.1] we show that each Ferrers ideal defines a small subscheme in the sense of Eisenbud, Green, Hulek, and Popescu [**22**], i.e. the free resolution of I_λ is 2-linear. More precisely, we give an explicit — but at the same time surprisingly simple — formula for the Betti numbers of the ideal I_λ; namely, we show that:

$$\beta_i(R/I_\lambda) = \binom{\lambda_1}{i} + \binom{\lambda_2+1}{i} + \binom{\lambda_3+2}{i} + \dots + \binom{\lambda_n+n-1}{i} - \binom{n}{i+1}$$

for $1 \le i \le \max\{\lambda_i + i - 1\}$. Furthermore, the Hilbert series is:

$$\sum_{k\ge 0} \dim_K[R/I_\lambda]_k \cdot t^k = \frac{1}{(1-t)^m} + \frac{t}{(1-t)^{m+n+1}} \cdot \sum_{j=1}^{n} (1-t)^{\lambda_j + j}.$$

Notice that the formula for the Betti numbers involves a minus sign: This is quite an unusual phenomenon for Betti numbers, as they tend, in general, to have an enumerative interpretation. In order to determine the Betti numbers it is essential to find a (not necessarily irredundant) primary decomposition of I_λ. We refine this decomposition into an irredundant one in [**13**, Corollary 2.5], where we observe, in particular, that the number of prime components is related to the outer corners of the Ferrers tableau. For instance, in the case of the ideal I_λ described in Figure 1 we have that it is the intersection of 5 (= 4 outer corners +1) components:

$$I_\lambda = (y_1, \dots, y_6) \cap (x_1, y_1, y_2, y_3, y_4) \cap (x_1, x_2, x_3, y_1, y_2) \cap (x_1, x_2, x_3, x_4, y_1) \cap (x_1, \dots, x_5).$$

We conclude Section 2 of [**13**] by identifying, in terms of the shape of the tableau, the unmixed ([**13**, Corollary 2.6]) and Cohen-Macaulay ([**13**, Corollary 2.7]) members in the family of Ferrers ideals. The latter result also follows from recent work of Herzog and Hibi [**32**].

There are relatively few general classes of ideals for which an explicit minimal free resolution is known: The most noteworthy such families include the Koszul complex, the Eagon-Northcott complex [**19**], and the resolution of generic monomial ideals [**3**] (see also [**4**]). In Section 3 of [**13**] we analyze even further the minimal free resolution of a Ferrers ideal I_λ and obtain a surprisingly elegant description of the differentials in the resolution in [**13**, Theorem 3.2]. In some sense, this is a prototypical result as it provides the minimal free resolution of several classes of ideals obtained from Ferrers ideals by appropriate specializations of the variables (see [**14**] for further details). Our description of the free resolution of a Ferrers ideal relies on the theory of cellular resolutions as developed by Bayer and Sturmfels in [**3**] (see also [**43**]). More precisely, let $\Delta_{n-1} \times \Delta_{m-1}$ denote the product of two simplices of dimensions $n-1$ and $m-1$, respectively. Given a Ferrers ideal I_λ, we associate to it the polyhedral cell complex X_λ consisting of the faces of $\Delta_{n-1} \times \Delta_{m-1}$ whose vertices are labeled by generators of I_λ. By the theory of Bayer and Sturmfels, X_λ determines a complex of free modules. Using an inductive argument we show in [**13**, Theorem 3.2] that this complex is in fact the multigraded minimal free resolution of the ideal I_λ. While leaving the details to [**13**], we illustrate the situation in the case of the partition $\lambda = (4, 3, 2, 1)$, which is the largest we can draw. In this case the polyhedral cell complex X_λ can actually be identified with the subdivision of the simplex Δ_3 pictured below (see [**14**] for additional details):

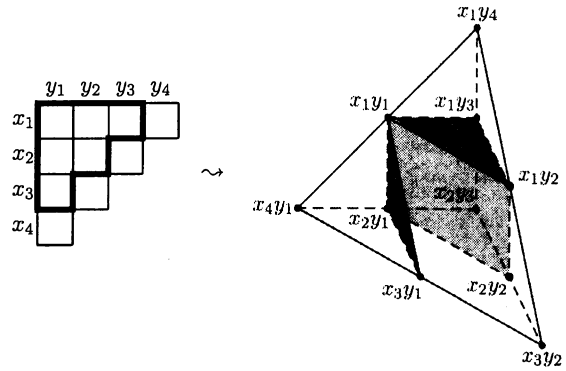

Figure 2: Ferrers tableau and associated polyhedral cell complex

In particular, we observe that X_λ has four 3-dimensional cells: Two of them are isomorphic to Δ_3 whereas the remaining two are isomorphic to either $\Delta_1 \times \Delta_2$ or $\Delta_2 \times \Delta_1$. A grey shading in the picture above also indicates how the polyhedral cell complex corresponding to the partition $(3, 2, 1)$ sits inside X_λ.

In Section 4 of [**13**] we prove the converse of [**13**, Theorem 2.1]. Namely, we show that any edge ideal of a bipartite graph with a 2-linear resolution necessarily arises from a Ferrers graph (see [**13**, Theorem 4.2]). One of the ingredients of the proof is a well-known characterization of edge ideals of graphs with a 2-linear resolution in terms of complementary graphs, due to Fröberg [**23**] (see also [**21**]).

The starting point of Section 5 of [**13**] is the observation that the *toric ring* of a Ferrers graph can be identified with a special *ladder determinantal ring*. We then proceed to recover/establish formulæ for the Hilbert series and other invariants associated with these rings. We remark that this is a highly investigated part of mathematics that has been the subject of the work of many researchers. Among the extensive, impressive and relevant literature we single out [**1, 8, 10, 11, 12, 28, 34, 36, 37, 38, 39, 40, 45, 46, 53**]. While most of these works involve — to a different extent — path counting arguments, we offer here a new and self-contained approach that yields easy proofs of explicit formulæ for the Hilbert series, the Castelnuovo-Mumford regularity, and the multiplicity of the toric rings of Ferrers graphs. This method, which is based on results from Gorenstein liaison theory (see [**41**] for a comprehensive introduction), has been pioneered in **[35]**, where it was proved that every standard determinantal ideal is *glicci*, i.e. it is in the Gorenstein liaison class of a complete intersection (see also **[42]**). Recently, Gorla **[26]** has considerably refined these arguments to show that all ladder determinantal ideals are glicci. This result can be used to establish first a simple recursive formula, which we then turn into an explicit formula that involves only positive summands.

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