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序

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ON MACAULAYFICATION OF NOETHERIAN SCHEMES

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1. Introduction

Let X be a Noetherian scheme. The desingularization of X is a birational proper morphism $Y \to X$ with Y non-singular. We know that X has a desingularization if X is a variety over a field of characteristic 0 or if $\dim X \leq 3$. Several authors are studying the desingularization of varieties over a field of positive characteristic.

In the present paper, we give a birational proper morphism $Y \to X$ with Y *Cohen-Macaulay.*

Theorem 1.1. *Let A be a Noetherian ring possessing a dualizing complex and X a separated, of finite type scheme over* $\operatorname{Spec} A$. *Then there is a birational proper morphism $Y \to X$ where Y is a Cohen-Macaulay scheme.*

Such a morphism $Y \to X$ is named to be a *Macaulayfication* of X by Faltings [3].

The birational morphism given by Theorem 1.1 is a blowing-up. Theorem 1.1 stands on the following

Theorem 1.2. *Let A be a Noetherian local ring of dimension d. If*

(1) *A has a dualizing complex*;

(2) $\dim A/\mathfrak{p} = d$ *for each associated prime $\mathfrak{p}$ of A,*

then there is an ideal $\mathfrak{b}$ of positive height such that $\operatorname{Proj} R(\mathfrak{b})$ *is a Cohen-Macaulay scheme. Here $R(\mathfrak{b})$ denotes the Rees algebra $\bigoplus_{n\geq 0} \mathfrak{b}^n$.*

Analyzing the proof of Theorem 1.2, we can answer to Sharp's conjecture [10] in affirmative.

Theorem 1.3. *Let A be a Noetherian local ring or a Noetherian integral domain. Then A has a dualizing complex if and only if A is a homomorphic image of a finite-dimensional Gorenstein ring.*

In this paper we give a brief proof of Theorem 1.2. Please refer to [5] for the complete proof of Theorem 1.1 and 1.3. From now on we agree that A denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$.

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2. A P-STANDARD SYSTEM OF PARAMETERS

In this section, we give the definition and properties of a *p-standard system of parameters*, which was introduced by N. T. Cuong [2]. Firstly we see the following definition and lemmas given by Schenzel [7], [8] and [9].

Definition 2.1. For a finitely generated A-module M, an ideal $\mathfrak{a}(M)$ is defined to be

$$\mathfrak{a}(M) = \prod_{i<\dim M} \operatorname{ann} H^i_{\mathfrak{m}}(M).$$

Of course $\mathfrak{a}(M) = A$ if and only if M is Cohen-Macaulay.

Lemma 2.2. *Let M be a finitely generated A-module. If A has a dualizing complex, then the following statements hold:*

(1) $\dim A/\mathfrak{a}(M) < \dim M$;
(2) *Let* $\mathfrak{p} \in \operatorname{Supp} M$. *Then* $\mathfrak{a}(M) \not\subset \mathfrak{p}$ *if and only if $M_{\mathfrak{p}}$ is a Cohen-Macaulay $A_{\mathfrak{p}}$-module and* $\dim A/\mathfrak{p} + \dim M_{\mathfrak{p}} = \dim M$.

Lemma 2.3. *Let M be a finitely generated A-module of dimension d and $x_1, \dots,$ x_d a system of parameters for M. Then*

$$(x_1, \dots, x_{i-1})M : x_i \subset (x_1, \dots, x_{i-1})M : \mathfrak{a}(M)$$

for any $1 \le i \le d$.

The p-standard system of parameters is defined by using the notion of $\mathfrak{a}(-)$.

Definition 2.4. Let M be a finitely generated A-module of dimension d and $x_1, \dots,$ x_d a system of parameters for M. We say that $x_1, \dots, x_d$ is a *p-standard system of parameters* if

$$\begin{cases} x_d \in \mathfrak{a}(M), \\ x_i \in \mathfrak{a}(M/(x_{i+1}, \dots, x_d)M), \quad \text{for } i < d. \end{cases}$$

As a consequence of Lemma 2.2, we have the following

Lemma 2.5. *Let M be a finitely generated A-module. If A has a dualizing complex, then M has a p-standard system of parameters.*

In particular, a Noetherian local ring A satisfying the assumption of Theorem 1.2 has a p-standard system of parameters.

The following theorem is the main theorem of this section.

Theorem 2.6. *Let M be a finitely generated A-module of dimension $d > 0$ and x_1, ..., x_d a p-standard system of parameters for M. Then, for any integer $1 \leq i \leq d$ and subsystem of parameters y_1, ..., y_{u-1} for $M/(x_i, \ldots, x_d)M$, we have*

$$(2.6.1) \quad (y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M : y_v y_u = (y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M : y_u$$

for $1 \leq v \leq u$ and $\Lambda \subset \{i+1, \ldots, d\}$. Here we agree that $y_u = x_i$.

Proof. If $i = d$, then the both side of (2.6.1) coincide with

$$(y_1, \ldots, y_{v-1})M : \mathfrak{a}(M)$$

by Lemma 2.3.

Therefore we assume that $i < d$ and work by descending induction of $\sharp\Lambda$. If $\Lambda = \{i+1, \ldots, d\}$, then the both sides of (2.6.1) are

$$(y_1, \ldots, y_{v-1}, x_{i+1}, \ldots, x_d)M : \mathfrak{a}(M/(x_{i+1}, \ldots, x_d)M).$$

Assume that $\Lambda \neq \{i+1, \ldots, d\}$ and let l be the largest integer in $\{i+1, \ldots, d\} \setminus \Lambda$. Let a be an element of the left hand side of (2.6.1). Then

$$\begin{aligned} a &\in (y_1, \ldots, y_{v-1}, x_l, x_\lambda \mid \lambda \in \Lambda)M : y_v y_u \\ &= (y_1, \ldots, y_{v-1}, x_l, x_\lambda \mid \lambda \in \Lambda)M : y_u \end{aligned}$$

by the induction hypothesis. We put $y_u a = x_l b + c$ with

$$c \in (y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M.$$

Then

$$\begin{aligned} b &\in (y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M : y_u x_l \\ &= (y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M : x_l \end{aligned}$$

by Lemma 2.3. Thus $x_l b$, $y_u a \in (y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M$. $\square$

Corollary 2.7. *With notation of Theorem 2.6, x_1, ..., x_d is a d-sequence on M.*

See [4] for the notion of d-sequences.

Corollary 2.8. *With notation of Theorem 2.6, let $\mathfrak{q}_i = (x_i, \ldots, x_d)$. Then*

$$[(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : x_{i-1}^{n_{i-1}} = [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : \mathfrak{q}_{i-1}$$

for any integers $2 \leq i \leq j \leq d$, n_1, ..., $n_j > 0$ and $\Lambda \subset \{1, \ldots, i-2\}$.

The proof of Corollary 2.8 is very long. We omit it.

3. The proof of Theorem 1.2

Let M be a finitely generated A-module of dimension $d > 0$ and $\mathfrak{b}$ an ideal. We agree that $R(\mathfrak{b})$ is the Rees algebra $\bigoplus_{n\geq 0} \mathfrak{b}^n$ and $R_M(\mathfrak{b})$ denotes a finitely generated $R(\mathfrak{b})$-module $\bigoplus_{n\geq 0} \mathfrak{b}^n M$.

To prove Theorem 1.2, we show the following theorem by induction on t. As a consequence of Corollary 2.7 and 2.8, a p-standard system of parameters satisfies the assumption of the theorem.

Theorem 3.1. *Let $x_t, \ldots, x_d$ be a subsystem of parameters for M where $t \leq d$. We put*

$$\mathfrak{q}_i = (x_i, \ldots, x_d),$$
$$\mathfrak{b}_i = \mathfrak{q}_i \cdots \mathfrak{q}_d,$$
$$Y_i = \operatorname{Proj} R(\mathfrak{b}_i)$$

and $\mathcal{F}_i$ the coherent sheaf $[R_M(\mathfrak{b}_i)]^{\sim}$ on Y_i for each $t \leq i \leq d$. If

(1) *$x_i, \ldots, x_d$ is a d-sequence on $M/(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M$ for any $t \leq i < d$, $n_t, \ldots, n_{i-1} > 0$ and $\Lambda \subset \{t, \ldots, i-1\}$;*

(2)
$$[(x_\lambda^{n_\lambda})M + \mathfrak{b}_i^{n_i} M] : x_{i-1}^{n_{i-1}} = [(x_\lambda^{n_\lambda})M + \mathfrak{b}_i^{n_i} M] : \mathfrak{q}_{i-1}$$
for any $t+1 \leq i \leq d$, $n_t, \ldots, n_{i-1} > 0$ and $\Lambda \subset \{t, \ldots, i-2\}$,

then

$$\operatorname{depth}(\mathcal{F}_i)_p \geq d - t + 1 \quad \textit{for any closed point } p \in Y_i.$$

Proof. First we note that $\dim(\mathcal{F}_t)_p = d$ for any closed point $p \in Y_t$. Let $\mathfrak{p}$ be an associated prime ideal of M such that $\dim A/\mathfrak{p} = d$. Then $\mathfrak{p}^* = \bigoplus_{n\geq 0} \mathfrak{p} \cap \mathfrak{b}_t^n$ is an associated prime ideal of $R_M(\mathfrak{b}_t)$. Since $x_t, \ldots, x_d$ is a subsystem of parameters for $A/\mathfrak{p}$, and hence analytically independent on $A/\mathfrak{p}$, $\mathfrak{p} \cap \mathfrak{b}_t^n \subset \mathfrak{m}\mathfrak{b}_t^n$ for all $n > 0$. Therefore any closed point on Y_t contains $\mathfrak{p}^*$. Furthermore $\dim R(\mathfrak{b}_t)/\mathfrak{p}^* = d + 1$.

Assume that $t = d$. Then $Y_d = \operatorname{Spec} A/H^0_{x_d}(A)$ and $\mathcal{F}_d = [M/H^0_{x_d}(M)]^{\sim}$. Therefore $\operatorname{depth}(\mathcal{F}_d)_p > 0$ where p is the unique closed point of Y_d.

Assume that $t < d$. Then Y_t is the blowing-up of Y_{t+1} with respect to $\mathfrak{q}_t \mathcal{O}_{Y_{t+1}}$. Let p be a closed point of Y_t and q its image under the blowing-up $Y_t \to Y_{t+1}$. Then q is also a closed point. Let $B = \mathcal{O}_{Y_{t+1}}$, $N = (\mathcal{F}_{t+1})_q$ and $\mathfrak{n}$ the maximal ideal of B. Since $\mathfrak{q}_{t+1}\mathcal{O}_{Y_{t+1}}$ is invertible, $\mathfrak{q}_{t+1}B = x_i B$ for some $t+1 \leq i \leq d$ and x_i is B-regular. Therefore $\mathcal{O}_{Y_t,p} = B[x_i/x_t]_{(\mathfrak{n}, f(x_i/x_t))}$ or $B[x_t/x_i]_{(\mathfrak{n}, f(x_t/x_i))}$, where f is a monic polynomial with coefficients in B.

We compute the local cohomology $H^q_{\mathfrak{q}_t}(N)$. It is clear that $H^q_{\mathfrak{q}_t}(N) = 0$ if $q \neq 1$, 2. Let $\mathcal{F}^{(l)}_{t+1}$ be the coherent sheaf $[R_{M/x_t^l M}(\mathfrak{b}_{t+1})]^{\sim}$ on Y_{t+1} and $N^{(l)} = (\mathcal{F}^{(l)}_{t+1})_q$. Then $\operatorname{depth} N$, $\operatorname{depth} N^{(l)} \geq d - t$ by the induction hypothesis.

There is an exact sequence

$$(3.1.1)\quad 0 \longrightarrow \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M : x_t^l}{\mathfrak{b}_{t+1}^n M + 0 :_M x_t^l} \longrightarrow \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M}{x_t^l \mathfrak{b}_{t+1}^n M} \longrightarrow \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M + x_t^l M}{x_t^l M} \longrightarrow 0$$

The left hand side of (3.1.1) is annihilated by x_i because of the assumption (2) and x_i is a regular element on the right hand side of (3.1.1). Indeed

$$x_t^l M : x_i \cap (x_t^l, x_i, \dots, x_d)M = x_t^l M$$

because $x_i, \dots, x_d$ is a d-sequence on $M/x_t^l M$. Therefore

$$H_{x_i}^1 \left(\bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M}{x_t^l \mathfrak{b}_{t+1}^n M} \right) = H_{x_i}^1 \left(\bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M + x_t^l M}{x_t^l M} \right)$$

and

$$H_{x_i}^0 \left(\bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M}{x_t^l \mathfrak{b}_{t+1}^n M} \right) = \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M : x_t}{\mathfrak{b}_{t+1}^n M + 0 :_M x_t}.$$

Taking localization, we have $H_{x_i}^1(N/x_t^l N) = H_{x_i}^1(N^{(l)})$. Since the local cohomology functor commutes with direct limit and there is an exact sequence

$$0 \longrightarrow H_{x_i}^1 H_{x_t}^{p-1}(-) \longrightarrow H_{(x_t,x_i)}^p(-) \longrightarrow H_{x_i}^0 H_{x_t}^p(-) \longrightarrow 0$$

we have

$$\begin{aligned} H_{\mathfrak{q}_t}^2(N) &= H_{x_i}^1 H_{x_t}^1(N) \\ &= \varinjlim_l H_{x_i}^1(N/x_t^l N) \\ &= \varinjlim_l H_{x_i}^1(N^{(l)}) \\ &= \varinjlim_{l,m} N^{(l)}/x_i^m N^{(l)}. \end{aligned}$$

Since depth $N^{(l)} \geq d-t$ and x_i is $N^{(l)}$-regular,

$$H_{\mathfrak{n}}^p H_{\mathfrak{q}_t}^2(N) = 0 \quad \text{for } p < d-t-1.$$

Similarly

$$\begin{aligned} H_{\mathfrak{q}_t}^1 \left(\bigoplus_{n>0} \mathfrak{b}_{t+1}^n M \right) &= H_{x_i}^0 H_{x_t}^1 \left(\bigoplus_{n>0} \mathfrak{b}_{t+1}^n M \right) \\ &= \varinjlim_l H_{x_i}^0 \left(\bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M}{x_t^l \mathfrak{b}_{t+1}^n M} \right) \\ &= \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M : x_t}{\mathfrak{b}_{t+1}^n M + 0 :_M x_t}. \end{aligned}$$

Indeed, $H^0_{x_t}(\mathfrak{b}^n_{t+1}M) = 0$ for all $n > 0$ because $x_t, \ldots, x_d$ is a d-sequence and hence $0 :_M x_t \cap \mathfrak{q}_{t+1}M = 0$. Therefore $\mathfrak{q}_t H^1_{\mathfrak{q}_t}(N) = 0$. Next we consider the spectral sequence

$$E_2^{pq} = H^p_{\mathfrak{n}} H^q_{\mathfrak{q}_t}(N) \Rightarrow H^{p+q}_{\mathfrak{n}}(N).$$

We already know that $E_2^{pq} = 0$ if $q \neq 1, 2$ and $E_2^{p2} = 0$ if $p < d - t - 1$. Since depth $N \geq d - t$, $E_2^{p1} = H^{p+1}_{\mathfrak{n}}(N) = 0$ for $p < d - t - 1$. Thus we obtain

$$H^p_{\mathfrak{n}} H^q_{\mathfrak{q}_t}(N) = 0 \quad \text{if } q \neq 1, 2 \text{ or } p < d - t - 1$$

and

$$\mathfrak{q}_t H^1_{\mathfrak{q}_t}(N) = 0.$$

Using this we compute depth$(\mathcal{F}_t)_p$. We assume that $\mathcal{O}_{Y_t,p} = B[x_i/x_t]_{(\mathfrak{n}, f(x_i/x_t))}$. Let $L = N[T]/(x_tT - x_i)N[T]$, where T is an indeterminate. Then

$$(\mathcal{F}_t)_p \cong [L/H^0_{\mathfrak{q}_t}(L)]_{(\mathfrak{n}, f(x_i/x_t))}.$$

Taking local cohomology of an exact sequence

$$0 \longrightarrow N[T] \xrightarrow{x_tT - x_i} N[T] \longrightarrow L \longrightarrow 0,$$

we obtain an exact sequence

$$0 \longrightarrow H^1_{\mathfrak{q}_t}(N[T]) \longrightarrow H^1_{\mathfrak{q}_t}(L) \longrightarrow H^2_{\mathfrak{q}_t}(N[T]) \longrightarrow H^2_{\mathfrak{q}_t}(N[T]) \longrightarrow 0.$$

By using an exact sequence

$$0 \longrightarrow H^1_{f(T)} H^{p-1}_{\mathfrak{n}}(-) \longrightarrow H^p_{(\mathfrak{n}, f(T))}(-) \longrightarrow H^0_{f(T)} H^p_{\mathfrak{n}}(-) \longrightarrow 0,$$

we find that

$$H^p_{(\mathfrak{n}, f(T))} H^q_{\mathfrak{q}_t}(N[T]) = 0 \quad \text{if } q \neq 1, 2 \text{ or } p < d - t.$$

Hence $H^p_{(\mathfrak{n}, f(T))} H^1_{\mathfrak{q}_t}(L) = 0$ for $p < d - t$. Since

$$H^1_{\mathfrak{q}_t}(L/H^0_{\mathfrak{q}_t}(L)) = H^1_{\mathfrak{q}_t}(L)$$

and

$$H^q_{\mathfrak{q}_t}(L/H^0_{\mathfrak{q}_t}(L)) = 0 \quad \text{if } q \neq 1,$$

we have

$$H^p_{(\mathfrak{n}, f(T))} H^q_{\mathfrak{q}_t}(L/H^0_{\mathfrak{q}_t}(L)) = 0 \quad \text{if } q \neq 1 \text{ or } p < d - t.$$

By the spectral sequence

$$E_2^{pq} = H^p_{(\mathfrak{n}, f(T))} H^q_{\mathfrak{q}_t}(-) \Rightarrow H^{p+q}_{(\mathfrak{n}, f(T))}(-),$$

we have depth$(\mathcal{F}_t)_p \geq d - t + 1$.

We can also show that depth$(\mathcal{F}_t)_p \geq d - t + 1$ when $\mathcal{O}_{Y_t,p} = B[x_t/x_i]_{(\mathfrak{n}, f(x_t/x_i))}$. The proof of Theorem 3.1 is competed. $\square$

4. Questions

We closed this paper by giving further questions.

Question 4.1. Let X be a Noetherian scheme. Is there a birational proper morphism $Y \to X$ such that Y is Cohen-Macaulay and *normal*?

When X is a quasi-projective variety over an algebraically closed field, whose characteristic zero or positive prime, and X has little non-Cohen-Macaulay points, Brodmann [1] gave such Y.

Question 4.2. Let A be a Noetherian local ring satisfying the assumption of Theorem 1.2. Is there an ideal $\mathfrak{b}$ such that $R(\mathfrak{b})$ itself is Cohen-Macaulay?

When $\dim A/\mathfrak{a}(A) = 0$, we know the existence of such an ideal. For example, see [4]. Recently the author [6] find such an ideal when $\dim A/\mathfrak{a}(A) = 1$.

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What makes a flat complex exact?

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1 Introduction

Throughout this paper, R denotes a noetherian commutative ring. The symbols $\otimes$ and Hom stand for $\otimes_R$ and Hom_R, respectively. For $\mathfrak{p} \in \operatorname{Spec} R$, the field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is denoted by $\kappa(\mathfrak{p})$. Moreover, the symbol $?(\mathfrak{p})$ stands for the functor $\kappa(\mathfrak{p})\otimes ?$.

For an R-algebra A, an A-module means a left A-module unless otherwise specified, and the category of A-modules is denoted by ${}_A\mathbb{M}$.

For an R-coalgebra C, a C-comodule means a right C-comodule, unless otherwise specified. The category of C-comodules is denoted by $\mathbb{M}^C$.

For a locally noetherian abelian category $\mathcal{C}$, the full subcategory of $\mathcal{C}$ consisting of all noetherian objects of $\mathcal{C}$ is denoted by $\mathcal{C}_f$.

We sometimes face to a difficulty when we want to generalize a result over a field to a result over arbitrary base. Let

$$\mathbb{F} : 0 \to F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} F^2 \to \cdots$$

be an R-flat complex. We say that $\mathbb{F}$ is *u-acyclic* if for any R-module M, $H^i(\mathbb{F}\otimes M) = 0$ $(i > 0)$ and the canonical map

$$\rho_M : H^0(\mathbb{F}) \otimes M \to H^0(\mathbb{F} \otimes M)$$

is isomorphic. We want to prove a theorem which guarantees the u-acyclicity of $\mathbb{F}$ and the finiteness of $H^0(\mathbb{F})$ from assumptions on $\mathbb{F}(\mathfrak{p})$ for various $\mathfrak{p} \in \operatorname{Spec} R$. Assume that $\mathbb{F}$ is u-acyclic and $H^0(\mathbb{F})$ is R-finite. Then, it is easy to see that $H^0(\mathbb{F})$ is R-finite projective. Hence, we have that $H^i(\mathbb{F}(\mathfrak{p})) = 0$ $(i > 0)$ and $h^0_\mathbb{F}(\mathfrak{p}) := \dim_{\kappa(\mathfrak{p})} H^0(\mathbb{F}(\mathfrak{p}))$ is finite for $\mathfrak{p} \in \operatorname{Spec} R$, and $h^0_\mathbb{F}$ is a locally constant function on $\operatorname{Spec} R$ by u-acyclicity. Are these conditions are sufficient to guarantee the converse? The answer is no, see Example 22. What is a good condition to impose to guarantee the converse? One possibility is to add some finiteness conditions on $H^i(\mathbb{F} \otimes R/\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec} R$, see (19) and (20) in Theorem 15. This assumption is really effective when we study higher direct images of proper morphisms, and studied extensively by Grothendieck [5] by using some different point of view and methods. Even if each term of $\mathbb{F}$ is not R-finite, flat complexes in commutative ring theory satisfies some finiteness conditions (e.g., metafiniteness), and (19) is satisfactory in this sense.

However, in studying integral representations of quantum groups, the author felt necessity of another good additional assumption to impose, because quantum groups are independent of geometric objects (at least in the classical sense) and hence (19) is useless, and quantum groups has been studied extensively over a field by many authors. Our main theorem states: the converse is true if we assume that F^0 is R-projective, see Theorem 15 (18). As an application, we show that Schur algebras (in the sense of Donkin [3]) for any poset ideal of the set of dominant weights are constructed over arbitrary base ring and a q-analogue of Donkin-Koppinen's bicomodule filtration theorem hold for $GL_q(n)$ and $SL_q(n)$ over arbitrary base R and for arbitrary $q \in R^\times$.

In section 2, we give a detailed proof of the main theorem and some counterexamples. In section 3, we give a survey on some applications of Theorem 15.

Note that all of section 2 and some part of section 3 are contained in [6]. For other applications of Theorem 15 such as generalizations of Ringel's approximations which we miss here, see [6].

2 The theorem

Let $f : M \to N$ be an R-linear map. We say that f is *R-pure* if $1_W \otimes f : W \otimes M \to W \otimes N$ is injective for any R-module W. Obviously, any R-pure map is injective. When $M \subset N$, then we say that M is a *pure submodule* of N if the inclusion map $M \hookrightarrow N$ is pure. Note that a split injection is pure.

Lemma 1 *Let P and F be flat R-modules, and $f : P \to F$ an R-linear map. Consider the following conditions.*

1 *f is injective and* Coker f *is R-flat.*

2 *f is pure.*

3 *For any $\mathfrak{p} \in \operatorname{Spec} R$, the map $f(\mathfrak{p}) : P(\mathfrak{p}) \to F(\mathfrak{p})$ is injective.*

4 *For any $\mathfrak{m} \in \operatorname{Spec} R$, the map $f(\mathfrak{m}) : P(\mathfrak{m}) \to F(\mathfrak{m})$ is injective.*

Then, **1,2,3** *are equivalent. If P is R-projective moreover, then* **1–4** *are equivalent.*

Proof The direction **1**⇒**2**⇒**3**⇒**4** is obvious.

First, we prove **1** assuming either **3** or that P is R-finite and **4** holds. We assume the contrary, and prove the contradiction. There exists some $\mathfrak{m} \in \operatorname{Max} R$ such that $f_{\mathfrak{m}}$ is not injective or $\operatorname{Coker} f_{\mathfrak{m}}$ is not $R_{\mathfrak{m}}$-flat. Hence, we may assume that $(R, \mathfrak{m})$ is local. As R is noetherian, there exists some ideal maximal with respect to the incidence relation among ideals I of R such that $R/I \otimes f$ is not injective or $R/I \otimes \operatorname{Coker} f$ is not R/I-flat. Replacing R by R/I with I maximal one as above, we may assume that $I = 0$. We set $C := \operatorname{Coker} f$ and $K := \operatorname{Ker} f$. Note that for a nonzero ideal J of R, $R/J \otimes C$ is R/J-flat and $R/J \otimes f$ is injective.

Let J be a nonzero ideal of R. Then, from the short exact sequence

$$0 \to P/K \to F \to C \to 0,$$

we have an exact sequence

$$0 \to \operatorname{Tor}_1^R(R/J, C) \to R/J \otimes P/K \to F/JF \to C/JC \to 0$$

and an isomorphism $\operatorname{Tor}_1^R(R/J, P/K) \cong \operatorname{Tor}_2^R(R/J, C)$. On the other hand, as the map $R/J \otimes f : P/JP \to F/JF$ is injective, the canonical map $R/J \otimes P \to R/J \otimes P/K$ is isomorphic and $\operatorname{Tor}_1^R(R/J, C) = 0$ for any nonzero ideal J of R. In particular, C is R-flat, and hence so is P/K. So the inclusion map $K \hookrightarrow P$ is R-pure, and K is R-flat. This shows that $R/J \otimes K = 0$ for any nonzero ideal J of R.

Now what we want to prove is $K = 0$. If R is not a domain, then any prime ideal $\mathfrak{p}$ of R is nonzero. As the R-module R has a filtration with successive subquotient is of the form $R/\mathfrak{p}$ with $\mathfrak{p} \in \operatorname{Spec} R$, we have that $K = 0$, since $R/\mathfrak{p} \otimes K = 0$ for any $\mathfrak{p}$. Hence, we may assume that R is a domain. Consider the case **3** holds. As the map $K = R \otimes K \to \kappa(0) \otimes K = 0$ is injective, we have that $K = 0$. Next, we consider the case P is R-finite and **4** holds. In this case, as we have $K/\mathfrak{m}K = 0$ and K is R-finite, we have $K = 0$ by Nakayama's lemma.

Finally, assuming that P is R-projective and **4** holds, we prove **1** holds, and this completes the proof of the lemma. We may assume that R is local as well, and hence P is R-free by Kaplansky's theorem [9]. Let B be a free basis of P, and we denote the set of finite subsets of B by Λ. For $\lambda \in \Lambda$, we denote the free summand of P generated by λ by P_λ. Let us denote the composite map $P_\lambda \hookrightarrow P \to F$ by f_λ. Then, we have $K = \varinjlim \operatorname{Ker} f_\lambda = 0$ and $C = \varinjlim \operatorname{Coker} f_\lambda$ is R-flat, as each P_λ is R-finite free. □

Corollary 2 *Let P be an R-flat module, and assume that $P(\mathfrak{p}) = 0$ for $\mathfrak{p} \in \operatorname{Spec} R$. Then, we have $P = 0$.*

Proof Apply the lemma to the map $P \to 0$. □

Lemma 3 *Let P be an R-projective module, and M an R-finite pure submodule of P. Then, $M \hookrightarrow P$ splits, and M and P/M are R-projective.*

Proof As P is a direct summand of a free module, we may assume that P is an R-free module with a basis B. As M is R-finite, there exists some finite subset B_0 of B such that M is contained in the R-span P_0 of B_0. Let us denote the R-span of $B \setminus B_0$ by P_1. Then, we have $P/M \cong P_0/M \oplus P_1$. Hence, replacing P by P_0, we may assume that P is R-finite, which case is easy. $\square$

Lemma 4 *Let $R \to K$ be an injective homomorphism of commutative rings, P an R-projective module, M_K an K-finite submodule of $P_K := K \otimes P$. Then, $M_K \cap P$ is R-finite.*

Proof As P is a direct summand of a free module, we may assume that P is an R-free module with a basis B. As M_K is K-finite, there exists some finite subset B_0 of B such that M_K is contained in the K-span $K \cdot B_0$ of B_0. As $R \to K$ is injective, we have

$$M_K \cap P \subset (K \cdot B_0) \cap P = R \cdot B_0,$$

and $M_K \cap P$ is R-finite.

Corollary 5 *Let R be reduced, P an R-projective module, M an R-submodule of P. If $\dim_{\kappa(\mathfrak{p})} M(\mathfrak{p}) < \infty$ for any $\mathfrak{p} \in \operatorname{Min} R$, then M is R-finite.*

Proof Let us set K to be the total quotient ring $\prod_{\mathfrak{p} \in \operatorname{Min} R} R_{\mathfrak{p}}$ of R and $M_K := M \otimes K$. As we have $M \subset M_K \cap P$, we have that M is R-finite by the lemma. $\square$

Lemma 6 (Universal Coefficient Theorem) *Let $\mathbb{F}$ be an R-flat complex and M an R-module. Assume that $\mathbb{F}$ is bounded above (i.e., $F^i = 0$ for $i \gg 0$) or M is of finite flat dimension. Then, there exists some spectral sequence*

$$E_2^{p,q} = \operatorname{Tor}_{-p}^R(H^q(\mathbb{F}) \otimes M) \Rightarrow H^{p+q}(\mathbb{F} \otimes M).$$

If we have $\operatorname{flat.dim}_R M \leq 1$, *then we have $E_2^{p,q} = E_\infty^{p,q}$, and there exists some exact sequence*

$$0 \to H^n(\mathbb{F}) \otimes M \to H^n(\mathbb{F} \otimes M) \to \operatorname{Tor}_1^R(H^{n+1}(\mathbb{F}), M) \to 0.$$

Proof Let $\mathbb{G}$ be a flat resolution of M of finite length. Consider the spectral sequence for the double complex $\mathbb{F} \otimes \mathbb{G}$. $\square$

Definition 7 We say that an R-complex

$$\mathbb{F} : 0 \to F^0 \to F^1 \to F^2 \to \cdots$$

is *u-acyclic* if for any R-module M, $H^i(\mathbb{F} \otimes M) = 0$ ($i > 0$) and the canonical map

$$\rho_M : H^0(\mathbb{F}) \otimes M \to H^0(\mathbb{F} \otimes M)$$

is isomorphic.

Lemma 8 *Let $\mathbb{F}$ be a u-acyclic R-complex. Then, $R' \otimes \mathbb{F}$ is a u-acyclic R'-complex for any base change $R \to R'$. Moreover, for any R-module M, $\mathbb{F} \otimes M$ is u-acyclic.*

Proof Easy. □

Lemma 9 *Let*

$$0 \to \mathbb{F} \xrightarrow{f} \mathbb{G} \to \mathbb{H} \to 0$$

be an exact sequence of R-complexes. Assume that $\mathbb{F}$ is u-acyclic and $f^i : F^i \to G^i$ is R-pure for $i \geq 0$. Then, $\mathbb{G}$ is u-acyclic if and only if $\mathbb{H}$ is u-acyclic.

Proof Easy. □

Lemma 10 *Let $(\mathbb{F}_\lambda)_{\lambda \in \Lambda}$ be a filtered inductive system of u-acyclic R-complexes. Then, $\varinjlim \mathbb{F}_\lambda$ is u-acyclic.*

Proof Obvious.

Lemma 11 *Let $(R, \mathfrak{m})$ be a local ring, and F an R-flat module. Then, there exists some exact sequence of R-modules*

$$0 \to P \to F \to G \to 0 \tag{1}$$

with P being R-free, G being R-flat, and $G/\mathfrak{m}G = 0$.

Proof Take an $\kappa(\mathfrak{m})$-basis B of $F(\mathfrak{m})$, and let P be the R-free module with the basis B. The canonical map $P = R \cdot B \to \kappa(\mathfrak{m}) \cdot B = F/\mathfrak{m}F$ is lift to a map $\varphi : P \to F$. When we set $G := \operatorname{Coker} \varphi$, then obviously we have $G/\mathfrak{m}G = 0$, and by Lemma 1, φ is injective and G is R-flat. □

Corollary 12 *Let $(R, \mathfrak{m})$ be a local ring, F an R-flat module, and c a non-negative integer. If $\dim_{\kappa(\mathfrak{p})} F(\mathfrak{p}) = c$ for any $\mathfrak{p} \in \operatorname{Spec} R$, then, we have $F \cong R^c$.*

Proof We take an exact sequence (1) as in the lemma. As we have $\dim_{\kappa(\mathfrak{m})} P(\mathfrak{m}) = c$, we have $P \cong R^c$. This shows that $G(\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \operatorname{Spec} R$ by dimension counting. By Corollary 2, we have $G = 0$, and hence we have $F \cong R^c$. $\square$

Lemma 13 *Let*

$$\mathbb{F} : F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} F^2$$

be an R-flat complex. Then, we have the following.

1 *If $H^1(\mathbb{F} \otimes R/\mathfrak{p})$ is R-finite (resp. 0) for any $\mathfrak{p} \in \operatorname{Spec} R$, then $H^1(\mathbb{F} \otimes M)$ is R-finite (resp. 0) for any R-module (resp. any R-finite module) M.*

2 *If $H^1(\mathbb{F} \otimes R/\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \operatorname{Spec} R$, then*

$$F^0 \to F^1 \to F^2 \to \operatorname{Coker} d^1 \to 0$$

is a u-acyclic R-flat complex, and $\operatorname{Ker} d^0$ *is an R-pure submodule of F^0 (hence is R-flat).*

Proof Easy. $\square$

Let M be an R-module. We say that M is *R-metafinite* if there exists some noetherian commutative R-algebra A and an A-finite module structure of M which induces the original R-module structure of M via restriction.

Lemma 14 *Let M be a metafinite R-module. Then, for any R-finite module N, $N \otimes M$ is R-metafinite. If S is a multiplicatively closed subset of R, then M_S is R_S-metafinite. If $M(\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \operatorname{Spec} R$, then we have $M = 0$.*

Proof Easy.

Our main theorem is as follows:

Theorem 15 *Let*

$$\mathbb{F} : 0 \to F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} F^2 \to \cdots$$

be an R-flat complex. Consider the following conditions.

1 *$H^i(\mathbb{F} \otimes R/\mathfrak{p}) = 0$ $(i > 0)$ and $H^0(\mathbb{F} \otimes R/\mathfrak{p})$ is R-finite for any $\mathfrak{p} \in \operatorname{Spec} R$.*

2 *$H^0(\mathbb{F})$ is R-finite projective and $\mathbb{F}$ is u-acyclic.*

3 *$H^0(\mathbb{F})$ is R-finite and $H^i(\mathbb{F}) = 0$ $(i > 0)$.*

4 *$H^i(\mathbb{F}(\mathfrak{p})) = 0$ $(i > 0)$ and $h^0_{\mathbb{F}}(\mathfrak{p}) := \dim_{\kappa(\mathfrak{p})} H^0(\mathbb{F}(\mathfrak{p}))$ is finite for $\mathfrak{p} \in \operatorname{Spec} R$, and $h^0_{\mathbb{F}}$ is a locally constant function on $\operatorname{Spec} R$.*

5 *$H^i(\mathbb{F}(\mathfrak{m})) = 0$ $(i > 0)$ for $\mathfrak{m} \in \operatorname{Max} R$.*

Then, we have the following.

(16) *We have* **5**$\Leftarrow$**4**$\Leftarrow$**1**$\Leftrightarrow$**2**$\Rightarrow$**3**.

(17) *If* $\mathrm{gl.dim}\, R < \infty$ *or* $\mathbb{F}$ *is bounded, then* **3**$\Rightarrow$**2**.

(18) *If* F^0 *is* R*-projective, then* **4**$\Rightarrow$**1**.

(19) *Let* n *be a nonnegative integer. Assume that* $H^i(\mathbb{F}(\mathfrak{p})) = 0$ $(i \geq n)$, *and* $H^n(\mathbb{F} \otimes R/\mathfrak{p})$ *is* R*-metafinite for any* $\mathfrak{p} \in \operatorname{Spec} R$. *Then,* $H^i(\mathbb{F} \otimes M) = 0$ $(i \geq n)$ *for any* R*-module* M. *In particular, if* $H^1(\mathbb{F} \otimes R/\mathfrak{p})$ *is* R*-metafinite for any* $\mathfrak{p} \in \operatorname{Spec} R$, *then we have* **4**$\Rightarrow$**1**.

(20) *Let* n *be a nonnegative integer. Assume that* $H^i(\mathbb{F}(\mathfrak{m})) = 0$ $(i \geq n)$, *and* $H^n(\mathbb{F} \otimes R/\mathfrak{p})$ *is* R*-finite for any* $\mathfrak{p} \in \operatorname{Spec} R$. *Then,* $H^i(\mathbb{F} \otimes M) = 0$ $(i \geq n)$ *for any* R*-module* M. *In particular, if* $H^1(\mathbb{F} \otimes R/\mathfrak{p})$ *is* R*-finite for any* $\mathfrak{p} \in \operatorname{Spec} R$ *and* $H^0(\mathbb{F})$ *is* R*-finite, then we have* **5**$\Rightarrow$**1**.

Proof The implication **1**$\Rightarrow$**2** is obvious from Lemma 13. It is trivial that **2** implies **1**,**3** and **4**. The implication **4**$\Rightarrow$**5** is also trivial. Hence, (16) follows.

The assertion (17) immediately follows from Lemma 6.

We prove (18). By Lemma 5, we have that $H^0(\mathbb{F} \otimes R/\mathfrak{p})$ is R-finite for any $\mathfrak{p} \in \operatorname{Spec} R$. So it suffices to show that $H^i(\mathbb{F} \otimes R/\mathfrak{p}) = 0$ $(i > 0)$ for $\mathfrak{p} \in \operatorname{Spec} R$. We may and shall assume that R is a domain. Moreover, we may localize at maximal ideals of R, and we may assume that $(R, \mathfrak{m})$ is local. So we shall assume that $(R, \mathfrak{m})$ is a d-dimensional local domain, and we proceed by induction on d. If $d = 0$, then R is a field and there is nothing to be proved. Hence, we consider the case $d > 0$.

We take an element $0 \neq x \in \mathfrak{m}$. By induction assumption, any proper localization of $\mathbb{F}$ is u-acyclic. Moreover, for any non-zero ideal I of R, $\mathbb{F} \otimes R/I$ is u-acyclic. Hence, it suffices to show $H^i(\mathbb{F}) = 0$ for $i > 0$. This shows that $\operatorname{supp} H^i(\mathbb{F}) \subset \{\mathfrak{m}\}$ for $i > 0$. As R is a domain, we have $\mathrm{proj.dim}_R\, R/Rx = 1$. By Lemma 6, we have an exact sequence

$$0 \to H^i(\mathbb{F}) \otimes R/Rx \to H^i(\mathbb{F} \otimes R/Rx) \to \operatorname{Tor}_1^R(H^{i+1}(\mathbb{F}), R/Rx) \to 0.$$

Hence, for $i \geq 2$, we have

$$\operatorname{soc} H^i(\mathbb{F}) = \operatorname{Hom}_R(R/\mathfrak{m}, H^i(\mathbb{F})) \subset \operatorname{Hom}_R(R/xR, H^i(\mathbb{F})) \cong \operatorname{Tor}_1^R(R/xR, H^i(\mathbb{F})) = 0.$$

As $\operatorname{supp} H^i(\mathbb{F}) \subset \{\mathfrak{m}\}$, we have $H^i(\mathbb{F}) = 0$ for $i \geq 2$. Hence, it suffices to show that $H^1(\mathbb{F}) = 0$. By Lemma 13, we have that $\operatorname{Ker} d^1$ is an R-pure submodule of F^1, and is R-flat. Hence, replacing $\mathbb{F}$ by the R-flat complex

$$0 \to F^0 \to \operatorname{Ker} d^1 \to 0,$$

we may assume that $F^i = 0$ $(i \geq 2)$ without loss of generality.

As F^0 is R-projective and R is local, F^0 is R-free by Kaplansky's theorem [9]. We take a basis B of F^0. As $\dim_{\kappa(\mathfrak{m})} H^0(\mathbb{F}(\mathfrak{m})) < \infty$, there exists some finite subset B_0 of B such that $H^0(\mathbb{F}(\mathfrak{m}))$ is contained in the $\kappa(\mathfrak{m})$-span of B_0 in $F^0(\mathfrak{m}) = \kappa(\mathfrak{m}) \cdot B$. Now we set $G^0 := R \cdot B_0$ and $Q := R \cdot (B \setminus B_0)$. When we denote the composite map

$$Q \hookrightarrow F^0 \xrightarrow{d^0} F^1$$

by φ, we have φ is injective and $G^1 := \operatorname{Coker} \varphi$ is R-flat by Lemma 1. The composite map

$$G^0 \hookrightarrow F^0 \xrightarrow{d^0} F^1 \to G^1$$

gives an R-flat complex $\mathbb{G}$ of length one, and we have a short exact sequence of R-flat complexes

$$0 \to (\mathrm{id}_Q : Q \xrightarrow{1_Q} Q) \to \mathbb{F} \xrightarrow{\pi} \mathbb{G} \to 0.$$

As π and $\pi(frakp)$ ($\mathfrak{p} \in \operatorname{Spec} R$) are quasi-isomorphisms, replacing $\mathbb{F}$ and $\mathbb{G}$, we may assume that F^0 is R-finite free without loss of generality.

As the sequence

$$0 \to H^0(\mathbb{F}(\mathfrak{p})) \to F^0(\mathfrak{p}) \to F^1(\mathfrak{p}) \to 0 = H^1(\mathbb{F}(\mathfrak{p}))$$

is exact, $\dim_{\kappa(\mathfrak{p})} F^1(\mathfrak{p})$ is finite and constant on $\operatorname{Spec} R$ by assumption. By Corollary 12, we have that F^1 is R-finite, and hence so is $H^1(\mathbb{F})$. As we have $H^1(\mathbb{F}) \otimes R/Rx \subset H^1(\mathbb{F} \otimes R/Rx) = 0$, $H^1(\mathbb{F}) = 0$ by Nakayama's lemma, and this completes the proof of (18).

(19) is proved using similar reduction steps (and is easy, because $H^n(\mathbb{F}) = 0$ follows from Lemma 14). The proof of (20) is also similar, and is proved using Nakayama's lemma. □

Remark 21 Let $f : X \to Y$ be a proper morphism of locally noetherian schemes, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module flat over Y. Then, $R^i f_* \mathcal{F} = 0$ $(i > 0)$ if and only if $H^i(f^{-1}(y), \mathcal{F} \otimes_{\mathcal{O}_Y} \kappa(y)) = 0$ $(i > 0)$. In this case, $f_*\mathcal{F}$ is a locally free $\mathcal{O}_Y$-module. To verify this, we may assume that $Y = \operatorname{Spec} R$ is affine, as the question is local on Y. Let $\mathbb{F}$ be a Čech complex $C^\bullet(\mathfrak{U}, \mathcal{F})$ for some finite affine open cover $\mathfrak{U}$ of X. Then, "if" part and the local freeness of $f_*\mathcal{F}$ follows from (20). The "only if" part is a consequence of (17). This result, which is a special case of the results due to Grothendieck, is found in [5, Corollaire III.7.9.10] (although "if" part is not shown). It seems that (19) and (20) has been known, at least as a folklore, and what is essentially new here is only (18).

Example 22 Projectivity of F^0 is really necessary to guarantee the implication **4⇒1**. Let $(R, \mathfrak{m})$ be a DVR, and we set $K := \kappa(0)$, and we denote the canonical inclusion $R \hookrightarrow K$ by i. Let $\mathbb{F}$ be the R-flat complex

$$0 \to R \oplus K \xrightarrow{{}^t(i,0)} K \to 0.$$

Then, it is easy to see that the condition **4** is satisfied, but $\mathbb{F}$ is not even acyclic. Note that $H^1(\mathbb{F}) = K/R$ is not metafinite.

Example 23 Even if the condition **4** is satisfied and $\mathbb{F}$ is u-acyclic, $H^0(\mathbb{F})$ is not necessarily R-finite. Let $R := \mathbb{Z}$, and we set $F^0 := \{a/b \mid a, b \in \mathbb{Z} \text{ and } b \text{ is square-free}\}$. We define $\mathbb{F}$ as F^0, concentrated in degree zero. Then, $\mathbb{F}$ is a u-acyclic flat complex, **4** is satisfied, but $H^0(\mathbb{F}) = F^0$ is not R-finite.

3 Application to comodule theory over arbitrary base

In this section, we briefly review some applications of Theorem 15. For detail, see [6, Chapter III].

Let X^+ be an ordered set. We use interval notation for X^+ freely. Namely, for $\lambda, \mu \in X^+$, we set

$$[\lambda, \mu) := \{\nu \in X^+ \mid \lambda \leq \nu < \mu\}$$
$$(-\infty, \lambda] := \{\nu \in X^+ \mid \nu \leq \lambda\}$$

and so on. For a subset Q of X^+, we say that Q is a *poset ideal* of X^+ if $\lambda \in Q$ implies $(-\infty, \lambda] \subset Q$ for $\lambda \in X^+$.

Lemma 24 *Let X^+ be an ordered set. Then, the following are equivalent.*

1 *There exists some order-preserving injective map $f : X^+ \to \mathbb{N}$ such that $f(X^+)$ is a poset ideal of $\mathbb{N}$.*

2 *$\#X^+ < \infty$, or there exists some order-preserving bijective map $g : X^+ \to \mathbb{N}$.*

3 *X^+ is countable, and $\#(-\infty, \lambda] < \infty$ for any $\lambda \in X^+$.*

Let $R = k$ be a field.

Definition 25 If C is a k-coalgebra and (X^+, Δ, ∇, L) satisfies the following condition, then we say that (X^+, Δ, ∇, L) is a *weak highest weight theory* (resp. *highest weight theory*) over C.

a X^+ is an ordered set which satisfies the equivalent conditions in Lemma 24.

b $\Delta = (\Delta_C(\lambda))$, $\nabla = (\nabla_C(\lambda))$ and $L = (L_C(\lambda))$ are families of finite dimensional C-comodules parameterized by X^+.

c $L_C(\lambda)$ is simple for $\lambda \in X^+$, and any simple C-comodule is isomorphic to some $L_C(\lambda)$.

d $\operatorname{top}(\Delta_C(\lambda)) \cong L_C(\lambda) \cong \operatorname{soc}(\nabla_C(\lambda))$ for $\lambda \in X^+$.

e Any simple subquotient of $\operatorname{rad}\Delta_C(\lambda)$ is isomorphic to $L_C(\mu)$ for some $\mu < \lambda$.

e* Any simple subquotient of $\nabla_C(\lambda)/\operatorname{soc}(\nabla_C(\lambda))$ is isomorphic to $L_C(\mu)$ for some $\mu < \lambda$.

f If $\lambda, \mu \in X^+$ and $\lambda \neq \mu$, then $\operatorname{Hom}_C(\Delta_C(\lambda), \nabla_C(\mu)) = 0$.

g For $\lambda, \mu \in X^+$ and $i = 1$ (resp. $i = 1, 2$), $\operatorname{Ext}^i_C(\Delta_C(\lambda), \nabla_C(\mu)) = 0$.

If $\operatorname{End}_C L_C(\lambda) \cong k$ for $\lambda \in X^+$ moreover, then we say that (X^+, Δ, ∇, L) is *split*.

Remark 26 Let k be a field. We say that $(\mathcal{C}, X^+, \nabla, L)$ is a highest weight category over k [1] if

i $\mathcal{C}$ is a locally finite k-category, where an abelian category $\mathcal{A}$ is called locally finite if it has a small set of generators consisting of objects of finite lengths and it has an exact inductive limit of arbitrary filtered inductive system.

ii X^+ is an *interval-finite* ordered set in the sense that X^+ is an ordered set such that for any $\lambda, \mu \in X^+$, $[\lambda, \mu]$ is finite.

ii $L = (L(\lambda))_{\lambda \in X^+}$ is a complete set of non-isomorphic simple objects of $\mathcal{C}$ parameterized by X^+.

iii $\nabla = (\nabla(\lambda))_{\lambda \in X^+}$ is a family of objects of $\mathcal{C}$, and for each $\lambda \in X^+$, a monomorphism $L(\lambda) \to \nabla(\lambda)$ is specified, and any simple subquotient of $\nabla(\lambda)/L(\lambda)$ is isomorphic to some $L(\mu)$ for $\mu < \lambda$.

iv For $\lambda, \mu \in X^+$, $\dim_k \operatorname{Hom}_{\mathcal{C}}(\nabla(\lambda), \nabla(\mu)) < \infty$.

v For any $\lambda, \mu \in X^+$, there exists some subobject N of finite length of $A(\lambda)$ such that $A(\lambda)/N$ does not have $L(\mu)$ as its subquotient.

vi For any $\lambda \in X^+$, the injective envelope $I(\lambda)$ of $L(\lambda)$ has a filtration

$$0 = F_0(\lambda) \subset F_1(\lambda) \subset F_2(\lambda) \subset \cdots$$

such that

A $F_1(\lambda) \cong \nabla(\lambda)$

B For $n > 1$, there exists some $\mu = \mu(n) > \lambda$ such that $F_n(\lambda)/F_{n-1}(\lambda) \cong \nabla(\mu)$.

C For $\mu \in X^+$, $\mu(n) = \mu$ for only finitely many n.

D $\varinjlim F_i(\lambda) = I(\lambda)$.

It is easy to see that if (X^+, Δ, ∇, L) is a highest weight theory over a k-coalgebra C, then $(\mathbb{M}^C, X^+, \nabla, L)$ is a highest weight category over k, see [6]. Conversely, if $(\mathcal{C}, X^+, \nabla, L)$ is a highest weight category over k and X^+ satisfies the equivalent conditions in Lemma 24, then there exists some k-coalgebra C, an equivalence of k-categories $F : \mathcal{C} \rightarrow \mathbb{M}^C$, and a family Δ' of C-comodules such that $(X^+, \Delta', F(\nabla), F(L))$ is a highest weight theory over C.

If X^+ is finite, then the construction of C is easy. It is easy to see that $I := \bigoplus_{\lambda \in X^+} I(\lambda)$ is of finite length this case, and C is defined to be the opposite coalgebra of the dual coalgebra of the endomorphism algebra $\operatorname{End}_{\mathcal{C}}(I)$ (finite dimensional!). The case X^+ infinite is done using stratification argument (in this case, C may be infinite dimensional).

To construct such an F, it suffices to construct an equivalence $F_f : \mathcal{C}_f \rightarrow \mathbb{M}_f^C$ of such type, because $\nabla(\lambda)$ is finite length for $\lambda \in X^+$ under the condition. Note that a locally noetherian k-category $\mathcal{A}$ is completely determined by $\mathcal{A}_f$ because it is equivalent to the functor category $\operatorname{Sex}_k(\mathcal{A}_f^{\mathrm{op}}, {}_k\mathbb{M})$,[1] where Sex_k denotes the category of left exact k-functors, see [4]. Now F_f is given by $(\operatorname{Hom}_{\mathcal{C}}(?, \bigoplus_{\lambda \in X^+} I(\lambda)))^*$.

Thus, if X^+ satisfies the conditions in Lemma 24, then a highest weight category is always expressed in terms of a highest weight theory over a coalgebra. The reason we work on coalgebras is that the coalgebra case admits a very easy generalization to the work over arbitrary base, and the conditions in Lemma 24 is satisfied by many examples.

Note that if (X^+, Δ, ∇, L) is a highest weight theory over C, then any of Δ, ∇ and L determine others.

From now on, let R be a noetherian commutative ring. Let C be an R-flat coalgebra.

[1] This notation is due to Gabriel and explained thus: sinister **ex**act.

Definition 27 We say that (X^+, Δ, ∇) is a semisplit highest weight theory over C if the following conditions are satisfied.

a X^+ is an ordered set which satisfies the equivalent conditions in Lemma 24.

b $\Delta = (\Delta_C(\lambda))_{\lambda \in X^+}$ and $\nabla = (\nabla_C(\lambda))_{\lambda \in X^+}$ are families of R-finite projective C-comodules indexed by X^+.

c For any $\mathfrak{p} \in \operatorname{Spec} R$, there exists some family of finite dimensional $C(\mathfrak{p})$-comodules $\Delta(\mathfrak{p})$ indexed by X^+ such that $(X^+, \Delta(\mathfrak{p}), \nabla(\mathfrak{p}), L(\mathfrak{p}))$ is a split highest weight theory over $C(\mathfrak{p})$, where $\nabla(\mathfrak{p}) := (\nabla_C(\lambda)(\mathfrak{p}))$ and $L(\mathfrak{p}) := (\operatorname{soc}_{C(\mathfrak{p})}(\nabla_C(\lambda)(\mathfrak{p})))$.

Definition 28 Let C be an R-coalgebra. We say that $D \subset C$ is an *R-subcoalgebra* of C if D is an R-pure submodule of C, and $\Delta_C(D) \subset D \otimes D$ holds.

If D is an R-subcoalgebra of an R-coalgebra C, then D itself is an R-coalgebra, and the inclusion map $D \hookrightarrow C$ is an R-coalgebra map. If C is R-flat, then so is D, and the restriction functor res^D_C is fully faithful exact. A D-comodule is identified with a C-comodule M such that $\omega_M(M) \subset M \otimes D$.

Definition 29 Let (X^+, Δ, ∇) be a semisplit highest weight theory over C. We denote the set of poset ideals (resp. finite poset ideals) of X^+ by Π (resp. Π_f). A family $(C_\pi)_{\pi \in \Pi_f}$ of is called a *Donkin system* associated to (X^+, Δ, ∇) if the following conditions are satisfied.

1 $C_\emptyset = 0$.

2 For $\pi, \pi' \in \Pi_f$, $C_\pi \subset C_{\pi'}$ if $\pi \subset \pi'$.

3 Let $\pi \in \Pi_f$, and λ a maximal element in π, and we set $\pi' := \pi \setminus \{\lambda\}$. Then, we have an isomorphism of (C, C)-bicomodules

$$C_\pi / C_{\pi'} \cong R(\lambda)^* \otimes \Delta_C(\lambda)^* \otimes \nabla_C(\lambda),$$

where $R(\lambda) := \operatorname{Hom}_C(\Delta_C(\lambda), \nabla_C(\lambda))$.

Lemma 30 *Let (X^+, Δ, ∇) be a semisplit highest weight theory over C, and $R \to R'$ a homomorphism of commutative rings. Then, (X^+, Δ', ∇') is a semisplit highest weight theory over C', where $(?)'$ denotes the functor $R' \otimes ?$. If (C_π) is a Donkin system associated to (X^+, Δ, ∇) moreover, then (C'_π) is a Donkin system associated to (X^+, Δ', ∇').*

Lemma 31 *Let (X^+, Δ, ∇) be a semisplit highest weight theory over C. Then, $(X^+, \nabla^*, \Delta^*)$ is a semisplit highest weight theory over the opposite coalgebra C^{op}. If (C_π) is a Donkin system associated to (X^+, Δ, ∇) moreover, then (C_π^{op}) is a Donkin system associated to $(X^+, \nabla^*, \Delta^*)$.*

As an application of Theorem 15, we have the following

Theorem 32 *Let R be a noetherian commutative ring, C an R-flat coalgebra, and (X^+, Δ, ∇) a semisplit highest weight theory over C. Then, the following are equivalent.*

1 *There exists some Donkin system associated to (X^+, Δ, ∇).*

2 *There exists some Donkin system associated to (X^+, Δ, ∇) uniquely.*

3 *C is projective as an R-module.*

By definition of Donkin system, the theorem is viewed as a generalization of Donkin-Koppinen's bimodule filtration theorem [11].

From now on, let C be an R-projective R-coalgebra, (X^+, Δ, ∇) a semisplit highest weight theory over C, and (C_π) the associated Donkin system. For $\pi \in \Pi$, we define $C_\pi := \varinjlim C_\rho$, where ρ runs through all finite poset ideals of π.

As in Donkin's paper [3], we have the following.

Theorem 33 *Let $\pi \in \Pi$, and V and W be C_π-comodules. Then, the canonical map*

$$\mathrm{Ext}^i_{C_\pi}(V, W) \to \mathrm{Ext}^i_C(V, W)$$

is an isomorphism for $i \geq 0$.

Corollary 34 *If $\pi \in \Pi$, then $(\pi, \Delta(\pi), \nabla(\pi))$ is a semisplit highest weight theory of C_π, and (C_ρ) is its associated Donkin system, where $\Delta(\pi) := (\Delta_C(\lambda))_{\lambda \in \pi}$, $\nabla(\pi) := (\nabla_C(\lambda))_{\lambda \in \pi}$, and ρ runs through all finite poset ideals of π.*

For $\pi \in \Pi_f$, C_π is R-finite projective. We define the *Schur algebra* with respect to π to be the dual algebra of C_π, and denote it by S_π. Note that $\mathbb{M}^{C_\pi}$ and ${}_{S_\pi}\mathbb{M}$ are equivalent. As in [1], we have the following.

Theorem 35 *Let V and W be R-finite C-comodules. Then, $\mathrm{Ext}^i_C(V, W)$ is R-finite for $i \geq 0$.*

4 Examples

The following is another application of Theorem 15.

Lemma 36 *Let $f : C \to D$ be an R-coalgebra map of R-flat coalgebras. Let V be a D-comodule. Assume*

1 *C and V are R-projective*

2 *For any $\mathfrak{p} \in \operatorname{Spec} R$, $R^i \operatorname{ind}_{D(\mathfrak{p})}^{C(\mathfrak{p})} V(\mathfrak{p}) = 0$ for $i > 0$.*

3 $\dim_{\kappa(\mathfrak{p})} \operatorname{ind}_{D(\mathfrak{p})}^{C(\mathfrak{p})} V(\mathfrak{p})$ *is finite and locally constant on* $\operatorname{Spec} R$.

Then, we have $\operatorname{ind}_D^C V$ is R-finite projective, and for any commutative R-algebra R' and any R'-module M, we have $R^i \operatorname{ind}_{D'}^{C'}(M \otimes V) = 0$ $(i > 0)$ and the canonical map

$$M \otimes \operatorname{ind}_D^C V \to \operatorname{ind}_{D'}^{C'}(M \otimes V)$$

is isomorphic, where $C' := R' \otimes C$ and $D' := R' \otimes D$.

Proof Note that if I is an R-injective module, then $I \otimes D$ is an injective D-comodule. For an R-module N, $I^\bullet \otimes D$ is a D-injective resolution of $N \otimes D$ for an R-injective resolution $I^\bullet$ of N, as D is R-flat. This shows that $R^i \operatorname{ind}_D^C(N \otimes D) = 0$ $(i > 0)$ and $\operatorname{ind}_D^C(N \otimes D) = N \otimes D$. In particular, we have that the cobar resolution

$$\operatorname{Cobar}_D V : 0 \to V \otimes D \to V \otimes D^{\otimes 2} \to \cdots$$

of V is an ind_D^C-acyclic resolution of V. Hence, we have that $R^i \operatorname{ind}_D^C V$ is the ith cohomology of the complex

$$\operatorname{ind}_D^C \operatorname{Cobar}_D V : 0 \to V \otimes C \to V \otimes D \otimes C \to \cdots.$$

Applying Theorem 15 to this R-flat complex, we are done. □

Let k be an algebraically closed field. An k-group scheme G is called reductive, if it is k-smooth (of finite type), affine, connected, and its radical (maximal connected normal solvable subgroup) is a torus (a finite direct product of $\mathbb{G}_m = GL_1$). Let R be a noetherian commutative ring.

An R-group scheme G is called reductive if it is R-smooth, affine of finite type over R and all geometric fibers of G is reductive in the sense of the last paragraph. A torus T over R is an affine flat R-group scheme of finite type whose geometric fibers are tori. A torus T is called split if it is isomorphic to a finite direct product of $\mathbb{G}_m$. For split torus $T = \mathbb{G}_m^n$, a T-module is identified with a $\mathbb{Z}^n$-graded R-module. A maximal torus T of G is a closed subgroup, which is a torus, such that all geometric fibers of T is a maximal (with respect to incidence relation) torus. A reductive R-group G is called split if there is a split maximal torus T of G such that each graded component $(\operatorname{Lie} G)_\alpha$ of the adjoint representation $\operatorname{Lie} G$ as a T-module is an R-free module. Note that any split reductive group is defined over $\mathbb{Z}$, and any reductive group over a strictly Henselian local ring is split.

We assume that G is non-trivial. The set of roots Σ of G forms an abstract root system, and we take a base Δ of Σ. Thus, the set of positive roots is determined, which we denote by Σ^+. For each root α, the root subgroup U_α of G is determined. We denote the product $\prod_{\alpha\in\Sigma^+} U_\alpha$ (in any order) by U. The semidirect product $UT = TU$ is denoted by B. Note that U is an affine space over R, and B is smooth affine of finite type over R.

Theorem 37 *Let G be an R-flat infinitesimally flat affine R-group scheme of finite type with connected fibers. Then, the coordinate ring $R[G]$ of G is R-projective.*

Proof See [6, Theorem II.2.2.5]. □

In particular, any reductive R-group scheme has a projective coordinate ring, see also [15] and [14].

Let us consider a split reductive R-group G. Let us denote the set of dominant weights of G by X_G^+. For $\lambda \in X_G^+$, we denote the corresponding rank-one R-free B-module by R_λ. We denote the induced module $\operatorname{ind}_B^G R_\lambda$ by $\nabla_G(\lambda)$, and we denote $\nabla_G(-w_0\lambda)^*$ by $\Delta_G(\lambda)$.

The following is an immediate consequence of Lemma 36.

Proposition 38 *Let $\lambda \in X^+$. Then, the following hold.*

1 (Kempf's vanishing). *We have $R^i \operatorname{ind}_B^G(R_\lambda) = 0$ for $i > 0$.*

2 (Universal freeness). *We have $\nabla_G(\lambda)$ is R-finite free. If R' is a commutative R-algebra, then we have that the canonical map $R' \otimes \nabla_G(\lambda) \to \nabla_{R'\otimes G}(\lambda)$ is an isomorphism.*

Proof Note that the assertion **1** is well-known as Kempf's vanishing theorem [10] when R is a field. Moreover, if R is a field, then $\dim_R \nabla_G(\lambda)$ is finite and independent of R by Weyl's character formula, see [8, p.250]. By Lemma 36, the assertions follows. □

Remark 39 As we know that G/B is proper over $\operatorname{Spec} R$, we can prove the proposition without using (18). In fact, as we know that $R^i \operatorname{ind}_B^G(R_\lambda)$ is R-finite for $i \geq 0$, it suffices to invoke (20), which is well-known and easier to prove.

By the proposition and by Theorem 37, immediately we have

Example 40 Let G be a split reductive R-group. Then, $(X_G^+, \Delta_G, \nabla_G)$ is a semisplit highest weight theory with a unique associated Donkin system.

Note that integral Schur algebras were constructed by S. Donkin [3] by different method.

There is an example to which (19) or (20) is not applicable, but our main theorem (18) is applicable.

Let q be a unit of R. Let $H := R[\mathrm{Mat}_q(n)]$ be the R-algebra generated by x_{ij} over R with the fundamental relation

$$\begin{aligned} &x_{ik}x_{il} = q^{-1}x_{il}x_{ik}, \qquad x_{ik}x_{jk} = q^{-1}x_{jk}x_{ik}, \qquad x_{il}x_{jk} = x_{jk}x_{il} \\ &x_{jl}x_{ik} - x_{ik}x_{jl} + (q^{-1} - q)x_{il}x_{jk} = 0 \qquad (1 \le i < j \le n,\ 1 \le k < l \le n). \end{aligned}$$

Defining R-algebra maps $\Delta : H \to H \otimes H$ and $\varepsilon : H \to R$ by $\Delta(x_{ij}) := \sum_{l=1}^{n} x_{il} \otimes x_{lj}$ and $\varepsilon(x_{ij}) = \delta_{ij}$, respectively, H is an R-bialgebra, where δ_{ij} denotes the Kronecker delta. The bialgebra H is called a *quantum matrix space*, see [13]. See also [7] for some discussion over arbitrary base ring. Note that $SE^{\vee}$ in [7] agrees with $R[\mathrm{Mat}_{q^{-1}}(n)]$. Note that H is not commutative or cocommutative in general. We have an identity

$$D := \sum_{\sigma \in \mathfrak{S}_n} (-q)^{-l(\sigma)} x_{\sigma(1)\,1} x_{\sigma(2)\,2} \cdots x_{\sigma(n)\,n} = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{-l(\sigma)} x_{1\,\sigma(1)} x_{2\,\sigma(2)} \cdots x_{n\,\sigma(n)},$$

see [13, p.42] and [7, Lemma 9.2]. It is known that D is a central group-like element in H [12]. Moreover, D is transcendental over R, and H is $R[D]$-free by the straightening formula [7, Theorem 9.6]. Now we define $R[SL_q(n)] := H/(D-1)$ and $R[GL_q(n)] := H[D^{-1}]$.

Lemma 41 *Both $R[SL_q(n)]$ and $R[GL_q(n)]$ are R-projective.*

Proof As H is $R[D]$-free, $R[SL_q(n)] = H/(D-1)$ is $R[D]/(D-1)$-free, hence is R-free. As an R-module, $R[GL_q(n)]$ is an inductive limit of the inductive system

$$\mathbb{H} : H \xrightarrow{D} H \xrightarrow{D} H \to \cdots,$$

and the multiplication by D on H is R-pure, since H is $R[D]$-free. As H is R-free, we have $R[GL_q(n)] = \varinjlim \mathbb{H}$ is R-Mittag-Leffler by [6, Lemma I.3.2.10]. As H is a finitely generated R-algebra, it is countably generated as an R-module. Hence, so is $R[GL_q(n)]$, as it is a countable union of H. As $R[GL_q(n)]$ is an R-Mittag-Leffler module of countable type, it is R-projective. $\square$

Induced modules and Weyl modules are defined for these quantized linear groups, see [13]. In [13, Chapter 10], a q-analogue of Kempf's vanishing and Weyl's character formula (over a field) are proved. Hence, we have Kempf's vanishing over arbitrary base and universal freeness of induced and Weyl modules as well. The proof is similar to that of Proposition 38. Explicit free basis for induced modules (only for polynomial representations, but this is not restrictive, essentially) are found in [7]. Now applying Theorem 32, we have a base ring independent q-independent bicomodule filtration also for $R[GL_q(n)]$ and $R[SL_q(n)]$, and we have Schur algebras for any poset ideal of the set of dominant weights. As a result, we know that $R[GL_q(n)]$ and $R[SL_q(n)]$ are R-free. As a special case, this family of Schur algebra includes the q-Schur algebra $\mathcal{S}_q(n,r)$ (see [13, 7], see also [2]) over arbitrary base R.

Note that for this case, we do not have any sufficiently powerful geometric machineries. The author for example does not know how to prove the finiteness of $\mathrm{Ext}^i_{SL_q(n)}(V,W)$ for R-finite $SL_q(n)$-modules V and W without using (18), which is new here.

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A note on sums of geometrically linked ideals

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In this note, we will give some results on Betti numbers of Gorenstein graded algebras constructed by sums of geometrically linked ideals, which are part of joint work in progress with A. V. Geramita and Y. S. Shin [3].

1 Two conjectures

It is a standard fact of linkage theory that the sum $I+J$ of two geometrically linked Cohen-Macaulay ideals I and J is an Gorenstein ideal of grade one greater [9]. This provides a way to construct Gorenstein ideals from Cohen-Macaulay ideals of grade one smaller. It seems natural to ask what properties are preserved under this construction, as we pass from I or J to $I+J$ ([12], for example). Firstly, we state two conjectures concerning Betti numbers of Gorenstein ideals which are sums of geometrically linked ideals.

1) Let I and J be two geometrically linked (with respect to $(\alpha) = (\alpha_1, \cdots, \alpha_r)$) Cohen-Macaulay homogeneous ideals in $R = k[x_0, x_1, \ldots, x_n]$ of grade $r(< n+1)$.
We consider the following property **(P)**.

(P) I (resp. $I+J$) has the maximal graded Betti numbers among all the C-M (resp. Gorenstein) homogeneous ideals with the same Hilbert function.

A. M. Bigatti [1] and H. A. Hulett [8], independently, showed that the lex-segment ideal with a given Hilbert function has the maximal Betti numbers among all homogeneous ideals with the same Hilbert function. We would like to consider a question what sort of Gorenstein ideals have the property **(P)**. Put

$$\sigma(R/I) = \mathrm{Min}\{i \mid \Delta^{\dim R/I} H(R/I, i) = 0\},$$

where $H(R/I,i)$ is the Hilbert function of R/I and $\Delta^j H(R/I,i) = \Delta^{j-1}H(R/I,i) - \Delta^{j-1}H(R/I,i-1)$, that is , $\sigma(R/I)$ =(the socle degree of R/I)+1.

Conjecture 1 ([3]) Assume that $2\sigma(R/I) \leq \sigma(R/(\alpha))$. If I has the property **(P)**, then $I+J$ also has the property **(P)**.

2) R/I has a minimal graded free resolution (as a graded R-module) of the form

$$\mathbf{F}_. : 0 \longrightarrow F_r \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 = R(0) \longrightarrow R/I \longrightarrow 0,$$

where

$$F_i = \bigoplus_{j=1}^{b_i} R(-a_{i,j}).$$

The integers b_i and $a_{i,j}$ are uniquely determined. We note that

$$b_i = \dim_k \operatorname{Tor}_i^R(R/I, k)$$

for all $1 \leq i \leq r$. We call $\beta_i = b_i$ the i-th graded Betti number of R/I. Furthermore we say that

$$\beta_{i,j} = \dim_k \operatorname{Tor}_i^R(R/I, k)_j$$

is the (i, j)-th graded Betti number of R/I. Also we call $\{a_{i,1}, a_{i,2}, \ldots, a_{i,b_i}\}$ the numerical characters of R/I. Put

$$c = \deg \alpha_1 + \deg \alpha_2 + \cdots + \deg \alpha_r.$$

Conjecture 2 ([3]) Assume that $2\sigma(R/I) \leq \sigma(R/(\alpha))$. Then the numerical characters of $R/I+J$ are as follows:

$$\begin{aligned}
1\text{-th}: &\quad \{a_{1,1}, a_{1,2}, \ldots, a_{1,b_1}, c - a_{r,1}, c - a_{r,2}, \ldots, c - a_{r,b_r}\};\\
2\text{-th}: &\quad \{a_{2,1}, a_{2,2}, \ldots, a_{2,b_2}, c - a_{r-1,1}, c - a_{r-1,2}, \ldots, c - a_{r-1,b_{r-1}}\};\\
\vdots\ &\quad \vdots\\
i\text{-th}: &\quad \{a_{i,1}, a_{i,2}, \ldots, a_{i,b_i}, c - a_{r-i+1,1}, c - a_{r-i+1,2}, \ldots, c - a_{r-i+1,b_{r-i+1}}\};\\
\vdots\ &\quad \vdots\\
r\text{-th}: &\quad \{a_{r,1}, a_{r,2}, \ldots, a_{r,b_r}, c - a_{1,1}, c - a_{1,2}, \ldots, c - a_{1,b_1}\};\\
(r+1)\text{-th}: &\quad \{c\}.
\end{aligned}$$

2 Some results

Let

$$\mathbf{K}. : 0 \longrightarrow K_r \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_0 = R(0) \longrightarrow R/(\alpha) \longrightarrow 0$$

be the Koszul complex of $R/(\alpha)$. Then the inclusion $(\alpha) \subset I$ can be lifted to a degree preserving map of complexes $\psi : \mathbf{K}. \longrightarrow \mathbf{F}.$. Let $[g_1, \ldots, g_{b_r}]$ be the matrix of the map $\psi_r : K_r \longrightarrow F_r$.

Peskine and Szpiro [9] showed a fundamental theorem of linkage theory: The mapping cone of the dual of $\psi : \mathbf{K}. \longrightarrow \mathbf{F}.$ is a free resolution of R/J. The following is obvious from this theorem.

Proposition 2.1 *Let $\{f_1, \ldots, f_{b_1}\}$ be a minimal generators of I. Then*

(1) *$I + J$ is generated by $\{f_1, \ldots, f_{b_1}, g_1, \ldots, g_{b_r}\}$.*

(2) $\nu(I+J) \le \nu(I) + r(R/I)$,

where $\nu(\ )$ *is the number of minimal generators of an ideal and* $r(\ )$ *is the C-M type of an C-M ring.*

First we prove the following theorem.

Theorem 2.2 *Assume that* $2\sigma(R/I) \le \sigma(R/(\alpha))$. *Then*

(1) $I+J$ *is minimally generated by* $\{f_1, \ldots, f_{b_1}, g_1, \ldots, g_{b_r}\}$.

(2) $\nu(I+J) = \nu(I) + r(R/I)$.

PROOF. First we note that $a_{1,b_1} \le \sigma(R/I)$, $\sigma(R/I) = a_{r,b_r} - r + 1$ and $\sigma(R/(\alpha)) = c - r + 1$. Hence since $2\sigma(R/I) \le \sigma(R/(\alpha))$, it follows that

$$2(a_{r,b_r} - r + 1) \le c - r + 1, \text{ i.e., } a_{1,b_1} \le a_{r,b_r} - r + 1 \le c - a_{r,b_r}.$$

Thus

$$a_{1,1} \le a_{1,2} \le \cdots \le a_{1,b_1} \le c - a_{r,b_r} \le c - a_{r,b_r-1} \le \cdots \le c - a_{r,1}.$$

Here we recall a result concerning Hilbert functions under linkage (cf. [7, Theorem 2.1(4)]). Hence it follows that

$$\Delta^d H(R/I+J, i) = \Delta^d H(R/I, i) \text{ for all } 0 \le i \le \sigma(R/(\alpha)) - \sigma(R/I) - 1,$$

where $d = \dim R/I + J$. Since $2\sigma(R/I) \le \sigma(R/(\alpha))$, it follows that

$$\sigma(R/I) - 1 \le \sigma(R/(\alpha)) - \sigma(R/I) - 1.$$

Hence

$$\Delta^d H(R/I+J, i) = \Delta^d H(R/I, i) \text{ for all } 0 \le i < \sigma(R/I).$$

Thus

$$H(R/I+J, i) = H(R/I, i) \text{ for all } 0 \le i < \sigma(R/I).$$

Therefore

$$\dim_k (I+J)_i = \dim_k I_i$$

for every $0 \le i < \sigma(R/I)$, i.e., $(I+J)_i = I_i$ for such i since $I \subset I+J$. Hence it follows that $\{f_i \mid \deg f_i < \sigma(R/I)\}$ is a part of a minimal generators of $I+J$. Furthermore we note that if $a_{1,b_1} = \sigma(R/I)$ then

$$\{f_i \mid \deg f_i = \sigma(R/I)\} = \{f_t, f_{t+1}, \ldots, f_{b_1}\}$$

is linearly independent in $(I+J)_{\sigma(R/I)}/R_1(I+J)_{\sigma(R/I)-1}$. In fact, if $\{f_i \mid \deg f_i = \sigma(R/I)\}$ is not linearly independent in $(I+J)_{\sigma(R/I)}/R_1(I+J)_{\sigma(R/I)-1}$, then

$$\sum_{i=t}^{b_1} y_i f_i \in R_1(I+J)_{\sigma(R/I)-1}$$

for some $y_i \in k$. We may assume that $y_t \neq 0$. Hence

$$f_t + \sum_{i=t+1}^{b_1} y'_i f_i \in R_1(I+J)_{\sigma(R/I)-1}$$

for some $y'_i \in k$. Here we note that $R_1(I+J)_{\sigma(R/I)-1} = R_1 I_{\sigma(R/I)-1}$. Thus

$$f_t \in (f_1, \ldots, f_{t-1}, f_{t+1}, \ldots, f_{b_1}).$$

This is a contradiction. Hence it follows that $\{f_i \mid \deg f_i = \sigma(R/I)\}$ is linearly independent in $(I+J)_{\sigma(R/I)}/R_1(I+J)_{\sigma(R/I)-1}$. Therefore the minimal generators $\{f_1, \ldots, f_{b_1}\}$ of I is a part of a minimal generators of $I+J$.

Now assume that

$$g_j \in (f_1, \ldots, f_{b_1}, g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_{b_r}).$$

Then obviously $\nu(I+J/I) < b_r$, that is, $\nu(I+J/I) < r(R/I)$. On the other hand, we know that $\nu(I+J/I) = r(R/I)$ (cf. [12, Proposition 1.2(c)]). This is a contradiction. Thus our assertion (1) holds. Also (2) is obvious from (1). q.e.d.

As a corollary of this theorem, we obtain the following theorem.

Theorem 2.3 *Conjecture 2 is true for the* 1*-th numerical characters.*

Remark 2.4 It follows from [7, Theorem 2.1(3)] that Conjecture 2 is also true for the $(r+1)$-th numerical characters.

In the following remark we recall a result from the proof of [11, Theorem 4.1].

Remark 2.5 Let

$$0 \longrightarrow R(-s) \longrightarrow M_r \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 = R(0) \longrightarrow R/I+J \longrightarrow 0$$

be a minimal graded free resolution of $R/I+J$, where

$$M_i = \bigoplus_{j=1}^{q_i} R(-p_{i,j}) \text{ and } p_{i,1} \leq p_{i,2} \leq \cdots \leq p_{i,q_i}.$$

Then by the duality of a Gorenstein algebra, we have

- $q_i = q_{r+1-i}$ for $1 \leq i \leq r$ and

- $p_{i,j} + p_{r+1-i,q_{r+1-i}-j+1} = s$ for $1 \le i \le r$ and $1 \le j \le q_i$.

Theorem 2.6 *Conjecture 2 is true in the case $r = 2$.*

PROOF. This follows from Theorem 2.3, Remark 2.4 and Remark 2.5. q.e.d.

Theorem 2.7 *Conjecture 2 is true in the case $r = 3$.*

We need a lemma to prove Theorem 2.7.

Lemma 2.8 *Let I and J be two C-M homogeneous ideals in $R = k[x_0, x_1, \ldots, x_n]$ which are geometrically linked. Then*

$$F(R/I+J,\lambda) = F(R/I,\lambda) + (-1)^d \lambda^{\sigma-d-2} F(R/I,\frac{1}{\lambda}),$$

where $d = \dim R/I+J$ and $\sigma = \sigma(R/I \cap J)$.

PROOF. First we note that

$$F(R/I+J,\lambda) = \frac{\sum \Delta^{d+1} H(R/I+J,i)\lambda^i}{(1-\lambda)^{d+1}} \quad \text{and} \quad F(R/I,\lambda) = \frac{\sum \Delta^{d+1} H(R/I,i)\lambda^i}{(1-\lambda)^{d+1}}.$$

Furthermore, from [7, Theorem 2.1(2)], we have

$$\Delta^d H(R/I+J,i) = \Delta^d H(R/I,i) + \Delta^d H(R/I,\sigma-2-i) - e(R/I),$$

where $e(R/I)$ is the multiplicity of R/I. Hence

$$\Delta^{d+1} H(R/I+J,i) = \Delta^{d+1} H(R/I,i) - \Delta^{d+1} H(R/I,\sigma-1-i).$$

Thus

$$\begin{aligned}
& F(R/I,\lambda) + (-1)^d \lambda^{\sigma-d-2} F(R/I,\tfrac{1}{\lambda}) \\
= \; & \frac{\sum \Delta^{d+1} H(R/I,i)\lambda^i}{(1-\lambda)^{d+1}} + (-1)^d \lambda^{\sigma-d-2} \frac{\sum \Delta^{d+1} H(R/I,i)\frac{1}{\lambda^i}}{(1-\frac{1}{\lambda})^{d+1}} \\
= \; & \frac{\sum \Delta^{d+1} H(R/I,i)\lambda^i}{(1-\lambda)^{d+1}} - \frac{\sum \Delta^{d+1} H(R/I,i)\lambda^{\sigma-1-i}}{(1-\lambda)^{d+1}} \\
= \; & \frac{\sum \{\Delta^{d+1} H(R/I,i) - \Delta^{d+1} H(R/I,\sigma-1-i)\}\lambda^i}{(1-\lambda)^{d+1}} \\
= \; & \frac{\sum \Delta^{d+1} H(R/I+J,i)\lambda^i}{(1-\lambda)^{d+1}} \\
= \; & F(R/I+J,\lambda).
\end{aligned}$$

q.e.d.

Remark 2.9 We recall that $I \cap J = (\alpha_1, \dots, \alpha_r)$ and $c = \deg \alpha_1 + \cdots + \deg \alpha_r$. Since $\mathrm{ht}(I+J) = r+1$, we have $d = (n+1) - (r+1) = n - r$. Furthermore since $\mathrm{ht}(I \cap J) = r$, we can check that $\sigma = c - r + 1$. Hence $\sigma - d - 2 = c - n - 1$.

PROOF OF THEOREM 2.7. Let

$$0 \longrightarrow M_4 \longrightarrow M_3 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 = R(0) \longrightarrow R/I+J \longrightarrow 0$$

be a minimal graded free resolution of $R/I+J$. Then it follows from Theorem 2.3, Remark 2.4 and Remark 2.5 that

$$\begin{aligned} M_1 &= \{\textstyle\bigoplus_{j=1}^{b_1} R(-a_{1,j})\} \bigoplus \{\bigoplus_{j=1}^{b_3} R(-(c-a_{3,j}))\} \\ M_3 &= \{\textstyle\bigoplus_{j=1}^{b_3} R(-a_{3,j})\} \bigoplus \{\bigoplus_{j=1}^{b_1} R(-(c-a_{1,j}))\} \\ M_4 &= R(-c). \end{aligned}$$

Put

$$M_2 = \bigoplus_{j=1}^{q_2} R(-p_{2,j}).$$

Hence

$$\begin{aligned} F(R/I+J,\lambda) = \tfrac{1}{(1-\lambda)^{n+1}}\{1 - (\textstyle\sum_{j=1}^{b_1} \lambda^{a_{1,j}} + \sum_{j=1}^{b_3} \lambda^{c-a_{3,j}}) \\ +(\textstyle\sum_{j=1}^{q_2} \lambda^{p_{2,j}}) - (\sum_{j=1}^{b_3} \lambda^{a_{3,j}} + \sum_{j=1}^{b_1} \lambda^{c-a_{1,j}}) + \lambda^c\}. \end{aligned}$$

On the other hand, it follows from Lemma 2.8 and Remark 2.9 that

$$F(R/I+J,\lambda) = F(R/I,\lambda) + (-1)^{n-r}\lambda^{c-n-1}F(R/I,\frac{1}{\lambda}).$$

Hence

$$\begin{aligned} &F(R/I+J,\lambda) \\ = &\ \tfrac{1-\sum_{j=1}^{b_1}\lambda^{a_{1,j}}+\sum_{j=1}^{b_2}\lambda^{a_{2,j}}-\sum_{j=1}^{b_3}\lambda^{a_{3,j}}}{(1-\lambda)^{n+1}} \\ &+(-1)^{n-3}\lambda^{c-n-1}\tfrac{1-\sum_{j=1}^{b_1}\frac{1}{\lambda^{a_{1,j}}}+\sum_{j=1}^{b_2}\frac{1}{\lambda^{a_{2,j}}}-\sum_{j=1}^{b_3}\frac{1}{\lambda^{a_{3,j}}}}{(1-\frac{1}{\lambda})^{n+1}} \\ = &\ \tfrac{1}{(1-\lambda)^{n+1}}\{1 - (\textstyle\sum_{j=1}^{b_1}\lambda^{a_{1,j}} + \sum_{j=1}^{b_3}\lambda^{c-a_{3,j}}) + (\sum_{j=1}^{b_2}\lambda^{a_{2,j}} + \sum_{j=1}^{b_2}\lambda^{c-a_{2,j}}) \\ &-(\textstyle\sum_{j=1}^{b_3}\lambda^{a_{3,j}} + \sum_{j=1}^{b_1}\lambda^{c-a_{1,j}}) + \lambda^c\}. \end{aligned}$$

Thus

$$\sum_{j=1}^{q_2} \lambda^{p_{2,j}} = \sum_{j=1}^{b_2} \lambda^{a_{2,j}} + \sum_{j=1}^{b_2} \lambda^{c-a_{2,j}}.$$

Therefore we can easily check that

$$M_2 = \{\bigoplus_{j=1}^{b_2} R(-a_{2,j})\} \bigoplus \{\bigoplus_{j=1}^{b_2} R(-(c - a_{2,j}))\}.$$

q.e.d.

Theorem 2.10 *Conjecture 1 is true in the case $r = 2$.*

PROOF. Put

$$\alpha(R/I) = \mathrm{Min}\{i \mid \Delta^{\dim R/I} H(R/I, i) \ \text{ is not generic}\},$$

i.e., $\alpha(R/I)$ is the initial degree of I. It follows from an inequality of Dubreil that if I has the property (P), then

$$\nu(I) = \alpha(R/I) + 1.$$

Hence noting that $r(R/I) = \nu(I) - 1 = \alpha(R/I)$, we can check from Theorem 2.2(2) that

$$\nu(I + J) = 2\alpha(R/I) + 1.$$

Furthermore we can check from [7, Theorem 2.1(4)] that $\alpha(R/I + J) = \alpha(R/I)$. Hence

$$\nu(I + J) = 2\alpha(R/I + J) + 1.$$

Thus it follows by virtue of [2, Theorem 3.3] that $I + J$ also has the property (P). q.e.d.

3 Minimal free resolutions of sums of geometrically linked height two ideals

In this section, we would like to add some additional observations concerning the two conjectures for the case of height two (we must omit the proofs).

Let I and J be two geometrically linked (with respect to $(\alpha) = (\alpha_1, \alpha_2)$) C-M homogeneous ideals in $R = k[x_0, x_1, \ldots, x_n]$ of height 2 ($< n + 1$), and put $c = \deg \alpha_1 + \deg \alpha_2$. Furthermore let $\{f_1, \ldots, f_{b_1}\}$ be a minimal generators of I and

$$\mathbf{F}. : 0 \longrightarrow \bigoplus_{j=1}^{b_2} R(-a_{2,j}) \longrightarrow \bigoplus_{j=1}^{b_1} R(-a_{1,j}) \longrightarrow R(0) \longrightarrow R/I \longrightarrow 0$$

the minimal free resolution of R/I.

Theorem 3.1 *Assume that $\nu(I) \leq \nu(J)$. Then the minimal free resolution of $R/I + J$ is*

$$\begin{aligned} 0 \longrightarrow \quad & R(-c) \longrightarrow [\textstyle\bigoplus_{j=1}^{b_2} R(-a_{2,j})] \bigoplus [\bigoplus_{j=1}^{b_1} R(-(c - a_{1,j}))] \longrightarrow \\ & [\textstyle\bigoplus_{j=1}^{b_1} R(-a_{1,j})] \bigoplus [\bigoplus_{j=1}^{b_2} R(-(c - a_{2,j}))] \longrightarrow R(0) \longrightarrow R/I + J \longrightarrow 0. \end{aligned}$$

We need some lemmas to prove this theorem.

Lemma 3.2 *In the same notation as in Section 2, we have the following.*

(1) *J is generated by* $\{g_1, \dots, g_{b_r}, \alpha_1, \dots, \alpha_r\}$.

(2) $\{g_1, \dots, g_{b_r}\}$ *is a part of a minimal generators of* J.

(3) $\{f_\ell, \dots, f_{b_1}, g_1, \dots, g_{b_r}\}$ *is a part of a minimal generators of* $I+J$, *where* $\{f_\ell, \dots, f_{b_1}\}$ *is a minimal generators of* I *modulo* (α).

Lemma 3.3 *In the same notation as above, we have the following.*

(1) $\nu(I/(\alpha)) = b_2 - 1, b_2$ *or* $b_2 + 1$.

(2) *If* $\nu(I/(\alpha)) = b_2 - 1$, *then* $\{g_1, \dots, g_{b_2}\}$ *is a minimal generators of* J.

(3) *If* $\nu(I/(\alpha)) = b_2$, *then* $\{g_1, \dots, g_{b_2}, \alpha_1\}$ *or* $\{g_1, \dots, g_{b_2}, \alpha_2\}$ *is a minimal generators of* J.

(4) *If* $\nu(I/(\alpha)) = b_2 + 1$, *then* $\{g_1, \dots, g_{b_2}, \alpha_1, \alpha_2\}$ *is a minimal generators of* J.

(5) $\nu(I) - 1 \leq \nu(J) \leq \nu(I) + 1$.

Lemma 3.4 *In the same notation as above, we have the following.*

(1) *If* $\nu(I/(\alpha)) = b_2 - 1$, *then* $\{f_3, \dots, f_{b_1}, g_1, \dots, g_{b_2}\}$ *is a minimal generators of* $I + J$, *where* $\{f_3, \dots, f_{b_1}\}$ *is a minimal generators of* I *modulo* (α).

(2) *If* $\nu(I/(\alpha)) = b_2$ *or* $b_2 + 1$, *then* $\{f_1, \dots, f_{b_1}, g_1, \dots, g_{b_2}\}$ *is a minimal generators of* $I + J$.

(3) $\nu(I + J) = 2\min\{\nu(I), \nu(J)\} - 1$.

Question. Is Conjecture 2 true in the case $r = 3$ (and further, in general), without assuming $2\sigma(R/I) \leq \sigma(R/(\alpha))$?

Next we would like to consider a question whether Conjecture 1 is true in the case $r = 2$, without assuming $2\sigma(R/I) \leq \sigma(R/(\alpha))$.

Theorem 3.5 *Assume that* $\nu(I/(\alpha)) \geq b_2$. *If* I *has the property* $(\mathbf{P})$, *then* $I + J$ *also has the property* $(\mathbf{P})$.

Theorem 3.6 *Assume that* $\nu(I/(\alpha)) = b_2 - 1$. *If* J *has the property* $(\mathbf{P})$, *then* $I + J$ *also has the property* $(\mathbf{P})$.

Remark 3.7 If $\nu(I/(\alpha)) = b_2 - 1$, then $2\sigma(R/I) > \sigma(R/(\alpha))$. In fact since $\nu(I/(\alpha)) = b_2 - 1$, it follows that $\{\alpha_1, \alpha_2\}$ is a part of a minimal resolution of I. Hence $2\sigma(R/I) \geq \deg \alpha_1 + \deg \alpha_2 = c$. Thus since $\sigma(R/(\alpha)) = c-1$, we have $2\sigma(R/I) > \sigma(R/(\alpha))$. Therefore the two conditions $\nu(I/(\alpha)) = b_2 - 1$ and $2\sigma(R/I) \leq \sigma(R/(\alpha))$ don't occur at the same time.

The following is an example satisfying

- $2\sigma(R/I) > \sigma(R/(\alpha))$,
- $\nu(I/(\alpha)) = b_2$ and
- I has the property $(\mathbf{P})$.

Thus by Theorem 3.5, it follows that $I+J$ has the property $(\mathbf{P})$. Here we note that Theorem 2.10 can not apply to this example.

Example 3.8 Let X and Y be the two geometrically linked sets of points in P^2 as follows:

$$\begin{array}{ccccc} & \bullet & * & * & Y \\ & \bullet & * & * & \\ X & \bullet & \bullet & * & \end{array}$$

Let I and J be the two ideals of X and Y in $R = k[x_0, x_1, x_2]$, respectively. Then we can check the following: $\sigma(R/I) = 3$, $\sigma(R/(\alpha)) = 5$, $\nu(I) = 3$, $b_2 = 2$ and $\nu(I/(\alpha)) = 2$. Furthermore since $\alpha(R/I) = 2$, we have $\nu(I) = \alpha(R/I) + 1$. Hence I has the property $(\mathbf{P})$.

The following is an example satisfying

- $2\sigma(R/I) > \sigma(R/(\alpha))$, $2\sigma(R/J) > \sigma(R/(\alpha))$,
- $\nu(I/(\alpha)) = b_2 - 1$,
- I has the property $(\mathbf{P})$ and
- J and $I + J$ don't have the property $(\mathbf{P})$.

Thus it follows that Conjecture 1 is not necessarily true in the case of $r = 2$, without assuming $2\sigma(R/I) \leq \sigma(R/(\alpha))$.

Example 3.9 Let X and Y be the two geometrically linked sets of points in P^2 as follows:

$$\begin{array}{ccccc} & \bullet & * & * & Y \\ & \bullet & * & * & \\ X & \bullet & \bullet & \bullet & \end{array}$$

Let I and J be the two ideals of X and Y in $R = k[x_0, x_1, x_2]$, respectively. Then we can check the following: $\sigma(R/I) = 3$, $\sigma(R/(\alpha)) = 5$, $\nu(I) = 3$, $b_2 = 2$ and $\nu(I/(\alpha)) = 1$. Furthermore since $\alpha(R/I) = 2$, we have $\nu(I) = \alpha(R/I) + 1$. Hence I has the property

(**P**). On the other hand we can check that $\nu(J) = 2$, $\alpha(R/I) = 2$, $\nu(I+J) = 3$ and $\alpha(R/I+J) = 2$. Hence since $\nu(J) < \alpha(R/I)+1$ and $\nu(I+J) < 2\alpha(R/I+J)+1$, it follows that J and $I+J$ don't have the property (**P**).

Theorem 3.10 *Let I and J be two geometrically linked C-M ideals.*

(1) *If $\nu(I) \leq \nu(J)$ and I has* (**P**), *then $I+J$ also have* (**P**).

(2) *If I and J have* (**P**), *then $I+J$ also have* (**P**).

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CELLULAR RESOLUTIONS

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This lecture is about recent developments in the study of minimal free resolutions of monomial ideals and toric ideals. All results presented are joint with Dave Bayer and Irena Peeva; details can be found in our papers *Monomial Resolutions* (with D. Bayer and I. Peeva), *Generic Lattice Ideals* (with I. Peeva), and *Cellular Resolutions of Monomial Modules* (with D. Bayer).

Consider a monomial ideal $M = \langle m_1, \ldots, m_r \rangle$ in the polynomial ring $S = k[x_1, \ldots, x_n]$. A well-known resolution of M over S is constructed as follows. For a subset I of $\{1, \ldots, r\}$ we set $m_I := lcm(m_i \mid i \in I)$. Let $a_I \in \mathbf{N}^n$ be the exponent vector of m_I and $S(-a_I)$ the free S-module with one generator in multidegree a_I. The *Taylor resolution* of M is the $\mathbf{Z}^n$-graded module $\mathbf{F} = \bigoplus_{I \subseteq \{1,\ldots,r\}} S(-a_I)$ with basis denoted by $\{e_I\}_{I \subseteq \{1,\ldots,r\}}$ and equipped with the differential

$$d(e_I) \;=\; \sum\nolimits_{i \in I} sign(i, I) \cdot \frac{m_I}{m_{I \setminus i}} \cdot e_{I \setminus i} \, , \tag{1}$$

where $sign(i, I)$ is $(-1)^{j+1}$ if i is the jth element in the ordering of I. Combinatorially this resolution corresponds to the inclusion-exclusion formula for the $\mathbf{N}^n$-graded Hilbert series of S/M:

$$\text{The sum over all monomials not in } M \quad = \quad \frac{\sum_{I \subset \{1,2,\ldots,r\}} (-1)^{|I|} \cdot m_I}{(1 - x_1) \cdots (1 - x_n)}. \tag{2}$$

If $r \gg n$ then Taylor's resolution is far from minimal and most terms in the numerator of (2) cancel.

We define the following simplicial complex the set of generators of M:

$$\Delta_M \;:=\; \big\{ I \subseteq \{1, \ldots, r\} \mid \; m_I \neq m_J \;\text{ for all }\; J \subseteq \{1, \ldots, r\} \;\text{ other than }\; I \big\} \tag{3}$$

We call Δ_M the *Scarf complex* of M. The mathematical economist Herbert Scarf at Yale University introduced it in his work on game theory in the early 1970's. The Scarf complex defines a submodule $\mathbf{F}_{\Delta_M} := \bigoplus_{I \in \Delta_M} S(-a_I)$ of the Taylor resolution $\mathbf{F}$ which is closed under the differential (1).

The minimal free resolution of M always contains the complex $\mathbf{F}_{\Delta_M}$, but it can be larger. However, $\mathbf{F}_{\Delta_M}$ is exact for "almost all monomial ideals" in the following sense. We call a monomial ideal M *generic* if no variable x_i appears to the same degree in two minimal generators of M.

Theorem 1.. *If M is generic, then the Scarf complex $\mathbf{F}_{\Delta_M}$ is a minimal free resolution of M.*

Corollary 2.. *Let M be a generic monomial ideal.*

(1) The number of j-faces of Δ_M equals the total Betti number $\beta_j(M) = dim_k Tor_j^S(M, k)$.

(2) The minimal free resolution of M is characteristic free.

(3) The numerator of the Hilbert series (2) equals $\sum_{I \in \Delta_M} (-1)^{|I|} \cdot m_I$, the $\mathbf{N}^n$-graded Euler characteristic of the Scarf complex, and there are no cancellations in this alternating sum.

Example. Let $M := \langle\, xyz,\, x^4y^3,\, x^3y^5,\, y^4z^3,\, y^2z^4,\, x^2z^2 \,\rangle$. The Scarf complex of this generic monomial ideal consists of two triangles and an edge meeting at a vertex. Hence M has 6 generators, 7 first syzygies and 2 second syzygies. The complex Δ_M is not shellable, but it is contractible.

Theorem 3.. *The Scarf complex Δ_M of a generic monomial ideal M is contractible.*

One technique to apply the Scarf complex to all monomial ideals is *deformation of exponents.* Suppose M is not generic. Let $a_i = (a_{i1}, \ldots, a_{in})$ denote the exponent vector of m_i. Choose vectors $\epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{in}) \in \mathbf{R}^n$ for $1 \leq i \leq r$ such that, for all i and all $s \neq t$, the numbers $a_{is} + \epsilon_{is}$ and $a_{it} + \epsilon_{it}$ are distinct, and $a_{is} + \epsilon_{is} < a_{it} + \epsilon_{it}$ implies $a_{is} \leq a_{it}$. Each vector ϵ_i defines a monomial $\mathbf{x}^{\epsilon_i} = x_1^{\epsilon_{i1}} \cdots x_n^{\epsilon_{in}}$ with real exponents. We formally introduce the generic monomial ideal

$$M_\epsilon \quad := \quad \langle\, m_1 \cdot \mathbf{x}^{\epsilon_1},\, m_2 \cdot \mathbf{x}^{\epsilon_2},\, \ldots,\, m_r \cdot \mathbf{x}^{\epsilon_r} \,\rangle.$$

Let Δ_{M_ϵ} be the Scarf complex of M_ϵ. Let $\mathbf{F}_\epsilon$ be the restriction of Taylor's resolution of M to Δ_{M_ϵ}.

Theorem 4.. *The complex $\mathbf{F}_\epsilon$ is a free resolution of M over S.*

Convex polytopes are a powerful tool for structuring combinatorial data appearing in many branches of algebra. Also the nice properties of the Scarf complex are best understood by looking at a certain polytope P_M in $\mathbf{R}^n$. We fix a sufficiently large real number $T \gg 0$ and define

$$P_M \quad := \quad \text{the convex hull of } \left\{\ (T^{a_{i1}}, T^{a_{i2}}, \ldots, T^{a_{in}}) \ \in \mathbf{R}^n \ \middle|\ 1 \leq i \leq r \ \right\}. \tag{4}$$

The combinatorial type of P_M is independent of the choice of T for large T. Theorem 3 is a consequence of the following convexity result, which is essentially due to Herbert Scarf.

Proposition 5.. *Let M be a generic monomial ideal. Then Δ_M is isomorphic to the subcomplex of the boundary of P_M consisting of all faces supported by a strictly positive inner normal vector.*

Corollary 6.. *If M is artinian and generic, then Δ_M is a regular triangulation of the $(n-1)$-simplex.*

Corollary 2 (1), Theorem 4 and Proposition 5 imply that the Betti numbers $\beta_i(M)$ of *any* monomial ideal M satisfy the inequalities of the *Upper Bound Theorem for Convex Polytopes.*

Corollary 7.. *Let M be any monomial ideal with r generators in n variables. Then*

$$\begin{array}{rrcll} & \beta_i(M) & \leq & c_i(n,r) & \quad \textit{for } 1 \leq i \leq n-2, \\ \textit{and} & \beta_{n-1}(M) & \leq & c_{n-1}(n,r) - 1. & \end{array}$$

where $c_i(n,r)$ denotes the number of i-dimensional faces of the cyclic n-polytope with r vertices.

These inequalities are not tight in general. For instance, for $n = 4$ and $r = 13$, Agnarsson showed that $\beta_1(M) \leq 77$, and this bound is tight, while $c_1(4, 13) = 78$. This gap is not yet well understood. There are fascinating connections to the dimension theory for partially ordered sets.

Here is a second approach to resolving non-generic monomial ideals which preserves given structures (e.g. symmetry) much better than Theorem 4. For any monomial ideal M define the

Scarf polytope P_M as in (4). Let H_M be the polyhedral complex consisting of all bounded faces of the polyhedron $P_M + \mathbf{R}^n_+$. For a face $F \in H_M$ we let m_F be the least common multiple of monomials indexed by vertices of F, and let a_F be the exponent vector of m_F. On pairs of faces there is an incidence function $\epsilon(F, F')$ which takes values $\{0, 1, -1\}$, is 0 unless F' is a facet of F, and compares orientations of F' and F when nonzero. The *hull resolution* of M is the $\mathbf{Z}^n$-graded module $\bigoplus_{F \in H_X} S(-a_F)$ with basis denoted by $\{e_F\}_{F \in H_X}$ and equipped with the differential

$$d(e_F) \quad = \quad \sum\nolimits_{F' \in H_X} \epsilon(F', F) \cdot \frac{m_F}{m_{F'}} \cdot e_{F'} \,. \tag{5}$$

Theorem 8.. *The hull resolution of any monomial ideal M is a free resolution.*

The hull resolution may be minimal even if M is non-generic. This happens, for instance, for the ideal M generated by the monomials $x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_n^{\sigma_n}$ where σ runs over all permutations of $\{1, 2, \ldots, n\}$. Here P_M is the permutohedron, and the i-th syzygies of M are just the i-faces of P_M.

The idea of the hull resolution and its minimality for generic objects can be extended to the setting of *toric varieties* as follows. Let $\mathcal{L}$ be any sublattice of $\mathbf{Z}^n$. Its associated *lattice ideal* is

$$I_{\mathcal{L}} \quad := \quad \langle \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \, : \, \mathbf{a}, \mathbf{b} \in \mathbf{N}^n \text{ and } \mathbf{a} - \mathbf{b} \in \mathcal{L} \rangle,$$

where monomials are denoted $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ for $\mathbf{a} = (a_1, \ldots, a_n)$. We call a lattice $\mathcal{L}$ generic if the lattice ideal $I_{\mathcal{L}}$ it is generated by binomials with full support, i.e.,

$$I_{\mathcal{L}} \quad = \quad \langle \mathbf{x}^{\mathbf{a}_1} - \mathbf{x}^{\mathbf{b}_1}, \mathbf{x}^{\mathbf{a}_2} - \mathbf{x}^{\mathbf{b}_2}, \ldots, \mathbf{x}^{\mathbf{a}_r} - \mathbf{x}^{\mathbf{b}_r} \rangle \tag{1.1}$$

where none of the r vectors $\mathbf{a}_i - \mathbf{b}_i$ has a zero coordinate. The term "generic" is justified by a recent result in integer programming due to Barany and Scarf. In our setting it translates to:

Theorem 9.. (Barany and Scarf 1996) *The set*

$$\{ \lambda \cdot B \, : \, \lambda > 0, B \in \mathbf{Z}^{n \times d} \text{ and the lattice spanned by the columns of } B \text{ is generic} \}$$

is dense in $\mathbf{R}^{n \times d}$ in the classical topology.

Suppose that $\mathcal{L}$ contains no nonnegative vectors. This ensures that $I_{\mathcal{L}}$ is positively graded. For any finite subset J of $\mathcal{L}$ we define $max(J)$ to be the vector which is the coordinatewise maximum of J. We extend the definition (3) to the infinite subset $\mathcal{L}$ of $\mathbf{Z}^n$ as follows:

$$\Delta_{\mathcal{L}} \, := \, \{ \, J \subset \mathcal{L} \, : \, max(J) \neq max(J') \quad \text{for all finite subset } J' \subset \mathcal{L} \text{ different from } J \}.$$

This is an infinite simplicial complex of dimension $\leq n - 1$. There is a natural action of the lattice $\mathcal{L}$ on $\Delta_{\mathcal{L}}$, since $J \in \Delta_{\mathcal{L}}$ if and only if $J + \mathbf{a} \in \Delta_{\mathcal{L}}$ for any $\mathbf{a} \in \mathcal{L}$. We identify $\Delta_{\mathcal{L}}$ with its poset of non-empty faces, and we form the quotient $\Delta_{\mathcal{L}}/\mathcal{L}$. This poset is called the *Scarf complex* of $\mathcal{L}$.

Lemma 10.. *Let $\mathcal{L}$ be any sublattice of $\mathbf{Z}^n$.*

(a) The simplicial complex $\Delta_{\mathcal{L}}$ is locally finite, i.e., the link of every vertex in $\Delta_{\mathcal{L}}$ is finite.

(b) The Scarf complex $\Delta_{\mathcal{L}}/\mathcal{L}$ is a finite poset.

Consider the quotient poset $\mathbf{N}^n/\mathcal{L}$. Its elements are the congruence classes of monomials modulo $I_{\mathcal{L}}$; they are called *fibers*. The partial order on fibers is $C_2 \preceq C_1$ if and only if $\mathbf{x}^{\mathbf{r}} \cdot C_2 \subseteq C_1$ for some monomial $\mathbf{x}^{\mathbf{r}}$. If C is a fiber then $gcd(C)$ denotes the greatest common divisor of all monomials in C. A fiber C is called *basic* if $gcd(C) = 1$ and $gcd(C \backslash \{\mathbf{x}^{\mathbf{a}}\}) \neq 1$ for all $\mathbf{x}^{\mathbf{a}} \in C$.

Lemma 11.. *Let C be a basic fiber and m a monomial in C. The monomials in $C\backslash\{m\}$ divided by their greatest common divisor form a basic fiber.*

We consider the finite subposet of $(\mathbf{N}^n/\mathcal{L}, \preceq)$ whose elements are all basic fibers.

Theorem 12.. *The poset of basic fibers is isomorphic to the Scarf complex $\Delta_\mathcal{L}/\mathcal{L}$.*

The *algebraic Scarf complex* of a lattice ideal $I_\mathcal{L}$ is the complex of free S-modules

$$\mathbf{F}_\mathcal{L} \quad = \quad \bigoplus_{C \in \Delta_\mathcal{L}/\mathcal{L}} S \cdot E_C\,,$$

where E_C denotes a basis vector in homological degree $|C| - 1$, and the sum runs over all basic fibers C. By Lemma 11 it is well defined to take the differential acting as in (1), namely,

$$\partial(E_C) \quad := \quad \sum\nolimits_{m \in C} sign(m, C) \cdot gcd(C\backslash\{m\}) \cdot E_{C\backslash\{m\}}\,,$$

where $sign(m, C)$ is $(-1)^{l+1}$ if m is in the l'th position in the lexicographic ordering of C.

Theorem 13.. *The complex $\mathbf{F}_\mathcal{L}$ is contained in the minimal free resolution of $I_\mathcal{L}$ over S. If the lattice $\mathcal{L}$ is generic then $\mathbf{F}_\mathcal{L}$ coincides with the minimal free resolution of $I_\mathcal{L}$ over S.*

On a monomial (or lattice) ideal whose minimal free resolution is defined by a simplicial complex

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Abstract

Let $M = (m_1, \dots, m_r)$ be a monomial ideal of $S = k[x_1, \dots, x_n]$. Bayer-Peeva-Sturmfels [1] studied a subcomplex F_Δ of the Taylor resolution, defined by a simplicial complex $\Delta \subset 2^r$. They proved that if M is *generic* (i.e., no variable x_i appears with the same non-zero exponent in two distinct monomials which are minimal generators), F_{Δ_M} is the *minimal* free resolution of S/M, where Δ_M is the *Scarf complex* of M.

In this paper, we prove the following: for a generic (in the above sense) monomial ideal M and each integer $\operatorname{depth} S/M \leq i < \dim S/M$, there is an *embedded* prime $P \in \operatorname{Ass}(S/M)$ of $\dim S/P = i$. Thus a generic monomial ideal with no embedded primes is Cohen-Macaulay (in this case, Δ_M is shellable). We also study a non-generic monomial ideal M whose *minimal* free resolution is F_Δ for some Δ. Especially, we prove that if M is pure dimensional (i.e., all associated primes of M have the same height) then M is Cohen-Macaulay, and Δ is pure and strongly connected.

In Section 3, we will study *lattice ideals.* For a lattice ideal $I_\mathcal{L} \subset S$, Peeva and Sturmfels ([6, 7]) constructed a subcomplex $F_\mathcal{L}$ of the minimal free resolution, which is defined by a simplicial complex. $F_\mathcal{L}$ is analogous to F_{Δ_M} of a monomial ideal M, and has a very simple structure. If $\operatorname{ht} I_\mathcal{L} = 2$ but not a complete intersection, or $I_\mathcal{L}$ is *generic* (see Definition 3.2), then $F_\mathcal{L}$ is acyclic and coincides with the minimal free resolution. In this paper, we will see that a lattice ideal $I_\mathcal{L}$ has some remarkable properties if $F_\mathcal{L}$ is acyclic.

1 Preliminary results

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. S has a natural $\mathbf{N}^n$-grading such that each homogeneous component is a 1-dimensional k-vector space spanned by a single monomial. Let M be a monomial ideal (not necessarily minimally) generated by *monomials* $m_1, \ldots, m_r$ ($m_1, \ldots, m_r$ and M are used in this sense throughout this paper). For a subset $I \subset \{1, \ldots, r\}$ we set $m_I = lcm\{m_i \mid i \in I\}$. Let $a_I \in \mathbf{N}^n$ be the exponent vector of m_I and $S(-a_I)$ the free S-module with one generator in multidegree a_I. *Taylor resolution* of S/M is the $\mathbf{N}^n$ graded module $F = \bigoplus_{I \subset \{1,\ldots,r\}} S(-a_I)$ with basis denoted by $\{e_I\}_{I \subset \{1,\ldots,r\}}$ and equipped with the differential

$$d(e_I) = \sum_{i \in I} \operatorname{sign}(i, I) \cdot \frac{m_I}{m_{I \setminus i}} \cdot e_{I \setminus i},$$

where $\operatorname{sign}(i, I)$ is $(-1)^{j+1}$ if i is the j-th element in the ordering of I. This is an $\mathbf{N}^n$ graded free resolution of S/M over S having length r and 2^r terms. The minimal free resolution of S/M is always an $\mathbf{N}^n$ graded subcomplex of the Taylor resolution F, but F is far from minimal when $r \gg n$.

We say that $\Delta \subset 2^{\{1,\ldots,r\}}$ is a *simplicial complex*, if $I \in \Delta$ and $J \subset I$ always imply $J \in \Delta$. An element of Δ is called a *face*, and the dimension of a face I is defined by $\dim I = |I| - 1$. The dimension of the simplicial complex Δ is $\dim \Delta = \max\{\dim I \mid I \in \Delta\}$. Note that the empty set ϕ is a face (of dimension -1) of any non-empty simplicial complex. Faces of dimension 0 (resp. 1) are called *vertices* (resp. *edges*). Maximal faces under inclusion are called *facets*. A simplicial complex with only one facet is called a *simplex*.

For a simplicial complex $\Delta \subset 2^{\{1,\ldots,r\}}$, we define a submodule $F_\Delta := \bigoplus_{I \in \Delta} S(-a_I)$ of the Taylor resolution F. Since F_Δ is closed under the differential d, F_Δ is a subcomplex of the Taylor resolution.

Definition 1.1 (Bayer-Peeva-Sturmfels) Let $M = (m_1, \ldots, m_r)$ be a monomial ideal. We define a simplicial complex:

$$\Delta_M := \{I \subset \{1, \ldots, r\} \mid m_I \neq m_J \text{ for all } J \subset \{1, \ldots, r\} \text{ other than } I\}.$$

We call Δ_M the *Scarf complex* of M.

For each $1 \leq i \leq r$, $\{i\} \in \Delta_M$ if and only if m_i is a minimal generator of M. It is easy to see that F_{Δ_M} is always contained in the minimal free resolution of S/M as a subcomplex (moreover, a direct summand of F_{Δ_M} is a direct summand of the minimal free resolution too). But F_{Δ_M} is not acyclic in general. For example, if $M = (xy, yz, zx)$ then Δ_M is of the form • • •

and F_{Δ_M} is of the form $0 \to S^3 \to S \to 0$. This is clearly non-acyclic. If $\Delta \subset 2^{\{1,2,3\}}$ is a simplicial complex whose facets are two edges (of course this Δ is not unique), then F_Δ is the minimal free resolution.

Definition 1.2 (Bayer-Peeva-Sturmfels) A monomial ideal M is called *generic* if no variable x_i appears with the same non-zero exponent in two distinct monomials which are minimal generators of M.

$(x^2y^3, x^3z^2, xyz, y^2)$ is a generic monomial ideal, but (xy, xz) is not.

Theorem 1.3 (Bayer-Peeva-Sturmfels) *If M is generic, then the complex F_{Δ_M} defined by the Scarf complex Δ_M is acyclic and gives the minimal free resolution of S/M over S.*

Example 1.4 (1) Any monomial ideal $M \subset k[x, y]$ is always generic and can be written as

$$M = (x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, \ldots, x^{a_r}y^{b_r}),$$

where $a_1 > a_2 > \cdots > a_r$ and $b_1 < b_2 < \cdots < b_r$. The facets of Δ_M are edges $\{1, 2\}, \{2, 3\}, \ldots, \{r-1, r\}$. So the combinatoric property of Δ_M depends only on r. The minimal free resolution F_{Δ_M} is of the form $0 \to S^{r-1} \to S^r \to S \to 0$. When $r \geq 2$, we have that

$$S/M \text{ is Cohen-Macaulay} \iff \dim S/M = 0 \iff a_r = b_1 = 0.$$

Even if M is generic, Δ_M does not determine fundamental properties of M such as $\operatorname{ht} M$, $\dim S/M$ and Cohen-Macaulayness.

(2) Set $M := (xy, xz) \subset k[x, y, z]$. M is not generic, but F_{Δ_M} is the minimal free resolution of S/M (in this case, Δ_M is a 1-simplex). A similar phenomenon occurs when $M = (x^3y^2, y^3z^2, z^3x^2, x^2y^2z^2)$. In this case, the Scarf complex Δ_M is three edges joined at one vertex, and M is Cohen-Macaulay.

If M is a monomial ideal which is not generic, then typically, the minimal free resolution of S/M cannot be written as F_Δ for any Δ. Though Bayer-Peeva-Sturmfels [1] proved that there is a simplicial complex Δ of $\dim \Delta \leq n-1$ such that F_Δ is acyclic.

2 Main Results

Roughly speaking, we first show that a generic monomial ideal M has many *embedded* associated primes if M is not Cohen-Macaulay.

Definition 2.1 Let Δ be a simplicial complex. We say Δ is pure if all its facets are of the same dimension. A pure simplicial complex is *shellable* if the facets of Δ can be given a linear order $I_1, \ldots, I_t$ satisfying the following condition: for all i, j, $1 \leq j < i \leq t$, there exist some $v \in I_i \setminus I_j$ and some $s \in \{1, 2, \ldots, i-1\}$ with $I_i \setminus I_s = \{v\}$. A linear order satisfying the above condition is called a *shelling* of Δ.

Let Δ be a pure simplicial complex. We say Δ is *strongly connected*, if Δ satisfies the following condition: for any two facets I and I', there is a sequence of facets $I_1, I_2, \cdots, I_s$ such that $I = I_1$, $I' = I_s$ and $\dim(I_i \cap I_{i+1}) = \dim \Delta - 1$ for each $1 \leq i \leq s-1$. It is easy to see that a shellable simplicial complex is always strongly connected.

Lemma 2.2 *Let $M = (m_1, \ldots, m_r)$ be a generic monomial ideal, and let P, P' be associated primes of M such that $\operatorname{ht} P < \operatorname{ht} P'$. Then for each integer s such that $\operatorname{ht} P < s \leq \operatorname{ht} P'$, there is an embedded associated prime $Q \in \operatorname{Ass}(S/M)$ of $\operatorname{ht} Q = s$.*

Proof. Choose an integer D larger than the (total) degree of any minimal generator of M. Consider an artinian monomial ideal

$$M^* := M + (x_1^D, x_2^D, \ldots, x_n^D).$$

Here we consider $m_{r+i} = x_i^D$ for $1 \leq i \leq n$, and Δ_{M^*} is a simplicial complex on $\{1, \ldots, r+n\}$. M^* is also generic and the Scarf complex Δ_{M^*} is pure $(n-1)$-dimensional (see [1], also Theorem 2.10 below). If $x_i^d \in M$ for some d, x_i^D is not a minimal generator of M^* and $\{r+i\} \notin \Delta_{M^*}$. Bayer-Peeva-Sturmfels [1] also showed that Δ_{M^*} is shellable (this is a consequence from convex geometry). In particular, Δ_{M^*} is strongly connected.

For a facet I of Δ_{M^*}, we set

$$P_I := (x_i \mid 1 \leq i \leq n \text{ such that } r+i \notin I).$$

By [1, Theorem 8.1], we have $\operatorname{Ass}(S/M) = \{P_I \mid I \text{ is a facet of } \Delta_{M^*}\}$. Since Δ_M is pure $(n-1)$-dimensional, we have $\dim(S/P_I) = |I \cap W| = n - |I \cap V|$ where $W = \{r+1, \cdots, r+n\}$ and $V = \{1, \ldots, r\}$. Hence $\operatorname{ht}(P_I) = |I \cap V|$. There are facets I and I' of Δ_{M^*} such that $P_I = P$ and $P_{I'} = P'$. Since Δ_{M^*} is strongly connected, there is a sequence of facets $I_1, I_2, \cdots, I_s$ such that $I = I_1$, $I' = I_s$ and $\dim(I_i \cap I_{i+1}) = \dim \Delta - 1$ for each $1 \leq i \leq s-1$.

Let i be an integer such that $1 \leq i \leq s-1$. Set $\{c\} := I_i \setminus I_{i+1}$ and $\{d\} := I_{i+1} \setminus I_i$. If $\operatorname{ht} P_{I_i} > \operatorname{ht} P_{I_{i+1}}$, then $c \in V$ and $d \notin V$ (i.e., $c \notin W$ and $d \in W$). Hence we have $\operatorname{ht} P_{I_i} = \operatorname{ht} P_{I_{i+1}} + 1$ and $P_{I_i} \supset P_{I_{i+1}}$. If $\operatorname{ht} P_{I_i} < \operatorname{ht} P_{I_{i+1}}$, then $\operatorname{ht} P_{I_i} = \operatorname{ht} P_{I_{i+1}} - 1$ and $P_{I_i} \subset P_{I_{i+1}}$. So we can prove the assertion.

Theorem 2.3 *Let $M = (m_1, \ldots, m_r)$ be a generic monomial ideal. Then there is an embedded associated prime $P \in \mathrm{Ass}(S/M)$ of $\dim S/P = i$ for each integer i such that* $\operatorname{depth} S/M \leq i < \dim S/M$.

Proof. Let M^* be an artinian monomial ideal defined in the proof of Lemma 2.2, and let $J \in \Delta_M$ be a facet of $\dim J = \dim \Delta_M$. Since Δ_M is a subcomplex of Δ_{M^*}, there is a facet I of Δ_{M^*} such that $J = I \cap \{1, \ldots, r\}$. Let $P_I \in \mathrm{Ass}(S/M)$ be an associated prime defined in the proof of Lemma 2.2. Since F_Δ is the minimal free resolution, we have

$$\dim(S/P_I) = n - |J| = n - (\dim \Delta_M + 1) = n - \mathrm{proj.dim}(S/M) = \mathrm{depth}(S/M).$$

On the other hand, there clearly exists a prime ideal $P \in \mathrm{Ass}(S/M)$ with $\dim S/M = \dim S/P$. So the assertion follows from Lemma 2.2.

Bayer-Peeva-Sturmfels [1] proved that a generic monomial ideal M is Cohen-Macaulay, if M is pure dimensional, i.e., all associated primes of M have the same height. But we can prove a stronger result.

Corollary 2.4 *Let $M = (m_1, \ldots, m_r)$ be a generic monomial ideal. If M has no embedded associated primes, then M is Cohen-Macaulay. In this case, Δ_M is shellable.*

Proof. The former statement immediately follows from Theorem 2.3. So it suffices to prove the shellability of Δ_M. Let M^* be as in the proof of Lemma 2.2, and $I \in \Delta_{M^*}$ a facet. Let $P_I \in \mathrm{Ass}(S/M)$ be as in the proof of Lemma 2.2. We see that $\mathrm{ht}(P_I) = |I \cap \{1, \ldots, r\}|$. Since M is Cohen-Macaulay, we have $|I \cap \{1, \ldots, r\}| = \mathrm{ht}\, M$. In particular, the cardinarity $|I \cap \{1, \ldots, r\}|$ does not depend on the choise of a facet I. So Δ_M is shellable by [3, Theorem 11.13]

If M is not Cohen-Macaulay, Δ_M may be non pure.

Corollary 2.5 *Let M be a generic monomial ideal with* $\dim S/M \geq 2$. *If S/M has FLC (e.g., Buchsbaum), then S/M must be Cohen-Macaulay.*

Proof. If S/M has FLC and $\mathrm{depth}(S/M) \geq 1$, then M is pure dimensional. Hence S/M is Cohen-Macaulay by Corollary 2.4. So we may assume that $\mathrm{depth}(S/M) = 0$ (i.e., $(x_1, \ldots, x_n) \in \mathrm{Ass}(S/M)$). By Theorem 2.3, for each $0 < i < \dim S/M$, there is an associated prime $P \in \mathrm{Ass}(S/M)$ of $\dim S/P = i$. This is a contradiction (in fact, $\ell(H^i_{\mathbf{m}}(S/M)) = \infty$ for all $1 \leq i \leq \dim S/M - 1$, where $\mathbf{m} := (x_1, \ldots, x_n)$ is a graded maximal ideal).

Remark 2.6 The definition of a *generic monomial ideal* can be weakened in the following way (c.f. [2]): If there are distinct minimal (monomial) generators $m_s, m_t \in M$ such that $\deg_{x_i} m_s = \deg_{x_i} m_t > 0$ for some i, then there exists the third monomial $m \in M$ which divides $m' := lcm\{m_s, m_t\}$ and satisfies $\mathrm{supp}(m'/m) = \mathrm{supp}\, m'$. Here we set $\deg_{x_i}(\prod x_i^{a_i}) := a_i$ and $\mathrm{supp}(\prod x_i^{a_i}) := \{i \mid a_i > 0\}$.

If a monomial ideal M satisfies the above condition, then Δ_{M_ϵ} coincides with Δ_M, where M_ϵ is a *degeneration* of M (c.f. [1]). Hence F_{Δ_M} is acyclic. Moreover, Theorem 2.3 and Corollary 2.4 hold under this weaker assumption.

We now study a non-generic monomial ideal M whose minimal free resolution can be written as F_Δ for some Δ. It is easy to see that Δ always contains the Scarf complex Δ_M as a subcomplex. Some results on generic monomial ideals remains valid for M in slightly weaker form.

Remark 2.7 Let $I_\mathcal{L}$ be a *generic lattice ideal* (this notion is introduced by [7]. See Definition 3.2 of this note), and let $M := \mathrm{in}(I_\mathcal{L})$ be the initial ideal of $I_\mathcal{L}$ under a degree reverse lexicographic order. Gasharov, Peeva and Welker [5] proved that F_{Δ_M} is acyclic, although M is not a generic monomial ideal in general (see [7]).

Theorem 2.8 *Let $M = (m_1, \dots, m_r)$ be a (not necessarily generic) monomial ideal. Suppose that there is a simplicial complex Δ on $\{1, \dots, r\}$ such that F_Δ is the minimal free resolution of S/M. If Δ has a facet of dimension $i-1$, then there is an associated prime $P \in \mathrm{Ass}(S/M)$ with $\mathrm{ht}\, P = i$.*

Proof. It is well known that $\dim \mathrm{Ext}^i_S(S/M, S) \leq n - i$, and the equality holds iff there is an associated prime $P \in \mathrm{Ass}(S/M)$ with $\mathrm{ht}\, M = i$ (c.f. [4, Theorem 8.1.1.]).

On the other hand, $\mathrm{Ext}^i_S(S/M, S)$ is the i-th cohomology of the cochain complex $F^*_\Delta := \mathrm{Hom}(F_\Delta, S)$. Let $I \in \Delta$ be a facet of dimension $i-1$, and $e^*_I \in (F^*_\Delta)^i$ the dual base of $e_I \in (F_\Delta)_i$. Since I is a facet, e^*_I is a cocycle of F^*_Δ. So we can regard $e^*_I \in \mathrm{Ext}^c_S(S/M, S)$. For some ideal $L \subset S$, we have

$$S/L \simeq S \cdot e^*_I \subset \mathrm{Ext}^i_S(S/M, S).$$

Note that $|I| = i$ and

$$d(e_I) = \sum_{j \in I} m'_j \cdot e_{I \setminus j},$$

for some monomials $m'_1 \dots, m'_i$. These monomials are non-constant, since F_Δ is minimal. We have $L' := (m'_1, \dots, m'_i) \supset L$, and

$$\dim \mathrm{Ext}^i_S(S/M, S) \geq \dim S/L \geq \dim S/L' \geq n - i,$$

by Krull's theorem. By the remark above, $\dim \operatorname{Ext}_S^i(S/M, S) = n - i$ and there is an associated prime $P \in \operatorname{Ass}(S/M)$ with $\operatorname{ht} P = i$.

Even if M is generic, there is the case that the Scarf complex Δ_M dose not have a facet of dimension $i-1$ though we have $\operatorname{ht} P = i$ for some $P \in \operatorname{Ass}(S/M)$. In fact, there is a generic monomial ideal $M \subset S$ of $\operatorname{ht} M = 1$ such that Δ_M is an $(n-1)$-simplex.

Corollary 2.9 *Let $M = (m_1, \ldots, m_r)$ be a monomial ideal. Suppose that there is a simplicial complex Δ on $\{1, \ldots, r\}$ such that F_Δ is the minimal free resolution of S/M. Then there is an associated prime $P \in \operatorname{Ass}(S/M)$ of $\dim S/P = \operatorname{depth} S/M$.*

Proof. By the previous theorem, there is an associated prime $P \in \operatorname{Ass}(S/M)$ of $\operatorname{ht}(P) = \dim \Delta + 1$. Since F_Δ is minimal, we have $\dim S/P = \operatorname{depth} S/M$ by the same argument as the proof of Theorem 2.3.

Set $M := (xw, yw, zw) \subset k[x, y, z, w]$. Then the Scarf complex Δ_M of M is a 2-simplex, and F_{Δ_M} is the minimal free resolution of S/M. We have $M = (x, y, z) \cap (w)$ and $\operatorname{depth} S/M = 1$. Note that M has no embedded associated primes nor associated primes of height 2. So Theorem 2.3 does not hold for a non-generic monomial ideal M, even if F_{Δ_M} is acyclic.

Corollary 2.10 *Let $M = (m_1, \ldots, m_r)$ be a monomial ideal. Suppose that there is a simplicial complex Δ on $\{1, \ldots, r\}$ such that F_Δ is the minimal free resolution of S/M. If M is pure dimensional, then S/M is Cohen-Macaulay. In this case, Δ is pure and strongly connected.*

Proof. The former assertion immediately follows from Corollary 2.9. The purity of Δ easily follows from Theorem 2.8. We now prove the strongly connectedness. Let $F_\Delta^* := \operatorname{Hom}(F_\Delta, S)$ be the S-dual of the complex F_Δ. By the local duality, F_Δ^* gives the minimal free resolution of the canonical module $\omega_{S/M}$ up to degree shifting (c.f., [4]). If $I \in \Delta$ is a facet, then the dual base e_I^* corresponds to a minimal generator of $\omega_{S/M}$. If a face $L \in \Delta$ of $\dim L = \dim \Delta - 1$ is contained in facets $I_1, \ldots, I_t \in \Delta$, then e_L^* corresponds a relation among $e_{I_1}^* \ldots, e_{I_t}^*$. Conversely, if $G_1 \to G_0 \to \omega_{S/M} \to 0$ is the minimal representation, then G_1 is generated by

$$\{e_L^* \mid L \in \Delta \text{ of } \dim L = \dim \Delta - 1\}.$$

Hence, if Δ is not strongly connected, then $\omega_{S/M}$ is not indecomposable. This is a contradiction.

Corollary 2.11 *Let M be a monomial ideal. Suppose that there is a simplicial complex $\Delta \subset 2^{\{1,\dots,r\}}$ such that F_Δ is the minimal free resolution of S/M. If M is Gorenstein, then it is complete intersection. In particular, when M is generic, S/M is Gorenstein if and only if it is complete intersection.*

Proof. By the Corollary 2.10, Δ is pure. Since F_Δ is the minimal free resolution of S/M, the number of the facets of Δ is equal to the Cohen-Macaulay type of S/M. So Δ is a simplex in this case. Since S/M is Cohen-Macaulay, we have

$$\begin{aligned} \mathrm{ht}(M) = \mathrm{proj.dim}\, S/M &= \dim \Delta + 1 \\ &= \text{the number of vertices of } \Delta \\ &= \text{the minimal number of generators of } M. \end{aligned}$$

Even if M is generic, there is the case that the Scarf complex Δ_M is a simplex but M is not complete intersection. For example, $M = (x^2, xy)$ or $M = (x^2y, y^2z, z^2x)$. That is, there is a non-complete intersection (generic) monomial ideal whose minimal free resolution coincides with the Taylor resolution. The next result follows from Corollary 2.10 immediately (this is maybe a well-known result).

Proposition 2.12 *Let $M = (m_1, \dots, m_r)$ be a (not necessarily generic) monomial ideal. Suppose that the Taylor resolution of M gives the minimal free resolution of S/M. If M is pure dimensional, then M is complete intersection (of course, the Taylor resolution coincides with the Koszul complex in this case).*

Remark 2.13 If F_Δ is acyclic, then it has a structure of a DG-algebra (skew commutative associative differential graded S-algebra) structure (see [1]). Hence the Taylor resolution itself and the minimal free resolution of a generic monomial ideal have DG-algebra structures. But there is a monomial ideal M whose minimal free resolution does not have a DG-algebra structure. The minimal free resolution of this ideal cannot be written as F_Δ for any Δ.

Let M be a codimension 3 Gorenstein monomial ideal. It is well known that the minimal free resolution of S/M has a DG-algebra structure. But if M is not complete intersection, F_Δ is not the minimal free resolution of S/M for any Δ by Theorem 2.11. Thus a DG-algebra structure of the minimal free resolution $\mathbf{F}$ is not a sufficient condition for the existence of a simplicial complex Δ such that $\mathbf{F} \simeq F_\Delta$.

3 Generic Lattice Ideals

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring. Throughout this section, $\mathcal{L}$ is a sublattice of $\mathbf{Z}^n$ which contains no nonnegative vectors. The *lattice ideal* $I_\mathcal{L}$ associated to $\mathcal{L}$ is defined by

$$I_\mathcal{L} := (\, \mathbf{x}^\mathbf{a} - \mathbf{x}^\mathbf{b} \mid \mathbf{a}, \mathbf{b} \in \mathbf{N}^n \;\text{ and }\; \mathbf{a} - \mathbf{b} \in \mathcal{L}\,),$$

where $\mathbf{x}^\mathbf{a} = x_1^{a_1} \cdots x_n^{a_n}$ for $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{N}^n$.

Since $\mathcal{L}$ contains no nonnegative vectors, $I_\mathcal{L}$ is homogeneous with respect to some grading where $\deg(x_i)$ is a *positive* integer for each i. Set $\Gamma := \mathbf{Z}^n/\mathcal{L}$. Then $S/I_\mathcal{L}$ is a Γ-graded ring, and the dimension of each homogeneous component is at most one. It is easy to see that $\operatorname{ht} I_\mathcal{L} = \operatorname{rank} \mathcal{L}$.

Set $\mathcal{L}^{sat} := \{a \in \mathbf{Z}^n \,|\, n \cdot a \in \mathcal{L} \text{ for } n \gg 0\}$. Then the following three conditions are equivalent; (1) $\mathcal{L} = \mathcal{L}^{sat}$, i.e., Γ is torsion free, (2) $I_\mathcal{L}$ is a prime ideal, and (3) $I_\mathcal{L}$ is a toric ideal (i.e., $S/I_\mathcal{L}$ is an *affine* semigroup ring). Even if $I_\mathcal{L}$ is not a prime ideal, all monomials are non-zero divisors in $S/I_\mathcal{L}$, and all associated primes of $I_\mathcal{L}$ have the same height.

Example 3.1 Let $\mathcal{L} \subset \mathbf{Z}^2$ be a lattice generated by $(2, -2)$. Then $I_\mathcal{L} = (x^2 - y^2)$. This is not a prime ideal, i.e., $I_\mathcal{L}$ is a lattice ideal which is not a toric ideal. $S/I_\mathcal{L}$ is a $\mathbf{Z}^2/\langle(2,-2)\rangle$-graded ring, and not an affine semigroup ring. The associated primes of $I_\mathcal{L}$ are $(x - y)$ and $(x + y)$.

Definition 3.2 (Peeva-Sturmfels [7]) We call a lattice ideal $I_\mathcal{L}$ *generic* if it is generated by binomials with full support, i.e.,

$$I_\mathcal{L} = (\, \mathbf{x}^{a_1} - \mathbf{x}^{b_1},\, \mathbf{x}^{a_2} - \mathbf{x}^{b_2},\, \dots,\, \mathbf{x}^{a_r} - \mathbf{x}^{b_r} \,)$$

where none of the r vectors $a_i - b_i \in \mathbf{N}^n$ has a zero coordinate.

The term "generic" is justified by [7, Theorem 4.1].

Example 3.3 The *defining ideal* of $k[t^{20}, t^{24}, t^{25}, t^{31}]$ is

$$(a^4 - bcd, a^3c^2 - b^2d^2, a^2b^3 - c^2d^2, ab^2c - d^3, a^4 - a^2cd,\, b^3c^2 - a^3d^2, c^3 - abd)$$

in $k[a, b, c, d]$, so this is a generic lattice ideal. The corresponding lattice in $\mathbf{Z}^4$ is spanned by $(4, -1, -1, -1)$, $(3, -2, 2, -2)$ and $(2, 3, -2, -2)$.

Peeva-Sturmfels [6] defined the Scarf complex $\Delta_\mathcal{L}$ of a lattice ideal $I_\mathcal{L}$, and constructed a Γ-graded complex (of free S-modules) $F_\mathcal{L}$ associated to $\Delta_\mathcal{L}$. They called it the *algebraic Scarf complex* of $I_\mathcal{L}$. $F_\mathcal{L}$ is not acyclic in general, but it is always contained in the minimal free resolution as a subcomplex. If $F_{\mathcal{L}}$ is acyclic, then it gives a minimal free resolution.

I do not give the construction of $F_{\mathcal{L}}$ here. I only note that $F_{\mathcal{L}} : \cdots \to F_2 \to F_1 \to F_0 (\simeq S) \to 0$ has the following remarkable property: Let $e \in F_l$, $l \geq 1$, be a base of a free S-module F_l. Then

$$d(e) = \sum_{i=1}^{l+1} \pm 1 \cdot m_i \cdot e_i$$

where $m_i \in S$ is a monomial and e_i is a base of F_{l-1}. In other words, $d(e)$ is a linear combination of $l+1$ "monomials", (in F_{Δ_M} of a monomial ideal M, $d(e)$ is a linear combination of l "monomials").

The algebraic Scarf complex $F_{\mathcal{L}}$ of a lattice ideal $I_{\mathcal{L}}$ is acyclic (hence $F_{\mathcal{L}}$ is a minimal free resolution of $S/I_{\mathcal{L}}$) in the following cases.

- $I_{\mathcal{L}}$ is not a complete intersection and $\operatorname{ht} I_{\mathcal{L}} = 2$ ([6]).
- $I_{\mathcal{L}}$ is generic ([7]).

Next we will recall a result from the local duality (c.f. [4, 8]).

Proposition 3.4 *Let $I \subset S = k[x_1, \cdots, x_n]$ be a homogeneous ideal (under certain grading such that $\deg(x_i)$ is a positive integer for each i) with $\dim S/I = d$. If all associated primes of I have the same height (e.g., I is a lattice ideal), then $\dim \operatorname{Ext}_S^{n-i}(S/I, S) \leq i-1$ for all $i < d$. I satisfies the S_2 condition iff $\dim \operatorname{Ext}_S^{n-i}(S/I, S) < i-1$ for all $i < d$.*

Theorem 3.5 *Suppose that $I_{\mathcal{L}}$ is a lattice ideal whose algebraic Scarf complex $F_{\mathcal{L}}$ is acyclic (e.g., $I_{\mathcal{L}}$ is generic). Then $\dim \operatorname{Ext}_S^{n-e}(S/I_{\mathcal{L}}, S) \geq e-1$, where $e := \operatorname{depth} S/I_{\mathcal{L}}$.*

Proof. We can prove this by an argument similar to Theorem 2.8 and Corollary 2.9.

The next result follows from Proposition 3.4 and Theorem 3.5 immediately.

Corollary 3.6 *Suppose that $I_{\mathcal{L}}$ is a lattice ideal whose algebraic Scarf complex $F_{\mathcal{L}}$ is acyclic (e.g., $I_{\mathcal{L}}$ is generic). If $S/I_{\mathcal{L}}$ satisfies Serre's condition S_2, then $S/I_{\mathcal{L}}$ is Cohen-Macaulay.*

Remark 3.7 (1) If $F_{\mathcal{L}}$ is not acyclic, then the above statement is false. The S_2-ness of an affine semigroup ring does not depend on the characteristic of k, although the Cohen-Macaulayness may depend on the characteristic (see [9]).

(2) Lemma 2.4 states that a generic *monomial* ideal which satisfies the S_1-condition is Cohen-Macaulay.

Proposition 3.8 *Let $I_{\mathcal{L}}$ be a generic lattice ideal, and M the initial ideal of $I_{\mathcal{L}}$ under a degree reverse lexicographic order. If $S/I_{\mathcal{L}}$ is not Cohen-Macaulay (of course, S/M is not Cohen-Macaulay in this case), then M has an embedded associated prime P with* $\dim S/P = \operatorname{depth} S/M$.

Proof. Gasharov-Peeva-Welker [5] shows that F_{Δ_M} is acyclic. By Corollary 2.9, there is an associated prime $P \in \operatorname{Ass}(S/M)$ with $\dim S/P = \operatorname{depth} S/M$. Since $I_{\mathcal{L}}$ is pure dimensional, P must be an embedded prime.

Proposition 3.9 *Let $I_{\mathcal{L}}$ be a lattice ideal whose algebraic Scarf complex $F_{\mathcal{L}}$ is acyclic. If $S/I_{\mathcal{L}}$ is Gorenstein, then it is a hypersurface.*

Proof. If $S/I_{\mathcal{L}}$ is Gorenstein, we have $\operatorname{Hom}(F_{\mathcal{L}}, S) \simeq F_{\mathcal{L}}$. So $\operatorname{Hom}(F_{\mathcal{L}}, S)$ gives the minimal free resolution of $S/I_{\mathcal{L}}$ again. But if $\operatorname{ht} I_{\mathcal{L}} \geq 2$, then $I_{\mathcal{L}}$ must be a monomial ideal by the structure of $\operatorname{Hom}(F_{\mathcal{L}}, S)$. This is a contradiction.

Corollary 3.10 (c.f. [5, Theorem 1.1 (2)]) *A generic lattice ideal is a Gorenstein ideal if and only if it is a principal ideal.*

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Stanley-Reisner rings of Alexander dual complexes

NAOKI TERAI

Introduction

Recenty Alexander duality theorem plays an important role in the study on a minimal free resolution of Stanley-Reisner rings. (See [Br-He$_2$], [Te-Hi$_1$], [Te-Hi$_2$], for example.) In particular, Eagon and Reiner introduced Alexander dual complexes and proved the following interesting theorem:

THEOREM 0.1 ([Ea-Re]). *Let k be a field. and let Δ be a simplicial complex and Δ^* its Alexander dual complex. Then $k[\Delta]$ has a linear resolution if and only if $k[\Delta^*]$ is Cohen-Macaulay.*

The above result is a starting point of this article. We generalize it in the following way.

THEOREM 0.2. *Let k be a field. Let Δ be a $(d-1)$-dimensional complex on the vertex set $[n]$. Suppose $d \leq n-2$. Then*

$$\text{reg } I_\Delta - \text{indeg } I_\Delta = \dim k[\Delta^*] - \text{ depth } k[\Delta^*].$$

Note that Theorem 0.2 corresponds to Theorem 0.1 in the case that either side of the equality is 0.

Using the Auslander-Buchsbam formula, we have the following corollary:

COROLLARY 0.3. *Let k be a field. Let Δ be a $(d-1)$-dimensional complex on the vertex set $[n]$. Suppose $d \leq n-2$. Then*

$$\text{reg } I_\Delta = \text{prodim } k[\Delta^*].$$

On the other hand, it is one of important problems to characterize the h-vectors of a good class of homogeneous k-algebras (i.e., noetherian graded k-algebras gererated by elements in degree one and degree 0 part is k) for a field k. This kind of a problem was originated in Macaulay's work (see Theorem 1.1 in §1), and developed by a lot of mathematicians in algebraic, geometric, and/or combinatoric methods. See, for example, [St$_1$] and [St$_3$] to survey this topic. In this article we give a necessary and sufficient condition for a sequence of integers to be the h-vector of a homogeneous k-algebra $R = k[x_1, x_2, \ldots, x_n]/I$ with $\mathrm{reg}\ I - \mathrm{indeg}\ I \leq c$ for a fixed $c \geq 0$, as an application of the above theorem using the Gröbner basis theory.

As another application, we give some upper bound for the multiplicities of homogeneous k-algebras. In [He-Sr] Herzog and Srinivasan give a conjecture for the upper bound for multiplicities as follows:

Conjecture 0.4 ([He-Sr, Conjecture 2]). Let k be a field and let R be a homogeneous k-algebra of codimension h_1. Then

$$\mathrm{e}(R) \leq \frac{\prod_{i=1}^{h_1} \mathrm{d}_i(R)}{h_1!},$$

where $\mathrm{d}_i(R) := \max\{j \mid \beta_{i,j}(R) \neq 0\}$.

We obtain a bound as follows:

THEOREM 0.5. *Let k be a field and let $R = k[x_1, x_2, \ldots, x_n]/I$ be a homogeneous k-algebra of codimension $h_1 \geq 2$. Then*

$$\mathrm{e}(R) \leq \binom{\mathrm{reg}\ I + h_1 - 1}{h_1} - \binom{\mathrm{reg}\ I - \mathrm{indeg}\ I + h_1 - 1}{h_1}.$$

As a corollary we obtain some partial affirmative result on Conjecture 0.4 as in [He-Sr].

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§1. Preliminaries

We first fix notation. Let $\mathbf{N}$(resp. $\mathbf{Z}$) denote the set of nonnegative integers (resp. integers). Let $\mid S \mid$ denote the cardinality of a set S.

We recall some notation on simplicial complexes and Stanley-Reisner rings according to [St$_1$]. We refer the reader to, e.g., [Br–He], [Hi], [Ho] and [St$_1$] for the detailed information about combinatorial and algebraic background.

A *simplicial complex* Δ on the *vertex set* $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$ such that (i) $\{i\} \in \Delta$ for every $1 \leq i \leq n$ and (ii) $F \in \Delta$, $G \subset F \Rightarrow G \in \Delta$. Each element F of Δ is called a *face* of Δ. We call $F \in \Delta$ an *i-face* if $\mid F \mid = i + 1$ We set $d = \max\{\mid F \mid \mid F \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d - 1$.

Let $f_i = f_i(\Delta)$, $0 \leq i \leq d-1$, denote the number of i-faces in Δ. We define $f_{-1} = 1$. We call $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ the *f-vector* of Δ. Define the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ by

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

If F is a face of Δ, then we define a subcomplex $\mathrm{link}_\Delta F$ as follows:

$$\mathrm{link}_\Delta F = \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}.$$

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k.

Let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n-variables over a field k. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, with $\{i_1, i_2, \ldots, i_r\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k.

Next we summarize basic facts on the Hilbert series. Let k be a field and R a homogeneous k-algebra. We means a homogeneous k-algebra R by a noetherian graded ring $R = \bigoplus_{i \geq 0} R_i$ generated by R_1 with $R_0 = k$. In this case R can be written as a quotient algebra $k[x_1, x_2, \ldots, x_n]/I$, where $\deg x_i = 1$. In this article we always use the representatation A/I with $A = k[x_1, x_2, \ldots, x_n]$ a polynomial ring and with $I_1 = (0)$.

Let M be a graded R-module with $\dim_k M_i < \infty$ for all $i \in \mathbf{Z}$, where $\dim_k M_i$ denotes the dimension of M_i as a k-vector space.

The *Hilbert series* of M is defined by

$$F(M, t) = \sum_{i \in \mathbf{Z}} (\dim_k M_i) t^i.$$

It is well known that the Hilbert series $F(R,t)$ of R can be written in the form

$$F(R,t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^{\dim R}},$$

where $h_0(=1), h_1, \ldots, h_s$ are integers with $\mathrm{e}(R) := h_0 + h_1 + \cdots + h_s \geq 1$. The vector $h(R) = (h_0, h_1, \ldots, h_s)$ is called the *h-vector* of R and the number $\mathrm{e}(R)$ the multiplicity of R.

Let f and i be positive integers. Then f can be uniquely written in the form

$$f = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. Define

$$\begin{aligned} f^{<i>} &= \binom{n_i + 1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j + 1}{j+1}, \\ 0^{<i>} &= 0. \end{aligned}$$

THEOREM 1.1 (Macaulay, Stanley [St3, Theorem 2.2]). *Let $h = (h_i)_{i\geq 0}$ be a sequence of integers. Then the following conditions are equivalent:*
(1) There exists a homogeneous k-algebra R with $F(R,t) = \sum_{i\geq 0}(\dim_k R_i)t^i$.
(2) $h_0 = 1$ and $0 \leq h_{i+1} \leq h_i^{<i>}$ for $i \geq 1$.

We say that a sequence $h = (h_i)_{i\geq 0}$ of integers is an *O-sequence* if it satisfies the equivalent conditions in Theorem 1.1.

For a finite sequence $(h_0, h_1, \ldots, h_s)$, we identify it with the infinite sequence $(h_0, h_1, \ldots, h_s, 0, 0, \ldots)$.

We consider $k[\Delta]$ as the graded algebra $k[\Delta] = \bigoplus_{i\geq 0} k[\Delta]_i$ with $\deg x_j = 1$ for $1 \leq j \leq n$. The Hilbert series $F(k[\Delta], t)$ of a Stanley-Reisner ring $k[\Delta]$ can be written as follows:

$$\begin{aligned} F(k[\Delta], t) &= 1 + \sum_{i=1}^{d} \frac{f_{i-1} t^i}{(1-t)^i} \\ &= \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}, \end{aligned}$$

where $\dim \Delta = d-1$, $(f_0, f_1, \ldots, f_{d-1})$ is the f-vector of Δ, and $(h_0, h_1, \ldots, h_d)$ is the h-vector of Δ.

THEOREM 1.2 (Hochster's formula on the local cohomology modules (cf. [St_1, Theorem 4.1])).

$$F(H^i_{\mathfrak{m}}(k[\Delta]),\ t) = \sum_{F\in\Delta} \dim_k \tilde{H}_{i-|F|-1}(\mathrm{link}_\Delta F; k)\left(\frac{t^{-1}}{1-t^{-1}}\right)^{|F|}.$$

where $H^i_{\mathfrak{m}}(k[\Delta])$ denote the i-th local cohomology module of $k[\Delta]$ with respect to the graded maximal ideal $\mathfrak{m}$.

Let A be the polynomial ring $k[x_1, x_2, \ldots, x_n]$ for a field k. Let M be a finitely generated graded A-module and let

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h,j}(M)} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{0,j}(M)} \longrightarrow M \longrightarrow 0$$

be a graded minimal free resolution of M over A. We call $\beta_i(M) = \sum_{j\in\mathbf{Z}} \beta_{i,j}(M)$ the i-th Betti number of M over A. We sometimes denote $\beta_i^A(M)$ for $\beta_i(M)$ to emphasize the base ring A. We define a *Castelnuovo-Mumford regularity* reg M of M by

$$\mathrm{reg}\ M = \max\ \{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

We define an *initial degree* indeg M of M by

$$\mathrm{indeg}\ M = \min\ \{i \mid M_i \neq 0\} = \min\ \{j \mid \beta_{0,j}(M) \neq 0\}.$$

THEOREM 1.3**(Hochster's formula on the Betti numbers[Hoc, Theorem 5.1]).**

$$\beta_{i,j}(k[\Delta]) = \sum_{F\subset[n],\ |F|=j} \dim_k \tilde{H}_{j-i-1}(\Delta_F; k),$$

where

$$\Delta_F = \{G \in \Delta \mid G \subset F\}.$$

Finally we quote some result on Gröbner bases we use later. See [Ei, Chapter 15] for complete explanation.

Let A be the polynomial ring $k[x_1, x_2, \ldots, x_n]$ for a infinite field k. Let I be a homogeneous ideal in A. We denote Gin (I) to be a *generic initial ideal* of I with respect to the reverse lexicographic order. It is well known that $h(A/\mathrm{Gin}\ (I)) = h(A/I)$ and, in particular, $\mathrm{e}(A/\mathrm{Gin}\ (I)) = \mathrm{e}(A/I)$.

Further we have:

THEOREM 1.4 ([Ba-St]).

$$\text{depth } A/\text{Gin }(I) = \text{depth } A/I$$

and

$$\text{reg Gin }(I) = \text{reg } I.$$

§2. Alexander duality and some generalization of the Eagon-Reiner theorem

First we recall the definition of Alexander dual complexes.

Definition ([Ea-Re]). For a simplicial complex Δ on the vertex set $[n]$, we define an *Alexander dual complex* Δ^* as follows:

$$\Delta^* = \{F \subset [n] : [n] \setminus F \notin \Delta\}.$$

If $\dim \Delta \leq n-3$, then Δ^* is also a simplicial complex on the vertex set $[n]$.

In the rest of the paper we always assume $\dim k[\Delta] = d$ and $\dim k[\Delta^*] = d^*$ for a fixed field k.

Now we give some generalization of the Eagon-Reiner theorem.

THEOREM 2.1. *Let Δ be a $(d-1)$-dimensional complex on the vertex set $[n]$. Suppose $d \leq n-2$. Then*

$$\text{reg } I_\Delta - \text{indeg } I_\Delta = \dim k[\Delta^*] - \text{ depth } k[\Delta^*].$$

Proof. Put depth $k[\Delta^*] = p^*$. By Hochster's formula on the local cohomology modules, we have

$$F(H^l_{\mathfrak{m}}(k[\Delta^*]),\ t) = \sum_{F \in \Delta^*} \dim_k \tilde{H}_{l-|F|-1}(\mathrm{link}_{\Delta^*} F; k) \left(\frac{t^{-1}}{1-t^{-1}}\right)^{|F|}.$$

Hence if $l < p^*$, then $\tilde{H}_{l-|F|-1}(\mathrm{link}_{\Delta^*} F; k) = (0)$ for all $F \in \Delta^*$. By the proof in [Ea-Re, Proposition 1], we have $\tilde{H}_{n-l-2}(\Delta_F; k) = (0)$ for all $F \subset [n]$. By

Hochster's formula on the Betti numbers this means that $\beta_{i,i+n-l-1}(k[\Delta]) = 0$ for $i \geq 1$. Hence

$$\beta_{i,i+n}(I_\Delta) = \beta_{i,i+n-1}(I_\Delta) = \cdots = \beta_{i,i+n-p^*+1}(I_\Delta) = 0$$

for $i \geq 0$. Similarly, since $\tilde{H}_{n-p^*-2}(\Delta_{[n]\setminus F}; k) \cong \tilde{H}_{p^*-|F|-1}(\mathrm{link}_{\Delta^*} F; k) \neq (0)$ for some $F \in \Delta$, we have $\beta_{i,i+n-p^*}(I_\Delta) \neq 0$ for some $i \geq 0$. Hence $\mathrm{reg}\ I_\Delta = n - p^*$. By the definition of the Alexander dual complex we have $\mathrm{indeg}\ I_\Delta = n - d^*$. Therefore, we have $\mathrm{reg}\ I_\Delta - \mathrm{indeg}\ I_\Delta = d^* - p^*$. Q.E.D.

Let $h = (h_0, h_1, \ldots, h_s)$ be a finite seqence with $h_0 = 1$ and $h_1 \geq 1$. Put $p := \min\{i \geq 1 \mid h_i \neq \binom{h_1+i-1}{i}\}$. We define the *dual sequence* $h^* = (h_i^*)_{i\geq 0}$ by

$$\sum_{i\geq 0} h_i^* t^i = \frac{1 - t^{h_1}(h_0 + h_1(1-t) + \cdots + h_s(1-t)^s)}{(1-t)^p}.$$

LEMMA 2.2. *$h_i^* = 0$ for $i > h_1 + s$.*

Proof. We have

$$\begin{aligned}
\sum_{i\geq 0} h_i^* t^i &= \frac{1 - t^{h_1}(h_0 + h_1(1-t) + \cdots + h_s(1-t)^s)}{(1-t)^p} \\
&= \frac{1 - t^{h_1}\sum_{i=0}^{s} h_i(\sum_{j=0}^{i}(-1)^j\binom{i}{j}t^j)}{(1-t)^p} \\
&= \frac{1 - \sum_{j=0}^{s}(-1)^j(\sum_{i=j}^{s} h_i\binom{i}{j})t^{h_1+j}}{(1-t)^p}.
\end{aligned}$$

For $l > h_1 + s$, we have

$$\begin{aligned}
h_i^* &= \binom{p+l-1}{l} - \sum_{(j+h_1)+m=l}((-1)^j\sum_{i=j}^{s} h_i\binom{i}{j})\binom{p+m-1}{m} \\
&= \binom{p+l-1}{l} - \sum_{j=0}^{l-h_1}((-1)^j\sum_{i=j}^{s} h_i\binom{i}{j})\binom{p+l-h_1-j-1}{l-h_1-j-1} \\
&= \binom{p+l-1}{l} - \sum_{i=0}^{s} h_i\sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{p+l-h_1-j-1}{l-h_1-j-1} \\
&= \binom{p+l-1}{l} - \sum_{i=0}^{s} h_i\binom{p+l-h_1-i-1}{l-h_1} \quad \text{([Ri, Page8 (5)])}
\end{aligned}$$

$$\begin{aligned}
&= \binom{p+l-1}{l} - \sum_{i=0}^{p-1} \binom{h_1+i-1}{i}\binom{p+l-h_1-i-1}{l-h_1} \\
&= \binom{p+l-1}{l} - \binom{p+l-h_1-1+h_1}{l-h_1+h_1} \quad \text{([Ri, Page8 (3b)])} \\
&= \binom{p+l-1}{l} - \binom{p+l-1}{l} \\
&= 0.
\end{aligned}$$

Q.E.D.

By the above lemma we can define h^* by

$$\begin{aligned}
& t^{h_1}(h_0 + h_1(1-t) + \cdots + h_s(1-t)^s) \\
= \;& 1 - (1-t)^p(h_0^* + h_1^* t + \cdots + h_{h_1+s}^* t^{h_1+s}).
\end{aligned}$$

We justify the notation h^* by the following lemma:

LEMMA 2.3. Let Δ be a simplicial complex on the vertex set $[n]$. Then we have

$$h^*(\Delta) = h(\Delta^*).$$

Proof. By the definition we have $f_i(\Delta^*) = \binom{n}{i+1} - f_{n-i-2}(\Delta)$. Put $\tau = 1-t$. Then we have

$$\begin{aligned}
& \frac{h_0(\Delta) + h_1(\Delta)t + \cdots + h_d(\Delta)t^d}{(1-t)^d} \\
= \;& \sum_{i=0}^{d} \frac{f_{i-1}(\Delta)t^i}{(1-t)^i} \\
= \;& \sum_{i=0}^{n} \frac{(\binom{n}{i} - f_{n-i-1}(\Delta^*))t^i}{(1-t)^i} \\
= \;& \sum_{i=0}^{n} \frac{\binom{n}{i}t^i}{(1-t)^i} - \sum_{i=0}^{n} \frac{f_{n-i-1}(\Delta^*)t^i}{(1-t)^i} \\
= \;& \left(1 + \frac{t}{1-t}\right)^n - \left(\frac{t}{1-t}\right)^n \sum_{i=0}^{n} \frac{f_{n-i-1}(\Delta^*)\tau^{n-i}}{(1-\tau)^{n-i}} \\
= \;& \frac{1}{(1-t)^n} - \left(\frac{t}{1-t}\right)^n \frac{h_0(\Delta^*) + h_1(\Delta^*)\tau + \cdots + h_{d^*}(\Delta^*)\tau^{d^*}}{(1-\tau)^{d^*}}
\end{aligned}$$

$$= \frac{1}{(1-t)^n} - \frac{t^{n-d^*}(h_0(\Delta^*) + h_1(\Delta^*)(1-t) + \cdots + h_{d^*}(\Delta^*)(1-t)^{d^*})}{(1-t)^n}.$$

Therefore, since $p = n - d^*$ we have

$$\begin{aligned} & (1-t)^{n-d}(h_0(\Delta) + h_1(\Delta)t + \cdots + h_d(\Delta)t^d) \\ = \ & 1 - t^{n-d^*}(h_0(\Delta^*) + h_1(\Delta^*)(1-t) + \cdots + h_{d^*}(\Delta^*)(1-t)^{d^*}) \end{aligned}$$

Q.E.D.

§3. Application to the h-vectors of homogeneous rings

For a sequence $h = (h_i)_{i \geq 0}$ of integers, we define the *partial sum sequence* Sh of h by

$$Sh = (h_0, h_0 + h_1, h_0 + h_1 + h_2, \ldots, \sum_{j=0}^{i} h_j, \ldots).$$

And inductively we define the i-th iterated partial sum sequence $S^i h$ by $S^i h = S(S^{i-1}h)$.

The next proposition is a variation of Stanley.

PROPOSITION 3.1(cf. [St$_3$, Corollary 3.11]). *Let $h = (h_0, h_1, \cdots, h_s)$ be a seqence of integers with $h_0 + h_1 + \cdots + h_s > 0$. We fix an integer $c \geq 0$. Then the following conditions are equivalent:*
(1) There exists a simplicial complex Δ with $\dim k[\Delta] - \operatorname{depth} k[\Delta] \leq c$ such that $h = h(k[\Delta])$.
(2) There exists a homogeneous k-algebra R with $\dim R - \operatorname{depth} R \leq c$ such that $h = h(R)$.
(3) The c-th iterated partial sum sequence $S^c h$ of h is an O-sequence.

Proof. We may assume $|k| = \infty$. Put $\dim R = d$.

(1)$\Rightarrow$ (2). Trivial.

(2)$\Rightarrow$ (3). (A)Case $d - c \leq 0$. The c-th iterated partial sum sequence $S^c h$ of h is the $(c-d)$-th iterated partial sum sequence of $(\dim_k R_i)_{i \geq 0}$. Then $S^c h$ is an O-sequence.

(B)Case $d - c > 0$. We have depth$R \geq d - c$. Let $\{y_1, y_2, \ldots y_{d-c}\}$ be a regular sequence in $k[\Delta]_1$. Then the c-th iterated partial sum sequence $S^c h$ of h is $(\dim(R/(y_1, y_2, \ldots y_{d-c}))_i)_{i \geq 0}$, which is an O-sequence.

(3)$\Rightarrow$ (1). There exists a monomial ring R (i.e., $R = A/I$, where I is generated by monomials) whose Hilbert function is $S^c h$. Note that $\dim R = c$. Let $k[\Delta]$ be a polarization of R (See [St-Vo] for the definition and basic properties of the polarization). Then

$$\begin{aligned} \dim k[\Delta] - \operatorname{depth} k[\Delta] &= \dim R - \operatorname{depth} R \\ &\leq \dim R \\ &= c. \end{aligned}$$

Q.E.D.

We have the following theorem which gives a characterization of h-vector of homogeneous k-algebras $R = A/I$ with $\operatorname{reg} I - \operatorname{indeg} I \leq c$.

THEOREM 3.2. *Let $h = (h_0,\ h_1, \cdots, h_s)$ be an integer sequence with $h_1 \geq 2$, and $h_0 + h_1 + \cdots + h_s > 0$. We fix an integer $c \geq 0$. Then the following conditions are equivalent:*
(1) There exists a homogeneous k-algebra $R = A/I$ with

$$\operatorname{reg} I - \operatorname{indeg} I \leq c$$

such that $h = h(R)$, where A is a polynomial ring and I is a homogeneous ideal with $I_1 = (0)$.
(2) There exists a simplicial complex Δ with

$$\operatorname{reg} I_\Delta - \operatorname{indeg} I_\Delta \leq c$$

such that $h = h(\Delta)$.
(3) The c-th iterated partial sum sequence $S^c(h^*)$ of the dual sequence h^* of h is an O-sequence.

Proof. We may assume $|\ k\ | = \infty$. (1)$\Rightarrow$(2). Let $R = A/I$ be a k-algebra satisfying the conditions in (1). Since we have $\operatorname{reg} \operatorname{Gin}(I) = \operatorname{reg} I$, we have

$$\operatorname{reg} \operatorname{Gin}(I) - \operatorname{indeg} \operatorname{Gin}(I) \leq c$$

and $h = h(A/\operatorname{Gin}(I))$. Considering the polarization, we obtain a Stanley-Reisner ring $k[\Delta]$ satisfying the conditions in (2).

(2)$\Rightarrow$(1). Trivial.

(2)$\Rightarrow$(3). If Δ is a simplicial complex with the conditions in (2), then Theorem 2.1 we have

$$\dim k[\Delta^*] - \operatorname{depth} k[\Delta^*] \leq c.$$

And by Lemma 2.5 we have $h^* = h(k[\Delta^*])$. Hence by Proposition 3.1, the condition (3) holds for h.

(3)$\Rightarrow$(2). If h^* satisfies the condition (3), there exists a simplicial complex Δ such that for its Alexander dual complex Δ^*, $h^* = h(k[\Delta^*])$ and

$$\dim k[\Delta^*] - \operatorname{depth} k[\Delta^*] \leq c.$$

then we have $h = h(k[\Delta])$ and

$$\operatorname{reg} I_\Delta - \operatorname{indeg} I_\Delta \leq c.$$

Q.E.D.

Remark. The inequality $\operatorname{reg} I_\Delta - \operatorname{indeg} I_\Delta \leq c$ means that at most ($\operatorname{indeg} I_\Delta$, $\operatorname{indeg} I_\Delta + 1, \ldots, \operatorname{indeg} I_\Delta + c$)-linear parts appear in the minimal free resolution of I_Δ.

§4. On upper bounds for multiplicities

In this section we give some upper bound for the multiplicities of homogeneous k-algebras. And we deduce some partial affermative result on the Herzog-Srinivasan conjecture.

First we prove the following lemma:

LEMMA 4.1.

$$\mathrm{e}(k[\Delta]) = \beta_{1,h_1}(k[\Delta^*]).$$

Proof. We have

$$\begin{aligned} & h_0(\Delta) + h_1(\Delta)(1-t) + \cdots + h_d(\Delta)(1-t)^d && (1)\\ = & \frac{1-(1-t)^{n-d^*}(h_0(\Delta^*) + h_1(\Delta^*)t + \cdots + h_{d^*}(\Delta^*)t^{d^*})}{t^{n-d}}, && (2) \end{aligned}$$

by Lemma 2.5. Since $\operatorname{indeg} I_{\Delta^*} = n - d = h_1$, we have

$$\begin{aligned} & \beta_{1,n-d}(k[\Delta^*]) \\ = & \text{(the coefficient of } t^{n-d} \text{ in } -(1-t)^{n-d^*}(h_0(\Delta^*) + h_1(\Delta^*)t + \cdots + h_{d^*}(\Delta^*)t^{d^*})) \\ = & \text{(the coefficient of } t^{n-d} \text{ in the numerator in (2))} \\ = & \lim_{t\to 0}(h_0(\Delta) + h_1(\Delta)(1-t) + \cdots + h_d(\Delta)(1-t)^d) \\ = & \mathrm{e}(k[\Delta]). \end{aligned}$$

Q.E.D.

THEOREM 4.2. *Let $R = A/I$ be a homogeneous k-algebra of codimension $h_1 \geq 2$. Then*

$$\mathrm{e}(R) \leq \binom{\mathrm{reg}\ I + h_1 - 1}{h_1} - \binom{\mathrm{reg}\ I - \mathrm{indeg}\ I + h_1 - 1}{h_1}.$$

Proof. We may assume $|\ k\ |= \infty$. By Theorem 1.4, we have $\mathrm{reg}\ \mathrm{Gin}(I) = \mathrm{reg}\ I$ and $h(A/I) = h(A/\mathrm{Gin}(I))$. Considering the polarization, we obtain a Stanley-Reisner ring $k[\Delta] = B/I_\Delta$ with $\mathrm{e}(A/I) = \mathrm{e}(k[\Delta])$ and $\mathrm{reg}\ I = \mathrm{reg}\ I_\Delta$. Put $p^* = \mathrm{depth}\ k[\Delta^*]$. By Theorem 2.1, we have $d^* - p^* = \mathrm{reg}\ I - (n - d^*)$, where $n = \mathrm{embdim}\ k[\Delta^*]$. Hence $\mathrm{reg}\ I = n - p^*$.

Let $y_1, y_2, \ldots, y_{p^*}$ be a regular sequence in $k[\Delta^*]_1$, and let $z_1, z_2, \ldots, z_{d^*-p^*} \in (k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*}))_1$ be a system of parameters of $k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*})$. We have $k[z_1, z_2, \ldots, z_{d^*-p^*}] \subset k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*})$. Since $k[z_1, z_2, \ldots, z_{d^*-p^*}]$ is isomorphic to the polynomial ring with $d^* - p^*$ variables, we have $\dim_k(k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*}))_{h_1} \geq \binom{d^*-p^*+h_1-1}{h_1}$. By Lemma 4.2 we have

$$\begin{aligned} \mathrm{e}(k[\Delta]) &= \beta_{1,h_1}(k[\Delta^*]) \\ &= \beta_{1,h_1}^{B/(y_1,y_2,\ldots,y_{p^*})}(k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*})) \\ &= \dim_k(B/(y_1, y_2, \ldots, y_{p^*}))_{h_1} - \dim_k(k[\Delta^*]/(y_1, y_2, \ldots, y_{p^*}))_{h_1} \\ &\leq \binom{n - p^* + h_1 - 1}{h_1} - \binom{d^* - p^* + h_1 - 1}{h_1}. \end{aligned}$$

Q.E.D.

COROLLARY 4.3 ([He-Sr]). *Let $R = A/I$ be a homogeneous k-algebra of codimension $h_1 \geq 2$ with $\beta_{0,\mathrm{reg}\ I} \neq 0$. Then Conjecture 0.3 holds.*

Proof. Since $\mathrm{d}_i(R) = \mathrm{reg}\ I + i - 1$ for $1 \leq i \leq h_1$, we have

$$\mathrm{e}(R) \leq \binom{\mathrm{reg}\ I + h_1 - 1}{h_1} = \frac{\prod_{i=1}^{h_1} \mathrm{d}_i(R)}{h_1!}.$$

Q.E.D.

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NON-COMMUTATIVE GRÖBNER BASES FOR COMMUTATIVE ALGEBRAS

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1. Introduction

Let k be a field, let $k[x] = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables and $k\langle X\rangle = k\langle X_1, \ldots, X_n\rangle$ the free associative algebra in n variables. Consider the natural map $\gamma : k\langle X\rangle \to k[x]$ taking X_i to x_i. It is sometimes useful to regard a commutative algebra $k[x]/I$ through its non-commutative presentation $k[x]/I \cong k\langle X\rangle/J$, where $J = \gamma^{-1}(I)$. This is especially true in the construction of free resolutions as in [An]. Non-commutative presentations have been exploited in [AR] and [PRS] to study homology of coordinate rings of Grassmannians and toric varieties. These applications all make use of Gröbner bases for J (see [Mo] for non-commutative Gröbner bases.) In this note we give an explicit description (Theorem 2.1) of the minimal Gröbner bases for J with respect to monomial orders on $k\langle X\rangle$ that are lexicographic extensions of monomial orders on $k[x]$.

Non-commutative Gröbner bases are usually infinite; for example, if $n = 3$ and $I = (x_1x_2x_3)$ then $\gamma^{-1}(I)$ does not have a finite Gröbner basis for any monomial order on $k\langle X\rangle$. (There are only two ways of choosing leading terms for the three commutators, and both cases are easy to analyze by hand.) However, after a linear change of variables the ideal becomes $I' = (X_1(X_1 + X_2)(X_1 + X_3))$, and we shall see in Theorem 2.1 that $X_1(X_1 + X_2)(X_1 + X_3)$ and the three commutators $X_iX_j - X_jX_i$ are a Gröbner basis for $\gamma^{-1}(I')$ with respect to a suitable order. This situation is rather general: Theorems 2.1 and 3.1 imply the following result:

Corollary 1.1. *Let k be an infinite field and $I \subset k[x]$ be an ideal. After a general linear change of variables, the ideal $\gamma^{-1}(I)$ in $k\langle X\rangle$ has a finite Gröbner basis. In characteristic 0, if I is homogeneous, such a basis can be found with degree at most $max\{2, regularity(I)\}$.*

The behavior of $\gamma^{-1}(I)$ is in sharp contrast to what happens for arbitrary ideals in $k\langle X\rangle$. For example if one takes a defining ideal in $k\langle X\rangle$ of the group algebra of a group with undecidable word problem, then there exists no finite Gröbner basis.

In characteristic 0 the Gröbner basis of $\gamma^{-1}(I)$ in Corollary 1.1 may be obtained by lifting the Gröbner basis of I, but this is not so in characteristic p; see Example 4.2. Furthermore, $\gamma^{-1}(I)$ might have no finite Gröbner basis at all if the field is finite; see Example 4.1.

In the next section we present the basic computation of the initial ideal and Gröbner basis for $J = \gamma^{-1}(I)$. In §3 we give the application to finiteness and liftability of Gröbner bases.

This is a joint work with D. Eisenbud and I. Peeva.

2. The Gröbner basis of $\gamma^{-1}(I)$

Throughout this paper we fix an ideal $I \subset k[x]$ and $J := \gamma^{-1}(I) \subset k\langle X\rangle$. We shall make use of the *lexicographic splitting* of γ which is defined as the k-linear map

$$\delta \;:\; k[x] \to k\langle X\rangle\,, \quad x_{i_1}x_{i_2}\cdots x_{i_r} \mapsto X_{i_1}X_{i_2}\cdots X_{i_r} \quad \text{if} \quad i_1 \leq i_2 \leq \cdots \leq i_r.$$

Fix a monomial order $\prec$ on $k[x]$. The *lexicographic extension* $\prec\!\!\!\prec$ of $\prec$ to $k\langle X\rangle$ is defined for monomials $M, N \in k\langle X\rangle$ by

$$M \prec\!\!\!\prec N \quad \text{if} \quad \begin{cases} \gamma(M) \prec \gamma(N) & \text{or} \\ \gamma(M) = \gamma(N) & \text{and } M \text{ is lexicographically smaller than } N. \end{cases}$$

Thus for example $X_iX_j \prec\!\!\!\prec X_jX_i$ if $i < j$.

To describe the $\prec\!\!\!\prec$-initial ideal of J we use the following construction: Let L be any monomial ideal in $k[x]$. If $m = x_{i_1}\cdots x_{i_r} \in L$ and $i_1 \leq \cdots \leq i_r$ denote by $\mathcal{U}_L(m)$ the set of all monomials $u \in k[x_{i_1+1}, \ldots, x_{i_r-1}]$ such that neither $u\frac{m}{x_{i_1}}$ nor $u\frac{m}{x_{i_r}}$ lies in L. For instance, if $L = (x_1x_2x_3,\, x_2^d)$ then $\mathcal{U}_L(x_1x_2x_3) = \{\, x_2^j \mid j < d\,\}$.

Theorem 2.1. *The non-commutative initial ideal $in_{\prec\!\!\!\prec}(J)$ is minimally generated by the set $\{\, X_iX_j \mid j < i\,\}$ together with the set*

$$\{\; \delta(u \cdot m) \mid m \text{ is a generator of } in_\prec(I) \text{ and } u \in \mathcal{U}_{in_\prec(I)}(m) \;\}.$$

In particular, a minimal $\prec\!\!\!\prec$-Gröbner basis for J consists of $\{X_iX_j - X_jX_i : j < i\}$ together with the elements $\delta(u \cdot f)$ for each polynomial f in a minimal $\prec$-Gröbner basis for I and each monomial $u \in \mathcal{U}_{in_\prec(I)}(in_\prec(f))$.

Proof. We first argue that a non-commutative monomial $M = X_{i_1}X_{i_2}\cdots X_{i_r}$ lies in $in_{\prec\!\!\!\prec}(J)$ if and only if its commutative image $\gamma(M)$ is in $in_\prec(I)$ or $i_j > i_{j+1}$ for some j. Indeed, if $i_j > i_{j+1}$ then $M \in in_{\prec\!\!\!\prec}(J)$ because $X_sX_t - X_tX_s \in J$ has initial term X_sX_t with $s > t$. If on the contrary $i_1 \leq \cdots \leq i_r$ but $\gamma(M) \in in_\prec(I)$ then there exists $f \in I$ with $in_\prec(f) = \gamma(M)$. The non-commutative polynomial $F = \delta(f)$ satisfies $in_{\prec\!\!\!\prec}(F) = M$. The opposite implication follows because γ induces an isomorphism $k[x]/I \cong k\langle X\rangle/\gamma^{-1}(I)$.

Now let $m' = u\cdot m$, where $m = x_{i_1}\cdots x_{i_r}$ is a minimal generator of $in_\prec(I)$ with $i_1 \leq \cdots \leq i_r$. We must show that $\delta(u \cdot m)$ is a minimal generator of $in_{\prec\!\!\!\prec}(J)$ if and only if $u \in \mathcal{U}_{in_\prec(I)}(m)$.

For the "only if" direction suppose that $\delta(u \cdot m)$ is a minimal generator of $in_{\prec\!\!\!\prec}(J)$. Suppose that u contains the variable x_j. We must have $j > i_1$ since else, taking j minimal, we would have $\delta(u \cdot m) = X_j \cdot \delta(\frac{u}{x_j}m)$. Similarly $j < i_r$. Thus $u \in k[x_{i_1+1}, \ldots, x_{i_r-1}]$. This implies $\delta(u \cdot m) = X_{i_1} \cdot \delta(u\frac{m}{x_{i_1}}) = \delta(u \cdot \frac{m}{x_{i_r}}) \cdot X_{i_r}$. Therefore neither $\delta(u\frac{m}{x_{i_1}})$ nor $\delta(u\frac{m}{x_{i_r}})$ lies in $in_{\prec\!\!\!\prec}(J)$ and hence neither $u\frac{m}{x_{i_1}}$ nor $u\frac{m}{x_{i_r}}$ lies in $in_\prec(I)$.

For the "if" direction we reverse the last few implications. If $u \in \mathcal{U}_{in_\prec(I)}(m)$ then neither $\delta(u\frac{m}{x_{i_1}})$ nor $\delta(u\frac{m}{x_{i_r}})$ lies in $in_{\prec\!\!\!\prec}(J)$ and therefore $\delta(u\cdot m)$ is a minimal generator of $in_{\prec\!\!\!\prec}(J)$. $\square$

3. Finiteness and lifting of non-commutative Gröbner bases

We maintain the notation described above. Recall that for a prime number p the *Gauss order* on the natural numbers is described by

$$s \leq_p t \quad \text{if} \quad \binom{t}{s} \not\equiv 0 \pmod p.$$

We write $\leq_0 = \leq$ for the usual order on the natural numbers. A monomial ideal L is called *p-Borel-fixed* if it satisfies the following condition: For each monomial generator m of L, if m is divisible by x_j^t but no higher power of x_j, then $(x_i/x_j)^s m \in L$ for all $i < j$ and $s \leq_p t$.

Theorem 3.1. *With notation as in Section 2:*

(a) If $in_\prec(I)$ is 0-Borel fixed, then a minimal $\prec\!\!\prec$-Gröbner basis of J is obtained by applying δ to a minimal $\prec$-Gröbner basis of I and adding commutators.

(b) If $in_\prec(I)$ is p-Borel-fixed for any p, then J has a finite $\prec\!\!\prec$-Gröbner basis.

Proof. Suppose that the monomial ideal $L := in_\prec(I)$ is p-Borel-fixed for some p. Let $m = x_{i_1} \cdots x_{i_r}$ be any generator of L, where $i_1 \leq \cdots \leq i_r$, and let $x_{i_r}^t$ be the highest power of x_{i_r} dividing m. Since $t \leq_p t$ we have $x_l^t m / x_{i_r}^t \in L$ for each $l < i_r$. This implies $x_l^t m / x_{i_r} \in L$ for $l < i_r$, and hence every monomial $u \in \mathcal{U}_L(m)$ satisfies $deg_{x_l}(u) < t$ for $i_1 < l < i_r$. We conclude that $\mathcal{U}_L(m)$ is a finite set. If $p = 0$ then $\mathcal{U}_L(m)$ consists of 1 alone since $x_l m / x_{i_r} \in L$ for all $l < i_r$. Theorem 3.1 now follows from Theorem 2.1. □

Proof of Corollary 1.1. We apply Theorem 3.1 together with the following results due to Galligo, Bayer-Stillman and Pardue which can be found in [Ei, Section 15.9]: if the field k is infinite, then after a generic change of variables, the initial ideal of I with respect to any order $\prec$ on $k[x]$ is fixed under the Borel group of upper triangular matrices. This implies that $in_\prec(I)$ is p-Borel-fixed in characteristic $p \geq 0$ in the sense above. If the characteristic of k is 0 and I is homogeneous then, taking the reverse lexicographic order in generic coordinates, we get a Gröbner basis whose maximal degree equals the regularity of I. □

We call the monomial ideal L *squeezed* if $\mathcal{U}_L(m) = \{1\}$ for all generators m of L or if, equivalently, $m = x_{i_1} \cdots x_{i_r} \in L$ and $i_1 \leq \cdots \leq i_r$ imply $x_l \frac{m}{x_{i_1}} \in L$ or $x_l \frac{m}{x_{i_r}} \in L$ for every index l with $i_1 < l < i_r$. Thus Theorem 2.1 implies that a minimal $\prec$-Gröbner basis of I lifts to a Gröbner basis of J if and only if the initial ideal $in_\prec(I)$ is squeezed. Monomial ideals that are 0-Borel-fixed, and more generally stable ideals (in the sense of [EK]), are squeezed. Squeezed ideals appear naturally in algebraic combinatorics:

Proposition 3.2. *A square-free monomial ideal L is squeezed if and only if the simplicial complex associated with L is the complex of chains in a poset.*

Proof. This follows from Lemma 3.1 in [PRS]. □

4. Examples in characteristic p

Over a finite field Corollary 1.1 fails even for very simple ideals:

Example 4.1. *Let k be a finite field and $n = 3$. If I is the principal ideal generated by the product of all linear forms in $k[x_1, x_2, x_3]$, then $\gamma^{-1}(I)$ has no finite Gröbner basis, even after a linear change of variables.*

Proof. The ideal I is invariant under all linear changes of variables. The $\lll$-Gröbner basis for J is computed by Theorem 2.1, and is infinite. That no other monomial order on $k\langle X\rangle$ yields a finite Gröbner basis can be shown by direct computation as in the example in the second paragraph of the introduction. $\square$

Sometimes in characteristic $p > 0$ no Gröbner basis for a commutative algebra can be lifted to a non-commutative Gröbner basis, even after a change of variables:

Example 4.2. *Let k be an infinite field of characteristic $p > 0$, and consider the Frobenius power*

$$L \;:=\; \big((x_1,\ x_2,\ x_3)^3\big)^{[p]} \quad \subset \quad k[x_1, x_2, x_3]$$

of the cube of the maximal ideal in 3 variables. No minimal Gröbner basis of L lifts to a Gröbner basis of $\gamma^{-1}(L)$, and this is true even after any linear change of variables.

Proof. The ideal L is invariant under linear changes of variable, so it suffices to consider L itself. Since L is a monomial ideal, it is its own initial ideal, so by Corollary 3.2 it suffices to show that L is not squeezed, that is, that neither of $x_1^{p-1}x_2^{p+1}x_3^p$ and $x_1^p x_2^{p+1} x_3^{p-1}$ is in L. This is obvious, since the power of each variable occurring in a generator of L is divisible by p and has total degree $3p$. $\square$

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Normal polytopes arising from finite graphs

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1 Background

(1.1) Let G be a finite connected simple graph on the vertex set $V(G) = [d] = \{1, \ldots, d\}$ and $E(G) = \{e_1, \ldots, e_n\}$ the set of edges of G. Here, a graph G is simple if G has no loop and no multiple edge. The *vertex-edge incidence matrix* of G is the $n \times d$ matrix $A(G) = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq d}$ with $a_{ij} = 1$ if $e_i \in E(G)$ is adjacent to $j \in [d]$ and with $a_{ij} = 0$ otherwise. Identifying each row vector $(a_{i1}, \ldots, a_{id})$ to be a point of $\mathbf{R}^d$, we may regard $A(G)$ to be a configuration of n points in $\mathbf{R}^d$. Let $\mathcal{P}_G$ denote the convex hull of $A(G)$ in $\mathbf{R}^d$. The convex polytope $\mathcal{P}_G \subset \mathbf{R}^d$ is called the *edge polytope* of G. Then, $A(G)$ coincides with the vertex set of $\mathcal{P}_G$. Moreover, $\dim \mathcal{P}_G = d - 2$ if G is bipartite, i.e., has no odd cycle, and $\dim \mathcal{P}_G = d - 1$ if G has at least one odd cycle.

(1.2) Let $K[t_1, \ldots, t_d]$ denote the polynomial ring in d indeterminates over a field K and $K[G]$ the subalgebra of $K[t_1, \ldots, t_d]$ generated by those squarefree quadratic monomials $t_p t_q$ with $\{p, q\} \in E(G)$. The affine semigroup ring $K[G]$ is called the *edge ring* of G. Then, Krull $-$ $\dim K[G] = \dim \mathcal{P}_G$. Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ denote the polynomial ring in n indeterminates and define the surjective homomorphism of semigroup rings $\hat{\pi} : K[\mathbf{x}] \to K[G]$ by $\hat{\pi}(x_i) = t_p t_q$ if $e_i = \{p, q\}$. Let I_G denote the kernel of $\hat{\pi}$ and call I_G the *toric ideal* of $K[G]$.

We will assume that the reader is familiar with fundamental material related with Gröbner bases discussed in, e.g., [Stu]. First, recall that a *term order* on $K[\mathbf{x}]$ is a linear order $\prec$ on the set of monomials of $K[\mathbf{x}]$ such that

$1 \prec u$ for all monomials $u \neq 1$ and that if u and v are monomials and if $u \prec v$ then $uw \prec vw$ for all monomials w. Given a term order $\prec$ on $K[\mathbf{x}]$, the *initial monomial* of a non-zero polynomial $f \in K[\mathbf{x}]$ is the monomial $in_{\prec}(f)$ which is a unique maximal monomial with respect to $\prec$ among the monomials appearing in f with non-zero coefficients. Now, the *initial ideal* of I_G with respect to $\prec$ is the ideal $in_{\prec}(I_G)$ of $K[\mathbf{x}]$ which is generated by all initial monomials $in_{\prec}(f)$ with $f \in I_G$. A finite set $\mathcal{G} \subset I_G$ is a *Gröbner basis* for I_G with respect to $\prec$ if $in_{\prec}(I_G)$ is generated by $\{in_{\prec}(g)\,;\, g \in \mathcal{G}\}$.

(1.3) A *triangulation* of $\mathcal{P}_G \subset \mathbf{R}^d$ is a collection Δ of subgraphs G' of G satisfying the following conditions:

(i) if $G' \in \Delta$, then $\mathcal{P}_{G'}$ is a simplex in $\mathbf{R}^d$ with $\dim \mathcal{P}_{G'} = \dim \mathcal{P}_G$;

(ii) if $G', G'' \in \Delta$, then $\mathcal{P}_{G'} \cap \mathcal{P}_{G''}$ is a face of $\mathcal{P}_{G'}$ and of $\mathcal{P}_{G''}$;

(iii) $\mathcal{P}_G = \cup_{G' \in \Delta} \mathcal{P}_{G'}$.

A triangulation Δ of $\mathcal{P}_G$ is called *unimodular* if the normalized volume [Stu, p. 36] of $\mathcal{P}_{G'}$ is equal to 1 for every subgraph G' belonging to Δ. A *unimodular covering* of $\mathcal{P}_G$ is a collection Δ of subgraphs G' of G satisfying the above conditions (i) and (iii) such that the normalized volume of $\mathcal{P}_{G'}$ is equal to 1 for every subgraph G' belonging to Δ.

(1.4) Fix a term order $\prec$ on $K[\mathbf{x}]$ and let $in_{\prec}(I_G)$ be the initial ideal of the toric ideal I_G of $K[G]$ with respect to $\prec$. We write $\Delta(in_{\prec}(I_G))$ for the set of subgraphs G' of G with $\dim \mathcal{P}_{G'} = \dim \mathcal{P}_G$ such that $\prod_{e_i \in E(G')} x_i \notin \sqrt{in_{\prec}(I_G)}$. It then follows that $\Delta(in_{\prec}(I_G))$ is a *regular* triangulation [Stu, pp. 64 – 65] of $\mathcal{P}_G \subset \mathbf{R}^d$ and that every regular triangulation of $\mathcal{P}_G$ is of the form $\Delta(in_{\prec}(I_G))$ for some term order $\prec$ on $K[\mathbf{x}]$. Moreover, a regular triangulation $\Delta(in_{\prec}(I_G))$ is unimodular if and only if $in_{\prec}(I_G)$ is square-free, i.e., generated by square-free monomials ([Stu, Corollary 8.9]).

(1.5) We say that an edge polytope $\mathcal{P}_G$ is *normal* if its edge ring $K[G]$ is normal, i.e., integrally closed in its quotient field. It is a fundamental fact that $\mathcal{P}_G$ is normal if $\mathcal{P}_G$ possesses a unimodular covering.

(1.6) Many fundamental problems from viewpoints of both commutative algebra and combinatorics now arise. For example, given a finite connected simple graph G, it is reasonable to present the questions below:

(1.6.1) Is $\mathcal{P}_G$ normal ?

(1.6.2) Does $\mathcal{P}_G$ possess a unimodular covering ?

(1.6.3) Does $\mathcal{P}_G$ possess a unimodular triangulation ?

(1.6.4) Does $\mathcal{P}_G$ possess a regular unimodular triangulation ?

If G is the complete graph on $[d]$, then the answer to all questions (1.6.1) –

(1.6.4) is yes. When G is bipartite, since the vertex-edge incidence matrix of G is unimodular, the answer of the questions (1.6.1) – (1.6.4) is yes.

2 Normal polytopes and unimodular coverings

We now study the problem (1.6.1) when $\mathcal{P}_G$ is normal and the problem (1.6.2) when $\mathcal{P}_G$ possesses a unimodular covering. Let, as before, G be a finite connected simple graph on the vertex set $V(G) = [d] = \{1, \ldots, d\}$. Recall that, when G is bipartite, the answer of the questions (1.6.1) – (1.6.4) is yes. A fundamental observation for the study on triangulations of edge polytopes of non-bipartite graphs is

Lemma 2.1 ([Stu, Lemma 9.5]). *Suppose that G has an odd cycle, i.e., $\dim \mathcal{P}_G = d - 1$ and let G' be a subgraph of G. Then, $\mathcal{P}_{G'}$ is a simplex with $\dim \mathcal{P}_{G'} = d - 1$ if and only if* (i) *G' is a spanning subgraph of G having d edges and* (ii) *every connected component has exactly one odd cycle and it is a unique cycle in the component. Moreover, the normalized volume of a simplex $\mathcal{P}_{G'}$ of dimension $d-1$ is equal to 2^{h-1}, where h is the number of the connected components of G'.*

In general, when two subgraphs G_1 and G_2 of G are disjoint, i.e., have no common vertex, a *bridge* of G_1 and G_2 is an edge combining a vertex of G_1 with a vertex of G_2. Following [Sta], we say that a finite graph G is an *FHM-graph* if an arbitrary pair of disjoint two odd cycles C and C' in G have a bridge. It is shown implicitly in [F–H–M] that the edge polytope $\mathcal{P}_G$ of a finite connected graph G is normal if and only if G is an FHM-graph.

Theorem 2.2 ([O–H$_1$]). *Let G be a finite connected simple graph. Then, the following conditions are equivalent:*
(i) *the edge polytope $\mathcal{P}_G$ is normal;*
(ii) *the edge polytope $\mathcal{P}_G$ possesses a unimodular covering;*
(iii) *G is an FHM-graph.*

Sketch of Proof. First, if G has two odd cycles with no common vertex and with no bridge, then the edge ring $K[G]$ is not normal since G is connected. Thus (i) $\Rightarrow$ (iii); while (ii) $\Rightarrow$ (i) is known. Now, to see why (iii) $\Rightarrow$ (ii) is true, suppose that G is a non-bipartite graph on $[d]$ and let Δ denote the set of all spanning subgraphs G' of G having d edges such that every connected component of G' has exactly one odd cycle and it is a unique cycle in the component. Then, by Lemma 2.1, we have $\mathcal{P}_G = \cup_{G' \in \Delta} \mathcal{P}_{G'}$. Let Γ denote the subset of Δ which consists of all *connected* subgraphs $G' \in \Delta$. Now, [O–H$_1$, Lemma 2.5] guarantees that, in an FHM-graph G, $\mathcal{P}_G = \cup_{G' \in \Gamma} \mathcal{P}_{G'}$. Since the normalized volume of $\mathcal{P}_{G'}$ with $G' \in \Delta$ is 1 if and only if $G' \in \Gamma$, we have (iii) $\Rightarrow$ (ii) as required. Q. E. D.

At present, we do not know an example of a normal $(0,1)$-polytope which possess no unimodular covering.

We now describe the normalization of the edge ring $K[G]$ of an arbitrary graph G explicitly. Let, as before, G be a finite connected simple graph on the vertex $[d]$. A pair $\Pi = \{C, C'\}$ of disjoint two odd cycles in G is called *exceptional* if neither C nor C' has a chord and if Π has no bridge. Given an exceptional pair $\Pi = \{C, C'\}$ in G, we write $\mathcal{M}_\Pi$ for the monomial $(\prod_{i \in V(C)} t_i)(\prod_{j \in V(C')} t_j)$ in $K[t_1, \ldots, t_d]$, where $V(C)$ is the vertex set of C.

Theorem 2.3 ([O–H_1]). *Let G be a finite connected simple graph on the vertex set $[d]$ and $K[G]$ the edge ring of G. Let $\Pi_1 = \{C_1, C_1'\}, \ldots, \Pi_q = \{C_q, C_q'\}$ denote the exceptional pairs in G. Then, the normalization of $K[G]$ is generated by the monomials $\mathcal{M}_{\Pi_1}, \mathcal{M}_{\Pi_2}, \ldots, \mathcal{M}_{\Pi_q}$ as an algebra over $K[G]$. More precisely, as a module over $K[G]$, the normalization of $K[G]$ is generated by those (square-free) monomials of the form $\mathcal{M}_{\Pi_{i_1}} \mathcal{M}_{\Pi_{i_2}} \cdots \mathcal{M}_{\Pi_{i_\ell}}$ with $1 \leq i_1 < i_2 < \cdots < i_\ell \leq q$ such that $(V(C_{i_f}) \bigcup V(C_{i_f}')) \bigcap (V(C_{i_g}) \bigcup V(C_{i_g}')) = \emptyset$ for all $1 \leq f < g \leq \ell$.*

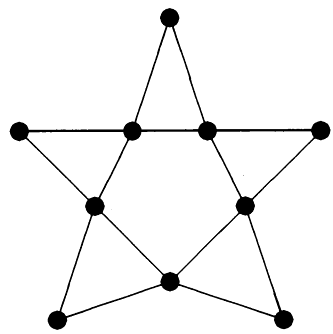
Sketch of Proof. The normalization of $K[G]$ is isomorphic to the Ehrhart ring [O–H_2] of the edge polytope $\mathcal{P}_G$ of G. The technique appearing in the sketch of proof of Theorem 2.2 enables us to describe the Ehrhart ring of the edge polytope $\mathcal{P}_G$ explicitly. Q. E. D.

In [O–H_1], we study a finite connected graph G allowing loops and having no multiple edge and its edge ring $K[G]$. Then, the above Theorem 2.3 is also valid if we regard a loop is an odd cycle of length 1. Moreover, if G is disconnected and if $G_1, \ldots, G_k$ are the connected components of G, then the normalization of $K[G]$ is the tensor product of the normalizations of $K[G_1], \ldots, K[G_k]$ over K. Hence, the normalizations of all affine semigroup rings generated by quadratic monomials can be obtained. To describe the normalizations of edge rings is of interest, however, the highlight of this section is the existence of a unimodular covering of a normal edge polytope. We also remark that Theorem 2.3 is also obtained in [S–V–V] independently by purely ring-theoretical technique without the notion of Ehrhart rings. However, in [S–V–V], they never discuss unimodular coverings of edge polytopes.

3 An exciting example

It was conjectured that $\mathcal{P}_G$ is normal if and only if $\mathcal{P}_G$ possesses a regular unimodular triangulation. On the other hand, for a long time, it was unknown if there exists a lattice polytope (not necessarily $(0,1)$-polytope) which possesses a unimodular triangulations and none of whose regular triangulations is unimodular.

We are now in the position to discuss an exciting example of normal edge polytope none of whose regular triangulations is unimodular. The first example of a normal $(0,1)$-polytope none of whose regular triangulations is unimodular given in [O–H$_2$] is the edge polytope of the following graph below.



It then turns out that this edge polytope *does* possesses a unimodular triangulation (Firla and Ziegler, De Loera). No other example of a lattice polytope which possesses a unimodular triangulation and none of whose regular triangulations is unimodular must be known. On the other hand, at present, we do not know an example of a normal edge polytope which possesses no unimodular triangulation.

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Duality of modules over Gorenstein rings

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Let $(R, \mathfrak{m}, k)$ be a complete Gorenstein local ring. We consider the following subcategories of the category mod R consisting of finitely generated R-modules:

$\mathcal{CM}(R)$: the category of maximal Cohen-Macaulay modules.
$\mathcal{F}(R)$: the category of modules with finite projective dimensions.

Let us remind the definition of Cohen-Macaulay approximation.

Theorem 1 (Auslander-Buchweitz [2], Kato [3]) *An arbitrary $M \in$ mod R has the following exact sequences.*

$$\begin{array}{ll} 0 \to Y_M \to X_M \to M \to 0 & \textit{(Cohen-Macaulay approximation)}, \\ 0 \to M \to Y^M \to X^M \to 0 & \textit{(finite projective hull)}, \\ 0 \to X \to M \oplus P \to Y \to 0 & \textit{(origin extension)} \end{array}$$

where $Y_M, Y^M, Y \in \mathcal{F}(R)$, $X_M, X^M, X \in \mathcal{CM}(R)$, and P is a free module.

Two modules M and M' are called stablly equivalent and denoted as $M \overset{st}{\cong} M'$ if projective modules P and P' exist such that $M \oplus P \cong M' \oplus P'$. A stable module refers to a module without non-zero free summand. For a full subcategory $\mathcal{C}$ of mod R, the projctive stabilization $\underline{\mathcal{C}}$ is defined as follows.

- Each object of $\underline{\mathcal{C}}$ is an object of $\mathcal{C}$.

- For $M, N \in \mathcal{C}$, a set of morphisms from M to M' is $\mathrm{Hom}_R(M,N)/\mathcal{P}(M,N)$ where $\mathcal{P}(M,N) := \{f \in \mathrm{Hom}_R(M,N) \mid$ f factors through some projective module$\}$.

We can identify an element of $\underline{\mathcal{C}}$ as a stable equivalence class of a module in $\mathcal{C}$.

We want to know

$$\mathcal{M}_Y := \{M \in \mathrm{mod}\ R \mid Y^M \overset{st}{\cong} Y\}$$

for a fixed $Y \in \mathcal{F}(R)$.

It is easy to see the following. For the proof, see [4] for example.

Remark 2 *Let M and M' be an element of $\mathcal{M}_{\mathcal{Y}}$ with the minimal finite projective hulls*

$$0 \to Y \oplus P \overset{(f\,*)}{\to} X^M \to 0,$$
$$0 \to Y \oplus P \overset{(f'\,*)}{\to} X^{M'} \to 0.$$

The following conditions are equivalent.

1) $M \overset{st}{\cong} M'$.

2) There exist isomorphisms $i_X \in \mathrm{Hom}_R(X^M, X^{M'})$ and $i_Y \in \mathrm{Aut}_R\ (Y)$ such that $i_X f - g i_Y$ factors through a projective module.

For two homomorphisms $f : M \to N$ and $g : M' \to N'$, we write $f \sim g$ if isomorphisms $i_M : M \to M'$ and $i_N : N \to N'$ exist such that $i_M g - f i_N$ factors through projective. Each element of $\underline{\mathcal{M}_Y}$ bijectively corresponds to that of $\bigcup_{X \in \mathcal{CM}\ (R)} \mathrm{Hom}_R(Y,X)/\sim$. From the viewpoint of Cohen-Macaulay approximations, naturally we feel like regarding $\mathcal{M}_Y$ as a subcategory of $\mathcal{CM}(R)$. We shall do this by using a functor $T : \underline{\mathrm{mod}\ R} \to \underline{\mathcal{CM}(R)}$.

1 Functor T.

For $M \in \mathrm{mod}\ R$, $TM := \Omega_R^{d+1} \mathrm{tr}\, \Omega_R^{d+1} \mathrm{tr} M$ with $d := \dim R$ is a stable Cohen Macaulay module. There is a natural map $\varphi_M : M \to TM$ for each $M \in \mathrm{mod}\ R$. To obtain this, let $F_{M\bullet}$ be an R-free complex such that

$$\cdots \to F_{M2} \overset{d_{F_{M}2}}{\to} F_{M1} \overset{d_{F_{M}1}}{\to} F_{M0} \to M$$

is the minimal free resolution of M and

$$\cdots \to F_{M-1}^* \overset{d_{F_{M0}}{}^*}{\to} F_{M0}^* \overset{d_{F_{M1}}{}^*}{\to} F_{M1}^* \to \mathrm{tr}\ M$$

is the minimal free resolution of tr M. Since R is Gorenstein, $\Omega_R^{d+1}(\mathrm{tr}\ M)$ is a stable Cohen-Macaulay module. We can take an exact free R-complex

$$E_{M\bullet} : \cdots \to E_{Mn} \overset{d_{E_{M}n}}{\to} E_{Mn-1} \to \cdots$$

with $\mathrm{Coker}\, d_{E_{M-d}}{}^* \cong \Omega_R^{d+1}(\mathrm{tr}\ M)$. The identity map on $\Omega_R^{d+1}(\mathrm{tr}\ M)$ induces a chain map $\varphi_{M\bullet} : F_{M\bullet} \to E_{M\bullet}$ with $\varphi_{Mn} = \mathrm{id}_{F_{Mn}}$ for $n \leq -d$.

$$\begin{array}{cccccccccccc}
F_{M\bullet}: & \cdots \to & F_{M1} & \to & F_{M0} & \to & F_{M-1} & \to \cdots \to & F_{M-d+1} & \to & F_{M-d} & \to \cdots \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_{-1}} & & \downarrow{\scriptstyle \varphi_{-d+1}} & & \| & \\
E_{M\bullet}: & \cdots \to & E_{M1} & \to & E_{M0} & \to & E_{M-1} & \to \cdots \to & E_{M-d+1} & \to & E_{M-d} & \to \cdots
\end{array}$$

Obviously $TM = \mathrm{Coker}\, d_{E_M}$, and φ_M is induced by $\varphi_{M\bullet}$. Remember that $E_{M\bullet}$ is completely exact, that is, both $E_{M\bullet}$ and $E_{M\bullet}^*$ dualized by R are exact. The chain map φ_M becomes isomorphism on sufficiently small degree, and is unique up to chain homotopy with respect to this property.

Lemma 3 1) *$M \overset{st}{\cong} TM$ if and only if $M \in \mathcal{CM}(R)$.*

2) *TM is zero if and only if tr $M \in \mathcal{F}(R)$.*

3) *Each homomorphism $f : M \to M'$ of modules induces a homomorphism $Tf : TM \to TM'$such that $Tf\, \varphi_M \sim \varphi_{M'} f$. In other words, T defines a functor $\underline{\mathrm{mod}\ R} \to \underline{\mathcal{CM}(R)}$.*

proof. From the construction above, it is easy to see $\mathrm{Coker}\, d_{E_{M1}}{}^* \overset{st}{\cong} X_{\mathrm{tr}\ M}$ and $\varphi_\bullet{}^*$ induces the stable part of the map $X_{\mathrm{tr}\ M} \to \mathrm{tr}\ M$. Hence 1) is obvious. While we have $E_{M\bullet} = 0$ if and only if $X_{\mathrm{tr}\ M}$ is free, which means tr $M \in \mathcal{F}(R)$.

We shall prove 3). Given homomorphism f induces a chain map $f_\bullet$: $\tau_0 F_{M\bullet} \to \tau_0 F_{M'\bullet}$, which again induces the lower part $f_\bullet : F_{M\bullet} \to F_{M'\bullet}$. The negative part

$$\begin{array}{cccc}
f_{F-d}: & F_{M-d} & \to & F_{M'-d} \\
 & \| & & \| \\
 & E_{M-d} & & E_{M'-d}
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
f_{F-d-1}: & F_{M-d-1} & \to & F_{M'-d-1} \\
 & \| & & \| \\
 & E_{M-d-1} & & E_{M'-d-1}
\end{array}$$

induce a chain map $f_{E_\bullet} : E_{M\bullet} \to E_{M'\bullet}$. Since $\varphi_{M'i} f_{Fi} - f_{Ei}\varphi_{Mi} = 0$ for $i \le -d$, $\varphi_{M'\bullet} f_{F\bullet} - f_{E\bullet}\varphi_{M\bullet}$ is homotopic to zero. As the chain map $f_{E_\bullet}$ induces Tf, $\varphi_{M'}f - Tf\varphi_M$ factors through projective. (q.e.d.)

Proposition 4 *Let $\theta : 0 \to A \to B \to C \to 0$ be an exact sequence of R-modules such that the sequence $0 \to C^* \to B^* \to A^* \to 0$ dualized by R is also exact. The given sequence induces the following commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow{\scriptstyle \varphi_A} & & \downarrow{\scriptstyle \tilde{\varphi}_B} & & \downarrow{\scriptstyle \varphi_C} & & \\ 0 & \to & TA & \to & TB \oplus P & \to & TC & \to & 0 \end{array}$$

where P a free module and the stable part of $\tilde{\varphi}_B$ is φ_B.

proof. The assumption says that $\mathrm{tr}\ \theta : 0 \to \mathrm{tr}\ C \to \mathrm{tr}\ B \oplus Q \to \mathrm{tr}\ A \to 0$ with some free module Q is exact; [2], Lemma (3.9). It follows that the chain map $\theta_{F_\bullet} : \tau_1 F_{C\bullet+1} \to \tau_0 F_{A\bullet}$ corresponding to θ is extended on the negative part to the chain map $\theta_{F_\bullet} : F_{C\bullet+1} \to F_{A\bullet}$. Similarly as in the proof of 2) of Lemma 3 , we get a chain map $\theta_{E_\bullet} : F_{C\bullet+1} \to F_{A\bullet}$ such that the diagram

$$\begin{array}{ccc} F_{C\bullet} & \overset{\theta_{F\bullet}}{\to} & F_{A\bullet-1} \\ \downarrow{\scriptstyle \varphi_{C\bullet}} & & \downarrow{\scriptstyle \varphi_{A\bullet-1}} \\ E_{C\bullet} & \overset{\theta_{E\bullet}}{\to} & E_{A\bullet-1} \end{array}$$

commutes up to homotopy; $\theta_{E\bullet}\varphi_{C\bullet} - \varphi_{A\bullet}\theta_{F\bullet} = d_{E_{A\bullet}} h_\bullet - h_\bullet d_{F_{C\bullet}}$. This leads us to the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & F_{A\bullet} & \to & Cone\ (\theta_F)_\bullet & \to & F_{C\bullet} & \to & 0 \\ & & \downarrow{\scriptstyle \varphi_{A\bullet}} & & \downarrow{\scriptstyle \varphi_{\tilde{B}\bullet}} & & \downarrow{\scriptstyle \varphi_{C\bullet}} & & \\ 0 & \to & E_{A\bullet} & \to & Cone\ (\theta_E)_\bullet & \to & E_{C\bullet} & \to & 0 \end{array}$$

where

$$\widetilde{\varphi_{B_\bullet}} = \begin{pmatrix} \varphi_{A\bullet} & h_\bullet \\ 0 & \varphi_{C\bullet} \end{pmatrix}.$$

We easily observe the following:

- $Cone\ (\theta_F)_\bullet$ is a direct sum of $F_{B\bullet}$ and the split exact sequence of free modules.
- $Cone\ (\theta_E)_\bullet$ is completely exact as well as $E_{A\bullet}$ and $E_{C\bullet}$.
- $\widetilde{\varphi_{B_\bullet}}$ is an isomorphism for $i \le -d$.

In other words, $Cone\ (\theta_F)_\bullet \overset{st}{\cong} F_{B\bullet}$ with $\operatorname{Coker} d_{Cone\ (\theta_F)_1} \cong B$, $Cone\ (\theta_E)_\bullet \cong E_{B\bullet}$, and $\widetilde{\varphi_B} \overset{st}{\cong} \varphi_B$. Taking the 0-th truncation homology of the diagram, we get the required diagram. (q.e.d.)

The next Lemma gives us a key to the classification by the functor T.

Lemma 5 *Let $M \in mod\ R$ and $X \in \mathcal{CM}(R)$. For every $f \in \operatorname{Hom}_R(M, X)$, there exists $f' \in \operatorname{Hom}_R(TM, X)$ such that $f \sim f'\ \varphi_M$.*

proof. From 3) of Lemma 3, $Tf\varphi_M - \varphi_X f$ factors through projective. The first statement 1) of Lemma 3 tells us φ_X is an isomorphism on stable part. Applied ${\varphi_X}^{-1}$, ${\varphi_X}^{-1}\ Tf\varphi_M - f$ factors through projective. So we may take $f' = {\varphi_X}^{-1}\ Tf$. (q.e.d.)

Now fix a stable module $Y \in \mathcal{F}(R)$. As we mentioned in Remark 2, each element of $M \in \underline{\mathcal{M}}$ is determined by a stable part f of the surjective map on the minimal finite projective hull

$$0 \to M \to Y \oplus P \overset{(f*)}{\to} X^M \to 0.$$

In terms of the above lemma, there exists $f' \in \operatorname{Hom}_R(TY, X^M)$ such that $f \sim f'\varphi_M$; in other words, the natural map $\varphi_Y : Y \to TY$ yields a generator of $\mathcal{M}_Y$. For each element of $\mathcal{M}_Y$, we can find an element of $\bigcup_{X \in \mathcal{CM}(R)} \operatorname{Hom}_R(TY, X)$. Our next step is to get the relation that determines the class corresponding to each element of $\underline{\mathcal{M}_Y}$.

Proposition 6 The following are equivalent for $f, g \in \bigcup_{X \in \mathcal{CM}(R)} \operatorname{Hom}_R(Y, X)$.

1) $f \sim g$.

2) $Tf \sim Tg$.

By Lemma 5, $f \sim Tf\ \varphi_Y$ and $g \sim Tg\ \varphi_Y$. So Proposition 6 is equivalent to the following.

Proposition 7 The following are equivalent for $f', g' \in \bigcup_{X \in \mathcal{CM}(R)} \operatorname{Hom}_R(TY, X)$.

1) $f' \sim g'$.

2) $f'\varphi_Y \sim g'\varphi_Y$.

proof. 2) $\Rightarrow$ 1). Let $f' \in \mathrm{Hom}_R(Y,X)$ and $g' \in \mathrm{Hom}_R(Y,X')$ for $X, X' \in \mathcal{CM}(R)$. We may assume X and X' are stable. By assumption, isomorphisms $i_X \in \mathrm{Hom}_R(X,X')$ and $i_Y \in \mathrm{Aut}_R\ Y$ exist such that $i_X f' \varphi_Y \sim g' \varphi_Y i_Y$. The isomorphism i_Y induces isomorphism $Ti_Y \in \mathrm{Aut}_R\ Y$. We claim that $i_X f' \sim g' T i_Y$. Homomorphism f', g', i_Y, i_X and Ti_Y of modules induce chain maps $f'_\bullet$, $g'_\bullet$, $i_{F_{Y_\bullet}}$, $i_{F_{X_\bullet}}$ and $i_{E_{Y_\bullet}}$.

$$\begin{array}{ccccc}
F_{Y_\bullet} & \stackrel{\varphi_{Y_\bullet}}{\rightarrow} & E_{Y_\bullet} & \stackrel{f'_\bullet}{\rightarrow} & F_{X_\bullet} \\
\downarrow{\scriptstyle i_{F_{Y_\bullet}}} & & \downarrow{\scriptstyle i_{E_{Y_\bullet}}} & & \downarrow{\scriptstyle i_{F_{X_\bullet}}} \\
F_{Y_\bullet} & \stackrel{\varphi_{Y_\bullet}}{\rightarrow} & E_{Y_\bullet} & \stackrel{g'_\bullet}{\rightarrow} & F_{X'_\bullet}
\end{array}$$

It is easy to see $g'_\bullet \varphi_{Y_\bullet} i_{F_{Y_\bullet}} = i_{X_\bullet} f'_\bullet \varphi_{Y_\bullet}$ up to chain homotopy. Also we have $\varphi_{Y_\bullet} i_{F_{Y_\bullet}} = i_{E_{Y_\bullet}} \varphi_{Y_\bullet}$ up to chain homptopy, and φ_Y is isomorphic on degree less than $-d+1$. Therefore $g'_\bullet i_{E_{Y_\bullet}} = i_{X_\bullet} f'_\bullet$ on degree less than $-d+1$, hence on every degree, since $F_{Y_\bullet}$, $F_{X_\bullet}$ and $F_{X'_\bullet}$ are all completely exact.

Now for the harder part 1) $\Rightarrow$ 2). For given $f' \in \mathrm{Hom}_R(TY,X)$, clearly $i_X f' \varphi_Y \sim f' \varphi_Y$. So we have only to show $f' \varphi_Y \sim f' i_{TY} \varphi_Y$ for arbitrary $i_{TY} \in \mathrm{Aut}_R TY$. Using Lemma 8, we can take some homomorphism p and p' from free modules to Y such that both of $(f' \varphi_Y\, p)$ and $(f' i_{TY} \varphi_Y\, p')$ are surjective and $\mathrm{Ker}\,(f' \varphi_Y\, p) \stackrel{st}{\cong} \mathrm{Ker}\,(f' i_{TY} \varphi_Y\, p')$. The last relation implies $f' \varphi_Y \sim f' i_{TY} \varphi_Y$ from Remark 2. (q.e.d.)

Lemma 8 *Let $X, Y, Z \in$ mod R and let $\varphi \in \mathrm{Hom}_R(Y,Z)$, $f \in \mathrm{Hom}_R(Z,X)$ and $i \in \mathrm{Aut}_R Z$. Take homomorphisms p and p' from some free module to X such that $(f \varphi\, p)$ and $(f i \varphi\, p)$. Then* $\mathrm{Ker}\,(f \varphi\, p) \stackrel{st}{\cong} \mathrm{Ker}\,(f i \varphi\, p)$.

proof. Take $\pi : P \to Z$ with a free module P to make $Y \oplus P \stackrel{(\varphi \pi)}{\rightarrow} Z$ surjective and put $N := \mathrm{Ker}\,(\varphi \pi)$. Take $\rho : Q \to X$ with a free module Q to make $Y \oplus Q \stackrel{(f \rho)}{\rightarrow} X$ surjective and put $U := \mathrm{Ker}\,(f \rho)$. Putting $M := \mathrm{Ker}\,(f \varphi\, f \pi\, \rho)$

and $M' := \mathrm{Ker}\,(fi\varphi\; fi\pi\; \rho)$, we shall show $M \overset{st}{\cong} M'$.

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & N & \to & Y \oplus P \oplus Q & \overset{\left(\begin{smallmatrix}\varphi & \pi & 0\\ 0 & 0 & 1\end{smallmatrix}\right)}{\longrightarrow} & Z \oplus Q & \to & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle (f\;\rho)} & & \\
0 & \to & M & \to & Y \oplus P \oplus Q & \overset{(f\varphi\; f\pi\; \rho)}{\longrightarrow} & X & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & U & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

Considering the mapping cone of $f_\bullet : F_{Z\bullet} \to F_{X\bullet}$ induced by f and $\varphi_\bullet : F_{Y\bullet} \to F_{Z\bullet}$ induced by φ, we have

$$
U \overset{st}{\cong} \mathrm{Coker}\;
\begin{array}{cc}
 & \begin{array}{cc} F_{X2} & F_{Z1} \end{array} \\
\begin{array}{c} F_{X1} \\ F_{Z0} \end{array} & \begin{pmatrix} d_{F_{X2}} & f_1 \\ 0 & d_{F_{Z1}} \end{pmatrix}
\end{array},
\quad
N \overset{st}{\cong} \mathrm{Coker}\;
\begin{array}{cc}
 & \begin{array}{cc} F_{Z2} & F_{Y1} \end{array} \\
\begin{array}{c} F_{Z1} \\ F_{Y0} \end{array} & \begin{pmatrix} d_{F_{Z2}} & \varphi_1 \\ 0 & d_{F_{Y1}} \end{pmatrix}
\end{array}.
$$

It is easy to see

$$
M \overset{st}{\cong} \mathrm{Coker}\;
\begin{array}{cc}
 & \begin{array}{cc} F_{X2} & F_{Y1} \end{array} \\
\begin{array}{c} F_{X1} \\ F_{Y0} \end{array} & \begin{pmatrix} d_{F_{X2}} & f_1\varphi_1 \\ 0 & d_{F_{Y1}} \end{pmatrix}
\end{array}
\overset{st}{\cong} \mathrm{Coker}\;
\begin{array}{cc}
 & \begin{array}{cccc} F_{Z2} & F_{Y1} & F_{X2} & F_{Z1} \end{array} \\
\begin{array}{c} F_{Z1} \\ F_{Y0} \\ F_{X1} \\ F_{Z0} \end{array} & \begin{pmatrix} d_{F_{Z2}} & \varphi_1 & 0 & 1 \\ 0 & d_{F_{Y1}} & 0 & 0 \\ 0 & 0 & d_{F_{X2}} & f_1 \\ 0 & 0 & 0 & d_{F_{Z1}} \end{pmatrix}
\end{array}.
$$

As an element of $\mathrm{Ext}^1_R(U, N)$, the right-handside is isomorphic to

$$
\mathrm{Coker}\;
\begin{array}{cc}
 & \begin{array}{cccc} F_{Z2} & F_{Y1} & F_{X2} & F_{Z1} \end{array} \\
\begin{array}{c} F_{Z1} \\ F_{Y0} \\ F_{X1} \\ F_{Z0} \end{array} & \begin{pmatrix} d_{F_{Z2}} & \varphi_1 & 0 & i_1^{-1} \\ 0 & d_{F_{Y1}} & 0 & 0 \\ 0 & 0 & d_{F_{X2}} & f_1 \\ 0 & 0 & 0 & d_{F_{Z1}} \end{pmatrix}
\end{array}
\overset{st}{\cong} \mathrm{Coker}\;
\begin{array}{cc}
 & \begin{array}{cc} F_{X2} & F_{Y1} \end{array} \\
\begin{array}{c} F_{X1} \\ F_{Y0} \end{array} & \begin{pmatrix} d_{F_{X2}} & f_1 i_1 \varphi_1 \\ 0 & d_{F_{Y1}} \end{pmatrix}
\end{array}
\overset{st}{\cong} M'.
$$

(q.e.d.)

We are now to state the conclusion of this section. The proof is straightforward from Proposition 6 and Proposition 7.

Theorem 9 *There is a one-to-one correspondence between isomorphic classes of* $\underline{\mathcal{M}_Y}$ *and those of* $\bigcup_{X \in \mathcal{CM}(R)} \mathrm{Hom}_R(TY, X)/\sim$.

Example 10 *Let* $R := k[[x,y]]/(xy)$*. All the* $\mathcal{M}_Y$*'s for any indecomposable* $Y \in \mathcal{F}(R)$ *are isomorphic to each other, since* $TY \cong R/xR \oplus R/yR$.

2 Approximations.

Let $M \in \mathrm{mod}\, R$. The natural homomorphism $\varphi_M : M \to TM$ is not surjective in general. The minimal projective cover $P_M \to \mathrm{Coker}\, \varphi_M$ induces a homomorphism $\pi_M : P_M \to TM$ to make $\tilde{\varphi_M} : M \oplus P_M \overset{(\varphi_M\ \pi_M)}{\to} TM$ surjective. Denote $\mathrm{Ker}\, \tilde{\varphi_M} := NM$.

Now let $Y \in \mathcal{F}(R)$ and $M \in \mathcal{M}_Y$. Lemma 5 is described as the following commutative diagram that commutes up to projective modules

$$\begin{array}{ccccccccc}
0 & \to & NY & \to & Y \oplus P_Y & \to & TY & \to & 0 \\
 & & \downarrow & & \downarrow \tilde{\mathrm{id}}_Y & & \downarrow f' & & \\
0 & \to & M & \to & Y^M & \to & X^M & \to & 0
\end{array}$$

where the stable part of $\tilde{\mathrm{id}}_Y$ is an identity map on Y. In other words, $NY \in \mathcal{M}_Y$ is a generating element of $\mathcal{M}_Y$. What is the difference between a generic element of $\mathcal{M}_Y$ and this NY?

Since the finite projective hull of M remains exact when dualized by R, from Proposition , we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & M \oplus P_M & \to & Y^M \oplus P_M & \to & X^M & \to & 0 \\
 & & \downarrow \widetilde{\varphi_M} & & \downarrow \widetilde{\varphi_{Y^M}} & & \downarrow \varphi_{X^M} & & \\
0 & \to & TM & \to & TY^M \oplus P & \to & TX^M & \to & 0
\end{array}$$

with free modules P and P_M. We may take as $\widetilde{\varphi_M}$ and $\widetilde{\varphi_{Y^M}}$ are surjective. Notice that $\varphi_{X\ M}$ is isomorphic. Observing this, we have the following Lemma.

Lemma 11 *For* $Y, Y' \in \mathcal{F}(R)$*,* $NY \cong NY'$ *if and only if* $Y \overset{st}{\cong} Y'$.

proof. On the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & NY & \to & Y \oplus P_Y & \to & TY & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & Y' \oplus P_{Y'} & \to & W & \to & TY & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & TY' & = & TY' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

the second row and the second column both split; $Y' \oplus P_{Y'} \oplus TY \cong Y' \oplus P_Y \oplus TY'$. Since TY and TY' are stable Cohen-Macaulay, $Y \oplus P_{Y'} \cong Y \oplus P_Y$. The converse is clear. (q.e.d.)

Next we introduce a new type of approximation that is a dual notion of the origin extension. Let us consider a full subcategory of mod R; $\mathcal{N}(R) := \{M \in \text{mod } R \mid TM = 0\}$ which is a dual of $\mathcal{F}(R)$. A Cohen-Macaulay module with positive codimension is contained in $\mathcal{N}(R)$.

Theorem 12 *For an $M \in$ mod R, there is an exact sequence*

$$0 \to Z \to M \oplus P \to X \to 0$$

with $Z \in \mathcal{N}(R)$, $X \in \mathcal{CM}(R)$, and a free module P. We call this exact sequence a NCM-approximation of M.

Definition and Theorem 13 *A NCM approximation $0 \to Z \to M \oplus P \to X \to 0$ of M with minimal rank of P is called a minimal NCM approximation of M. Each NCM approximation of M is a direct sum of a minimal NCM approximation of M and some split exact sequence of free modules.*

As we see in next example, two categories $\mathcal{N}(R)$ and $\mathcal{CM}(R)$ do not annihilate each other. In other words, NCM approximation is an approximation without Auslander-Buchweitz context.

Example 14 *It is obvious that $k \in \mathcal{N}(R)$. However $\text{Ext}^1_R(k, X) \neq 0$ for any $X \in \mathcal{CM}(R)$ if $\dim R = 1$.*

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Examples of Local Rings

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INTRODUCTION

良く知られているように、

1. １次元ネーター整域 A の商体 $Q(A)$ の有限次代数拡大体 L に含まれる A の拡大環 B $(A \subset B \subset L)$ はいつもネーター環である。
2. ２次元ネーター整域 A の整閉包 $\bar{A}$ はネーター環だが、A と $\bar{A}$ の中間環 B $(A \subset B \subset \bar{A})$ は必ずしもネーター環ではない。
3. ３次元ネーター整域 A の整閉包 $\bar{A}$ はKrull 環だが、必ずしもネーター環ではない。

ここでは、

1. ２次元ネーター局所整域 A で、A と整閉包 $\bar{A}$ との中間環 B がネーター環でない例、
2. ２次元ネーター正規局所整域 A で解析的被約でない例、
3. ３次元ネーター局所整域 A で A の整閉包 $\bar{A}$ がネーター環でない例、

を与える。

これから構成する例は、体の標数が２である。しかし、０を含む一般標数で同様の例が構成できる (はず)。

Example 0.1. k が標数 2 の体、$R_i = k[X_i^2, X_i^3]$, $P_i = (X_i^2, X_i^3)R_i$ $(i = 1, 2, \dots)$ が R_i の極大イデアルであるとき、$R' = \otimes_k R_i$ と、その積閉集合 $S = R' \setminus \bigcup P_i R'$ により、１次元ネーター整域 $R = R'_S$ を得る。

さて、$\bar{R} = R[X_1, \dots, X_i, \dots]$ を R の商体における整閉包、$C = R[[T]]$, $D = \bar{R}[[T]]$ で、それぞれ、R-係数一変数べき級数環、$\bar{R}$-係数一変数べき級数環を表す。

D の元 $\omega = \sum_i X_i T^i$ をとると、C-係数一変数多項式環 $C[W]$ の素元 $f(W) = W^2 - \omega^2 \in C[W]$ を得る。

２次元ネーター整域 $C[w] = C[W]/(f(W))$ の極大イデアル M での局所化 $A = C[w]_M$ の非単元 a を採り、その商体における整閉包 $\bar{A}$ と $A[\frac{1}{a}]$ との共通部分 $\bar{A} \cap A[\frac{1}{a}]$ を B と定める。この中間環 B はネーター環ではない。

Example 0.2. R, $\bar{R}$ は、上と同様。$C = R[[T, U]]]$, $D = \bar{R}[[T, U]]$ で、それぞれ、R-係数二変数べき級数環、$\bar{R}$-係数二変数べき級数環を表す。D の元 $\omega_1 = \sum_i X_{2i-1} T^i$、$\omega_2 = \sum_i X_{2j} U^j$ をとり、$\omega = \omega_1 + \omega_2$ と

書く。ω は C の商体 $Q(C)$ には含まれない。しかし、C の素イデアル $P = (T, U)C$ に於ける C の局所化 C_P は、2次元正則局所環で、完備化 $C_P{}^{\wedge}$ が ω を含む。ここで、C-係数一変数多項式環 $C[W]$ の元 $f(W) = W^2 - \omega^2 \in C[W]$ を考えると、上と同様に、$f(W)$ は $C[W]$ の素元。

さて、3次元ネーター整域 $C[w] = C[W]/(f(W))$ を、素イデアル Q で P の上にある、即ち $Q \cap C = P$ である、ものにより局所化した2次元ネーター局所整域 $C[w]_Q$ を A で表す。このとき、「標準的」議論により A は正規、かつ完備化 $\hat{A}$ が被約でない。

Example 0.3. さらに、3次元ネーター整域 $C[w]$ の任意の極大イデアル M での局所化 $C[w]_M$ を、改めて A、その商体における整閉包を $\bar{A}$ と記す。この $\bar{A}$ はネーター環ではない。

1. Two-dimensional local domain which has non Neotherian intermediate domains within its derived normal ring.

Let k be a field of characteristic 2. Taking indeterminates X_i, we set $R_i = k[X_i^2, X_i^3]$ with (fixed) maximal ideal $\mathfrak{p}_i = (X_i^2, X_i^3)$, $i = 1, 2, \ldots$. Put $R' = \otimes_k R_i$ and $S = R' \setminus \bigcup_i \mathfrak{p}_i R'$. Let $R = R'_S$. Then

- Maximal ideals $\mathfrak{q}_i$ of R are in one-to-one correspondence with $\mathbf{N}$ via $i \mapsto \mathfrak{q}_i = \mathfrak{p}_i R$.
- $R_{\mathfrak{q}_i} = (K_i \otimes_k R_i)_{\tilde{\mathfrak{q}}_i}$, where $K_i = k(\ldots, X_{i-1}, X_{i+1}, \ldots)$ is an extension field of k and $\tilde{\mathfrak{q}}_i = \mathfrak{q}_i(K_i \otimes_k R_i)$. (Note that $\mathfrak{p}_i$ is an absolutely prime ideal.)
- Any non-zero element of R is contained in only a finite number of maximal ideals.

Hence R is a one-dimensional Noetherian domain with field of quotients $K = k(X_1, \ldots, X_i, \ldots)$ [1]. Let $\bar{R}$ be the derived normal ring of R. Then $\bar{R} = R[X_1, \ldots, X_i, \ldots]$ and the set $\bar{R}^2 = \{r^2 \mid r \in \bar{R}\}$ is contained in R. Moreover, for any non-zero prime $\mathfrak{q}_i$ of R, $R_{\mathfrak{q}_i}$ is not normal, but its derived normal ring $\overline{R_{\mathfrak{q}_i}}$ is a finite $R_{\mathfrak{q}_i}$-module.

Now taking two more indeterminates T and W, we set $B = R[[T]]$, $C = \bar{R}[[T]]$. Let

$$\omega = \sum_{i=1}^{\infty} X_i T^i \in C, \tag{1.1}$$

$$f(W) = W^2 - \omega^2 \in B[W] \text{ and } g(W) = W - \omega \in C[W]. \tag{1.2}$$

Then $f(W) = g(W)^2$ in $C[W]$ and the set $C[W]^2 = \{h^2 \mid h \in C[W]\}$ is contained in $B[W]$. We claim

Lemma 1.1. *$f(W)$ is a prime element of $B[W]$.*

Proof. We note that $g(W)$ is a prime element of $C[W]$, because $C \xrightarrow{\sim} C[W]/gC[W]$. Put $Q = gC[W] \cap B[W]$ a prime ideal of $B[W]$. Then the set $\{q^2 \mid q \in Q\}$ is contained in $fB[W]$. Hence $Q^\nu \subset fB[W]$ for some sufficiently large $\nu > 0$. Consequently, Q is the only minimal prime of the principal ideal $fB[W]$. Moreover, as $B[W]$ is (locally) Cohen-Macaulay, Q is the only associated prime of $fB[W]$. Therefore, to get the assertion, it is sufficient to show

$$(1.3) \qquad f(W) \text{ is a prime element of } B[W]_Q.$$

Let L be the field of quotients of B. Then, as $Q \cap B = (0)$, (1.3) is equivalent to

$$(1.4) \qquad f(W) \text{ is irreducible in } L[W].$$

Proof of (1.4). Suppose that $f(W)$ is reducible in $L[W]$. Then $f(W) = (W + \omega)^2$ in $L[W]$. Hence, we find α, $\beta \in B$ such that $\omega = \beta/\alpha$, that is, $\alpha\omega = \beta$. Let

$$(1.5) \qquad \alpha = \sum_{m=0}^{\infty} a_m T^m \text{ and } \beta = \sum_{n=0}^{\infty} b_n T^n \text{ with } a_m,\ b_n \in R.$$

We compare the coefficients of T^n in the relation above. Then

$$(1.6) \qquad \sum_{m+j=n} a_m X_j = b_n \text{ for any } n > 0.$$

So, if we write $a = a_{m_0}$ where $m_0 = \min\{m \mid a_m \neq 0\}$, we have $a^j X_j \in R$ for any $j > 0$. Consequently

$$(1.7) \qquad \bar{R} = R[X_1, X_2, \dots, X_j, \dots] \subset R_a.$$

This contradicts our first remarks. Thus (1.4) is proved. □

By Lemma 1.1, $B[w] = B[W]/fB[W]$ is a two-dimensional Noetherian domain, which is canonically contained in C. Take a maximal ideal $\mathfrak{m}$ of B and a maximal ideal $\mathfrak{q} = \mathfrak{m} \cap R$ of R. Let $B^*_\mathfrak{m}$ be the $TB_\mathfrak{m}$-adic completion of $B_\mathfrak{m}$. Then $B^*_\mathfrak{m} = R_\mathfrak{q}[[T]]$.

Let $\mathfrak{n}$ be the corresponding maximal ideal of C and $C^*_\mathfrak{n}$ be the $TC_\mathfrak{n}$-adic completion of $C_\mathfrak{n}$. Then $C^*_\mathfrak{n} = \overline{R_\mathfrak{q}}[[T]] = B^*_\mathfrak{m} \otimes_R \bar{R}$, because $\overline{R_\mathfrak{q}}$ is a finite $R_\mathfrak{q}$-module. Hence $C^*_\mathfrak{n} \otimes_R K = B^*_\mathfrak{m} \otimes_R K$.

Now put $M = \mathfrak{n} \cap B[w]$. Then the $TB[w]_M$-adic completion $(B[w]_M)^*$ of $B[w]_M$ is isomorphic to $B^*_\mathfrak{m}[W]/fB^*_\mathfrak{m}[W]$. Thus

$$(1.8) \qquad (B[w]_M)^* \otimes_R K = (C^*_\mathfrak{n} \otimes_R K)[W]/f(C^*_\mathfrak{n} \otimes_R K)[W].$$

On the other hand, we have already seen that $f(W) = g(W)^2$ in $C[W]$ (cf. (1.2)). Therefore

$$(B[w]_M)^* \text{ is not reduced.} \tag{1.9}$$

Lemma 1.2. *With notation above, let a be a non-zero element of $MB[w]_M$ and $(B[w]_M)^{**}$ the $aB[w]_M$-adic completion of $B[w]_M$. Then $(B[w]_M)^{**}$ is not reduced.*

Proof. Suppose that $(B[w]_M)^{**}$ is reduced. Then, as $B[w]_M/aB[w]_M$ is a Nagata ring by our first remarks, Marot's Theorem [2] implies that $(B[w]_M)^{**}$ is a reduced Nagata ring. Hence, $B[w]_M$ is analytically unramified. Contradiction (cf. (1.9)). □

Let P be a height-one prime ideal of $B[w]_M$. Since the regular local ring C_n is integral over $B[w]_M$, there exists a prime element π of C_n such that $\pi C_n \cap B[w]_M = P$. Let $a = \pi^2 \in B[w]_M$ and $(B[w]_M)^{**}$ the $aB[w]_M$-adic completion of $B[w]_M$. Then, as $aB[w]_M$ is P-primary, we have a canonical injection: $(B[w]_M)^{**} \hookrightarrow (B[w]_P)^{\wedge}$. Hence

$$(B[w]_P)^{\wedge} \text{ is not reduced (cf. Lemma 1.2).} \tag{1.10}$$

Lemma 1.3. *With notation above, let $\overline{B[w]_M}$ be the derived normal ring of $B[w]_M$ and $(\overline{B[w]_M})^*$ the $T\overline{B[w]_M}$-adic completion of $\overline{B[w]_M}$. Then $(\overline{B[w]_M})^* = C_n^*$. Hence $(\overline{B[w]_M})^*$ is a regular local ring.*

Proof. First we note that, because $B[w]_M$ is two-dimensional, $\overline{B[w]_M}$ is Noetherian [3, (33.12)]. Next, as C_n is integral over the normal domain $\overline{B[w]_M}$, we see that $T^\nu C_n \cap \overline{B[w]_M} = T^\nu \overline{B[w]_M}$ for any $\nu > 0$.

Moreover, there exist canonical injections

$$\overline{R_q} \hookrightarrow \overline{B[w]_M}/T\overline{B[w]_M} \hookrightarrow C_n/TC_n = \overline{R_q}. \tag{1.11}$$

Thus we have an isomorphism: $\overline{B[w]_M}/T^\nu\overline{B[w]_M} \xrightarrow{\sim} C_n/T^\nu C_n$ for any $\nu > 0$. □

Example 1.4. With notation above, take a non-zero element a of $MB[w]_M$ and let $D = \overline{B[w]_M} \cap (B[w]_M)[1/a]$ the integral closure of $B[w]_M$ in $(B[w]_M)[1/a]$. Then D is not Noetherian.

This gives an example of a two-dimensional local domain which has an infinite number of non Noetherian quasi-local over-rings between the domain and its derived normal ring.

Proof. Suppose that D is Noetherian. Because $a^\nu\overline{B[w]_M} \cap D = a^\nu D$ for any $\nu > 0$, we have a canonical injection

$$D/a^\nu D \hookrightarrow \overline{B[w]_M}/a^\nu\overline{B[w]_M} \text{ for any } \nu > 0. \tag{1.12}$$

Then the aD-adic completion D^{**} of D is reduced (cf. Lemma 1.3).

Now take any prime ideal Q' of D which contains aD. Let $\bar{Q}$ be the prime ideal of $\overline{B[w]_M}$ such that $\bar{Q} \cap D = Q'$ and let $Q = Q' \cap B[w]_M$. Note that the residue field $\kappa(\bar{Q})$ is a finite algebraic extension of $\kappa(Q)$ (cf. [3, (33.10)]). Then, because $B[w]_M/Q$ is a Nagata ring, D/Q' is a finite $(B[w]_M/Q)$-module, and D/Q' also is a Nagata ring. Consequently, D/aD is a Nagata ring. Therefore, D^{**} is a reduced Nagata ring by Marot's Theorem [2]. Hence D is analytically unramified. Thus, for any prime ideal P of $D[1/a] = (B[w]_M)[1/a]$, $D_P(= B[w]_P$ is also analytically unramified. This contradicts (1.10). □

2. Two-dimensional local domain which is analytically ramified and Three-dimensional local domain whose derived normal ring is not Noetherian.

Let k be a field of characteristic 2 as above. Taking indeterminates Y_i, Z_j, we set $R_{1i} = k[Y_i^2, Y_i^3]$ with (fixed) maximal ideal $\mathfrak{p}_{1i} = (Y_i^2, Y_i^3)$ and $R_{2j} = k[Z_j^2, Z_j^3]$ with (fixed) maximal ideal $\mathfrak{p}_{2j} = (Z_j^2, Z_j^3)$ for $i, j = 1, 2, \ldots$. Put $R' = \otimes_k R_{1i} \otimes_k \otimes_k R_{2j}$ and $S = R' \setminus \bigcup_{\varepsilon,\ell} \mathfrak{p}_{\varepsilon\ell} R'$ with $\varepsilon = 1, 2$ and $\ell = 1, 2, \ldots$. Let $R = R'_S$. Then

- Maximal ideals $\mathfrak{q}_{1i}$ and $\mathfrak{q}_{2j}$ of R are in one-to-one correspondence with $\{1, 2\} \times \mathbf{N}$ via $(\varepsilon, \ell) \mapsto \mathfrak{q}_{\varepsilon\ell} = \mathfrak{p}_{\varepsilon\ell} R$.
- $R_{\mathfrak{q}_{1i}} = (K_{1i} \otimes_k R_{1i})_{\tilde{\mathfrak{q}}_{1i}}$ with $\tilde{\mathfrak{q}}_{1i} = \mathfrak{q}_{1i}(K_{1i} \otimes_k R_{1i})$, where $K_{1i} = k(\ldots, Y_{i-1}, Z_{i-1}, Z_i, Y_{i+1}, Z_{i+1}, \ldots)$ is an extension field of k.
- $R_{\mathfrak{q}_{2j}} = (K_{2j} \otimes_k R_{2j})_{\tilde{\mathfrak{q}}_{2j}}$ with $\tilde{\mathfrak{q}}_{2j} = \mathfrak{q}_{2j}(K_{2j} \otimes_k R_{2j})$, where $K_{2j} = k(\ldots, Y_{j-1}, Z_{j-1}, Y_j, Y_{j+1}, Z_{j+1}, \ldots)$ is an extension field of k.
- Any non-zero element of R is contained in only a finite number of maximal ideals.

Hence R is a one-dimensional Noetherian domain with field of quotients $K = k(Y_1, Z_1, \ldots, Y_i, Z_i, \ldots)$. Let $\bar{R}$ be the derived normal ring of R. Then $\bar{R} = R[Y_1, Z_1, \ldots, Y_i, Z_i, \ldots]$ and the set $\bar{R}^2 = \{r^2 \mid r \in \bar{R}\}$ is contained in R. Moreover, for any non-zero prime $\mathfrak{q}$ of R, $R_{\mathfrak{q}}$ is not normal, but its derived normal ring $\overline{R_{\mathfrak{q}}}$ is a finite $R_{\mathfrak{q}}$-module.

Now taking three more indeterminates T, U and W, we set a three-dimensional Noetherian domain $B = R[[T]]$, a three-dimensional regular ring $C = \bar{R}[[T]]$ and let

$$(2.1) \qquad \omega_1 = \sum_{i=1}^{\infty} Y_i T^i, \ \omega_2 = \sum_{j=1}^{\infty} Z_j U^j \in C \text{ and } \omega = \omega_1 + \omega_2,$$

$$(2.2) \qquad f(W) = W^2 - \omega^2 \in B[W] \text{ and } g(W) = W - \omega \in C[W].$$

Then $f(W) = g(W)^2$ in $C[W]$ and the set $C[W]^2 = \{h^2 \mid h \in C[W]\}$ is contained in $B[W]$.

Let L be the field of quotients of B. Then a similar argument as in the proof of Lemma 1.1 shows that

$$(2.3) \qquad \omega \notin L, \text{ and } f(W) \text{ is a prime element of } B[W].$$

Take a prime ideal $P = (T, U)B$ of B, then B_P is a two-dimensional regular local ring with its completion $B_P{}^{\wedge} = K[[T, U]]$. We get

Example 2.1. With notation above, let $B[w] = B[W]/fB[W]$ and let Q be the prime ideal of $B[w]$ such that $Q \cap B = P$. Then

$$(2.4) \qquad B[w]_Q = B_P[w] \text{ is normal.}$$

This gives an example of a two-dimensional normal local ring which is analytically ramified.

Proof of (2.4). Let α be an element of $L(w)$. Then α can be expressed as $\beta + \gamma w$, where β, $\gamma \in L$. So the element α is integral over $B_P[w]$ if and only if

$$(2.5) \qquad \beta^2 + \gamma^2 w^2 = \beta^2 + \gamma^2 \omega^2 \in B_P.$$

Let $\beta = b/a$ and $\gamma = c/a$ with a, b, $c \in B_P$. Then (2.5) is expressed as

$$(2.6) \qquad \frac{b^2 + c^2\omega^2}{a^2} = d \in B_P, \text{ that is, } b^2 + c^2\omega^2 = da^2.$$

We claim

$$(2.7) \qquad c \in aB_P.$$

Before we show (2.7), we make some more remarks in fixing notation. First, we note that B_P is the ring of quotients of B with respect to the muultiplicatively closed set

$$S = \{\sum_{i,j}^{\infty} r_{ij} T^i U^j \in R[[T, U]] \text{ with } r_{00} \neq 0\}.$$

Then, for any element x of B_P, we can find a non-zero element s of R such that $x \in R_s[[T, U]]$. Hence we may assume that there exists a non-zero elements s of R such that

$$(2.8) \qquad \begin{array}{l} a, b, c, d \in R_s[[T, U]] \text{ and that} \\ a, b, c \text{ have no common divisor in } B_P. \end{array}$$

Thus, we can express a, b, c, d as formal power series in T and U with coefficients in R_s:

$$
\begin{aligned}
&a=\sum_{i,j}^{\infty} a_{ij}T^iU^j, \quad b=\sum_{i,j}^{\infty} b_{ij}T^iU^j, \quad c=\sum_{i,j}^{\infty} c_{ij}T^iU^j, \\
&\text{and } d=\sum_{i,j}^{\infty} d_{ij}T^{2i}U^{2j}
\end{aligned} \tag{2.9}
$$

where a_{ij}, b_{ij}, c_{ij}, $d_{ij} \in R_s$ $(i,\, j=0,1,2,\dots)$.

Next, comparing the coefficients of $T^{2i}U^{2j}$ in (2.6), we have

$$
b_{ij}^2+\sum_{m_1+m_2=i} c_{m_1 j}^2 Y_{m_2}^2+\sum_{n_1+n_2=i} c_{in_1}^2 Z_{n_2}^2=\sum_{\substack{k_1+k_2=i\\ \ell_1+\ell_2=j}} d_{k_1\ell_1}a_{k_2\ell_2}^2 \tag{2.10}
$$

for any non-negative integers i, j.

If $a \notin PB_P$, our claim (2.7) is clear. Hence we may assume that $a \in PB_P$. This means that $a_{00}=0$ in (2.9). Moreover, we remark

(2.11) There exists an $i>0$ such that $a_{i0}\neq 0$.

Proof of (2.11). Suppose that $a_{i0}=0$ for any $i \geq 0$. Consider the relation (2.6) modulo $UR_s[[T,U]]$. Then, denoting by $\bar{x}$ the class of an element x of $R_s[[T,U]]$ modulo $UR_s[[T,U]]$, we have $\bar{b}^2+\bar{c}^2\omega^2\equiv 0$.

Consequently, as $\omega^2\equiv\omega_1^2$ and $\bar{b}\bar{c}\neq 0$ (cf. (2)), ω_1 is contained in the field of quotients of $R_s[[T]]$. The proof of (1.4) shows that this is impossible. □

Finally, because $B_P{}^\wedge=K[[T,U]]$ is faithfully flat over B_P, to get our assertion, it is sufficient to show

(2.12) $c\in aR_s[[T,U]]$ for some non-zero element s of R.

Proof of (2.7). With assumptions and notation above, suppose that $c\notin aB_P$. Let $i_0=\min\{i \mid a_{i0}\neq 0\}$ (cf. (2.11)). By (2.12) above, we may also assume

$$
a_{i_0 0}=1. \tag{2.13}
$$

Now let $j_0=\min\{j \mid c_{ij}\neq 0 \text{ for some } i\geq 0\}$ and $i_1=\min\{i \mid c_{ij_0}\neq 0\}$. Then, by adding a suitable multiple of a to c if necessary, we may further assume

(2.14) $i_1<i_0$ and $c_{i_1j_0}=1$ (cf. (2.12)).

From now on, we fix a non-zero element s of R which ensures the assumptions (2), (2.13) and (2.14). Under these assumptions, we shall show

$$Z_j \in R_s[Y_1, Y_2, \dots] \text{ for any } j > 0. \tag{2.15}$$

Proof of (2.15). First we check two preliminary steps.
Step 1. Let $n \geq -1$. Putting $R_s[Y_1, Y_2, \dots]^2 = \{f^2 \mid f \in R_s[Y_1, Y_2, \dots]\}$, we suppose that

$$d_{ij} \in R_s[Y_1, Y_2, \dots]^2 \text{ for any } j \leq n \text{ and } i \geq 0 \text{ and that} \tag{2.16}$$

$$Z_k \in R_s[Y_1, Y_2, \dots] \text{ for any } k \leq n - j_0 + 1. \tag{2.17}$$

Under these assumptions, we compare the coefficients of $T^{2(i_0+i)}U^{2(n+1)}$ in (2.6). Then

$$\begin{aligned} b^2_{(i_0+i)(n+1)} &+ \sum_{m_1+m_2=i_0+i} c^2_{m_1(n+1)} Y^2_{m_2} + \sum_{\substack{n_1+n_2=n+1 \\ (n_2 \leq n-j_0+1)}} c^2_{(i_0+i)n_1} Z^2_{n_2} \\ &= d_{i(n+1)} + \sum_{\substack{k_1+k_2=i_0+i \\ \ell_1+\ell_2=n+1 \\ (k_1<i \text{ or } \ell_1<n+1)}} d_{k_1\ell_1} a^2_{k_2\ell_2} \text{ (cf. (2.10), (2.13), (2.14)).} \end{aligned}$$

By induction on $i \geq 0$, we get $d_{i(n+1)} \in R_s[Y_1, Y_2, \dots]^2$ for any $i \geq 0$.
Step 2. Let n be a non negative integer. Suppose that

$$d_{ij} \in R_s[Y_1, Y_2, \dots]^2 \text{ for any } j \leq n \text{ and } i \geq 0 \text{ and that} \tag{2.18}$$

$$Z_k \in R_s[Y_1, Y_2, \dots] \text{ for any } k \leq n - j_0. \tag{2.19}$$

Under these assumptions, we compare the coefficients of $T^{2i_1}U^{2(n+1)}$ in (2.6). Then

$$\begin{aligned} b^2_{i_1(n+1)} &+ \sum_{m_1+m_2=i_1} c^2_{m_1(n+1)} Y^2_{m_2} + \sum_{\substack{n_1+n_2=n+1 \\ (n_2 \leq n-j_0)}} c^2_{i_1 n_1} Z^2_{n_2} \\ &= \sum_{\substack{k_1+k_2=i_1 \\ \ell_1+\ell_2=n+1 \\ (\ell_2>0)}} d_{k_1\ell_1} a^2_{k_2\ell_2} \quad \text{(cf. (2.10), (2.14)).} \end{aligned}$$

Hence $Z_{(n-j_0+1)} \in R_s[Y_1, Y_2, \dots]$.

Now we prove (2.15) by induction. Let m be a non negative integer and suppose that

$$Z_k \in R_s[Y_1, Y_2, \dots] \text{ for any } k \leq m \text{ with } Z_0 = 0. \tag{2.20}$$

By double induction on i and j, we see that $d_{ij} \in R_s[Y_1, Y_2, \dots]^2$ for any $j \leq m + j_0$ and for any $i \geq 0$ (cf. Step 1).

Hence, the assumptions (2.18) and (2.19) in Step 2 for $n = m + j_0$ are fulfilled. Therefore $Z_{m+1} \in R_s[Y_1, Y_2, \dots]$. This completes the proof of (2.15). □

Final step of the proof of (2.7). By (2.15), we have

$$\bar{R} = R[Y_1, Z_1, \dots, Y_i, Z_i, \dots] \subset R_s[Y_1, Y_2, \dots]. \tag{2.21}$$

This contradicts our first remarks of this section. □

Final step of the proof of (2.4). We have already shown that $\gamma \in B_P$ (cf. (2.7)). Then β becomes also integral over B_P. Hence $\beta \in B_P$, because B_P is regular. Thus $\alpha = \beta + \gamma w \in B_P[w]$. □

Example 2.2. With notation above, take a maximal ideal M of $B[w]$. Then $B[w]_M$ is a three-dimensional Noetherian local domain. Let $\overline{B[w]_M}$ be the derived normal ring of $B[w]_M$. Then

$$\overline{B[w]_M} \text{ is not Noetherian.} \tag{2.22}$$

This gives an example of a three-dimensional local domain whose derived normal ring is not Noetherian.

Proof of (2.22). Suppose that $\overline{B[w]_M}$ is Noetherian. Let $\mathfrak{n}$ be the maximal ideal of C such that $\mathfrak{n} \cap B[w] = M$ and let $M \cap R = \mathfrak{q}$, say $\mathfrak{q}_{2j}$. Then, because $C_\mathfrak{n}$ is integral over the normal domain $\overline{B[w]_M}$, we have

$$Z_j^\nu C_\mathfrak{n} \cap \overline{B[w]_M} = Z_j^\nu \overline{B[w]_M} \text{ for any } \nu > 0. \tag{2.23}$$

Further, there exist canonical injections:

$$K_{2j}[[T, U]] = B/\mathfrak{q}_{2j}B \hookrightarrow \overline{B[w]_M}/Z_j\overline{B[w]_M} \hookrightarrow C_\mathfrak{n}/Z_jC_\mathfrak{n} = K_{2j}[[T, U]].$$

Hence, we have an isomorphism

$$\overline{B[w]_M}/Z_j^\nu\overline{B[w]_M} \xrightarrow{\sim} C_\mathfrak{n}/Z_j^\nu C_\mathfrak{n} \text{ for any } \nu > 0. \tag{2.24}$$

Now let $(\overline{B[w]_M})^{**}$ be the $Z_j\overline{B[w]_M}$-adic completion of $\overline{B[w]_M}$ and $C_\mathfrak{n}^{**}$ the $Z_jC_\mathfrak{n}$-adic completion of $C_\mathfrak{n}$. Then

$$(\overline{B[w]_M})^{**} = C_\mathfrak{n}^{**} \quad \text{(cf. (2.24))}. \tag{2.25}$$

Then, $\overline{B[w]_M}$ is regular, because $C = \bar{R}[[T]]$ is a three-dimensional regular ring. Thus, because $\overline{B[w]_M}_Q = B[w]_Q$, $B[w]_Q$ is analytically unramified, and this contradicts Example 2.1. □

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Étale endomorphisms of smooth affine surfaces

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1 Introduction

Let k be an algebraically closed field of characteristic zero. Let $k[x,y]$ be a polynomial ring in two variables x, y over k, let $A = k[x,y,f^{-1}]$ for an irreducible element f of $k[x,y]$ and let $X = \operatorname{Spec} A$. Let $\alpha : X \longrightarrow X$ be an étale endomorphism of X as an algebraic variety. For $\lambda \in k$, F_λ denotes an affine curve defined by $f = \lambda$. Then a family of curves $\Lambda = \{F_\lambda; f = \lambda\ (\lambda \in k)\}$ is called a linear pencil on $\mathbf{A}^2_k$ defined by f. Let $\phi_\Lambda : \mathbf{A}^2 \longrightarrow \mathbf{A}^1$ be the associated morphism. Then ϕ_Λ induces a morphism $X \longrightarrow \mathbf{A}^1_*$ which we denote by $\rho : X \longrightarrow \mathbf{A}^1_*$.

We recall the following well-known result due to [7, 6].

Lemma 1.1 *Let $\varphi : X \longrightarrow Y$ be a dominant morphism of algebraic varieties X, Y. Assume that a general fiber $X_y = \varphi^{-1}(y)$ of φ over a general point $y \in Y$ is an irreducible curve. Then we have the following inequalities for the logarithmic Kodaira dimensions:*

$$\overline{\kappa}(X) \geq \overline{\kappa}(Y) + \overline{\kappa}(X_y) \tag{1}$$
$$\dim Y + \overline{\kappa}(X_y) \geq \overline{\kappa}(Y) \tag{2}$$

Applying lemma 1.1 to the morphism $\rho : X \longrightarrow \mathbf{A}^1_*$, we obtain the inequalities

$$1 + \overline{\kappa}(F) \geq \overline{\kappa}(X) \geq \overline{\kappa}(\mathbf{A}^1_*) + \overline{\kappa}(F)$$

The purpose of the present article is to determine a pair (X, f) with $X = \operatorname{Spec} k[x,y,f^{-1}]$ and an étale endomorphism $\alpha : X \to X$ which is not an automorphism if it exists. Since α is an étale endomorphism and since α is not an automorphism, $\overline{\kappa} < 2$ by the following result of Iitaka [6]:

Lemma 1.2 *Let X be a nonsingular algebraic variety with the logarithmic Kodaira dimension equal to* $\dim X$*. Let $f : X \to X$ be a quasi-finite endomorphism. Then f is an automorphism.*

Main results of the present article are the following results:

Theorem 1.3 *Assume that $\overline{\kappa}(X) = -\infty$. After a suitable change of coordinates of $k[x,y]$, we have $f = x$. Then an étale endomorphism α of X has the following associated algebra homomorphism α^* : $k[x,y,1/x] \to k[x,y,1/x]$:*

$$\alpha^*(x) = cx^n, \quad \alpha^*(y) = x^r y + dg(x)$$

where $c, d \in k, c \neq 0, r \in \mathbf{Z}, r \geq 0$ and $g(x) \in k[x]$. The endomorphism α is then a finite étale endomorphism.

Theorem 1.4 *Suppose $\overline{\kappa}(X) \geq 0$. Then X has no étale finite endomorphisms which are not automorphisms, unless X is isomorphic to one of the folowing surfaces:*

(1) $\mathbf{A}^1_* \times \mathbf{A}^1_*$,

(2) $\mathbf{A}^2 - F_0$, *where F_0 is a curve on $\mathbf{A}^2$ defined by $y^m - x^n = 0$ for positive integers m, n with* $\gcd(m,n) = 1$.

For any of these two surfaces, there exist étale finite endomorphisms of X whose degree exceeds an arbitrarily given integer.

2 Case $\overline{\kappa}(X) = -\infty$

First of all, we remark that the assumption that f be irreducible is not necessary in the present case $\overline{\kappa}(X) = -\infty$. In fact, we have the following result due to Miyanishi [10].

Lemma 2.1 *Suppose $\overline{\kappa}(X) = -\infty$ and that one of the following conditions is satisfied:*

(1) *X is irrational but not elleptic ruled,*

(2) *$\Gamma(X, \vartheta_X)^* \neq k^*$ and* rank $(\Gamma(X, \vartheta_X)^*/k^*) \geq 2$ *if X is rational.*

Then an étale endomorphism $f : X \to X$ is an automorphism.

Since $X = \mathbf{A}^2_k - F_0$, X is a rational surface and $\Gamma(X, \mathcal{O}_X)^*/k^*$ is generated by prime factors of f. By the above lemma, rank $(\Gamma(X, \vartheta_X)^*/k^*) = 1$ provided α is not an automorphism. So, f is irreducible. Furthermore, since $\mathbf{P}^1 - \{0, \infty\} = \mathbf{A}^1_*$ and $\overline{\kappa}(\mathbf{A}^1_*) = 0$, we have $\overline{\kappa}(C) = -\infty$ by the formula (1), where C is a general fiber of $\rho : X \to \mathbf{A}^1_*$. Furthermore, the second theorem of Bertini implies that C is a smooth curve. We shall then show the following:

Claim. $C \cong \mathbf{A}^1$

Proof. Let $\overline{C}$ be a smooth compactification of C and let D be the reduced boundary divisor. Then $\overline{C} - D = C$ and $\kappa(C) = \kappa(\overline{C}, K_{\overline{C}} + D)$. If $g = g(\overline{C}) \geq 1$, there exists a section which belongs to $\Gamma(\overline{C}, K_{\overline{C}})$. So, $K_{\overline{C}} + D$ is linealy equivalent to some effective divisor. Therefore, $h^0(\overline{C}, n(K_{\overline{C}} + D)) \geq 1$ for $n \geq 1$. Hence $\overline{\kappa}(C) \geq 0$. Since $\overline{\kappa}(C) = -\infty$, this is a contradiction. So, $g = 0$ and $\overline{C} \cong \mathbf{P}^1$. Then, D consists of one point. Since $K_{\overline{C}} \sim -2P$, if D consists of two or more points, $h^0(\overline{C}, \mathcal{O}_{\overline{C}}(K_{\overline{C}} + D)) \geq h^0(\overline{C}, \mathcal{O}_{\overline{C}}) = 1$. we have $\overline{\kappa}(C) \geq 0$. Thus, D must consists of one point. So, $C \cong \mathbf{P}^1 -$ one point $= \mathbf{A}^1$.

Here, we shall use the embedded line theorem of Abhyanker-Moh [1].

Lemma 2.2 *Let $C = \mathbf{A}^1 \subset X \subset \mathbf{A}^2$. Then there exist coordinates u, v on $\mathbf{A}^2_k$ such that $k[x, y] = k[u, v]$ and C is defined by $u = 0$.*

By the above lemma, we may assume that a general fiber C of ρ, that is to say F_λ for $\lambda \in k$, is definded by $u = 0$. Then, the curve F_0 defined by $f = 0$ is defined by $u + \lambda = 0$. So, $F_0 \cong \mathbf{A}^1$. We may assume that $f = x$. Hence, after a change of variables, the coordinate ring A of X is isomorphic to $k[x, y, 1/x]$. Then, $X \cong \mathbf{A}^1_* \times \mathbf{A}^1$.

We shall show that α^* is written as stated in Theorem 1.3, where α^* is the algebra homomorphism of the coordinate ring corresponding to a given étale endomorphism α of X. Note that the multiplicative group A^* of invertible elements of A is generated by k^* and the element x. Since α^* induces a multiplicative group homomorphism from A^* to A^*, we may write

$$\alpha^*(x) = cx^n, \qquad \alpha^*(y) = \frac{\varphi(x, y)}{x^m},$$

where $\varphi(x, y) \in k[x, y], c \in k^*, n, m \in \mathbf{Z}, m \geq 0$. Then we obtain the following matrix relation for the differentials

$$\begin{pmatrix} d\alpha^*(x) \\ d\alpha^*(y) \end{pmatrix} = \begin{pmatrix} ncx^{n-1} & 0 \\ \dfrac{\varphi_x x^m - m\varphi x^{m-1}}{x^2 m} & \dfrac{\varphi_y}{x^m} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Since α^* is unramified, the determinant of this matrix must be an invertible element of A. Hence we have

$$ncx^{n-1}\frac{\varphi_y}{x^m} = c' x^\ell \quad \text{for some } c' \in k^* \text{ and } \ell \in \mathbf{Z}.$$

Namely, we have

$$\varphi_y = \frac{c'}{nc} x^{l+m-n+1} \quad \text{with} \quad l + m - n + 1 \geq 0$$

This implies that

$$\varphi \sim x^r y + dg(x) \quad \text{for some } d \in k, r = l + m - n - 1 \geq 0 \text{ and } g(x) \in k[x],$$

where $\sim$ means that $\varphi(x, y)$ is determined by the right hand side up to an element of k^*. Furthermore, we shall show that $k[x, y, 1/x]$ is integral over $k[\alpha^*(x), \alpha^*(y), 1/\alpha^*(x)]$. From the above arguments, we have

$$k[\alpha^*(x), \alpha^*(y), \frac{1}{\alpha^*(x)}] = k[x^n, \frac{x^r y + dg(x)}{x^m}, \frac{1}{x^n}] .$$

Since $y = (x^r y + dg(x))/x^r - dg(x)/x^r$, we have $k[x, y, 1/x] = k[x, x^r y + dg(x), 1/x]$. Since $k[x, x^r y + dg(x), 1/x]$ is integral over $k[x^n, x^r y + dg(x), 1/x^n]$ and since

$$k[x^n, x^r y + dg(x), \frac{1}{x^n}] \subset k[x^n, \frac{x^r y + dg(x)}{x^m}, \frac{1}{x^n}] \subset k[x, x^r y + dg(x), \frac{1}{x}]$$

we know that $k[x, y, 1/x]$ is integral over $k[\alpha^*(x), \alpha^*(y), 1/\alpha^*(x)]$. So, α^* is a finite étale endomorphism. This completes a proof of Theorem 1.3.

3 Case $\overline{\kappa}(X) = 0$

Since $X = \mathbf{A}^2 - F_0$, X is a smooth affine rational surface. Furthermore, since $A = k[x, y, 1/f]$ is a UFD, it follows that $\mathrm{Pic}\,(X) = 0$. We recall first the following crucial result of Fujita [3, (8.64)]:

Lemma 3.1 *Let X be an smooth affine rational surface with $\overline{\kappa}(X) = 0$ and $\mathrm{Pic}\,(X) = 0$. Then X is isomorphic to either one of the following two surfaces:*

(1) *$X = \mathbf{P}^2 - D$, where D is a sum of three non confluent lines.*

(2) *$X = \mathbf{P}^2 - D$, where D is a sum of a line and a smooth conic meeting each other transversally.*

In the case (1), $X = \mathbf{P}^2 - D$ with $D = \ell_0 + \ell_1 + \ell_2$, where ℓ_i $(i = 1, 2, 3)$ are lines. Then $X \cong \mathbf{A}^1_* \times \mathbf{A}^1_*$ and the coordinate ring of X is written as $A = k[x, y, 1/xy]$.

In the case (2) $X = \mathbf{P}^2 - D$ with $D = \ell_\infty + C$, where ℓ is a line and C is a smooth conic. Then X is isomorphic to $\mathbf{A}^2 - C$, where the present $\mathbf{A}^2$ might differ from the $\mathbf{A}^2$ we started with. Hence we may assume that f is an irreducible polynomial of degree 2. By an affine change of coordinates $\{x, y\}$ of $\mathbf{A}^2$, we obtain

$$f(x, y) \sim x^2 + by + c \quad \text{or} \quad f(x, y) \sim xy + c.$$

The first case is excluded because the conic on $\mathbf{P}^2$ defined by a homogeneous equation $x_0^2 + bx_1x_2 + cx_2^2 = 0$ is tangent to the line at infinity ℓ_∞, which is defined by $x_2 = 0$, at $(0, 1, 0)$. So, the second case remains. we may clearly assume $c = 1$.

Summarizing the above arguments, we know that, after a suitable change of coordinates x, y on $\mathbf{A}^2$, the surface $X = \mathbf{A}^2 - V(f)$ is isomorphic to one of the following surfaces:

(1) $f = xy$,

(2) $f = xy + 1$.

We shall start with the case (1). Let $\alpha^* : A \to A$ be the endomorphism of A attached to an étale endomorphism α of X. Since α^* sends the invertible elements of A to the invertible elements of A, we may write

$$\alpha^*(x) = cx^s y^t, \qquad \alpha^*(y) = dx^u y^v \tag{1}$$

with $s, t, u, v \in \mathbf{Z}$ and $c, d \in k^*$. Then, taking the differentials, we have the following matrix relation:

$$\begin{pmatrix} d\alpha^*(x) \\ d\alpha^*(y) \end{pmatrix} = \begin{pmatrix} csx^{s-1}y^t & ctx^s y^{t-1} \\ dux^{u-1}y^v & dvx^u y^{v-1} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

The determinant of this matrix, which is equal to

$$cd(sv - tu)x^{s+u-1}y^{v+t-1},$$

must be an invertible element of A because α is étale. In particular, $sv \neq tu$. Conversely, if $sv \neq tu$, the endomorphism α^* defined by (1) is an étale endomorphism. Then, $k[x, y, 1/x, 1/y]$ is integral over $k[\alpha^*(x), \alpha^*(y), 1/\alpha^*(x),$

$1/\alpha^*(y)]$. In fact, since the lattice subgroup $G := \langle(s,t),(u,v)\rangle$ of $\mathbf{Z}\oplus\mathbf{Z} = \langle(1,0),(0,1)\rangle$ is a free abelian group of rank 2, $G\otimes_{\mathbf{Z}}\mathbf{Q}\cong\mathbf{Q}\oplus\mathbf{Q}$. So, there exists an integer $m\in\mathbf{Z}$ such that $(mp,mq)\in G$ for every $(p,q)\in\mathbf{Z}\oplus\mathbf{Z}$. Then, for any $(p,q)\in\mathbf{Z}\oplus\mathbf{Z}$, we have

$$(x^py^q)^m = \frac{(cx^sy^t)^a(dx^uy^v)^b}{c^ad^b}$$

for some $a,b\in\mathbf{Z}$. Hence α is a finite morphism.

4 Case $\overline{\kappa}(X) = 1$

We need the following lemma of Kawamata [8] ; see also Gurjar-Miyanishi [4, Lemma 10] :

Lemma 4.1 *Let X be an affine smooth rational surface with $\overline{\kappa}(X) = 1$. Then there exists a morphism $\rho : X\to B$ onto a nonsingular rational curve B which defines a twisted or untwisted $\mathbf{A}^1_*$-fibration.*

In the case where ρ is a twisted $\mathbf{A}^1_*$-fibration, Pic (X) contains a nonzero 2-torsion element. Since Pic $(X) = (0)$ in our case, the $\mathbf{A}^1_*$-fibration ρ must be untwisted.

We shall show that

$$B\cong\mathbf{P}^1,\mathbf{A}^1,\ \text{or}\ \mathbf{A}^1_*.$$

Since B is rational, B is isomorphic to $\mathbf{P}^1$ minus n points. Since f is irreducible, $\operatorname{rank}\Gamma(X,\mathcal{O}_X)^*/k^* = 1$. If $n\geq 3$, then $\operatorname{rank}\Gamma(B,\mathcal{O}_B)^*/k^*\geq 2$. Since $\Gamma(B,\mathcal{O}_B)^*/k^*$ is a subgroup of $\Gamma(X,\mathcal{O}_X)^*/k^*$, this is a contradiction. So, $n\leq 2$. This implies the above assertion.

Here we have two cases which we consider below separately.

(I) Case ρ extends to an $\mathbf{A}^1_*$-fibration,

(II) Case ρ does not extend to an $\mathbf{A}^1_*$-fibration.

In the case (I), let $\widetilde{\rho} : \mathbf{A}^2\to\widetilde{B}$ be the extension of ρ. Then F_0 can not meet the general fibers of ρ. Indeed, if it meets a general fiber, some more points are deleted from $\mathbf{A}^1_*$), which is not the case. So, F_0 is isomorphic to either $\mathbf{A}^1_*$ or an irreducible component of a singular fiber of $\widetilde{\rho}$. Assume that $F_0\cong\mathbf{A}_*$. We recall the following lemma of Saito [9, Theorem 2.3] [11]:

Lemma 4.2 *Let f be a generically rational polynomial in $k[x,y]$ with two places at infinity. Then, after a suitable change of coordinates, f is reduced to either one of the following two forms:*

(1) *$f\sim x^\alpha y^\beta + 1$, where $\alpha,\beta > 0$ and $\gcd(\alpha,\beta) = 1$.*

(2) *$f\sim x^\alpha(x^ly + p(x))^\beta + 1$, where $\alpha,\beta,l > 0, \gcd(\alpha,\beta) = 1$ and $p(x)\in k[x]$ with $\deg p(x) < l$ and $p(0)\neq 0$.*

These two cases in the above lemma give two cases to consider in the case (I).

Next suppose that F_0 is isomorphic to an irreducible component of a singular fiber of $\widetilde{\rho}$. Given an $\mathbf{A}^1_*$-fibration, we can classify all possible types of singular fibers by the following of Miyanishi [10]

Lemma 4.3 *Let $\rho : X\to B$ be an $\mathbf{A}^1_*$-fibration on an affine nonsingular surface X over a nonsingular curve B, and let S be a singular fiber of ρ. Then S is written as a divisor in the form $S = \Gamma + \triangle$, where*

(1) *$\Gamma = 0$ or $\Gamma = \alpha\Gamma_1$ with $\alpha\geq 1$ and $\Gamma_1\cong\mathbf{A}^1_*$ or $\Gamma = \alpha_1\Gamma_1+\alpha_2\Gamma_2$, where $\alpha_1\geq 1,\alpha_2\geq 1,\Gamma_1\cong\Gamma_2\cong\mathbf{A}^1$ and Γ_1 and Γ_2 meet each other transversally in one point.*

(2) *$\triangle\geq 0$ and $\operatorname{Supp}\triangle$ is a disjoint union of connected components isomorphic to $\mathbf{A}^1$ provided $\triangle > 0$.*

From the above result, F_0 is isomorphic to $\mathbf{A}^1_*$ or $\mathbf{A}^1$. If $F_0\cong\mathbf{A}^1_*$, then f is a generically rational polynomial and we are done by Lemma of Saito. If $F_0\cong\mathbf{A}^1$, we have $A = k[x,y,1/x]$ by Theorem of Abhyankar-Moh and $X\cong\mathbf{A}^1\times\mathbf{A}^1_* = \mathbf{P}^1\times\mathbf{P}^1 - D$, where $D = \ell_0 + m_0 + m_\infty$ with a fiber ℓ_0 of a ruling on $\mathbf{P}^1\times\mathbf{P}^1$ and fibers m_0, m_1 of another ruling. So, we have $\overline{\kappa}(X) = -\infty$, which is a contradiction. So, the case $F_0\cong\mathbf{A}^1$ is excluded.

Next we consider the case (II) where $\widetilde{\rho}$ does not extend to an $\mathbf{A}^1_*$-fibration on $\mathbf{A}^2$. We have the following two cases to consider:

(1) Case the closures of the general fibers of ρ in $\mathbf{A}^2$ have one common point P which lies on the curve F_0. Let $\overline{S}$ be the closure of a general fiber S of ρ. Then $\overline{S} = S \cup \{P\}$, where the point P is a one-place point of $\overline{S}$. Therefore $\overline{S}$ is a topologically contractible curve. Then the following lemma of Gurjar and Miyanishi [5] is available:

Lemma 4.4 *Let $C \subset \mathbf{A}^2$ be an irreducible algebraic curve which is topologically contractible. Then there exist affine coordinates x, y on $\mathbf{A}^2$such that in terms of these coordinates C is defined by an equation $x^m = y^n$, where $\gcd(m, n) = 1$.*

So, we may assume that $\overline{S}$ is defined by $x^m - \lambda y^n = 0$, where $\lambda \in k$, and the point P is the point of origin. If λ moves in $\mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$, the curves $\{x^m - \lambda y^n = 0; \lambda \in \mathbf{P}^1\}$ is a linear pencil on $\mathbf{A}^2$ parametrized by $\mathbf{P}^1$. Since the curve F_0 is irreducible and reduced, we may assume that $f = x^m - y^n$.

(2) Case ρ can be extended to an $\mathbf{A}^1$-fibration $\widetilde{\rho} : \mathbf{A}^2 \to B$. Then F_0 is necessarily a cross-section of $\widetilde{\rho}$. Indeed, $\widetilde{\rho}|_{F_0} : F_0 \to B$ is injective and F_0 is smooth. Since the general fiber of $\widetilde{\rho}$ is isomorphic to $\mathbf{A}^1$, we may assume that it is defined by $x = \lambda$. The curve $x = \lambda$ meets the cross-section $f(x, y) = 0$ at one point. So, $f(\lambda, y) = 0$ has only one solution. Write

$$f(x, y) = a_0(x)y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$$

where $a_0(x), \ldots, a_n(x) \in k[x]$ and $a_0(x) \neq 0$. Then $a_0(\lambda) \neq 0$ for a general element $\lambda \in k$. Hence we have $n = 1$. So, $f(x, y) = a_0(x)y + a_1(x)$, where $gcd(a_0(x), a_1(x)) = 1$ because $f(x, y)$ is irreducible.

Summarizing the above arguments, we have the following:

Theorem 4.5 *Suppose that $\overline{\kappa}(X) = 1$. Then, after a suitable change of coordinates x, y, the polynomial $f(x, y)$ is reduced to one of the following forms:*

(I) *Case where the given $\mathbf{A}^1_*$-fibration $\rho : X \to B$ extends to an $\mathbf{A}^1_*$-fibration $\widetilde{\rho} : \mathbf{A}^2 \to \widetilde{B}$.*

(1) $f \sim x^\alpha y^\beta + 1$, *where $\alpha, \beta > 0$ and $\gcd(\alpha, \beta) = 1$. In this case, $\widetilde{B} \cong \mathbf{A}^1$ and $B \cong \mathbf{A}^1_*$.*

(2) $f \sim x^\alpha(x^l y + p(x))^\beta + 1$, *where $\alpha, \beta, l > 0, \gcd(\alpha, \beta) = 1$ and $p(x) \in k[x]$ with $\deg p(x) < l$ and $p(0) \neq 0$. In this case, $\widetilde{B} \cong \mathbf{A}^1$ and $B \cong \mathbf{A}^1_*$.*

(II) *Case where the given $\mathbf{A}^1_*$-fibration $\rho : X \to B$ is not extended to an $\mathbf{A}^1_*$-fibration on $\mathbf{A}^2$.*

(3) $f \sim a_0(x)y + a_1(x)$, $\gcd(a_0(x), a_1(x)) = 1$. *In this case, the $\mathbf{A}^1_*$-fibration $\rho : X \to B$ extends to an $\mathbf{A}^1$-fibration $\widetilde{\rho} : \mathbf{A}^2 \to \widetilde{B}$, where $\widetilde{B} = B \cong \mathbf{A}^1$.*

(4) $f \sim x^m - y^n$, $\gcd(m, n) = 1$. *In this case, the closures of the fibers of the $\mathbf{A}^1_*$-fibration $\rho : X \to B$ form a linear pencil $\{x^m - \lambda y^n\}$ parametrized by $\lambda \in \mathbf{P}^1 = k \cup \{\infty\}$, which has the point of origin as a base point. Furthermore, $B \cong \mathbf{A}^1$.*

5 Proof of Theorem 1.4

Let $\alpha : X \to X$ be an étale finite endomorphism. By the Lefschetz principle, we may and shall assume that the ground field k is the complex number field. First of all, we note the following easy result (cf. [10, p. 361]:

Lemma 5.1 *Let $\alpha : X \to X$ be as above and let $d = \deg \alpha$. Then the Euler number $E(X)$ must be zero provided $d > 1$.*

Proof. We have $E(X) = dE(X)$, whence $E(x) = 0$ if $d > 1$. Q.E.D.

In the case $\overline{\kappa}(X) = 0$, either $X \cong \mathbf{A}^1_* \times \mathbf{A}^1_*$ or $f \sim xy + 1$. In the latter case, $X = \mathbf{A}^2 - \{f = 0\}$ has the Euler number $E(X) = 1$. Hence we are done.

So, we consider the case $\overline{\kappa}(X) = 1$. We shall look into the four cases separately, i.e., the cases (1) and (2) of Lemma 4.2 and the cases (3) and (4) of Lemma 4.5. We shall start with the cases (1) and (2).

Then the curve $f = 0$ is isomorphic to $\mathbf{A}^1_*$ (cf. [9, Theorem 2.3]). Hence $X = \mathbf{A}^2 - \{f = 0\}$ has again the Euler number $E(X) = 1$. So, by the same reason as above, $\alpha : X \to X$ must be an automorphism.

Let us consider the case (3) in Theorem 4.5. Here we need the following lemma of Gurjar and Miyanishi [5].

Lemma 5.2 *Let X be a smooth affine surface and let $\rho : X \to B$ be a morphism onto a smooth curve B whose general fibers are irreducible and reduced. Then we have the following equality of the Euler numbers:*

$$E(X) = E(B)E(F) + \sum_i (E(F_i) - E(F))$$

where F is a general fiber of ρ and the summation is over all the singular fibers F_i of ρ. Furthermore,

$$E(F_i) \geq E(F)$$

for all i and the equality occurs if and only if either $F \cong \mathbf{A}^1$ or $F \cong \mathbf{A}^1_*$ and $(F_i)_{\mathrm{red}} \cong F$.

In the case (3), write $f \sim a_0(x)y + a_1(x)$, where x is a parameter of the base curve $B \cong \mathbf{A}^1$. Let n be the number of distinct roots of the equation $a_0(x) = 0$. If $f(c) = 0$ for $c \in k$. Then the fiber of ρ over the point $x = c$ is isomorphic to $\mathbf{A}^1$. Hence the fibration $\rho : X \to B$ has as many singular fibers as the distinct roots of $a_0(x) = 0$ which are isomorphic to $\mathbf{A}^1$, and the other fibers of ρ are isomorphic to $\mathbf{A}^1_*$. By Lemma 5.2, we then have $E(X) = n$. Hence the endomorphism $\alpha : X \to X$ is an automorphism unless $n = 0$. If $n = 0$ then $a_0(x)$ is a constant and the curve F_0 defined by $f = 0$ on $\mathbf{A}^2$ is isomorphic to $\mathbf{A}^1$. Then X is isomorphic to $\mathbf{A}^1 \times \mathbf{A}^1_*$ and $\overline{\kappa}(X) = -\infty$, which contradicts the hypothesis $\overline{\kappa}(X) = 1$.

It remains to consider the case (4). We shall show that, given an arbitrarily big integer N, X has a cyclic Galois étale covering $\alpha : X \to X$ whose order is bigger than N. For this purpose, we need some preparations.

In the case (4), we have $f \sim y^m - x^n$ with $\gcd(m,n) = 1$. Let $Y = \mathbf{A}^2 - \{0\}$. Then $X = \mathrm{Spec}\, k[x, y, f^{-1}]$ is a Zariski open set of Y. In fact, $X = Y - F_0$, where F_0 is the curve defined by $f = 0$. There is an $\mathbf{A}^1_*$-fibration $\widetilde{\rho} : Y \to \widetilde{B}$, where $\widetilde{B} \cong \mathbf{P}^1$, and the $\mathbf{A}^1_*$-fibration $\rho : X \to B$ is the restriction of $\widetilde{\rho}$ onto the open set X. The $\mathbf{A}^1_*$-fibration $\rho : X \to B$ is uniquely determined by the surface X by virtue of the following result.

Lemma 5.3 *Let X be a smooth algebraic curve with $\overline{\kappa}(X) = 1$, and let $\rho : X \to B$ be an $\mathbf{A}^1_*$-fibration. Let (V, D) be a pair of a smooth projective surface and an effective reduced divisor with simple normal crossings such that $V - D = X$ and the $\mathbf{A}^1_*$-fibration extends to a $\mathbf{P}^1$-fibration $p : V \to C$, where C is a smooth projective curve containing B as an open set. Then $\dim |n(D + K_V)| > 0$ for some $n > 0$ and the movable part of $|n(D + K_V)|$ is composed of the pencil associated with the fibration $p : V \to C$.*

Proof. By the hypothesis $\overline{\kappa}(X) = 1$ there exist a pair (V, D) as above and an integer $n > 0$ such that $\dim |n(D + K_V)| > 0$ and the movable part M of the linear system $|n(D + K_V)|$ is composed of a pencil. It suffices to show that $(\overline{F} \cdot G) = 0$ for a general fiber $\overline{F}$ of p and a general member G of M. Note that the fiber $\overline{F}$ is the closure of a general fiber F of ρ in V and that $\overline{F}$ consists of F and two smooth points on D. Hence $(\overline{F} \cdot D) = 2$. Since $(\overline{F}^2) = 0$, we have $(\overline{F} \cdot K_V) = -2$. Hence $(\overline{F} \cdot n(D + K_V)) = 0$. Write $|n(D + K_V)| = M + H$, where H is the fixed part. Then $(\overline{F} \cdot M) \geq 0$ and $(\overline{F} \cdot H) \geq 0$. Hence we have $(\overline{F} \cdot G) = 0$. Q.E.D.

As a corollary of this lemma we have:

Corollary 5.4 *Let $X = \mathrm{Spec}\, k[x, y, f^{-1}]$ and $\rho : X \to B$ be the same as above. Let $\alpha : X \to X$ be an étale finite endomorphism of degree d. Then there exists an endomorphism $\beta : B \to B$ such that $\rho \circ \alpha = \beta \circ \rho$.*

Proof. Let F be a general fiber of ρ. Since $F \cong \mathbf{A}^1_*$ and aince the pull-back of $\mathbf{A}^1_*$ by an étale finite morphism is a disjoint union of the curves isomorphic to $\mathbf{A}^1_*$, it follows from the uniqueness of the $\mathbf{A}^1*$-fibration on X that $\alpha^{-1}(F)$ consists of finitely many fibers of ρ for the upper X. [1] Hence a general fiber F_u on the upper X maps to a general fiber of ρ on the lower X. In other words, the endomorphism

[1] For an endomorphism $\alpha : X \to X$ we call the source X (resp. the target X) the upper (resp. the lower) X.

$\alpha : X \to X$ induces a rational endomorphism $\beta : B \to B$, which is a morphism because B is a smooth curve. It is clear that $\rho \circ \alpha = \beta \circ \rho$. Q.E.D.

The endomorphism $\beta : B \to B$ extends to an endomorphism $\widetilde{\beta} : \widetilde{B} \to \widetilde{B}$, where $B \cong \mathbf{A}^1$ and $\widetilde{B} \cong \mathbf{P}^1$. We denote the point $\widetilde{B} - B$ by P_∞. Then $\widetilde{\beta}^{-1}(P_\infty)$ consists of a single point P_∞. Hence $\widetilde{\beta}$ is totally ramified over P_∞. We shall look into the ramification of the morphism $\widetilde{\beta}$ on B. Note that $\widetilde{\beta}^{-1}(B) = \beta^{-1}(B) = B$.

Lemma 5.5 *Let the notation and the assumptions be the same as above. Then the following assertions hold:*

(1) *Let P be a point of B such that the fiber of ρ over P is reduced. Then β is unramified over P.*

(2) *Let P_0 and P_1 be the points of B such that the fibers of ρ over P_0 and P_1 are multiple fibers of respective multiplicities m and n. Then $\beta^{-1}(P_0)$ (resp. $\beta^{-1}(P_1)$) consists of one unramified point and points $\widetilde{P}_0^{(1)}, \ldots, \widetilde{P}_0^{(r)}$ with ramification index m (resp. one unramified point and points $\widetilde{P}_1^{(1)}, \ldots, \widetilde{P}_1^{(s)}$ with ramification index n).*

Proof. (1) Let t be a local parameter of B at the point P. Let $\widetilde{P}$ be a point of B which maps to P. Suppose $\beta^*(t) \sim \tau^a$ at the point $\widetilde{P}$ with $a > 1$. Let R be a smooth point of the fiber $\rho^{-1}(P)$ and let u be a local parameter of the curve $\rho^{-1}(P)$ at R. Then $\{t, u\}$ is a local system of parameters of X at the point R. Let $\widetilde{R}$ be a point on the fiber of ρ over the point $\tau = 0$ such that $\beta(\widetilde{R}) = R$. Since $\beta : X \to X$ is unramified at $\widetilde{R}$, the set $\{\beta^*(t), \beta^*(u)\}$ is a local system of parameters, while this is not the case because $\beta^*(t) \sim \tau^a$. This is a contradiction. Hence β is unramified over P.

(2) Let Q be a point of B which maps to P_0. Suppose that β ramifies at the point Q with ramification index e. Take a local parameter t of B at the point P_0 and a local parameter τ at the point Q. Then $\beta^*(t) \sim \tau^e$. Write $\rho^*(P_0) = m\Gamma_0$ with $\Gamma_0 \cong \mathbf{A}^1_*$. Take a smooth point R on Γ_0 and a local parameter u of Γ_0 at the point R. Then $t \sim u^m$. In the subsequent arguments, we denote $\beta^*(u)$ by the same letter u for the sake of simplifying the notation. Take a point $\widetilde{R}$ of the upper X which maps to Q by ρ and R by α. Then, near the point, we have $t^e \sim u^m$, where we note again that $t^e \sim u^m$ is synonimous to $t^e/u^m = \sigma$ with $\sigma(\widetilde{R}) \neq 0$. Let $\ell = \gcd(e, m)$, and write $e = e'\ell$ and $m = m'\ell$. Since X is normal at $\widetilde{R}$ the relation $(t^{e'}/u^{m'})^\ell = \sigma$ implies that $\sigma' = t^{e'}/u^{m'}$ is an invertible element of the local ring $\mathcal{O}_{X,\widetilde{R}}$. Then the Euclidean algorithm implies that there exists an element v of $\mathcal{O}_{X,\widetilde{R}}$ such that $t \sim v^{m'}$ and $u \sim v^{e'}$. Since u forms a local system of parameters at R with another element and since α is étale, it follows that $e' = 1$. Hence $e = \ell$ and $m = m'e$. Suppose that $m' > 1$. Then the fiber $\rho^{-1}(Q)$ is a multiple fiber. Hence its multiplicity is either m or n. Since $u \sim v$ and $t \sim v^{m'}$, it follows that m' is the multiplicity and $m = m'$. In this case $e = 1$. If $m' = 1$. Then $e = m$. This yields the assertion for the point P_0. The argument is the same for the point P_1. Q.E.D.

A consquence of the above lemma is the following:

Corollary 5.6 *Let $\beta : B \to B$ be the endomorphism defined as above. Then β is an automorphism.*

Proof. Let δ be the degree of β. Since $\widetilde{\beta}$ is totally unramified over the point P_∞. Hence, by the Riemann-Hurwitz formula, we have

$$1 + mr = \delta = 1 + ns \tag{1}$$

$$-2 = -2\delta + (\delta - 1) + (m-1)r + (n-1)s \tag{2}$$

By (1) we have $mr = ns$ and by (2) we have

$$\delta - 1 = mr + ns - (r + s)$$

Replacing δ by $\delta = 1 + ns$, we have $s = (m-1)$. Similarly, we have $r = (n-1)s$. Hence

$$s = (m-1)(n-1)s$$

Since $m > 1$ and $n > 1$, this implies $s = 0$. Similarly, $r = 0$. This implies $\delta = 1$. Hence β is an automorphism. Q.E.D.

Let K be the function field of B over k and let X_K be the generic fiber of ρ, i.e., $X_K = X \times_B \operatorname{Spec} K$. Then $X_K \cong \mathbf{A}^1_{*,K} = \operatorname{Spec} k[u, u^{-1}]$ because the $\mathbf{A}^1_*$-fibration ρ is untwisted. Set

$$t = \frac{y^m}{y^m - x^n} .$$

Then $B = \operatorname{Spec} k[t]$ and $K = k(t)$. The generic fiber X_K is the affine curve

$$y^m = \frac{t}{t-1} x^n \tag{3}$$

on $\mathbf{A}^2_K$ with the point $(x, y) = (0, 0)$ deleted off. Normalizing the curve (3) by taking the fractions of the powers of x and y, we may assume that $u = y^a / x^b$ and

$$y = \left(\frac{t}{t-1}\right)^c u^n, \quad x = \left(\frac{t}{t-1}\right)^d u^m$$

for some integers a, b, c, d. On the other hand, the étale endomorphism $\alpha : X \to X$ induces an étale finite endomorphism $\alpha_K : X_K \to X_K$, which must be a cyclic Galois covering. Indeed, $\alpha_K^*(u) = u^d$, where d is the degree of α. Let $G = \mathbf{Z}/d\mathbf{Z}$, a cyclic group of order d. Then G acts on X_K and the quotient X_K/G is the lower X_K.

Let ζ be a primitive d-th root of unity. Then the G-action is given by $u \mapsto \zeta u$, and hence we have

$$x \mapsto \zeta^m x, \quad y \mapsto \zeta^n y .$$

This implies that the G-action on X_K extends to a G-action on the upper X. Let $X' = X/G$ and let $\pi : X \to X'$ be the quotient morphism. Then $\alpha : X \to X$ factors as

$$\alpha : X \xrightarrow{\pi} X' \xrightarrow{\alpha'} X .$$

Since the G-action on X is free on the open set $\{x \neq 0, y \neq 0\}$, the degree of π is d and hence $\alpha' : X' \to X$ is a birational finite morphism. Since X is smooth and X' is normal, α' is an isomorphism by Zariski's Main Theorem. This implies that $\alpha : X \to X$ is a cyclic Galois covering with group G and the G-action on the upper X extends to the affine plane $\mathbf{A}^2_k$. Suppose that $\gcd(d, m) = \gcd(d, n) = 1$. Replacing a primitive d-th root ζ by ζ^ℓ with $m\ell \equiv 1 \pmod{d}$, we may normalize the G-action as

$$x \mapsto \zeta x, \quad y \mapsto \zeta^e y ,$$

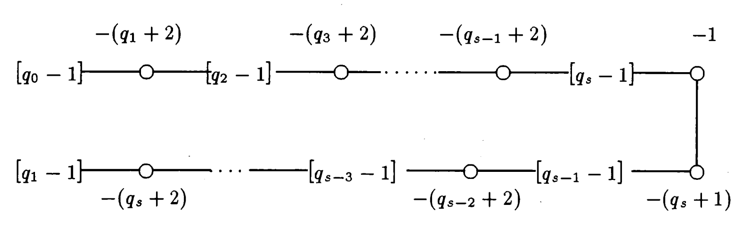
where e is the unique integer such that $0 < e < d$ and $em \equiv n \pmod{d}$.

Conversely, we shall construct an example of G-action on X which induces an étale finite endomorphism $\alpha : X \to X$. For this purpose, we define the integers $q_0, q_1, \ldots, q_2$ by the following Euclidean algorithm applied to the given positive integers m, n with $m < n$ and $\gcd(m, n) = 1$:

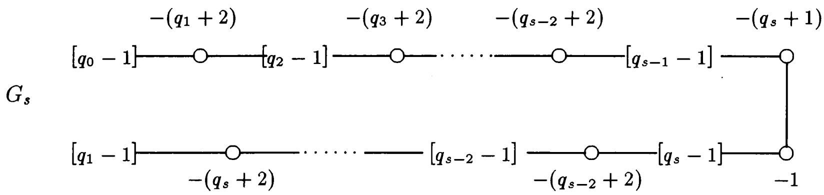
$$\begin{array}{rcl}
n & = & q_0 m_0 + m_1, \quad 0 < m_1 < m_0 \\
m_0 & = & q_1 m_1 + m_2, \quad 0 < m_1 < m_2 \\
\cdots & & \cdots \quad \cdots \\
m_{s-2} & = & q_{s-1} m_{s-1} + m_s, \quad 0 < m_{s-1} < m_s \\
m_{s-1} & = & q_s m_s, \quad m_s = 1
\end{array}$$

where $m_0 = m$. Consider a linear pencil $\Lambda = \{y^m = \lambda x^n\}_{\lambda \in \mathbf{P}^1}$, where the member is $x^n = 0$ if $\lambda = \infty$. The pencil Λ has base points at the point $(x, y) = (0, 0)$ and its infinitely near points. The blowing-ups with centers at these base points will give exceptional curves whose weighted dual graph is given as follows:

G_s



if $s \equiv 0 \pmod{2}$



if $s \equiv 1 \pmod 2$

where $[q_i - 1]$ means a linear chain of (-2) curves of length $q_i - 1$. The (-1) curve in the middle of the graph is a cross-section of the proper transform of the pencil Λ.The closure in $\mathbf{P}^2$ of a curve $y^m = \lambda x^n$ of the pencil Λ is defined by $Y^m Z^{n-m} = X^n$, where X, Y, Z are homogeneous coordinates such that $x = X/Z$ and $y = Y/Z$. We denote by the same symbol Λ the linear pencil formed by the closures in $\mathbf{P}^2$ of the members of Λ. The pencil Λ on $\mathbf{P}^2$ has base points at the point $(X, Y, Z) = (0, 1, 0)$ and its infinitely near points. If we eliminate these base points as well we have a nonsingular projective surface and a $\mathbf{P}^1$-fibration $p : V \longrightarrow C$ onto a complete curve C isomorphic to $\mathbf{P}^1$ such that

(1) the proper transform E_∞ of the curve $x = 0$ (or equivalently $X = 0$ on $\mathbf{P}^2$) on V is a (-1) curve and it is an irreducible component with multiplicity n of a degenerate fiber F_∞ of p whose weighted graph is a linear chain consisting of the curve E_∞, the left hand side (the upper side) Γ_∞ of the (-1) curve in the above weighted dual graph and a similar graph Δ_∞ determined uniquely by Γ_∞,

$${}^t\Gamma_\infty \;\text{———}\; E_\infty \;\text{———}\; \Delta_\infty$$

where ${}^t\Gamma_\infty$ is the reversed graph of Γ_∞, i.e., the vertex of Γ_∞ conected to (-1) curve in the above dual graph G_s is a tip (= an end vertex) of the dual graph of F_∞;

(2) the proper transform E_0 of the curve $y = 0$ (or equivalently $Y = 0$ on $\mathbf{P}^2$) on V is a (-1) curve and it is an irreducible component with multiplicity m of a degenerate fiber F_0 of p whose weighted dual graph is a linear chain consisting of the curve E_0, the right hand side (the lower side) ${}^t\Gamma_0$ of the (-1) curve in the above weighted dual graph and a similar graph Δ_0 determined uniquely by Γ_0;

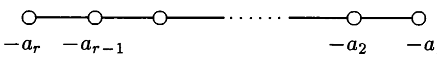
$${}^t\Gamma_0 \;\text{———}\; E_0 \;\text{———}\; \Delta_0$$

(3) the (-1) curves M_0 and M_∞ arising the elimination of the base points accumulated at the points $(X, Y, Z) = (0, 0, 1), (0, 1, 0)$ respectively are the cross-sections of the $\mathbf{P}^1$-fibration p.

Now we replace the (-1) curve in the graph G_s by a (-2) curve and denote the modified graph by $\widetilde{G_s}$. Then it is the resolution graph of a cyclic quotient singular point. More precisely, we have the following result. For the resolution of cyclic quotient singularity, see for example [2, p. 84].

Lemma 5.7 *With the above notations, let $d = mn + 1$ and $e = n(m - m_1) + q_0 + 1$. Let a cyclic group G of order d act on the affine plane $\mathbf{A}^2 = \operatorname{Spec} k[x, y]$ via $x \mapsto \zeta x$ and $y \mapsto \zeta^e y$, where ζ is a primitive d-th root of unity. Then the resolution graph of the quotient singular point P of $\mathbf{A}^2/G$ is the graph $\widetilde{G_s}$ given as above.*

Proof. Write the graph $\widetilde{G_s}$ as



where the vertex with weight $-a_r$ is the left-end vertex of $\widetilde{G_s}$. The assertion is then equivalent to showing that the continued fraction associated with the graph $\widetilde{G_s}$ is equal to a fraction d/e:

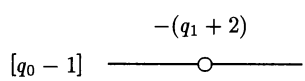
$$\frac{d}{e} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_{r-1} - \cfrac{1}{a_R}}}}$$

Namely, we have to show that if we write the continued fraction in the form of a single fraction d/e then the numerator d and the denominator e are given respectively by d and e specified in the statement. If we show the assertion that the numerator is equal to $mn+1$ then the denominator is obtained uniquely by solving the congruence equation

$$em \equiv n \pmod{mn+1} \quad \text{and} \quad 0 < e < mn+1 .$$

Indeed, $e = n(m - m_1) + q_0 + 1$ is a unique solution.

We prove the last assertion by induction on s. If $s = 1$ or 2 the assertion is straightforward. Suppose $s \geq 2$. Let $\widetilde{H}$ be the graph obtained by removig the following string from $\widetilde{G_s}$:



Then the reversed graph ${}^t\widetilde{H}$ is the dual graph obtained from a pair (m_1, m_0) instead of (m, n). Hence, by the induction hypothesis, the continued fraction associated with ${}^t\widetilde{H}$ is equal to a fraction

$$\frac{m_0 m_1 + 1}{m_0(m_1 - m_2) + q_1 + 1}$$

Then the continued fraction associated with ${}^t\widetilde{G_s}$ is obtained by making the following calcurations:

$$\begin{aligned} \frac{a}{b} &= q_1 + 2 - \frac{m_0(m_1 - m_2) + q_1 + 1}{m_0 m_1 + 1} \\ \frac{d'}{e'} &= \frac{q_0 a - (q_0 - 1)b}{(q_0 - 1)a - (q_0 - 2)b} \end{aligned}$$

where a, b, c, d are positive integers with $\gcd(a, b) = \gcd(d', e') = 1$. A straightforward computation shows that $d' = mn + 1$. Then we will be done if we note the well-known fact that the continued fractions associated with $\widetilde{G_s}$ and ${}^t\widetilde{G_s}$ have the same numerator if they are written as single fractions with coprime numerators and denominators. Q.E.D.

Let G and its action on $\mathbf{A}^2$ be the same as in Lemma 5.7. Note that $f = y^m - x^n$ is then a semi-invariant of weight n. Let $\rho : X \to B$ be the $\mathbf{A}^1_*$-fibration associated with the linear pencil $\{y^m - \lambda x^n\}_{\lambda \in \mathbf{P}^1}$, where $B \cong \mathbf{A}^1$. Since $B = \operatorname{Spec} k[t]$ with

$$t = \frac{y^m}{y^m - x^n}$$

and t is a invariant under the G-action, we know that the group G acts on X in such a way that G preserves each fiber of ρ. We shall show that X/G is isomorphic to X and that the quotient morphism $q : X \longrightarrow X/G$ thereby gives rise to an étale finite endomorphism $\alpha : X \longrightarrow X$ with degree $mn + 1$.

The above G-action on $\mathbf{A}^2$ extends to a G-action on $\mathbf{P}^2$ by setting

$$(X, Y, Z) \mapsto (\zeta X, \zeta^e Y, Z) \; .$$

The birational morphism $\sigma : V \longrightarrow \mathbf{P}^2$ which eliminates the base points of the linear pencil Λ is, in fact, a composite of the G-equivariant blowing-ups in the sense that the centers of the blowing-ups are the G-fixed points. Furthermore, the cross-sections M_0, M_∞ are pointwise G-fixed curves and the fibers of the $\mathbf{P}^1$-fibration p are preserved under the G-action. Each component of the fibers F_0 and F_∞ are G-stable. Let $\overline{V}$ be the quotient V/G and let $\pi : V \longrightarrow \overline{V}$ be the quotient morphism. Then $\overline{V}$ is a normal projective surface with a $\mathbf{P}^1$-fibration $\overline{p} : \overline{V} \longrightarrow C$, where $\overline{p} \cdot \pi = p$, and the morphism π is a finite morphism. Furthermore, π is totally ramified over $N_0 := \pi(M_0)$ and $N_\infty := \pi(M_\infty)$ and hence $\overline{V}$ is smooth in the neighborhoods of the curves N_0 and N_∞ which are the cross-sections of the $\mathbf{P}^1$-fibration $\overline{p}$. The morphism π is unramified along the fibers of p possibly except for the fibers F_0 and F_∞. Let $\overline{F_0}$ and $\overline{F_\infty}$ be the images of F_0 and F_∞ under π. Then there appear finitely many cyclic quotient singular points lying on the curves $\overline{F_0}$ and $\overline{F_\infty}$. Let $\tau : W \longrightarrow \overline{V}$ be the minimal resolution of these singular points and let $q = \overline{p} \cdot \tau : W \longrightarrow C$ be the induced $\mathbf{P}^1$-fibration.

Let A_0 and A_∞ denote respectively the set-theoretic inverse images by τ of the union of the images by π of the irreducible components of the fibers F_0 and F_∞ corresponding to the graphes Γ_0 and Γ_∞. Then A_0 and A_∞ are the linear chains of the curves isomorphic to $\mathbf{P}^1$ because they are the parts of the degenerate fibers of the $\mathbf{P}^1$-fibration q. Let $\widetilde{\Gamma_0}$ and $\widetilde{\Gamma_\infty}$ denote respectively the weighted dual graphes corresponding to A_0 and A_∞. By the abuse of notations, we may write

$$\widetilde{\Gamma_0} = \tau^{-1}(\pi(\Gamma_0)), \quad \widetilde{\Gamma_\infty} = \tau^{-1}(\pi(\Gamma_\infty)) \; .$$

Then Lemma 5.7 asserts that the graph

$$\begin{tikzcd} \widetilde{\Gamma_\infty} \arrow[r, dash] & \widetilde{N_0} \arrow[r, dash] & \widetilde{\Gamma_0} \end{tikzcd}$$

with all (-1) curves and subsequently contactible curves contracted down is the resolution graph $\widetilde{G_s}$ of the quotient singular point P of $\mathbf{A}^2/G$ under the G-action on $\mathbf{A}^2$ specified therein, where $\widetilde{N_0}$ is the proper transform of N_0 by τ. In particular, if we denote by $\overline{N_0}$ the image of $\widetilde{N_0}$ by this contraction, we have

$$(\overline{N_0}^2) = -2, \quad \text{while} \quad (\widetilde{N_0}^2) = ({N_0}^2) = -d \; .$$

Let B_0 and B_∞ denote respectively the inverse images by τ of the union of the images by π of the irreducible components of the fibers F_0 and F_∞ corresponding to the graphes Δ_0 and Δ_∞. We contract also the (-1) curves and subsequently contractible curves contained in B_0 and B_∞. We thus obtain a birational morphism $\overline{\tau} : W \longrightarrow \overline{W}$. Set $\overline{N_\infty} := \overline{\tau}(\widetilde{N_\infty})$, where $\widetilde{N_\infty}$ is the proper transform of N_∞ by τ, and let $\widetilde{H_s}$ be the weighted dual graph of $\overline{\tau}(B_0) + \overline{N_\infty} + \overline{\tau}(B_\infty)$.

$$\begin{tikzcd} X \subset V \arrow[d, "\sigma"'] \arrow[dr, "\pi"] & & W \arrow[dl, "\tau"'] \arrow[dr, "\overline{\tau}"] & & W' \arrow[dl, "\tau'"'] \\ X \subset \mathbf{P}^2 & \overline{V} = V/G & & \overline{W} & \end{tikzcd}$$

The $\mathbf{P}^1$-fibration $q : W \to C$ induces a $\mathbf{P}^1$-fibration $\overline{q} : \overline{W} \to C$ on the surface $\overline{W}$ for which $\overline{N_0}$ and $\overline{N_\infty}$ are the cross-sections of $\overline{q}$. Let F_1 be the fiber of $p : V \to C$ corresponding to the member $y^m = x^n$ of the pencil Λ on $\mathbf{P}^2$ and let $\overline{F_1}$ be the proper transform of $\pi(F_1)$ by $\overline{\tau} \cdot \tau^{-1}$. Then $\overline{F_1}$ is a smooth fiber of $\overline{q} : \overline{W} \to C$. Now apply an elementary transformation with center at the point $\overline{N_\infty} \cap \overline{F_1}$. Namely, blow up this point and contract the proper transform of $\overline{F_1}$. Let $\tau' : W' \to W$ be the elementary transformation and let F_1' be the fiber on W' corresponding to F_1 on $\overline{W}$. Let N_0' be the proper transform of $\overline{N_0}$. Then $(N_0'^2) = -1$ and the graph $\widetilde{G_s}$ changes back to the graph G_s. Now remove all the irreducible curves correspondingto the graphes $\widetilde{G_s}, \widetilde{H_s}$ and the fiber F_1' from W' to obtain an openset U. By the construction of W', the open set U is isomorphic to the surface X. More precisely, X is contained in the surfaces $\overline{V}, W, \overline{W}$ and W' and X is intact under the birational mappings $\tau, \overline{\tau}$ and τ'. Hence the restiction of the quotient morphism $\pi : V \to \overline{V}$ induces an étale finite endomorphism which is a Galois covering with group G.

We have thus proved the following result.

Theorem 5.8 *Let $X = \operatorname{Spec} k[x, y, f^{-1}]$ with $f = y^m - x^n$, where m, n are integers larger than 1 with $\gcd(m, n) = 1$. Then there exists an étale finite endomorphism $\alpha : X \to X$ of degree $mn + 1$ which is a cyclic Galois covering.*

REMARK **5.9** *In the above argument, we used the graph $\widetilde{G_s}$ which is obtained from the graph G_s by replacing the (-1) curve by a (-2) curve. The same argument as above applies to the graph $\widetilde{G_s}^{(t)}$ obtained from G_s by replacing the (-1) curve by a $(-t)$ curve with $t \geq 2$. We obtain thereby an étale finite endomorphism of X of degree different from $mn + 1$.*

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HOMOGENEITY CONDITION FOR NOETHERIAN R-ALGEBRAS

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1. *Cohen-Macaulay approximation*

Let Λ be a left and right Noetherian ring and ω a Λ-bimodule. Denote the category of finitely generated left (right) Λ-modules by $\mathrm{mod}\Lambda$ (Λ^{op}).

Put

$\mathcal{C}(\Lambda) = \mathcal{C}_\omega(\Lambda) := \{M \in \mathrm{mod}\Lambda : \mathrm{Ext}^i_\Lambda(M,\omega) = 0, \ \ i > 0\}$

$\mathrm{add}\omega := \{M \in \mathrm{mod}\Lambda : M \oplus M' \cong \omega^m, \ \ \exists m, \ \ \exists M'\}$

$\hat{\mathrm{add}}\omega := \{M \in \mathrm{mod}\Lambda : \exists 0 \to G_m \to \cdots \to G_0 \to M \to 0 \ \ \text{(exact)}, \ \ G_i \in \mathrm{add}\omega\}$

$\hat{\hat{\mathrm{add}}}\omega := \{M \in \mathrm{mod}\Lambda : \exists 0 \to M \to Y_0 \to \cdots \to Y_n \to 0 \ \ \text{(exact)}, \ \ Y_i \in \hat{\mathrm{add}}\omega\}$

$\mathcal{P} := \{M \in \mathrm{mod}\Lambda : \mathrm{pd}_\Lambda M < \infty\}$

$\mathcal{I} := \{M \in \mathrm{mod}\Lambda : \mathrm{id}_\Lambda M < \infty\}$.

Consider the following conditions on ω:

$$\text{(d1)} \quad \mathrm{Hom}_\Lambda(\omega,\omega) = \Lambda \qquad \text{(d2)} \quad \mathrm{Ext}^i_\Lambda(\omega,\omega) = 0 \ \ i > 1 \qquad \text{(d3)} \quad \mathrm{id}_\Lambda \omega < \infty$$

(di)^{op} (i=1,2,3) right version. (d) := (d1)+(d2)+(d3).

In this section, we prepare more or less well-known results about Cohen-Macaulay approximation (eg. [1, 4, 5, 7]) and characterize a module with finite injective dimension.

1.0 (Miyashita [4], Lemma 1.1) $T \in \mathrm{mod}\Lambda$. Consider the following exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X_{n-1} \arrow[rr] \arrow[dr] & & X_{n-2} \arrow[r] & \cdots \arrow[rr] \arrow[dr] & & X_0 \arrow[r] & N \arrow[r] \arrow[d, equal] & 0 \\
& Y_n & & Y_{n-1} \arrow[ur] & & & Y_1 \arrow[ur] & & Y_0 &
\end{tikzcd}$$

(r) $\mathrm{Ext}^j_\Lambda(T, X_k) = 0$ $(j > 0, \ \ 0 \le k \le n-1)$
$$\Longrightarrow \mathrm{Ext}^i_\Lambda(T,N) \cong \mathrm{Ext}^{i+1}_\Lambda(T,Y_1) \cong \cdots \cong \mathrm{Ext}^{i+n}_\Lambda(T,M) \ \ i > 0$$

(cor) $\mathrm{Ext}^j_\Lambda(X_k, T) = 0$ $(j > 0, \ \ 0 \le k \le n-1)$
$$\Longrightarrow \mathrm{Ext}^i_\Lambda(M,T) \cong \mathrm{Ext}^{i+1}_\Lambda(Y_{n-1},T) \cong \cdots \cong \mathrm{Ext}^{i+n}_\Lambda(N,T) \ \ i > 0$$

1.1. Assume (d)^{op}. Then $M \in \mathcal{C}(\Lambda) \Rightarrow M^* := \mathrm{Hom}_\Lambda(M,\omega) \in \mathcal{C}(\Lambda^{op})$ and $M^{**} \cong M$, canonically.

1.2. Asuume (d) and (d)^{op}. $M \in \mathrm{mod}\Lambda$. Then

$$\exists (C) \quad 0 \to Y \to X \to M \to 0 \text{ exact (Cohen-Macaulay approximation)}$$
$$\exists (I) \quad 0 \to M \to Y' \to X' \to 0 \text{ exact}$$
$$X, \ X' \in \mathcal{C}(\Lambda); \ \ Y, \ Y' \in \hat{\mathrm{add}}\omega.$$

Proof. By induction on $d(M)$ below (see [1, 7]).

1.2.1 Put $d(M) := \sup\{i : \mathrm{Ext}^i_\Lambda(M,\omega) \neq 0\}$. Then

The detailed version of this paper will be submitted for publication elsewhere.

(i) $0 \le d(M) \le \mathrm{id}_\Lambda \omega < \infty$,
(ii) $d(M) = 0 \iff M \in \mathcal{C}(\Lambda)$,
(iii) Let $0 \to X \to Y \to Z \to 0$ be exact. Then

$$d(Y) < d(Z) \Longrightarrow d(X) = d(Z) - 1.$$

1.3. Assume (d) and (d)^{op} and $\mathrm{id}_\Lambda \omega = \mathrm{id}_{\Lambda^{op}} \omega = e$. Then

$$\mathcal{I} = \hat{\hat{\mathrm{add}}}\omega$$

1.3.1. (i) $C \in \mathcal{C}(\Lambda)$, $X \in \hat{\mathrm{add}}\omega \Rightarrow \mathrm{Ext}^i_\Lambda(C, X) = 0 \;\; i > 0$.
(ii) $X \in \mathrm{mod}\Lambda$, $\exists m$ such that $\mathrm{Ext}^i_\Lambda(C, X) = 0$ for $i > m$, $C \in \mathcal{C}(\Lambda) \Rightarrow \mathrm{id}_\Lambda X \le m + e$.
(iii) $X \in \mathcal{C}(\Lambda)$ and $\mathrm{id}_\Lambda X = m < \infty \Rightarrow \exists\, 0 \to X \to W_0 \to \cdots \to W_m \to 0$ exact, $W_i \in \mathrm{add}\omega$.

1.3.2. Proof of 1.3. Let $M \in \mathcal{I}$. Take $(I)\quad 0 \to M \to Y_0 \to X \to 0$ (exact), $X \in \mathcal{C}(\Lambda)$, $Y_0 \in \hat{\mathrm{add}}\omega$. Then $\mathrm{id}_\Lambda X < \infty$. Thus there exists an exact sequense $0 \to X \to Y_1 \to \cdots \to Y_m \to 0$, $Y_i \in \mathrm{add}\omega$ by (iii). Combine two sequences.

Conversely, let $0 \to M \to Y_0 \to \cdots \to Y_n \to 0$, $Y_i \in \hat{\mathrm{add}}\omega$, be exact. Since we have $\mathrm{Ext}^i_\Lambda(X, Y_k) = 0$ for any $X \in \mathcal{C}(\Lambda)$, $i > 0$, $k \ge 0$ by (i), $\mathrm{Ext}^i_\Lambda(X, M) = 0$ for any $i > n$ by (r) 1.0. Hence $\mathrm{id}_\Lambda M \le n + e$ by (ii).

2. *On the condition (hc)*

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay (commutative) local ring with the canonical module K. Let Λ be an R-algebra such that Λ is a finitely generated R-module and a Cohen-Macaulay R-module with $\dim_R \Lambda = \dim R = d$. Let $M^* := \mathrm{Hom}_R(M, K)$ for $M \in \mathrm{mod}\Lambda$. Then $M^* \in \mathrm{mod}\Lambda^{op}$. Put $\omega = \Lambda^*$ a Λ-bimodule. Then $\mathrm{id}_\Lambda \omega = \mathrm{id}_{\Lambda^{op}} \omega = d$. We have

$$\mathcal{C}_\omega(\Lambda) = \mathcal{C}_d := \{M \in \mathrm{mod}\Lambda : M \text{ is a C-M } R\text{-module of } \dim_R M = d\}$$

All the results in §1 holds whenever we replace $\mathcal{C}(\Lambda)$ by $\mathcal{C}_d$ and $\mathrm{Hom}_\Lambda(-, \omega)$ by $\mathrm{Hom}_R(-, K)$ (Note $\mathrm{Hom}_\Lambda(-, \omega) \cong \mathrm{Hom}_R(-, K)$ on $\mathrm{mod}\Lambda$).

We study the following condition:

$$\text{(hc)} \quad \mathrm{id}_\Lambda M = d \text{ for every } M \in \mathcal{C}_d \cap \mathcal{I}$$

Let $M \in \mathrm{mod}\Lambda$.

2.0. $d \le \mathrm{id}_\Lambda M$ (by [3], Theorem 3.7).

2.1. $\mathrm{id}_\Lambda M < \infty \Leftrightarrow M \in \hat{\hat{\mathrm{add}}}\omega$, i.e. $\mathcal{I} = \hat{\hat{\mathrm{add}}}\omega$ (by 1.3).

2.2. $\mathrm{id}_\Lambda M = d \Leftrightarrow M \in \hat{\mathrm{add}}\omega$.

2.2.1. (i) $\mathrm{id}_\Lambda M = n, \mathrm{depth}_R X = t \Rightarrow \mathrm{Ext}^i_\Lambda(X, M) = 0 \;\; i > n - t$.
(ii) $M \in \mathcal{C}_d$, $\mathrm{id}_\Lambda M = n \Rightarrow \exists\, 0 \to M \to W_0 \to \cdots \to W_{n-d} \to 0$ exact, $W_i \in \mathrm{add}\omega$ (cf. 1.3.1(iii)).

2.2.2. $M \in \mathcal{C}_d$ and $\mathrm{id}_\Lambda M = d \iff M \in \mathrm{add}\omega$.

2.2.3. Proof of 2.2. $\Leftarrow$: We have $\mathrm{id}_\Lambda M \le d$ by 1.3.1 (i), (ii), hence equality holds by 2.0.

$\Rightarrow$: Let $\mathrm{id}_\Lambda M = d$. Take $(C)\quad 0 \to Y \to X \to M \to 0$ exact, $\ Y \in \widehat{\mathrm{add}}\omega,\ X \in \mathcal{C}_d$. Since $\mathrm{id}_\Lambda Y = d$ by the part $\Leftarrow$, $\mathrm{id}_\Lambda X = d$. Thus $X \in \mathrm{add}\omega$ by 2.2.2, hence $M \in \widehat{\mathrm{add}}\omega$.

2.2.4. Corollary. ((2)-(5) in [1], Proposition 4.6) The following are equivalent.

(1) Λ satisfies (hc).

(1′) $M \in \mathcal{I} \Rightarrow \mathrm{id}_\Lambda M = d$.

(2) $\widehat{\mathrm{add}}\omega = \hat{\hat{\mathrm{add}}}\omega$.

(3) $0 \to M \to W_0 \to W_1 \to 0$ exact, $W_0,\ W_1 \in \mathrm{add}\omega \Rightarrow M \in \mathrm{add}\omega$.

(4) $0 \to M \to W_0 \to W_1 \to \cdots \to W_n \to 0$ exact, $W_i \in \mathrm{add}\omega \Rightarrow M \in \mathrm{add}\omega$.

(5) $M \in \mathcal{C}_d$ and $\exists n; \mathrm{Ext}^i_\Lambda(C, M) = 0\ \ i > n,\ C \in \mathcal{C}_d \Rightarrow M \in \mathrm{add}\omega$.

2.3. The following are equivalent.

(1) Λ is Gorenstein.

(2) Λ is a Cohen-Macaulay R-module of $\dim_R\Lambda = d$, satisfies (hc), $\mathrm{id}_\Lambda\Lambda < \infty$.

(3) Λ is a Cohen-Macaulay R-module of $\dim_R\Lambda = d$, satisfies $(\mathrm{hc})^{op}$, $\mathrm{id}_{\Lambda^{op}}\Lambda < \infty$.

2.4. Remark. (1) '$\mu^d(Q, \Lambda) > 0\ \ \forall Q \in \mathrm{Max}\Lambda$' $\Rightarrow$ (hc), where μ^d is the *Bass number*[3].

All the cases; Gorenstein, $\Lambda/J(\Lambda)$ simple ring, commutative, satisfy '...', hence (hc).

(2) (hc) is not (left-right) symmetric.

3. *Category equivalence* $\mathcal{I} \sim \mathcal{P}$

Λ and R are the same as in section 2 and $\omega = \Lambda^*$ in 3.1, 3.2.

3.1. Assume that Λ satisfies (hc). $\mathcal{P}_0 := \{M \in \mathcal{P} : \mathrm{Tor}^\Lambda_i(\omega, M) = 0\ \ i > 0\}$.

Then there is a category equivalence $\mathcal{I} \sim \mathcal{P}_0$ induced from $F := \mathrm{Hom}_\Lambda(\omega, -) : \mathcal{I} \to \mathcal{P}_0$ and $G := \omega \otimes_\Lambda - : \mathcal{P}_0 \to \mathcal{I}$.

3.2. If $d = 0$, then $\omega = \mathrm{Hom}_R(\Lambda, E)$, $E = E_R(R/\mathfrak{m})$ an injective hull of an R-module $R/\mathfrak{m}$. Then we have:

$$\mathcal{P}_0 = \mathcal{P} \Leftrightarrow (hc)^{op} \Leftrightarrow fpd\Lambda = 0,$$

where $\mathrm{fpd}\Lambda = \sup\{\mathrm{pd}_\Lambda M : M \in \mathcal{P}\}$

3.3. Further developments. (a) In case $\omega = \Lambda^*$ and $d > 0$, the problem when $\mathcal{P} = \mathcal{P}_0$ is very valuable. Of course, if Λ is commutative, it holds that $\mathcal{P} = \mathcal{P}_0$ by Sharp[5]. This is the starting point of 3.1.

(b) (Assuming or not assuming (hc)) In case ω is a Λ-bimodule with (d), $(\mathrm{d})^{op}$, and $\mathrm{id}_\Lambda\omega = \mathrm{id}_{\Lambda^{op}}\omega \ge d$, the problem when $\mathcal{P} = \mathcal{P}_0$, or even what is $\mathcal{P}_0$, is also a valuable problem. Auslander and Reiten [2] study the case $d = 0$ (not assuming (hc)) and get the following:

F and G induce a category equivalence $\mathcal{I} \sim \mathcal{P}$ if and only if ω satisfies (d2), (d3), $(\mathrm{d2})^{op}$, $(\mathrm{d3})^{op}$, $\widehat{\mathrm{add}}_\Lambda\omega = \mathcal{I}(\Lambda)$, $\widehat{\mathrm{add}}_{\Lambda^{op}}\omega = \mathcal{I}(\Lambda^{op})$ (that is, ω is a strong cotilting module) and (d1).

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REGULARITY OF POLYNOMIAL IDEALS

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The Castelnuovo-Mumford regularity is one of the most important invariants in studying the minimal free resolution of the defining ideals of the projective varieties. There are some bounds on the Castelnuovo-Mumford regularity of the projective variety in terms of the other basic measures such as dimension, codimension and degree.

The purpose of this note is to provide a short overview of the recent developement on bounding the Castelnuovo-Mumford regularity and to encourage the readers to make familiar with this topic.

Let X be a projective scheme of $\mathbf{P}_K^N$ over a field K. Let $S = K[x_0, \cdots, x_N]$ be the polynomial ring and $\mathfrak{m} = (x_0, \cdots, x_N)$ be the irrelevant ideal. Then we put $\mathbf{P}_K^N = \mathrm{Proj}(S)$. We denote by $\mathcal{I}_X$ the ideal sheaf of X. Let m be an integer. Then X is said to be m-regular if $\mathrm{H}^i(\mathbf{P}_K^N, \mathcal{I}_X(m-i)) = 0$ for all $i \geq 1$. The Castelnuovo-Mumford regularity of $X \subseteq \mathbf{P}_K^N$, introduced by Mumford by generalizing ideas of Castelnuovo, is the least such integer m and is denoted by $\mathrm{reg}(X)$. The interest in this concept stems partly from the well-known fact that X is m-regular if and only if for every $p \geq 0$ the minimal generators of the p-th syzygy module of the defining ideal I of $X \subseteq \mathbf{P}_K^N$ occur in degree $\leq m + p$, see, e.g., [7, 12, 13]. In particular, $\mathrm{reg}(X) \geq 1$ for any projective scheme $X \subseteq \mathbf{P}_K^N$, and $\mathrm{reg}(X) \geq 2$ if the projective scheme X is nondegenerate, that is, X is not contained in any hyperplane of $\mathbf{P}_K^N$.

Let I be an ideal $\oplus_{\ell \in \mathbf{Z}} \Gamma(\mathbf{P}_K^N, \mathcal{I}_X(\ell))$ of S. We call I as the defining ideal of X. Let $in(I)$ be the initial ideal of I with respect to the reverse lexicographic order. In [8] they obtained that $\mathrm{reg}(X)$ is equal to the maximal degree of minimal generators of $in(I)$.

Let M be a finitely generated graded S-module with $\dim(M) = d+1 > 0$. We write $[M]_n$ for the n-th graded piece of M, and $M(p)$ for the graded module with $[M(p)]_n = [M]_{p+n}$. Then, for $i = 0, \cdots, d+1$, we set $a_i(M) = \max\{n \mid [\mathrm{H}_{\mathfrak{m}}^i(M)]_n \neq 0\}$ if there exists, and $a_i(M) = -\infty$ otherwise. The Castelnuovo-Mumford regularity of the S-module M is defined as $\mathrm{reg}(M) = \max\{a_i(M) + i \mid i = 0, \cdots, d+1\}$. For the projective scheme

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X and its defining ideal I, it is easy to show that $\mathrm{reg}(X) = \mathrm{reg}(I)$.

There are interesting topics on the behaviour of the regularity without assuming X is irreducible and reduced. The Gotzmann-type theorem, started in [19, 20], is generalized for exterior algebras in [3, 21] as an application of generic initial ideals. The asymptotic property, that is, how is $\mathrm{reg}(I^n)$ written as a function on n, has been studied in [10, 11, 15, 31, 57]. In [10, 15], $\mathrm{reg}(I^n) \leq n\,\mathrm{reg}(I)$ for the case $\dim(X) = 0$ and $\mathrm{reg}(I^n)^{\mathrm{sat}} \leq n\,\mathrm{reg}(I)$ for $\dim X = 1$ are shown, where J^{sat} is the saturation of the ideal J. In general case, she obtained in [57] that $\mathrm{reg}(I^n) \leq Kn\,\mathrm{reg}(I)$ for a constant K by using the uniform Artin-Rees Theorem. The further recent research is also founded in [11, 31]. The other interesting topic is the relationship with the arithmetic degree, which starts from [7]. The recent developement is also described in [28, 56, 40].

Now let us assume that X is irreduceble reduced. We will begin with stating the Eisenbud-Goto conjecture [13].

Conjecture. Let X be a nondegenerate projective variety in $\mathbf{P}_K^N$ over an algebraically closed field K. Then

$$\mathrm{reg}(X) \leq \deg(X) - \mathrm{codim}(X) + 1$$

The conjecture is solved for $\dim(X) = 1$ in [22]. Under the assumption that X is smooth over an algebraically closed field K of characteristic zero, the conjecture is solved for $\dim(X) = 2$ in [50, 30], and for $\Delta(X, \mathcal{O}(1)) \leq 5$, that is, the right hand side of the conjecture less than or equal to 7, in [2]. The toric case is also solved for $\mathrm{codim}(X) = 2$ in [48]. Recently a weaker bound $\mathrm{reg}(X) \leq \deg(X) - \mathrm{codim}(X) + 2$ is given in [32] for $\dim(X) = 3$ if X is smooth and $\mathrm{char}(K) = 0$.

Let us describe another attempt to getting the bound on the regularity for nondegenerate projective varieties. We will introduce an invariant measuring the intermediate cohomologies. Let k be a nonnegative integer. Then X is called k-Buchsbaum if the graded S-module $\mathrm{M}^i(X) = \oplus_{\ell \in \mathbf{Z}} \mathrm{H}^i(\mathbf{P}_K^N, \mathcal{I}_X(\ell))$, called the deficiency module of X, is annihilated by $\mathfrak{m}^k$ for $1 \leq i \leq \dim(X)$, see, e.g., [35, 36]. Further we call the minimal nonnegative integer n, if there exists, such that X is n-Buchsbaum, as the Ellia-Migliore-Miró Roig number of X and denote by $k(X)$. In case X is not k-Buchsbaum for all $k \geq 0$, then we put $k(X) = \infty$. It is known that the numbers $k(X)$ are invariant in a liaison class, see, e.g., [35, 55]. Note that $k(X) < \infty$ if and only if X is locally Cohen-Macaulay and equi-dimensional, and that $k = 0$ if and only if X is arithmetically Cohen-Macaulay.

In what follows, for a rational number $\ell \in \mathbf{Q}$, we write $\lceil \ell \rceil$ for the minimal integer which is larger than or equal to ℓ.

In recent years upper bounds on the Castelnuovo-Mumford regularity of a projective variety X have been given by several authors in terms of $\dim(X)$, $\deg(X)$, $\operatorname{codim}(X)$ and $k(X)$, see, e.g., [25, 26, 27, 39, 45, 46]. The following bound, which is firstly obtained in [46] and refined in [39], is the most optimal among the known results.

Theorem. Let X be a nondegenerate projective variety in $\mathbf{P}_K^N$ over an algebraically closed field K. Then

$$\operatorname{reg}(X) \leq \left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil + \max\{k(X)\dim(X), 1\}.$$

The problem is whether this bound is sharp or not. For arithmetically Cohen-Macaulay varieties, the varieties having such upper bounds are classified in [58] for $\operatorname{char}(K)$, and later in [42] for the positive characteristic case. Furthermore, such study is found in [60] for arithmetically Buchsbaum curves, that is, $k(X) = 1$ and $\dim(X) = 1$ and in [43] for arithmetically Buchsbaum varieties. We will state the result for general case, following [38].

Theorem. Let X be a nondegenerate projective variety in $\mathbf{P}_K^N$ over an algebraically closed field K of characteristic zero. Assume that $k(X) \geq 1$, $\deg(X) \geq \operatorname{codim}(X)^2 + 2\operatorname{codim}(X) + 2$ and

$$\operatorname{reg}(X) = \left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil + k(X)\dim(X).$$

Then $\dim(X) = 1$ and X is a curve on a rational ruled surface Y.
More precisely, X is a divisor on a rational ruled surface Y constructed as follows:

Let $\pi : Y = \mathbf{P}(\mathcal{E}) \to \mathbf{P}_K^1$ be a projective bundle, see, e.g., [24, (V.2)], where $\mathcal{E} = \mathcal{O}_{\mathbf{P}_K^1} \oplus \mathcal{O}_{\mathbf{P}_K^1}(-e)$ for some $e \geq 0$. Let Z be a minimal section of π corresponding to the natural map $\mathcal{E} \to \mathcal{O}_{\mathbf{P}_K^1}(-e)$ and F be a fibre corresponding to $\pi^*\mathcal{O}_{\mathbf{P}_K^1}(1)$. We have an embedding of Y in $\mathbf{P}_K^N$ by a very ample sheaf corresponding to a divisor $H = Z + n \cdot F$ $(n > e)$, where $N = 2n - e + 1$. Then X is a divisor on Y linearly equivalent to $a \cdot Z + b \cdot F$ such that $a \geq 1$ and $an + 2 \leq b \leq (a+2)n - e + 1$.

In this case, $\operatorname{codim}(X) = 2n - e$, $\deg(X) = a(n-e) + b$, $k(X) = \lfloor (b - an - 2)/(n-e) \rfloor + 1$ and $\operatorname{reg}(X) = \lfloor (b - an - 2)/(n-e) \rfloor + a + 2$.

These results motivate us to state the following problems.

Problem. Let X be a nondegenerate projective variety in $\mathbf{P}_K^N$ over an algebraically closed field K. Then is the inequality $\operatorname{reg}(X) \leq \lceil (\deg(X) -$

$1)/\operatorname{codim}(X)\rceil + \max\{k(X), 1\}$ true?

Problem. Describe the behaviour of k(X) for the projective variety X after a generic hyperplane section. For instance, for a projective variety X and a generic hyperplane H in $\mathbf{P}^N_K$ with $\dim(X) \geq 1$, is the inequality $2\,k(X \cap H) \geq k(X)$ true?

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Existence of homogeneous ideals fitting into long Bourbaki sequences and related topics

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§1. Main theorem

Given a homogeneous ideal I of height $p \geq 2$ in a polynomial ring $R := k[x_1, \ldots, x_r]$, there is a finitely generated torsion-free graded R-module M with no free direct summand satisfying $\mathrm{Ext}^i_R(M, R) = 0$ for $i = 1, \ldots, p-1$ that fits into an exact sequence of the form

$$0 \longrightarrow S_{p-1} \longrightarrow S_{p-2} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \oplus M \longrightarrow I(c) \longrightarrow 0,$$

where c is an integer and S_i $(0 \leq i \leq p-1)$ are finitely generated graded free R-modules (see e.g. [7], [17]). By this sequence one obtains

$$H^{i-1}_{\mathfrak{m}}(R/I)(c) \cong H^i_{\mathfrak{m}}(M) \quad \text{for} \quad i = 1, \ldots, \dim(R/I) = r - p,$$

Since $H^i_{\mathfrak{m}}(M) = 0$ for $i = r-p+1, \ldots, r-1$ and $i = 0$ by local duality, considering the local cohomologies of R/I is the same thing as considering those of M. When $p = 2$, the above sequence is often called a Bourbaki sequence and the map from $S_0 \oplus M$ to $I(c)$ is obtained essentially by taking maximal minors of the map from S_1 to $S_0 \oplus M$. Besides, it is well known that for any M as above there is a homogeneous ideal I of height two fitting into a Bourbaki sequnce if $p = 2$ (see [10, 11, 14, 16, 17]). Here a question arises. Given a $p \geq 2$ and an M, does there always exist an I connected with M by the above sequence? The following theorem answers to this question.

Theorem 1.1. *Let $p \geq 2$ be an integer and let M be a finitely generated torsion-free graded R-module with no free direct summand satisfying $\mathrm{Ext}^i_R(M, R) = 0$ for $i = 1, \ldots, p-1$. Then there exists a homogeneous ideal I in R of height p which fits into an exact sequence of the form*

$$(*) \qquad 0 \longrightarrow S_{p-1} \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \oplus M \longrightarrow I(c) \longrightarrow 0,$$

where c is an integer and S_i $(0 \leq i \leq p-1)$ are finitely generated graded free R-modules.

For the proof we refer the reader to [9]. The core of our proof is to take the mapping cones of successive chain maps from Koszul complexes to the finite free complex

$$F_\bullet : \cdots \xrightarrow{\varphi_{p+1}} F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} F_{-1} \xrightarrow{\varphi_{-1}} F_{-2} \xrightarrow{\varphi_{-2}} \cdots$$

such that

$$\cdots \xrightarrow{\varphi_{p+1}} F_p \xrightarrow{\varphi_p} F_{p-1} \xrightarrow{\varphi_{p-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0$$

is a minimal free resolution of M over R and

$$\cdots \xrightarrow{\varphi_{-2}^\vee} F_{-2}^\vee \xrightarrow{\varphi_{-1}^\vee} F_{-1}^\vee \xrightarrow{\varphi_0^\vee} F_0^\vee \xrightarrow{\varphi_1^\vee} \operatorname{Im}(\varphi_1^\vee) \longrightarrow 0$$

is a minimal free resolution of $\operatorname{Im}(\varphi_1^\vee)$ over R. The ideals I obtained by our method are far from good. Thus any general answer to the following problem is not given yet.

Problem 1.2. Let $p \geq 2$ and M be as in Theorem 1.1.

(1) Does there exist an I fitting into $(*)$ such thatR/I is reduced?

(2) Does there exist an I fitting into $(*)$ such thatR/I is an integral domain?

Of course it is well known that the answer is affirmative, if M is generalized Cohen-Macaulay, $H^1_{\mathfrak{m}}(M) = 0$, $p = 2$, and $r \geq 4$ (see e.g. [15, Section 3]).

§2. Project

Let $p \geq 2$ be an integer and let M be a finitely generated torsion-free graded R-module with no free direct summand satisfying $\operatorname{Ext}^i_R(M, R) = 0$ for $i = 1, \ldots, p-1$. Let further $\mathfrak{I}(M,p)$ denote the set of all homogeneous ideals I in R of height p fitting into exact sequences of the form $(*)$. Theorem 1.1 implies that $\mathfrak{I}(M,p) \neq \emptyset$. Perhaps the most popular way to study the structure of $\mathfrak{I}(M,p)$ is to do so in the framework of even linkage theory (see [10, 16, 17, 19]). But we would like to propose another approach based on the analysis of Weierstrass bases and basic sequences (see the next section or [4, 6]).

Let $I \in \mathfrak{I}(M,p)$. Then we can associate with M and I their basic sequences $B_R(M) = (\bar{\gamma}^1; \bar{\gamma}^2; \cdots; \bar{\gamma}^{r+1})$ and $B_R(I) = (\bar{n}^1; \bar{n}^2; \cdots; \bar{n}^{r+1})$. By the results of [7], they satisfy

$$\bar{n}^i = \bar{\gamma}^i + c \quad \text{for} \quad i = p+1, \ldots, r+1\,, \tag{2.1}$$

$$\bar{n}^p = (\bar{w}', \bar{\gamma}^p + c) \tag{2.2}$$

up to permutation with a suitable sequence of integers $\bar{w}'$, where $\bar{a}+c = (a_1+c, \ldots, a_l+c)$ for a sequence $\bar{a} = (a_1, \ldots, a_l)$. This means that the essential information on $B_R(I)$ lies in $B_R(M)$, $(\bar{n}^1; \bar{n}^2; \cdots; \bar{n}^p)$, and c. Assume that the variables $x_1, \ldots, x_r$ are chosen sufficiently generally. The ideal of the initial monomials of the elements of I with respect to the reverse lexicographic order has a Weierstrass basis $\{ \operatorname{in}(x, f_l^i) \mid 1 \leq i \leq r+1,\ 1 \leq l \leq m_i \}$, where $\operatorname{in}(x; f_l^i) \in k[x_1, \ldots, x_i]x_i$. Using them, let $\tilde{I} := \bigoplus_{i=1}^{p} \bigoplus_{l=1}^{m_i} \operatorname{in}(x; f_l^i) k[x_i, \ldots, x_r]$. Then

(2.3) $\tilde{I}$ is a Cohen-Macaulay Borel fixed monomial ideal in R of height p such that $B_R(\tilde{I}) = (\bar{n}^1; \cdots; \bar{n}^p)$.

The conditions described above, however, are not enough for characterizing all possible basic sequences of homogeneous ideals in $\mathfrak{I}(M, p)$. We have to find other ones to get an answer to the following

Problem 2.4. Let $p \geq 2$ and M be as above. Describe all sequences of integers $(\bar{n}^1; \bar{n}^2; \cdots; \bar{n}^{r+1})$ for which there exists a homogeneous ideal $I \in \mathfrak{I}(M, p)$ such that $B_R(I) = (\bar{n}^1; \bar{n}^2; \cdots; \bar{n}^{r+1})$.

There are two successful cases.

In the case M is Buchsbaum and $\operatorname{char}(k) = 0$, the conditions (2.1), (2.2) and (2.3) are necessary and sufficient for the existence of a homogeneous Buchsbaum ideal $I \in \mathfrak{I}(M, p)$ with $B_R(I) = (\bar{n}^1; \bar{n}^2; \cdots; \bar{n}^{r+1})$ (see [2, Section 3], [4, Sections 5 and 6]).

When $p = 2$, the situation is very clear. In fact, there is a unique integer σ_M and a unique sequence of integers $\bar{\beta}_M = (\beta_{M,1}, \ldots, \beta_{M,\lambda_M})$ which have the following properties, with only one minor exception in the case $\operatorname{rank}_R(M) = 1$. There exists a homogeneous ideal $I \in \mathfrak{I}(M, 2)$ with $B_R(I) = (\bar{n}^1; \bar{n}^2; \cdots; \bar{n}^{r+1})$ if and only if the conditions (2.1) and (2.3) hold, $c \geq \sigma_M + v$ for an integer $v \geq 0$, and $\bar{n}^2 = (\bar{w}, \bar{\beta}_M + c, \bar{\gamma}^2 + c)$ up to permutation, where $\bar{w} = (w_1, \ldots, w_v)$ is a sequence of integers (see [8, Theorems 2.4 and 2.5]). Note that $\bar{w}' = (\bar{w}, \bar{\beta}_M + c)$ up to permutation. This sequence $\bar{w}$ corresponds to the numerical function θ_X, defined by S. Nollet in [18] in the context of linkage theory of two-codimensional projective schemes, which plays an important role in the criterion of existence of prime ideals (see [3], [4, Section 7] also).

§3. Generic Gröbner basis, Weierstrass basis, and basic sequence

For the convenience of the readers, we describe here briefly the definitions of Weierstrass basis and basic sequence of a graded R-module. See [4, Section 1] and [6, Section 2] for the detail. Given a finitely generated graded R-module E, there exist a finitely generated graded $k[x_i, \ldots, x_r]$-submodule $E^{[i]} \subset E$ and a finitely generated graded free $k[x_i, \ldots, x_r]$-submodule $E^{\langle i \rangle} \subset E$ for each $i = 1, \ldots, r+1$ such that

(i) $E^{[1]} = E$, $E^{[r+1]} = E^{\langle r+1 \rangle}$,
(ii) $E^{[i]} = E^{\langle i \rangle} \oplus E^{[i+1]}$ as $k[x_{i+1}, \ldots, x_r]$-module and
(iii) $x_i E^{[i+1]} \subset (x_{i+1}, \ldots, x_r) E^{\langle i \rangle} \oplus E^{[i+1]}$

for all $i = 1, \dots, r$, if and only if

$$\mathfrak{m}^{t_i}(x_r, \dots, x_{i+1})E :_E x_i \subset (x_r, \dots, x_{i+1})E \quad \text{for some} \quad t_i \in \mathbf{N} \tag{3.1}$$

for all $i = 1, \dots, r$ (filter-regularity of $x_r, \dots, x_1$). When this is the case, the structures of $E^{\langle i \rangle}$ and $E^{[i]}$ are uniquely determined up to isomorphism over $k[x_i, \dots, x_r]$ for each $i = 1, \dots, r+1$ by the conditions (i), (ii) and (iii). If the submodules as above exist, denoting homogeneous free bases of $E^{\langle i \rangle}$ by e^i_l $(1 \le l \le m_i)$, we call $W := \{ e^i_l \mid 1 \le i \le r+1,\ 1 \le l \le m_i \}$ a weak Weierstrass basis of E. We may replace the condition (iii) with a stronger one, and in that strong formulation, we call W a Weierstrass basis of E. Now assume that $x_1, \dots, x_r$ are sufficiently general. Then the condition (3.1) must necessarily be satisfied and a Weierstrass basis of E always exists. We define the basic sequence $B_R(E)$ of E to be the sequence $(\bar{n}^1; \bar{n}^2; \cdots; \bar{n}^{r+1})$ made of the nondecreasing sequences of integers $\bar{n}^i$ $(1 \le i \le r+1)$ such that $\bar{n}^i = (\deg e^i_1, \dots, \deg e^i_{m_i})$ up to permutation. There are cases where $\bar{n}^i = \emptyset$. For instance

$$\operatorname{depth}_{\mathfrak{m}}(E) = r + 1 - \max\{ s \mid \bar{n}^s \neq \emptyset \}.$$

In the case E is a homogeneous ideal I and $x_1, \dots, x_r$ are sufficiently general, one can take as W a Gröbner basis with respect to reverse lexicographic order satisfying some additional conditions (see [6, Example 4.1]).

We also note that one can construct in a canonical way a free resolution of E starting with its Weierstrass basis, though it is not necessarily minimal (see [1, Section 2], [4, Section 3]).

§4. Examples obtained with the help of a cmputer

In this section let $R := \mathbf{Q}[v, w, x, y, z]$, $K_\bullet$ the Koszul complex of v, w, x, y, z with respect to R, $\mu_\bullet : K_\bullet \longrightarrow K_{\bullet-2}$, $C_\bullet$ its mapping cone and $(P_\bullet, \psi_\bullet)$ the minimal part of $C_\bullet$ (see [9, (1.1)]). Let further $N := \operatorname{Im}(\psi_2)$. Then N is a torsion-free graded quasi-Buchsbaum R-module such that $H^1_{\mathfrak{m}}(N) \cong R/\mathfrak{m}(-2)$, $H^2_{\mathfrak{m}}(N) \cong R/\mathfrak{m}$, and $H^i_{\mathfrak{m}}(N) = 0$ for $i \neq 1,\ 2,\ 5$. Although meaningless, we give here two examples of ideals in $\mathfrak{I}(N, 3)$ obtained by the method of the proof of our main theorem. The computation was done with the use of computer algebra system SINGULAR [13]. The module N in Example 4.1 and that in Example 4.2 are not isomorphic with each other.

Example 4.1. $\mu_2 = (0, 0, 0, 0, 1, 1, 1, 1, 1, 1)$. I is generated by

$$\begin{array}{lll} vw^2 - vwx, & -w^3 + 2w^2x - wx^2, & -w^3 + w^2x + v^2y, \\ -w^3 + w^2x + vwy, & -w^3 + w^2x + vxy, & -w^3 + w^2x + vy^2 - w^2z + wxz, \\ -vz + xz. & & \end{array}$$

Example 4.2. $\mu_2 = (0,0,1,1,1,1,1,1,1,1)$. I is generated by

$$\begin{array}{lll} x^3, & wx^2 - w^2y + v^2z, & vx^2, \\ w^2x - w^2y + v^2z - x^2z, & v^2x - w^2y - x^2y + v^2z, & w^3 - w^2y + v^2z, \\ vw^2 - w^2y + v^2z, & v^2w - w^2y + v^2z, & v^3 - w^2y + v^2z. \end{array}$$

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Tate-Vogel completions of functors

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1 Introduction

In the following R is always a commutative ring, and we denote the category of R-modules by $\mathcal{C}$. By a functor we mean an additve, R-linear, covariant and half-exact functor from $\mathcal{C}$ to itself, where a functor F is half-exact iff $F(A) \to F(B) \to F(C)$ is exact whenever $0 \to A \to B \to C \to 0$ is exact in $\mathcal{C}$. (E.g. $\mathrm{Ext}^i(M,\ )$, $\mathrm{Tor}_i(M,\ )$, $H^i_{\mathfrak{m}}$ and $\underline{\mathrm{Hom}}(M,\ )$ where $\underline{\mathrm{Hom}}(M, X)$ is defined to be the quotient module of $\mathrm{Hom}(M, X)$ by the submodule consisting of homomorphisms factoring through projectives.) For a given functor F, we may consider tow kind of homological variants of F. The first one is the derived functor: For an object $X \in \mathcal{C}$, the right derived functors R^nF and the left derived functors L_nF are defined respectively as the n-th homology of $F(I^\bullet_X)$ and $F(P^X_\bullet)$ where $I^\bullet_X$ (resp. $P^X_\bullet$) is an injective (resp. a projective) resolution of X. The second one is a more primitive one called satellite: For $X \in \mathcal{C}$, taking an n-th syzygy $0 \to \Omega_n(X) \to P^X_{n-1} \to \cdots \to P^X_0 \to X \to 0$, we define the left satellite $S_nF(X)$ as the kernel of $F(\Omega_n(X)) \to F(P^X_{n-1})$. Similarly, take the cosyzygy $0 \to X \to I^0_X \to \cdots \to I^{n-1}_X \to \Omega^n(X) \to 0$ and define the right satellite $S^nF(X)$ as the cokernel of $F(I^{n-1}_X) \to F(\Omega^n(X))$. It is easy to see that $S^{n+1}F = S^1(S^nF)$, $S_{n+1}F = S_1(S_nF)$ and they are half-exact again.

As shown in [1] there are natural transformations:

$$\begin{array}{l} F \to S_1S^1F \to \cdots \to S_nS^nF \to S_{n+1}S^{n+1}F \to \cdots, \\ F \leftarrow S^1S_1F \leftarrow \cdots \leftarrow S^nS_nF \leftarrow S^{n+1}S_{n+1}F \leftarrow \cdots. \end{array}$$

The following fact is one of our motivations to consider them.

Fact 1.1 *Let R be a Gorenstein local ring of dimension d and let M be a f.g. R-module. Then $S_nS^n\underline{\mathrm{Hom}}(M,\ )$ is stationary for $n \geqq d$ and is isomorphic to $\underline{\mathrm{Hom}}(X_M,\ )$, where X_M is the Cohen-Macaulay approximation of M.*

Naturally from this fact we are interested in the asymptotic properties of S_nS^nF and S^nS_nF.

Definition 1.2 Define $F^{\vee} = \varinjlim S_nS^nF$ and $F^{\wedge} = \varprojlim S_nS^nF$, and call them the Tate-Vogel completions of F.

Of course there are natural transformations $F \to F^{\vee}$ and $F^{\wedge} \to F$ which should be considered as generalizations of CM approximations.

2 Comparison with Tate-Vogel homologies

Let us denote the Tate-Vogel homologies by $\check{\mathrm{Tor}}_i(M,X)$ and $\check{\mathrm{Ext}}^i(M,X)$. See [2] for the Tate-Vogel homologies. Then one can easily see that there are natural transformations $\mathrm{Ext}^i(M,\ )^{\vee} \to \check{\mathrm{Ext}}^i(M,\ )$ and $\check{\mathrm{Tor}}_i(M,\ ) \to \mathrm{Tor}_i(M,\ )^{\wedge}$. It will be justified by the following theorems to call my functors the Tate-Vogel completions.

Theorem 2.1 *The natural transformation* $\mathrm{Ext}^i(M,\ )^{\vee} \to \check{\mathrm{Ext}}^i(M,\ )$ *is an isomorphism if* $i \geqq 0$. *(If* $i < 0$, *then this is not isomorphism.)*

Theorem 2.2 *Suppose that* R *is a Gorenstein ring or an Artinian ring. Then the natural transformation* $\check{\mathrm{Tor}}_i(M,\ ) \to \mathrm{Tor}_i(M,\ )^{\wedge}$ *is an isomorphism for* $i \geqq 0$.

As a general poperty of the Tata-Vogel completions, $F^{\vee}$ (resp. $F^{\wedge}$) annihilate modules of finite projective (resp. injective) dimension.

3 G-dimensions

The Tate-Voegel completions are easily computed in case that the functor has finite G-dimension.

Definition 3.1 [1] We say G-dim$F = 0$ if $L_0S^nF = R^0S_nF = 0$ for any $n \geqq 0$. And for an integer $n \geqq 1$, the right G-dimension rG-dim$F$ (resp. the left G-dimension ℓG-dim$F$) is equal to n iff G-dim$S^nF \neq 0$ and G-dim$S^{n+1}F = 0$ (resp. G-dim$S_nF \neq 0$ and G-dim$S_{n+1}F = 0$).

For an R-module M, G-dim$_RM$ is defined to be rG-dim $\underline{\mathrm{Hom}}(M,\ )$ which is known to equal ℓG-dim $\underline{\mathrm{Hom}}(\mathrm{tr}M,\ )$.

It is known that G-dim$_RM = 0$ iff M has a complete resolution.

Theorem 3.2 *The following two conditions are equivalent for a functor F.*
(1) G-dim$F = 0$,
(2) *For any $n \geqq 0$, the natural transformations $S^n F \to (S^n F)^\vee$ and $(S_n F)^\wedge \to S_n F$ are isomorphisms.*

If R is a Gorenstein ring, then the following condition is also equivalent to the above.
(3) *The natural transformations $F \to F^\vee$ and $F^\wedge \to F$ are isomorphisms.*

If the G-dimension of F is finite, one has the following

Theorem 3.3 *Suppose* rG-dim$F = n < \infty$, *then for any $i > n$ we have an isomorphism $F^\vee \cong S^i F \cdot \Omega_i$. Similarly, if* ℓG-dim$F = n < \infty$, *then $F^\wedge \cong S_i F \cdot \Omega^i$ for $i > n$.*

Using this theorem one can compute the Tate-Vogel completions of several functors. Particulary, if R is a Gorenstein ring, then the functors $\underline{\mathrm{Hom}}(M,\ ), \mathrm{Ext}^i(M,\ ), \mathrm{Tor}_i(M,\ )$ has finite left and right G-dimensions, and hence the theorem can be applied. Each formula obtained in this way will saggest us a different kind of approximation theory of modules over a Gorenstein ring.

4 ξ and η invariants

An advatage to consider the Tate-Vogel completions is that we can associate some invariants to a given functor.

Definition 4.1 Let $(R, \mathfrak{m}, k)$ be a local ring and let F be a functor. Then define $\xi(F)$ as the k-dimension of the kernel of the natural map $F(k) \to F^\vee(k)$. Likewise, $\eta(F)$ is the k-dimension of the cokernel of the natural map $F^\wedge(k) \to F(k)$.

The following is a mitivation of this definition.

Lemma 4.2 *If R is a Gorenstein local ring, then we have the following equalities for any f.g.R-module M and any $i \geqq 0$:*

$$\xi(\mathrm{Ext}^i(M,\ )) = \eta(\mathrm{Tor}_i(M,\ )) = \delta^i(M),$$

where the RHS is Auslander's higher delta invariant.

If M has finite projective dimension, then $\xi(\mathrm{Ext}^i(M,\ )) = \eta(\mathrm{Tor}_i(M,\ ))$ is just the i-th Betti number of M. On the other hand, one can show the following theorem which is essentially due to Martsinkovsky [4].

Theorem 4.3 (Auslander-Martsinkovsky vanishing) *Suppose* $(R, \mathfrak{m}, k)$ *is a non-regular local ring. Then we have* $\xi(\mathrm{Ext}^i(k,\ )) = \eta(\mathrm{Tor}_i(k,\ )) = 0$.

The following theorems are some of miscellaneous results in this direction.

Theorem 4.4 *Let* $(R, \mathfrak{m}, k)$ *be a local ring.*
(1) *For a f.g.R-module* M, *if* $\mathrm{grade}(M) > 0$ *(i.e.* $\mathrm{Hom}(M, R) = 0$*), then* $\xi(\mathrm{Tor}_i(M,\ )) = \eta(\mathrm{Ext}^i(M,\ )) = \beta_i(M)$. *(If* $\mathrm{grade}(M) = 0$, *then this is not true in general.)*
(2) *For* $0 \leqq i \leqq \dim R$, *we have* $\mathrm{rG\text{-}dim} H^i_{\mathfrak{m}} = \infty$, $(H^i_{\mathfrak{m}})^\vee = 0$ *and* $\xi(H^i_{\mathfrak{m}}) = \delta_{i0}$ *(Kronecker's delta).*
(3) *Assume that* R *is a CM ring. Then* $\ell\mathrm{G\text{-}dim} H^i_{\mathfrak{m}} = \infty$, $(H^i_{\mathfrak{m}})^\wedge = 0$ *and* $\eta(H^i_{\mathfrak{m}}) = \delta_{i0}$.

Questions 4.5 (1) *Does the set* $\{i \mid \xi(\mathrm{Ext}^i(M,\ )) \neq 0\}$ *have a finite bound? (Yes, if* R *is a Gorenstein local ring. In this case* $\dim R$ *is a bound for any* M.*)*
(2) *In the most cases,* $F^\wedge$ *is half-exact again. What condition leads the half-exactness of* $F^\wedge$ *?*
(3) *Related with* ξ *invariant, when is the natural morphism* $F \to F^\vee$ *a monomorphism ?*

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ON THE BUCHSBAUM GRADED RINGS ASSOCIATED TO IDEALS WITH $\ell_A(I^2/\mathfrak{q}I) = 1$

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ABSTRACT. Let I be an $\mathfrak{m}$-primary ideal in a Cohen-Macaulay local ring $(A, \mathfrak{m})$. In this paper, we investigate the Buchsbaum property of the associated graded ring of I. Because it is well known that $\mathrm{G}(I)$ is Cohen-Macaulay if $I^2 = \mathfrak{q}I$, we handle the ideals I that the length $\ell_A(I^2/\mathfrak{q}I) = 1$ for some minimal reduction $\mathfrak{q}$ of I, as one of the most simple cases that $I^2 \neq \mathfrak{q}I$.

1. INTRODUCTION.

Let I be an ideal of a Noetherian local ring $(A, \mathfrak{m})$. The associated graded ring $\mathrm{G}(I)$ of I is a graded A-algebra of the form $\mathrm{G}(I) = \oplus_{n\geq 0} I^n/I^{n+1}$. $\mathrm{G}(I)$ possesses the unique graded maximal ideal $M = \mathfrak{m}G(I) + G(I)_+$, where $G(I)_+ = \oplus_{n>0} I^n/I^{n+1}$. In this paper, we investigate the Buchsbaum property of the local ring $\mathrm{G}(I)_M$.

$\mathrm{G}(I)_M$ is not always a Buchsbaum ring, even if A is a Cohen-Macaulay and $I = \mathfrak{m}$. Besides, there are not so many results on its Buchsbaumness yet for even $\mathfrak{m}$-primary ideals in Cohen-Macaulay local rings. One can see recent results handling the Buchsbaumness in such a situation in [G]. In the article, we assume that A is a Cohen-Macaulay ring and I is an $\mathfrak{m}$-primary ideal. Suppose that there is a minimal reduction $\mathfrak{q}$ of I. It is well known that $\mathrm{G}(I)$ is Cohen-Macaulay if $I^2 = \mathfrak{q}I$ ([VV]). So, we need to think of ideals that I^2 does not coincide with $\mathfrak{q}I$. Which classes of ideals shall we see? We handle the ideals I that the length $\ell_A(I^2/\mathfrak{q}I)$ is exactly equal to 1, as one of the most simple cases that $I^2 \neq \mathfrak{q}I$. Note that this condition is independent of the choice of a minimal reduction ([V]).

The study of ideals I with $\ell_A(I^2/\mathfrak{q}I) = 1$ is begun by Sally ([S2]). It seems that she takes up this condition on her way researching the behavior of the coefficients of the Hilbert polynomial of I. She also proved in [S2] that the depth of $\mathrm{G}(I)$ is bigger than or equal to $\dim A - 1$ for such ideals I under an additional condition. In the case that $I = \mathfrak{m}$, Rossi-Valla ([RV]) and Wang ([W]) showed the same estimation with respect to the depth of $\mathrm{G}(\mathfrak{m})$ when $\ell_A(\mathfrak{m}^2/\mathfrak{q}\mathfrak{m}) = 1$. On the other hand, Goto ([G]) characterized the Buchsbaumness of $\mathrm{G}(I)_M$ in terms of the forms of local cohomology modules of $\mathrm{G}(I)$ and the reduction number of I with respect to

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$\mathfrak{q}$ for ideals I having minimal multiplicity, that is $\mathfrak{m}I = \mathfrak{m}\mathfrak{q}$, in a Cohen-Macaulay local ring. In view of the above observation, we try characterizing the Buchsbaum property of $\mathrm{G}(I)_M$ in terms of the local cohomology modules and the reduction number, and so on, for ideals with $\ell_A(I^2/\mathfrak{q}I) = 1$. Our main result is the following

Theorem (1.1). *Let $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field $A/\mathfrak{m}$. Let I be an $\mathfrak{m}$-primary ideal with $\ell_A(I^2/\mathfrak{q}I) = 1$ for a minimal reduction $\mathfrak{q}$ of I. Then the following conditions are equivalent:*

(1) *$\mathrm{G}(I)_M$ is a Buchsbaum ring.*
(2) *$\operatorname{depth} \mathrm{G}(I) \geq d-1$ and the $d-1$-th local cohomology module $\mathrm{H}^{d-1}_M(\mathrm{G}(I))$ is concentrated in only one homogeneous component.*

When this is the case, it holds that

(a) *the reduction number $\mathrm{r}_\mathfrak{q}(I)$ is at most 3,*
(b) *$\mathrm{H}^{d-1}_M(\mathrm{G}(I))$ is concentrated in degree $\mathrm{r}_\mathfrak{q}(I) - d - 1$, and*
(c) *the Buchsbaum invariant $\mathrm{I}(\mathrm{G}(I))$ is at most 1.*

We here give an explanation for some concepts appearing in the theorem. Let E be a finitely generated module over a Noetherian local ring A. We call E a Buchsbaum A-module if one of the following equivalent conditions is satisfied ([SV; Ch. I]).

(1) The difference $\ell_A(E/\mathfrak{q}E) - \mathrm{e}_\mathfrak{q}(E)$ is independent of the choice of a parameter ideal $\mathfrak{q}$ of E, where $\mathrm{e}_\mathfrak{q}(E)$ stands for the multiplicity of E with respect to $\mathfrak{q}$.
(2) Any system $a_1, a_2, ..., a_s$ of parameters of E forms a weak-sequence on E, i.e., the equality $(a_1, a_2, ..., a_{i-1})E : a_i = (a_1, a_2, ..., a_{i-1})E : \mathfrak{m}$ holds for all $1 \leq i \leq s$.
(3) Every system $a_1, a_2, ..., a_s$ of parameters of E forms a d-sequence on E ([H]), that is the equality $(a_1, ..., a_{i-1})E : a_i a_j = (a_1, ..., a_{i-1})E : a_j$ holds for all $1 \leq i \leq j \leq s$.

We say that A is a Buchsbaum ring if A is a Buchsbaum module over itself. We call the difference in (1) the Buchsbaum invariant of E and denote it by $\mathrm{I}(E)$. When E is Buchsbaum, E has finitely generated local cohomology modules, i.e., $\mathrm{H}^i_\mathfrak{m}(E)$ is a finitely generated A-module for $i < \dim E$. By [SV; Ch. I, (2.6)], $\mathrm{I}(E)$ is coincident with $\sum_{i=0}^{s-1} \binom{s-1}{i} \cdot \ell_A(\mathrm{H}^i_\mathfrak{m}(E))$ when E is Buchsbaum. Next we give some notations concerning with associated graded rings. Let I be an ideal of A and suppose that the field $A/\mathfrak{m}$ is infinite. An ideal $J \subseteq I$ is called a reduction of I, if $I^{n+1} = JI^n$ for some integer $n > 0$. If J is a minimal ideal among all of the reductions of I, then J is called a minimal reduction of I. When I is an $\mathfrak{m}$-primary ideal of d-dimensional local ring A, any minimal reduction of I is minimally generated by d-elements ([NR]). When J is a minimal reduction of I, we put $\mathrm{r}_J(I) = \min\{n \in \mathbb{Z} \mid I^{n+1} = JI^n\}$ and call it the reduction number of I with respect to J.

We state a brief orientation for this paper. In Section 2, we give the proof of Theorem (1.1). We there show that firstly assertion (1) in the theorem implies assertion (a), secondly (1) leads that the depth $\mathrm{G}(I) \geq \dim A - 1$ using the fact of (a). Then, we can reduce all of the situations to the case of dimension 1. Then, we show the equivalence between (1) and (2), and that both (b) and (c) follow under the case of dimension 1. When $\dim A = 1$, as can be seen in (2.6), it is not so hard to check the Buchsbaumness for certain classes of ideals. But, in higher dimensional cases, that may be a little quite hard even for the maximal ideal, because it is not easy to check that a homogeneous s.o.p. of $\mathrm{G}(I)$ forms a d^+-sequence on $\mathrm{G}(I)$ (see (2.7)). In section 3, we argue the case where $\dim A = 2$ and $I = \mathfrak{m}$. We shall there see a behavior of ideals in this situation.

Before closing this section, the author gratefully thanks Professor Shiro Goto for his many suggestions and encouragement throughout this research.

Throughout this paper, let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d. For a finitely generated A-module E we denote by $\ell_A(E)$, $\mathrm{e}_{\mathfrak{a}}(E)$ and $\mathrm{H}^i_{\mathfrak{m}}(E)$, respectively, the length of E, the multiplicity of E with respect to an ideal $\mathfrak{a}$ when $\ell_A(E/\mathfrak{a}E)$ is finite, and the i-th local cohomology module of E with respect to $\mathfrak{m}$.

2. Proof of Theorem (1.1).

Throughout this section, let $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field $A/\mathfrak{m}$. Let I be an $\mathfrak{m}$-primary ideal and $\mathfrak{q}$ a minimal reduction of I. We fix a system $a_1, a_2, ..., a_d$ of generators of $\mathfrak{q}$. The Rees algebra $\mathrm{R}(I)$ of I is a subring of a polynomial ring $A[t]$ of the form $\mathrm{R}(I) = A[It]$. We put $f_i = a_i t$ $(1 \le i \le d)$, which are elements in $\mathrm{R}(I)$. We can identify $\mathrm{G}(I)$ with the factor ring $\mathrm{R}(I)/I\mathrm{R}(I)$ and then $f_1, f_2, ..., f_d$ forms a homogeneous system of parameters for $\mathrm{G}(I)$ as a graded $\mathrm{R}(I)$-module. We put $\mathrm{R} = \mathrm{R}(I)$, $\mathrm{G} = \mathrm{G}(I)$ for simplicity, and denote the graded maximal ideal of R by M.

Because the equality $\ell_A(A/I^2) = \mathrm{e}_I(A) + d\cdot\ell_A(A/I) - \ell_A(I^2/\mathfrak{q}I)$ always holds (see [V, 1.1]), the length of $I^2/\mathfrak{q}I$ is independent of the choice of the minimal reduction $\mathfrak{q}$. We assume that $\ell_A(I^2/\mathfrak{q}I) = 1$ throughout the section. We put $r = \mathrm{r}_{\mathfrak{q}}(I)$. In our context, r must be bigger than 1.

The following lemma is originally given by Rossi-Valla ([RV]). Although they show similar assertions for only the maximal ideal, their proof is available to any $\mathfrak{m}$-primary ideal. For convenience, we give its brief proof below.

Lemma (2.1) (cf. [RV, 1.2]). *The following conditions hold.*

(1) *Either $I^3 = \mathfrak{q}I^2$ (i.e. $r = 2$) or there exists $c \in I$ such that $I^{j+1} = \mathfrak{q}I^j + (c^{j+1})$ for every $j \ge 1$.*

(2) $\ell_A(I^{j+1}/\mathfrak{q}I^j) = \begin{cases} 1 & \text{if } 1 \le j < r \\ 0 & \text{if } j \ge r \end{cases}$

(3) *$I^{j+2} \subseteq \mathfrak{q}I^j$ for every $j \ge 1$*

Proof. (1): We can choose $c, d \in I$ satisfying the equality $I^2 = \mathfrak{q}I + (cd)$. Here, c might be equal to d. If $c^2 \in \mathfrak{q}I$, then we have

$$I^3 = \mathfrak{q}I^2 + cdI \subseteq \mathfrak{q}I^2 + cI^2 = \mathfrak{q}I^2 + (c^2 d) \subseteq \mathfrak{q}I^2 + c^2 I = \mathfrak{q}I^2.$$

If $c^2 \notin \mathfrak{q}I$, then it holds that $I^2 = \mathfrak{q}I + (c^2)$. When this is the case, by induction on j, we get $I^{j+1} = \mathfrak{q}I^j + (c^{j+1})$ for every $j \ge 1$.

(2) and (3): These statements readily follows from (1). (Note that $\mathfrak{m}I^{j+1} \subseteq \mathfrak{q}I^j$ for all $j \ge 1$ because $\mathfrak{m}I^2 \subseteq \mathfrak{q}I$.)

For an element $\alpha \in \mathrm{R}$, let $\overline{\alpha}$ stand for the reduction $\alpha \mod I\mathrm{R}$. So, we can regard $\overline{\alpha}$ as a element of G.

Proposition (2.2). *If G_M is a Buchsbaum ring, then $r \leq 3$.*

Proof. Supposing $r \geq 4$, we shall lead a contradiction. Take $c \in I$ as the above lemma. We firstly show the following

Claim. $c^r \in \mathfrak{q}^2 I^{r-3}$.

Proof of Claim. By (2.1) (3), c^r is in $\mathfrak{q}I^{r-2}$. Since $r-2 \geq 2$, it follows from (2.1) (1) that

$$c^r \in \mathfrak{q}(\mathfrak{q}I^{r-3} + (c^{r-2})) = \mathfrak{q}^2 I^{r-3} + (a_1 c^{r-2}, a_2 c^{r-2}, \ldots, a_d c^{r-2}).$$

Suppose that $c^r \notin \mathfrak{q}^2 I^{r-3}$. Then for some $a_i c^{r-2}$, it holds that

$$a_i c^{r-2} \in \mathfrak{q}^2 I^{r-3} + (a_1 c^{r-2}, \ldots, \widehat{a_i c^{r-2}}, \ldots, a_d c^{r-2}) + (c^r).$$

Note here that $\mathfrak{q}I^{r-2}/\mathfrak{q}^2 I^{r-3}$ is a vector space by (2.1) (2). We may assume $i = d$ without loss of generality. Multiplying c to the above, we have $a_d c^{r-1} \in \mathfrak{q}^2 I^{r-2} + (a_1, \ldots, a_{d-1})c^{r-1}$, since $c^{r+1} \in \mathfrak{q}I^r$ by definition of r and since $\mathfrak{q}I^r \subseteq \mathfrak{q}^2 I^{r-2}$ by (2.1) (3). From a simple calculation it follows that $a_d c^{r-1} \in a_d^2 I^{r-2} + (a_1, \ldots, a_{d-1})I^{r-1}$. Multiplying the indeterminate t, putting $g = ct$, and considering them in R, we have $f_d \cdot g^{r-1} \in (f_1, \ldots, f_{d-1}, f_d^2)R$. Then there exists $h = xt^{r-2}$, with $x \in I^{r-2}$ such that $f_d \cdot g^{r-1} - f_d^2 \cdot h \in (f_1, \ldots, f_{d-1})R$. The Buchsbaum property of the local ring G_M yields that

$$\overline{g^{r-1} - f_d \cdot h} \in (f_1, \ldots, f_{d-1})\mathrm{G} :_{\mathrm{G}} f_d = (f_1, \ldots, f_{d-1})\mathrm{G} :_{\mathrm{G}} M$$

because $f_1, f_2, \ldots, f_d$ forms a homogeneous weak-sequence for G as an R-module, In particular, $g(\overline{g^{r-1} - f_d \cdot h}) \in (f_1, \ldots, f_{d-1})\mathrm{G}$. Hence $g^r \in (f_1, f_2, \ldots, f_d)\mathrm{G}$. This implies that $c^r \in \mathfrak{q}I^{r-1} + I^{r+1} = \mathfrak{q}I^{r-1}$ and that $I^r = \mathfrak{q}I^{r-1}$ by (2.1)(1). This contradicts to the definition of r and Claim follows.

By Claim, we have an expression $c^r = \sum_{i \leq j} c_{ij} a_i a_j$ with $c_{ij} \in I^{r-3}$. This yields the equation in G of the form $\sum_{i \leq j} \overline{c_{ij} t^{r-3}} f_i f_j = 0$. We put $c_{ij}^* = \overline{c_{ij} t^{r-3}}$. Because $f_1, f_2, \ldots, f_d$ is a d-sequence on G, a relation $\sum_{i \leq j} c_{ij}^* T_i T_j$ with respect to $f_1, f_2, \ldots, f_d$ in a polynomial extension $\mathrm{G}[T_1, T_2, \ldots, T_d]$ can be expressed by linear relations. In other words, there exist, in the polynomial extension, linear forms $\xi_{k1} T_1 + \xi_{k2} T_2 + \ldots + \xi_{kd} T_d$ $(1 \leq k \leq l)$ with $\xi_{k1} f_1 + \xi_{k2} f_2 + \ldots + \xi_{kd} f_d = 0$ and linear forms $\eta_{k1} T_1 + \eta_{k2} T_2 + \ldots + \eta_{kd} T_d$ $(1 \leq k \leq l)$ such that

$$\sum_{i \leq j} c_{ij}^* T_i T_j = \sum_{k=1}^{l} (\xi_{k1} T_1 + \xi_{k2} T_2 + \ldots + \xi_{kd} T_d)(\eta_{k1} T_1 + \eta_{k2} T_2 + \ldots + \eta_{kd} T_d).$$

Let $\xi_{ki}^{(p)}$ (resp. $\eta_{ki}^{(p)}$) denote the p-th homogeneous component of ξ_{ki} (resp. η_{ki}) in G. Let $u_{ki}^{(p)} \in I^p$ (resp. $v_{ki}^{(p)} \in I^p$) be a representative of $\xi_{ki}^{(p)}$ (resp. $\eta_{ki}^{(p)}$) in $[\mathrm{G}]_p \cong I^p/I^{p+1}$. Then, we have

$$c_{ii}^* = \sum_{k=1}^{l} \xi_{ki} \eta_{ki} = \sum_{k=1}^{l} \sum_{p+q=r-3} \xi_{ki}^{(p)} \eta_{ki}^{(q)} \quad \text{and}$$

$$c_{ij}^* = \sum_{k=1}^{l} (\xi_{ki} \eta_{kj} + \xi_{kj} \eta_{ki}) = \sum_{k=1}^{l} \sum_{p+q=r-3} (\xi_{ki}^{(p)} \eta_{kj}^{(q)} + \xi_{kj}^{(p)} \eta_{ki}^{(q)}),$$

where $i < j$. These yield that

$$c_{ii}-\sum_{k=1}^{l}\sum_{p+q=r-3} u_{ki}^{(p)}v_{ki}^{(q)} \in I^{r-2} \qquad \text{and}$$

$$c_{ij}-\sum_{k=1}^{l}\sum_{p+q=r-3} (u_{ki}^{(p)}v_{kj}^{(q)} + u_{kj}^{(p)}v_{ki}^{(q)}) \in I^{r-2}.$$

We put $d_{ii} = c_{ii} - \sum_{k=1}^{l}\sum_{p+q=r-3} u_{ki}^{(p)}v_{ki}^{(q)}$ for $1 \leq i \leq d$, and $d_{ij} = c_{ij} - \sum_{k=1}^{l}\sum_{p+q=r-3}(u_{ki}^{(p)}v_{kj}^{(q)} + u_{kj}^{(p)}v_{ki}^{(q)})$ for $1 \leq i < j \leq d$. Then, we obtain

$$\begin{aligned}
c^r &= \sum_{i\leq j} c_{ij}a_ia_j = \sum_{i=1}^{d} c_{ii}a_i^2 + \sum_{i<j} c_{ij}a_ia_j \\
&= \sum_{i=1}^{d} d_{ii}a_i^2 + \sum_{i=1}^{d}\sum_{k=1}^{l}\sum_{p+q=r-3} (u_{ki}^{(p)}a_i)(v_{ki}^{(q)}a_i) + \sum_{i<j} d_{ij}a_ia_j \\
&\qquad + \sum_{i<j}\sum_{k=1}^{l}\sum_{p+q=r-3} ((u_{ki}^{(p)}a_i)(v_{kj}^{(q)}a_j) + (u_{kj}^{(p)}a_j)(v_{ki}^{(q)}a_i)) \\
&= \sum_{i\leq j} d_{ij}a_ia_j + \sum_{k=1}^{l}\sum_{p+q=r-3} (u_{k1}^{(p)}a_1 + \dots + u_{kd}^{(p)}a_d)(v_{k1}^{(q)}a_1 + \dots + v_{kd}^{(q)}a_d).
\end{aligned}$$

Because $\xi_{k1}^{(p)}f_1+\dots+\xi_{kd}^{(p)}f_d = 0$ in G, it follows that $u_{k1}^{(p)}a_1+\dots+u_{kd}^{(p)}a_d \in I^{p+2}$, while we have $v_{k1}^{(q)}a_1+\dots+v_{kd}^{(q)}a_d \in \mathfrak{q}I^q$. Thus, c^r must be in $\mathfrak{q}I^{r-1}$ since $d_{ij}a_ia_j \in \mathfrak{q}^2I^{r-2}$ and $p+q=r-3$, which is a contradiction. This completes the proof.

For a graded R-module L, we put $a_j(L) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}_M^j(L)]_n \neq (0)\}$. In particular, $a_{\dim L}(L)$ is called the a-invariant of L, which is defined in [GW]. Let $Q = (f_1, f_2, \dots, f_d)\mathrm{G}$. This is a homogeneous parameter ideal of G. Besides, we put $\mathfrak{q}_j = (a_1, a_2, \dots, a_j)A$ and $Q_j = (f_1, f_2, \dots, f_j)\mathrm{G}$ for each $0 \leq j \leq d$. We call f_1, $f_2,\dots,f_d$ a d^+-sequence on G when $f_1^{n_1}, f_2^{n_2}, \dots, f_d^{n_d}$ is a d-sequence in any order and for all integers $n_1, n_2, \dots, n_d > 0$. One can get estimations $a_{j-k}(\mathrm{G}/Q_{d-j}) \leq r-j+k$ with $0 \leq k \leq j$ from a general theory of d^+-sequence. However, in our context, we have a little bit better one as follows, which is crucial in the sequel.

Lemma (2.3). *Suppose that f_1, $f_2,\dots,f_d$ forms a d^+-sequence on* G. *Then for each $0 \leq j \leq d$, we have*

(1) $a_j(\mathrm{G}/Q_{d-j}) = r - j$,
(2) $[\mathrm{H}_M^j(\mathrm{G}/Q_{d-j})]_{r-j} \cong A/\mathfrak{m}$,
(3) $a_{j-k}(\mathrm{G}/Q_{d-j}) \leq r - j - 1 + k$ *for each* $1 \leq k \leq j$.

Proof. Induction on j. The highest degree of G/Q_d is r, and the r-th component is isomorphic to $I^r/\mathfrak{q}I^{r-1}$, whose length is equal to 1 by (2.1)(2). Thus, the assertion is true when $j = 0$. Suppose that $j > 0$ and that it is true for $j-1$. Consider the exact sequence:

$$\mathrm{G}/Q_{d-j}(-1) \xrightarrow{f_{d-j+1}} \mathrm{G}/Q_{d-j} \longrightarrow \mathrm{G}/Q_{d-j+1} \longrightarrow 0.$$

It yields that the long exact sequence of local cohomology modules:

$$\begin{aligned}0 \longrightarrow & \mathrm{H}_M^{j-k}(\mathrm{G}/Q_{d-j}) \longrightarrow \mathrm{H}_M^{j-k}(\mathrm{G}/Q_{d-j+1}) \longrightarrow \\ & \mathrm{H}_M^{j-k+1}(\mathrm{G}/Q_{d-j})(-1) \xrightarrow{\alpha} \mathrm{H}_M^{j-k+1}(\mathrm{G}/Q_{d-j}).\end{aligned}$$

for each $1 \le k \le j$ because $f_{d-j+1} \cdot \mathrm{H}_M^{j-k}(\mathrm{G}/Q_{d-j}) = (0)$ (cf. [GY, 6.18]). If $k \ge 2$, then $a_{j-k}(\mathrm{G}/Q_{d-j+1}) \le r-j-1+k$ by hypothesis of induction. Thus, we get $a_{j-k}(\mathrm{G}/Q_{d-j}) \le r-j-1+k$. This implies that assertion (3) is true for $k \ge 2$. Assume that $k=1$. Since α is surjective map and since $a_{j-1}(\mathrm{G}/Q_{d-j+1}) = r-j+1$ by hypothesis of induction, we get $a_j(\mathrm{G}/Q_{d-j}) = r-j$. Hence (1) follows. Furthermore, the isomorphism $[\mathrm{H}_M^{j-1}(\mathrm{G}/Q_{d-j+1})]_{r-j+1} \cong A/\mathfrak{m}$, which also follows from the hypothesis of induction, yields that $[\mathrm{H}_M^{j}(\mathrm{G}/Q_{d-j})]_{r-j} \cong A/\mathfrak{m}$. Hence (2) follows. Besides, we have $[\mathrm{H}_M^{j-1}(\mathrm{G}/Q_{d-j})]_n = (0)$ for $n \ge r-j+1$. Thus, $a_{j-1}(\mathrm{G}/Q_{d-j}) \le r-j$. Consequently, (3) follows too when $k=1$.

Proposition (2.4). *Suppose that* G_M *is a Buchsbaum ring. Then* $\operatorname{depth}\mathrm{G} \ge d-1$.

Proof. Induction on d. We have nothing to say if $d=1$. Suppose that $d>1$ and that it is true for $d-1$. Take a generating system $a_1,a_2,\ldots,a_d$ of $\mathfrak{q}$ so that a_1 should be a superficial element for I. Let $\overline{A} = A/(a_1)$, $\overline{I} = I\overline{A}$, and $\overline{\mathfrak{q}} = \mathfrak{q}\overline{A}$. Note that $\ell_{\overline{A}}(\overline{I}^2/\overline{\mathfrak{q}}\overline{I}) \le 1$. If $\overline{I}^2 = \overline{\mathfrak{q}}\overline{I}$, then $\mathrm{G}(\overline{I})$ is a $d-1$ dimensional Cohen-Macaulay ring. In particular, $\operatorname{depth}\mathrm{G}(\overline{I}) > 0$. This leads that $\operatorname{depth}\mathrm{G} = \operatorname{depth}\mathrm{G}(\overline{I}) + 1 = d$ by [HM, 2.2]. So, we may assume that $\ell_{\overline{A}}(\overline{I}^2/\overline{\mathfrak{q}}\overline{I}) = 1$. Then $(a_1) \cap I^2 \subseteq \mathfrak{q}I$ since $\overline{I}^2/\overline{\mathfrak{q}}\overline{I} \cong I^2/\mathfrak{q}I + (a_1) \cap I^2$. A simple argument for a regular sequence leads that $(a_1) \cap I^2 = a_1 I$. Let L be the kernel of the canonical surjection $\mathrm{G}/(f_1) \to \mathrm{G}(\overline{I})$. L is a graded R-module of finite length, whose n-th homogeneous component, say L_n, is isomorphic to $((a_1) \cap I^n + I^{n+1})/(a_1 I^{n-1} + I^{n+1})$. Now, we have a injection $L \to \mathrm{H}_M^0(\mathrm{G}/(f_1))$, while it follows that $L_0 = L_1 = L_2 = (0)$. Furthermore, by (2.3)(3) it holds that $a_0(\mathrm{G}/(f_1)) \le r-1 \le 2$ since $r \le 3$ by (2.2). This implies that $L=(0)$. Hence f_1 is a regular element for G. By hypothesis of induction, $\operatorname{depth}\mathrm{G}(\overline{I}) \ge d-2$, so we conclude that $\operatorname{depth}\mathrm{G} \ge d-1$, as required.

We here argue the case of dimension 1 because the proof of Theorem (1.1) is showed after reducing to such a case.

Lemma (2.5). *Let $d=1$. Then, the following conditions are equivalent:*

(1) G_M *is a Buchsbaum ring.*
(2) $\mathrm{H}_M^0(\mathrm{G})$ *is concentrated in only one homogeneous component.*

When this is the case, $\mathrm{H}_M^0(\mathrm{G})$ *is concentrated in degree* $r-2$ *and* $\mathrm{I}(\mathrm{G}) \le 1$.

Proof. Let $\mathfrak{q} = (a)$. Note that G_M is a Buchsbaum ring if and only if $M \cdot \mathrm{H}_M^0(\mathrm{G}) = (0)$ ([SV, Ch. I (2.12)]).
(2) $\Longrightarrow$ (1): Suppose that $\mathrm{H}_M^0(\mathrm{G})$ is concentrated in degree n and take $\overline{xt^n} \in [\mathrm{H}_M^0(\mathrm{G})]_n$. Then, we have $ax \in I^{n+2}$ since $at \cdot \overline{xt^n} = 0$ in G, while $\mathfrak{m}I^{n+2} \subseteq aI^{n+1}$ by (2.1)(2), we thus get $\mathfrak{m}ax \subseteq aI^{n+1}$ and $\mathfrak{m}x \subseteq I^{n+1}$ since a is a non-zero-divisor. Hence, we have $\mathfrak{m} \cdot [\mathrm{H}_M^0(\mathrm{G})]_n = (0)$. Consequently, we obtain that $M \cdot \mathrm{H}_M^0(\mathrm{G}) = (0)$ because it obviously holds that $It \cdot \mathrm{H}_M^0(\mathrm{G}) = (0)$. Assertion (1) follows.

(1) $\Longrightarrow$ (2): We first show that $[\mathrm{H}_M^0(\mathrm{G})]_n = (0)$ for all $n \geq r-1$. Indeed, if $\overline{xt^n} \in [\mathrm{H}_M^0(\mathrm{G})]_n$, then as can be seen above we get $ax \in I^{n+2}$. For any $n \geq r-1$ we have $ax \in aI^{n+1}$, hence $x \in I^{n+1}$. This implies that $\overline{xt^n} = 0$ in G. When $r=2$, $\mathrm{H}_M^0(\mathrm{G})$ is necessarily concentrated in degree 0. So, the remaining is the case where $r=3$ by (2.2). We shall show that $\mathrm{H}_M^0(\mathrm{G})$ is concentrated in degree 1. To see it, it is enough to verify that $[\mathrm{H}_M^0(\mathrm{G})]_0 = (0)$. Suppose that $r=3$ and take an element $c \in I$ as can be chosen in (2.1)(1). Suppose that there exists $z \in A$ such that $0 \neq \overline{z} \in [\mathrm{H}_M^0(\mathrm{G})]_0$. Then $az \in I^2 = aI + (c^2)$. Taking a suitable representative of $\overline{z}$, we may assume that $az = wc^2$ for some $w \in A$. If $w \in \mathfrak{m}$, then $az \in aI$ since $wc^2 \in aI$ by (2.1)(2). Thus $w \notin \mathfrak{m}$, while $cz \in I^2$ since $ct \cdot \overline{z} = 0$. Hence, we get $c^3 = (1/w)acz \in aI^2$, and the equality $I^3 = aI^2$ follows. This is a contradiction.

We finally show that $\mathrm{I}(\mathrm{G}) \leq 1$. When $r=2$, the length of $[\mathrm{H}_M^0(\mathrm{G})]_0$ is at most 1 because there is a embedding $[\mathrm{H}_M^0(\mathrm{G})]_0 \to I^2 : a/I$ and there is a isomorphism $I^2 : a/I \cong I^2/aI$. When $r=3$, we also have $\ell_A([\mathrm{H}_M^0(\mathrm{G})]_1) \leq 1$ because $[\mathrm{H}_M^0(\mathrm{G})]_1$ can be embedded in $I^3 : a/I^2$ and there is an isomorphism $I^3 : a/I^2 \cong I^3/aI^2$, which is a module of length 1 by (2.1)(2). In any case, we obtain $\mathrm{I}(\mathrm{G}) \leq 1$.

From the above, the following corollary immediately follows.

Corollary (2.6). *Suppose $d=1$. Then*

(1) *if $r=2$, then G is Buchsbaum,*
(2) *if $I=\mathfrak{m}$, then G is Buchsbaum if and only if $r \leq 3$.*

Now, we give the proof of the theorem.

Proof of (1.1). Suppose that $\operatorname{depth} \mathrm{G} \geq d-1$ and that $\mathrm{H}_M^{d-1}(\mathrm{G})$ is concentrated in degree n for some $n \in \mathbb{Z}$. Then one can check, by induction on j, that $\operatorname{depth} \mathrm{G}/Q_j \geq d-j-1$ and $\mathrm{H}_M^{d-j-1}(\mathrm{G}/Q_j)$ is concentrated in degree $n+j$ for $0 \leq j \leq d-1$. Thus, the sequence f_1, ..., f_d of elements satisfies that $f_{j+1} \cdot \mathrm{H}_M^i(\mathrm{G}/Q_j) = (0)$ for $i+j<d$, which implies that f_1, ..., f_d forms a d^+-sequence on G (cf. [GY, 6.18]). Therefore, assuming each of (1) and (2) in the theorem guarantees that f_1, ..., f_d forms a d^+-sequence and $\operatorname{depth} \mathrm{G} \geq d-1$ (see (2.4)). Thus, f_1, ...,f_{d-1} is a G-regular sequence and we have an isomorphism $\mathrm{H}_M^{d-1}(\mathrm{G}) \cong \mathrm{H}_M^0(\mathrm{G}(\overline{I}))(1-d)$ as graded R-modules, where $\overline{I} = I/(a_1, ..., a_{d-1})$. Hence, to show the theorem, we can reduce the situation to the case of dimension 1. Then, the proof immediately follows from (2.5).

Corollary (2.7).

(1) *Suppose that $r=2$. Then G is a Buchsbaum ring if and only if f_1, ..., f_d forms a d^+-sequence on G.*
(2) *Suppose that $I=\mathfrak{m}$. Then G is a Buchsbaum ring if and only if $r \leq 3$ and f_1, ...,f_d forms a d^+-sequence on G.*

Proof. It is guaranteed that $\operatorname{depth} \mathrm{G} \geq d-1$ by [S2, (2.3)] if $r=2$. The same is guaranteed by [RV, (2.1)] or [W, (3.1)] if $I=\mathfrak{m}$. Thus, together with the fact that f_1, ...,f_d forms a d^+-sequence on G, we obtain that f_1, ...,f_{d-1} is a G-regular sequence and $\mathrm{H}_M^{d-1}(\mathrm{G}) \cong \mathrm{H}_M^0(\mathrm{G}(\overline{I}))(1-d)$, where $\overline{I} = I/(a_1, ..., a_{d-1})$. This implies that it is enough to give the proof in the case of dimension 1. That follows from (2.6).

Before closing this section, we give simple examples. Let k be an infinite field.

Example (2.8). Let $A = k[[x^4, x^5, x^6, x^7]]$ be a subring of a formal power series ring $k[[x]]$. For $n \geq 4$, let $I = (x^n, x^{n+1})$ and $J = (x^n, x^{n+1}, x^{n+2})$. Then $\mathrm{G}(J)_M$ is a Buchsbaum ring, but not so is $\mathrm{G}(I)_M$.

Proof. Let $\mathfrak{q} = (x^n)$. It is easy to check that $\mathfrak{q}$ is a minimal reduction for both I and J, $\mathrm{r}_\mathfrak{q}(I) = 3$ and $\mathrm{r}_\mathfrak{q}(J) = 2$. The Buchsbaumness of $\mathrm{G}(J)_M$ follows from (2.5). Now, $\overline{x^{n+2}} \in \mathrm{G}(I)$ belongs to $[\mathrm{H}^0_M(\mathrm{G}(I))]_0$ because $x^n \cdot x^{n+2} \in I^2$, while $\overline{x^{n+2}} \neq 0$ in $\mathrm{G}(I)$ since $x^{n+2} \notin I$. This implies that $[\mathrm{H}^0_M \mathrm{G}(I)]_0 \neq (0)$ and it contradicts to the Buchsbaumness of $\mathrm{G}(I)_M$ by (1.1)(b).

Example (2.9) (cf. [S1, 2.2]**).** Let $A = k[[x^4, x^5, x^7]]$ and $\mathfrak{q} = (x^4)$. Then $\mathfrak{q}$ is a minimal reduction of the maximal ideal $\mathfrak{m} = (x^4, x^5, x^7)$ of A. It is easy to check that $\mathrm{r}_\mathfrak{q}(\mathfrak{m}) = 3$. Thus $\mathrm{G}(\mathfrak{m})_M$ is a Buchsbaum ring by (2.5)

3. The case where $\dim A = 2$ and $I = \mathfrak{m}$.

We have seen examples in 1-dimensional cases in the previous section. To see higher dimensional cases, it is a little bit quite hard to check whether $f_1, \ldots, f_d$ forms a d^+-sequence on G. In this section, we assume that $\dim A = 2$ and $I = \mathfrak{m}$ as one of the most simple cases among higher dimensional cases. We shall see when f_1, f_2 is a d^+-sequence on $\mathrm{G} = \mathrm{G}(\mathfrak{m})$ and when G_M is a Buchsbaum ring in this situation. Let $(A, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring with infinite residue field and $\mathfrak{q}$ a minimal reduction of $\mathfrak{m}$. For a homogeneous s.o.p. α, β of G, we put $\mathrm{I}(\alpha, \beta; \mathrm{G}) = \ell_\mathrm{G}(\mathrm{G}/(\alpha, \beta)) - \mathrm{e}_{(\alpha,\beta)}(\mathrm{G})$. By virtue of [T; 2.1] and [GY; 6.18], it follows that α, β is a d^+-sequence if and only if the equality $\mathrm{I}(\alpha, \beta; \mathrm{G}) = \mathrm{I}(\alpha^2, \beta^2; \mathrm{G})$ holds.

Lemma (3.1). *Suppose that* $\mathfrak{m}^4 = \mathfrak{q}\mathfrak{m}$, $\ell_A(\mathfrak{m}^2/\mathfrak{q}\mathfrak{m}) = 1$, *and that there exists* $c \in \mathfrak{m}$ *such that* $\mathfrak{m}^2 = \mathfrak{q}\mathfrak{m} + (c^2)$. *Then, for any system* a, b *of generating set of* $\mathfrak{q}$, *the following conditions hold.*

(1) $\mathfrak{m}^n \cap (a^2, b^2) = (a^2, b^2)\mathfrak{m}^{n-2}$ *if either* $n \leq 3$ *or* $n \geq 6$.
(2) $\mathfrak{m}^5 = (a^2, b^2)\mathfrak{m}^3 + (abc^3)$.
(3) G_M *is a Buchsbaum ring if and only if* $\mathfrak{m}^4 \cap (a^2, b^2) = (a^2, b^2)\mathfrak{m}^2$.

Proof. (1): Let $\mathfrak{a} = (a^2, b^2)$. The assertion obviously holds for $n \leq 2$. If $n \geq 6$, it also follows since $\mathfrak{m}^n = \mathfrak{q}^3\mathfrak{m}^{n-3} = \mathfrak{a}\mathfrak{m}^{n-2}$. The remaining is the case that $n = 3$. Let $f = at$ and $g = bt$, which are elements in R. Take $\varphi \in \mathfrak{m}^3 \cap \mathfrak{a}$ and write $\varphi = xa^2 + yb^2$ with $x, y \in A$. Then φ yields a relation $\overline{x}f^2 + \overline{y}g^2 = 0$ in G. Thus $\overline{x} = \overline{y} = 0$ since f^2, g^2 forms a homogeneous s.o.p. for G. Hence $x, y \in \mathfrak{m}$ and we get $\varphi \in \mathfrak{a}\mathfrak{m}$.
(2): $\mathfrak{m}^5 = \mathfrak{q}^2\mathfrak{m}^3 = \mathfrak{q}^2(\mathfrak{q}\mathfrak{m}^2 + (c^3)) = \mathfrak{a}\mathfrak{m}^3 + (abc^3)$.
(3): ("only if" part) Take $\varphi \in \mathfrak{m}^4 \cap \mathfrak{a}$ and write $\varphi = xa^2 + yb^2$. As can be seen above, $x, y \in \mathfrak{m}$, so φ yields a relation $\overline{xt}f^2 + \overline{yt}g^2 = 0$ in G. Then it follows that

$$\overline{xt} \in (g^2\mathrm{G} :_\mathrm{G} f^2) = g(g\mathrm{G} :_\mathrm{G} f) + ((0) :_\mathrm{G} f)$$

from Goto's Lemma ([SV; p.139]). Now, we have $(0) :_\mathrm{G} f = (0)$ by (2.4) since $(0) :_\mathrm{G} f$ is of finite length, which follows from the Buchsbaumness of G_M, furthermore,

it obviously holds that $[g\mathrm{G} :_{\mathrm{G}} f]_0 = (0)$. Hence, we obtain $\overline{xt} = 0$ and $x \in \mathfrak{m}^2$. Similarly, $y \in \mathfrak{m}^2$ also follows.
("if" part) We shall show that f, g forms a d^+-sequence on G. To see it, it is enough to check that $\mathrm{I}(f^2, g^2; \mathrm{G}) \leq 1$ because the inequality $\mathrm{I}(f, g; \mathrm{G}) \leq \mathrm{I}(f^2, g^2; \mathrm{G})$ holds in general by [T, 2.2]. Let L be the kernel of the canonical surjective map $\mathrm{G}/(f^2, g^2) \to \mathrm{G}(I/\mathfrak{a})$. Then L is a graded G-module with the n-th component $L_n = (\mathfrak{m}^{n+1} + \mathfrak{m}^n \cap \mathfrak{a})/(\mathfrak{m}^{n+1} + \mathfrak{a}\mathfrak{m}^{n-2})$, so, L is concentrated in degree 5 by (1) and our hypothesis. On the other hand, we have the equality $\ell_{\mathrm{G}}(L) = \mathrm{I}(f^2, g^2; \mathrm{G})$ since

$$\ell_{\mathrm{G}}(\mathrm{G}(I/\mathfrak{a})) = \ell_A(A/\mathfrak{a}) = 4\mathrm{e}_I(A) = 4\mathrm{e}_{\mathrm{G}_+}(\mathrm{G}) = \mathrm{e}_{(f^2,g^2)}(\mathrm{G}).$$

Consequently, we obtain $\mathrm{I}(f^2, g^2; \mathrm{G}) = \ell_A(L_5)$, whence the inequality $\mathrm{I}(f^2, g^2; \mathrm{G}) \leq 1$ follows because L_5 is contained in a cyclic vector space $\mathfrak{m}^5/(\mathfrak{m}^6 + \mathfrak{a}\mathfrak{m}^3)$.

Corollary (3.2). *Let A, $\mathfrak{m}$, $\mathfrak{q}$ and $c \in \mathfrak{m}$ be as in (3.1). (Note that c^3 belongs to $\mathfrak{q}\mathfrak{m}$.) Suppose that $c^3 = ax$ where $x \in \mathfrak{m}$ and a is part of a minimal generating set of $\mathfrak{q}$. Then,*

(1) *if $x \in \mathfrak{m}^2$, then G is a Cohen-Macaulay ring,*
(2) *if $x \notin \mathfrak{m}^2$, then G_M is not a Buchsbaum ring.*

Proof. (1): Suppose that $c^3 \in \mathfrak{q}\mathfrak{m}^2$. It yields that $\mathfrak{m}^3 = \mathfrak{q}\mathfrak{m}^2$. Hence, G is Cohen-Macaulay by [S1, 2.1].
(2): Take $b \in \mathfrak{q}$ so that $\mathfrak{q} = (a, b)$, besides we put $\mathfrak{a} = (a^2, b^2)$. We shall show that $\mathfrak{m}^4 \cap \mathfrak{a} \neq \mathfrak{a}\mathfrak{m}^2$. Now, $ac^3 = a^2x \in \mathfrak{m}^4 \cap \mathfrak{a}$, but $a^2x \notin \mathfrak{a}\mathfrak{m}^2$. Indeed, if $a^2x \in \mathfrak{a}\mathfrak{m}^2$, then $x \in \mathfrak{m}^2$ since $\mathfrak{a}\mathfrak{m}^2 \cap (a^2) = a^2\mathfrak{m}^2$.

Proposition (3.2). *Let $B = k[x_1, x_2, ..., x_s]$ $(s \geq 4)$ be a 2-dimensional Cohen-Macaulay graded ring over an infinite field k with $\deg x_1 = \deg x_2 = 2$ and $\deg x_i = 1$ for $3 \leq i \leq s$. $\mathfrak{n}$ denotes the maximal homogeneous ideal of B. We assume that $\mathfrak{q} = (x_1, x_2)$ is a homogeneous parameter ideal of B satisfying that $\mathrm{r}_{\mathfrak{q}}(\mathfrak{n}) = 3$, $\ell_B(\mathfrak{n}^2/\mathfrak{q}\mathfrak{n}) = 1$ and $\mathfrak{n}^2 = \mathfrak{q}\mathfrak{n} + (x_3^2)$. We further write $x_3^3 = y_1x_1 + y_2x_2$ for some $y_1, y_2 \in [\mathfrak{n}]_1$, this is possible by our hypothesis. Let $A = B_{\mathfrak{n}}$ and $\mathfrak{m} = \mathfrak{n}A$. Then the following conditions are equivalent;*

(1) *$\mathrm{G}(\mathfrak{m})_M$ is a Buchsbaum ring,*
(2) *y_1, y_2 spans a 2-dimensional subspace in $\mathfrak{n}/\mathfrak{n}^2$.*

Proof. (1) $\Longrightarrow$ (2): Let $\overline{y_i}$ stand for the reduction of y_i in $\mathfrak{n}/\mathfrak{n}^2$. Suppose that the dimension of the subspace spanned by $\overline{y_1}$, $\overline{y_2}$ in $\mathfrak{n}/\mathfrak{n}^2$ is at most 1. We may assume that $y_2 = \rho y_1$ $(\rho \in k)$ without loss of generality. Then $x_3^3 = y_1(x_1 + \rho x_2)$. Applying (3.2) for the generating set $x_1 + \rho x_2$, x_2 of $\mathfrak{q}$, the Buchsbaumness of G_M leads that $y_1 \in \mathfrak{n}^2$, hence $y_1 = 0$. Applying (3.2) again, we also get $y_2 = 0$. Thus, $x_3^3 = 0$. This contradicts to the reduction number of $\mathfrak{m}$.
(2) $\Longrightarrow$ (1): Let us denote $a = x_1$, $b = x_2$, $c = x_3$ and $\mathfrak{a} = (a^2, b^2)B$. It is enough to show that $\mathfrak{n}^4 \cap \mathfrak{a} = \mathfrak{a}\mathfrak{n}^2$ by (3.1)(3). Note that $\mathfrak{n}^4 = \mathfrak{a}\mathfrak{n}^2 + (abc^2, ac^3, bc^3)B$. We take a homogeneous element $\varphi \in (abc^2, ac^3, bc^3)B \cap \mathfrak{a}$ and write $\varphi = uabc^2 + vac^3 + wbc^3$. Then $\deg \varphi \geq 5$ unless $\varphi = 0$. If $\deg \varphi \geq 7$, then u, v and w belong to $\mathfrak{n}$. Hence $\varphi \in \mathfrak{n}^5 \subseteq \mathfrak{a}\mathfrak{n}^2$ (note that $abc^3 \in ab\mathfrak{q}\mathfrak{n} \subseteq \mathfrak{a}\mathfrak{n}^2$). Let $\deg \varphi = 6$. Then, $u \in k$ and $v, w \in \mathfrak{n}$. So, we have $uabc^2 \in \mathfrak{a}$, whence $uc^2 \in \mathfrak{q}$ since a, b is a regular sequence on

B. On the other hand, it always holds that $\mathfrak{q} \cap \mathfrak{n}^2 = \mathfrak{q}\mathfrak{n}$. Hence $uc^2 = 0$ because $uc^2 \in [\mathfrak{n}]_2$ and $[\mathfrak{q}\mathfrak{n}]_2 = (0)$. Consequently, $\varphi = vac^3 + wbc^3 \in \mathfrak{n}^5$. Thus, $\varphi \in \mathfrak{a}\mathfrak{n}^2$, too. We finally assume that $\deg \varphi = 5$. Then φ can be written as $\varphi = vac^3 + wbc^3$ where $v, w \in k$. Substituting $c^3 = y_1 a + y_2 b$, we get $vy_2 + wy_1 \in \mathfrak{q}$ since a, b is a regular sequence, while $[\mathfrak{q}]_1 = (0)$, hence $vy_2 + wy_1 = 0$. We therefore obtain $v = w = 0$ because y_1, y_2 is linearly independent in $[\mathfrak{n}]_1$. Consequently, we have $\varphi = 0$. This completes the proof.

Example (3.4). Let $B = k[a, b, c, x, y]$ be a polynomial ring over an infinite field k where $\deg a = \deg b = 2$ and $\deg c = \deg x = \deg y = 1$. Let J be a homogeneous ideal of B of the form $J = (x, y)(x, y, c) + (c^4, c^3 - ax - by)$. Then we can check that $\overline{B} = B/J$ is a 2-dimensional Cohen-Macaulay graded ring, furthermore, for the maximal ideal $\mathfrak{n}$ of $\overline{B}$ and $\mathfrak{q} = (a, b)\overline{B}$, it holds that $\mathrm{r}_{\mathfrak{q}}(\mathfrak{n}) = 3$, $\ell_{\overline{B}}(\mathfrak{n}^2/\mathfrak{q}\mathfrak{n}) = 1$, and $\mathfrak{n}^2 = \mathfrak{q}\mathfrak{n} + c^2\overline{B}$. We have a relation $c^3 = ax + by$ in $\overline{B}$ and x, y is linearly independent in $[\overline{B}]_1$. So, by (3.3) $\mathrm{G}(\mathfrak{m})_M$ is a Buchsbaum ring for the maximal ideal $\mathfrak{m} = \mathfrak{n}\overline{B}_{\mathfrak{n}}$ in the local ring $\overline{B}_{\mathfrak{n}}$.

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ON THE ASSOCIATED GRADED RINGS OF POWERS OF PARAMETER IDEALS IN BUCHSBAUM RINGS

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In this note we shall discuss a few results on the associated graded rings of ideals in Buchsbaum rings.

Let $(A, \mathfrak{m})$ be a Noetherian local ring. For an ideal I of A let $\mathrm{G}(I)$ denote the associated graded ring of I, namely

$$\mathrm{G}(I) := \bigoplus_{n \geq 0} I^n / I^{n+1}.$$

Moreover M denotes its unique graded maximal ideal, i.e., $M = \mathfrak{m}\mathrm{G}(I) + \mathrm{G}(I)_+$.

Let A be a Buchsbaum ring of dimension $d > 0$. Then concerning the Buchsbaumness of the asscociated graded rings, in 1982/1983 S. Goto (Meiji University) showed the following two results: $\mathrm{G}(\mathfrak{m})_M$ is Buchsbaum if A has maximal embedding dimension [G1]; $\mathrm{G}(\mathfrak{q})_M$ is Buchsbaum for any parameter ideal $\mathfrak{q}$ [G2]. In 1996, these Goto's results are generalized by the work of Y. Nakamura [N] as follows: if $I^2 = \mathfrak{q}I$ holds for some parameter ideal $\mathfrak{q}$ contained in I, then $\mathrm{G}(I)_M$ is Buchsbaum if and only if $(a_1^2, a_2^2, \ldots, a_d^2) \cap I^n = (a_1^2, a_2^2, \ldots, a_d^2)I^{n-2}$ holds for each $3 \leq n \leq d+1$ (and hence for all $n \in \mathbb{Z}$), where $\mathfrak{q} = (a_1, a_2, \ldots, a_d)$.

On the other hand, in 1976, G. Valla studied the associated graded rings of powers of ideals generated by A-regular sequences, where A is Cohen-Macaulay, [V], cf. also [B].

Motivated by these works we shall pose the following questions:

Problem. *Let A be a Buchsbaum ring. (1) Is $\mathrm{G}(\mathfrak{q}^n)_M$ Buchsbaum for every parameter ideal $\mathfrak{q}$ and integer $n \geq 2$? (2) Suppose that A has maximal embedding dimension. Then is $\mathrm{G}(\mathfrak{m}^n)_M$ Buchsbaum for all $n \geq 2$?*

§1. Main results

Let $h^i(A)$ denote the length of the i-th local cohomology module of A, namely $h^i(A) := l_A(\mathrm{H}^i_{\mathfrak{m}}(A))$. We denote by $\mathbb{I}(A)$ the Buchsbaum invariant of A, namely

$$\mathbb{I}(A) := \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot h^i(A).$$

With these notations, our results are stated as follows.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Theorem 1. *Let A be a Buchsbaum ring and I an $\mathfrak{m}$-primary ideal. Let $\mathrm{G}(I)$ be the associated graded ring of I and $M = \mathfrak{m}\mathrm{G}(I)+\mathrm{G}(I)_+$. Let $\mathfrak{q} = (a_1, a_2, \dots, a_d)$ be a minimal reduction of I, i.e., $I^{r+1} = \mathfrak{q}I^r$ for some integer $r \geq 0$. Then the following statements are equivalent.*

(1) $\mathrm{G}(I)_M$ *is Buchsbaum and* $h^i(\mathrm{G}(I)_M) = h^i(A)$ *for each* $0 \leq i < d$.
(2) $\mathrm{G}(I)_M$ *is FLC (i.e.,* $h^i(\mathrm{G}(I)_M) < \infty$ *for all* $i \neq d$*) with* $\mathbb{I}(\mathrm{G}(I)_M) = \mathbb{I}(A)$.
(3) $(a_1^2, a_2^2, \dots, a_d^2) \cap I^n = (a_1^2, a_2^2, \dots, a_d^2)I^{n-2}$ *for every* $3 \leq n \leq r + d$.

When this is the case, every power I^n, $n \geq 2$, of the ideal I satisfies these equivalent conditions too.

Notice that for the cases $I = \mathfrak{m}$ and $r = 1$ our theorem is already given by S. Goto in [G3, Theorem (3.3)] and Y. Nakamura [N, Theorem (1.1)], respectively. The equivalence of (2) and (3) in our theorem is also established in [GY, Proposition (7.13)]. Moreover, even though $\mathrm{G}(I)_M$ is Buchsbaum, the equality $\mathbb{I}(\mathrm{G}(I)_M) = \mathbb{I}(A)$ does not necessarily hold. In [G4] we can find such Buchsbaum rings $\mathrm{G}(I)_M$, namely it occurs $\mathbb{I}(\mathrm{G}(I)_M) > \mathbb{I}(A) = 0$.

As consequences of our theorem we have the following corollaries.

Corollary 2. *Let A be a Buchsbaum ring. Then $\mathrm{G}(\mathfrak{q}^n)_M$ is Buchsbaum for every parameter ideal $\mathfrak{q}$ and integer $n \geq 2$.*

Corollary 3. *Suppose that a Buchsbaum ring A has maximal embedding dimension. Then $\mathrm{G}(\mathfrak{m}^n)_M$ is Buchsbaum for all $n \geq 2$.*

Throughout this note, let A be a Buchsbaum ring and I an $\mathfrak{m}$-primary ideal of A. We denote by $\mathrm{R}(I)$ the Rees algebra of an ideal I and by N its unique graded maximal ideal, namely

$$\mathrm{R}(I) := \bigoplus_{n\geq 0} I^n, \qquad N := \mathfrak{m}\mathrm{R}(I) + \mathrm{R}(I)_+.$$

When we set $I = (a_1, a_2, \dots, a_w)$, the Rees algebra $\mathrm{R}(I)$ is regared as the A-subalgera $A[a_1t, a_2t, \dots, a_wt]$ of the polynomial ring $A[t]$, where t is an indeterminate over A.

Here we introduce tow more useful notations. Let i, j be integers. We denote by $[i, j]$ the set of integers n such that $i \leq n \leq j$. Of course, $[i, j] = \emptyset$ if $i > j$. Moreover, for a set S, we sometimes use the notation $|S|$, instead of $\sharp S$, to indicate the number of all elements in S.

§2. Preliminaries

In this section we shall recall several arguments in Chapter 7 of [GY]; see also [T, §5].

Lemma 4 [GY, Lemma (2.5)]. *Let $\mathfrak{q} = (a_1, a_2, \dots, a_d)$ be a minimal reduction of I, i.e., $I^{r+1} = \mathfrak{q}I^r$ for some integer $r \geq 0$. Then the following statements are equivalent.*

(1) *The equality*

$$(a_l^{n_l} \mid l \in L) \cap I^n = \sum_{l\in L} a_l^{n_l} I^{n-n_l}$$

holds for all $L \subseteq [1, d], n_l > 0$ *and* $n \in \mathbb{Z}$.

(2) *The equality*

$$(a_1^2, a_2^2, \dots, a_d^2) \cap I^n = (a_1^2, a_2^2, \dots, a_d^2) I^{n-2}$$

holds for all $3 \le n \le r+d$.

When this is the case one has $\mathrm{G}(I)/a_i t \cdot \mathrm{G}(I) \cong \mathrm{G}(I/(a_i))$.

Proposition 5 [GY, Theorem (7.12), Proposition (7.13)]. *Let* $\mathfrak{q} = (a_1, a_2, \dots, a_d)$ *still be a minimal reduction of* I. *Then the following statements are equivalent.*

(1) $\mathrm{G}(I)_M$ *is quasi-Buchsbaum and* $h^i(\mathrm{G}(I)_M) = h^i(A)$ *for each* $0 \le i < d$.

(2) $\mathrm{G}(I)_M$ *is FLC (i.e.,* $h^i(\mathrm{G}(I)_M) < \infty$ *for all* $i \neq d$*) with* $\mathbb{I}(\mathrm{G}(I)_M) = \mathbb{I}(A)$.

(3) *The equlity*

$$(a_1^2, a_2^2, \dots, a_d^2) \cap I^n = (a_1^2, a_2^2, \dots, a_d^2) I^{n-2}$$

holds for all $3 \le n \le r+d$.

When this is the case $a_1 t, a_2 t, \dots, a_d t$ *form a u.s.d-sequence on* $\mathrm{G}(I)$.

§3. Sketch of proof of main results

By Proposition 5, the proof of Theorem 1 is completely covered by the following theorem. Namely we need

Theorem 6. *If* $\mathbb{I}(\mathrm{G}(I)_M) = \mathbb{I}(A)$, *then* $\mathrm{G}(I)_M$ *is a Buchsbaum ring.*

Let $v = \mu_A(\mathfrak{m})$, and $w = \mu_A(I)$.

Lemma 7. *There exist systems of elements in* A, *say* $x_1, x_2, \dots, x_v$ *and* $a_1, a_2, \dots, a_w$, *which satisfy the following conditions:*

(1) $a_1, a_2, \dots, a_w$ *is a minimal system of generators of* I;

(2) *any* d*-elements of* $a_1, a_2, \dots, a_w$ *form a minimal reduction of* I;

(3) $x_1, x_2, \dots, x_v$ *is a minimal system of generators of* $\mathfrak{m}$;

(4) *any* d*-elements of* $x_1, x_2, \dots, x_v, a_1, a_2, \dots, a_w$ *form a system of parameters for* A.

Let us assume that $x_1, x_2, \dots, x_v$ and $a_1, a_2, \dots, a_w$ be elements in A satisfying four conditions in Lemma 7 as above.

Let $\mathrm{K}^{\cdot}(\underline{x}, \underline{at}; \mathrm{G}(I))$ be the Koszul (co-)complex generated over R by the system $\underline{x} := x_1, x_2, \dots, x_v$, and $\underline{at} := a_1 t, a_2 t, \dots, a_w t$ with respect to $\mathrm{G}(I)$. Since $\underline{x}, \underline{at}$ is a minimal system of generators of N, this complex $\mathrm{K}^{\cdot}(\underline{x}, \underline{at}; \mathrm{G}(I))$ is uniquely determined by the ideal N upto isomorphisms not depending on the particular choice of a minimal system of generators, cf., [SV, $ 1 of Chapter 0, pp. 27]. Hence we denote it by $\mathrm{K}^{\cdot}(N; \mathrm{G}(I))$ simply.

Notice that the Koszul complex $\mathrm{K}^{\cdot}(N; \mathrm{G}(I))$ is a complex of direct sums of copies of a graded $\mathrm{R}(I)$-module $\mathrm{G}(I)$. Hence we have an expression of it as follows:

$$\mathrm{K}^{\cdot}(N; \mathrm{G}(I)) = \bigoplus_{\substack{L \subseteq [1,w] \\ J \subseteq [1,v]}} \mathrm{G}(I) \cdot e_J^L, \qquad \mathrm{K}^i(N; \mathrm{G}(I)) = \bigoplus_{|L|+|J|=i} \mathrm{G}(I) \cdot e_J^L$$

where $\{e_J^L \mid L \subseteq [1,w], J \subseteq [1,v]\}$ is the graded free basis with $\deg e_J^L = -|L|$.

Proof of Theorem 6. Assume that $\mathbb{I}(\mathrm{G}(I)_M) = \mathbb{I}(A)$ holds. Then we shall prove that the canonical map

$$\mathrm{H}^i(N; \mathrm{G}(I)) \longrightarrow \mathrm{H}^i_N(\mathrm{G}(I)) \tag{3.1}$$

is surjective for all $0 \leq i < d$. By the usual method of using induction on d, however, the surjectivity of the canonical maps (3.1) is led from the fact that the canonical map

$$\mathrm{H}^i(N; \mathrm{H}^0_N(\mathrm{G}(I))) \longrightarrow \mathrm{H}^i(N; \mathrm{G}(I)) \tag{3.2}$$

is injective for all $0 \leq i \leq d$, cf., [SV, Theorem (2.15) in Chap. 1]. Again by induction on d, it is enough to check only for $i = d$, namely we must show that the canonical map

$$\mathrm{H}^d(N; \mathrm{H}^0_N(\mathrm{G}(I))) \longrightarrow \mathrm{H}^d(N; \mathrm{G}(I)) \tag{3.3}$$

is injective.

To get the injectivity of the map (3.3) , look at the following commutative diagram

$$\begin{array}{ccc} \mathrm{K}^d(N; \mathrm{H}^0_N(\mathrm{G}(I))) & \longrightarrow & \mathrm{K}^d(N; \mathrm{G}(I)) \\ {\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle \partial} \\ \mathrm{K}^{d-1}(N; \mathrm{H}^0_N(\mathrm{G}(I))) & \longrightarrow & \mathrm{K}^{d-1}(N; \mathrm{G}(I)) \end{array}$$

and choose a homogeneous element η of $\mathrm{K}^{d-1}(N; \mathrm{G}(I))$ of degree $n \in \mathbb{Z}$ and put $\xi = \partial(\eta)$. Moreover, we assume that $\xi \in \mathrm{K}^d(N; \mathrm{H}^0_N(\mathrm{G}(I)))$. Our goal is to show $\xi = 0$. So we write ξ, η as follows:

$$\xi = \sum_{|L|+|J|=d} \xi^L_J \cdot e^L_J \,, \qquad \eta = \sum_{|P|+|Q|=d-1} \eta^P_Q \cdot e^P_Q \,,$$

where $\xi^L_J \in [\mathrm{G}(I)]_{n+|L|}$ and $\eta^P_Q \in [\mathrm{G}(I)]_{n+|P|}$. Then we have

$$\xi^L_J = \sum_{j \in J} (-1)^{J(j)} x_j \cdot \eta^L_{J\setminus\{j\}} + \sum_{l \in L} (-1)^{|J|+L(l)} a_l t \cdot \eta^{L\setminus\{l\}}_J .$$

where we define $J(j)$ as follows $J(j) := \sharp\{j' \in J \mid j' < j\}$, and $L(l)$ is also defined in the same way.

Then, our assumption means $\xi^L_J \in \mathrm{H}^0_N(\mathrm{G}(I))$ for each L, J with $|L| + |J| = d$, and our goal is to prove that $\xi^L_J = 0$ for all such L, J.

Since $\eta^P_Q \in [\mathrm{G}(I)]_{n+|P|}$ we can choose a representation $c^P_Q \in I^{n+|P|}$ of η^P_Q, i.e., $\overline{c^P_Q} = \eta^P_Q$, here we denote by $\bar{c}$ the homogeneous element $c \bmod I^{n+1}$ in $\mathrm{G}(I)$ of degree n for each $c \in I^n$. Now, using these representations c^P_Q's we define the element, say b^L_J, of A as follows:

$$b^L_J := \sum_{j \in J} (-1)^{J(j)} x_j \cdot c^L_{J\setminus\{j\}} + \sum_{l \in L} (-1)^{|J|+L(l)} a_l \cdot c^{L\setminus\{l\}}_J .$$

Then clearly we have $\overline{b^L_J} = \xi^L_J$, namely this element b^L_J is one of the representations of ξ^L_J. Unfortunately, this definition of b^L_J depends on a choice of the representations c^P_Q's.

Lemma 8 (Key-Lemma). *Let L, J be such that $L \subseteq [1,w]$, $J \subseteq [1,v]$ and $|L|+|J| = d$. Suppose that there exists another subset L' of $[1,w]$ such that $L' \supset L$, $|L'| = |L|+1$ and that $b^{L'}_{J'} = 0$ for all $J' \subset J$ with $|J'| = |J|-1$. Then it follows that $b^L_J = 0$, under a suitable exchange of the representations $c^{L\setminus\{l\}}_J$'s ($l \in L$). (Notice that, in the case $L = \emptyset$, the assertion $b^{\emptyset}_J = 0$ follows with no exchange of the representations.)*

Proof. First of all, we deal with the case $|L| = d$. Notice that in this case our hypothesis in Lemma is always considered to be fulfilled. Then, clearly $J = \emptyset$, and we have

$$\xi^L_{\emptyset} = \sum_{l \in L} (-1)^{L(l)} a_l t \cdot \eta^{L\setminus\{l\}}_{\emptyset}.$$

This implies

$$\xi^L_{\emptyset} \in (a_l t \mid l \in L) \cdot \mathrm{G}(I) \cap \mathrm{H}^0_N(\mathrm{G}(I)) = (0)$$

because $a_l t$'s from a u.s.d-sequence on $\mathrm{G}(I)$, by Proposition 5. Thus

$$\sum_{l \in L} (-1)^{L(l)} a_l c^{L\setminus\{l\}}_{\emptyset} \in (a_l \mid l \in L) \cap I^{n+d+1} = (a_l \mid l \in L) I^{n+d}$$

by Lemma 4. So, we can choose elements f_l's of I^{n+d} such that

$$\sum_{l \in L} (-1)^{L(l)} a_l c^{L\setminus\{l\}}_{\emptyset} = \sum_{l \in L} (-1)^{L(l)} a_l f_l$$

in A. Then we have

$$\eta^{L\setminus\{l\}}_{\emptyset} = \overline{c^{L\setminus\{l\}}_{\emptyset}} = \overline{c^{L\setminus\{l\}}_{\emptyset} - f_l}$$

because of $\eta^{L\setminus\{l\}}_{\emptyset} \in [\mathrm{G}(I)]_{n+d-1} = I^{n+d-1}/I^{n+d}$. Therefore, after exchanging a representation $c^{L\setminus\{l\}}_{\emptyset}$ of $\eta^{L\setminus\{l\}}_{\emptyset}$ for the element $c^{L\setminus\{l\}}_{\emptyset} - f_l$, we obtain that

$$b^L_{\emptyset} = \sum_{l \in L} (-1)^{L(l)} a_l c^{L\setminus\{l\}}_{\emptyset} = 0.$$

Next, let $|L| < d$. At first, we claim the following.

Claim. $\sum_{j \in J} (-1)^{J(j)} x_j c^L_{J\setminus\{j\}} \in (a_l \mid l \in L)$.

Proof of Claim. Let $j \in J$. Applying the hypothsis to each $J \setminus \{j\}$ we have

$$0 = b^{L'}_{J\setminus\{j\}} = \sum_{j' \in J\setminus\{j\}} (-1)^{J\setminus\{j\}(j')} x_{j'} c^{L'}_{J\setminus\{j,j'\}} + \sum_{l' \in L'} (-1)^{|J|-1+L'(l')} a_{l'} c^{L'\setminus\{l'\}}_{J\setminus\{j\}}.$$

Multiplying by $(-1)^{J(j)} x_j$ and taking the sum $\sum_{j \in J}$ we get that

$$\begin{aligned}
0 &= \sum_{j \in J} \sum_{j' \in J\setminus\{j\}} (-1)^{J(j)} (-1)^{J\setminus\{j\}(j')} x_j x_{j'} c^{L'}_{J\setminus\{j,j'\}} \\
&\quad + \sum_{l' \in L'} (-1)^{|J|-1+L'(l')} a_{l'} \Big(\sum_{j \in J} (-1)^{J(j)} x_j c^{L'\setminus\{l'\}}_{J\setminus\{j\}} \Big) \\
&= \sum_{l' \in L'} (-1)^{|J|-1+L'(l')} a_{l'} \Big(\sum_{j \in J} (-1)^{J(j)} x_j c^{L'\setminus\{l'\}}_{J\setminus\{j\}} \Big)
\end{aligned}$$

in A, because of $\sum_j \sum_{j'} \pm x_j x_{j'} c^{L'}_{J\setminus\{j,j'\}} = 0$. Choose the element $l'' \in L' \setminus L$. Then clearly $L = L' \setminus \{l''\}$. Hence we see from the above equation that

$$\sum_{j\in J}(-1)^{J(j)} x_j c^L_{J\setminus\{j\}} \in \big[(a_l \mid l \in L) : a_{l''}\big] \cap (x_j \mid j \in J) = (a_l \mid l \in L),$$

because x_j's and a_l's form a system of parameters for A by our choice of them (recall that $|L| + |J| = d$) and $(a_l \mid l \in L) : a_{l''} = (a_l \mid l \in L) : \mathfrak{m} = (a_l \mid l \in L) : x_j$ for some $j \in J$.

Since $c^L_{J\setminus\{j\}} \in I^{n+|L|}$ we see by Claim that

$$\sum_{j\in J}(-1)^{J(j)} x_j c^L_{J\setminus\{j\}} \in (a_l \mid l \in L) \cap I^{n+|L|} = (a_l \mid l \in L) I^{n+|L|-1}.$$

This means

$$\begin{aligned}\xi^L_J &= \overline{\sum_{j\in J}(-1)^{J(j)} x_j c^L_{J\setminus\{j\}} + \sum_{l\in L}(-1)^{|J|+L(l)} a_l c^{L\setminus\{l\}}_J} \\ &\in (a_l t \mid l \in L)\cdot \mathrm{G}(I) \cap \mathrm{H}^0_N(\mathrm{G}(I)) = (0)\end{aligned}$$

in $\mathrm{G}(I)$, because of the u.s.d-sequence property of $a_l t$'s. Thus we get

$$\begin{aligned}b^L_J &= \sum_{j\in J}(-1)^{J(j)} x_j c^L_{J\setminus\{j\}} + \sum_{l\in L}(-1)^{|J|+L(l)} a_l c^{L\setminus\{l\}}_J \\ &\in (a_l \mid l \in L) \cap I^{n+|L|+1} = (a_l \mid l \in L) I^{n+|L|}\end{aligned}$$

in A. Therefore, after exchanging each representation $c^{L\setminus\{l\}}_J$ of $\eta^{L\setminus\{l\}}_J \in I^{n+|L|-1}/I^{n+|L|}$, we finally conclude $b^L_J = 0$ in the same way described at the first step of our proof. This completes the proof of Lemma 8.

Now we finish our proof of Theorem 6. Let $\xi, \eta, \xi^L_J, \eta^P_Q, c^P_Q$ and b^L_J be the same notations described as above. In order to obtain $\xi^L_J = 0$, we show that $b^L_J = 0$, under a suitable exchange of the representations c^P_Q's.

Let L, J be fixed and put $k := d - |L|$. Then, we make such a nice choice of the representations c^P_Q's by an inductive way on k. For $k = 0$, our assertion is true by Lemma 8 as above. Now let $k > 0$. Then we can find a sequence of subsets of $[1, w]$, say

$$[1, w] \supseteq L_0 \supset L_1 \supset \cdots \supset L_k = L$$

such that $|L_i| = d - i$ for $0 \le i \le k$. Note that $|L_{i-1}| = |L_i| + 1$ for $1 \le i \le k$ and $|L_0| = d$. We define one more notation Δ_i, $0 \le i \le k$, as follows:

$$\Delta_i := \bigcup_{j=0}^{i} \{(P, Q) \mid P = L_j \setminus \{l\} \text{ where } l \in L_j, Q \subseteq J \text{ with } |Q| = j\}$$

Note that Δ_k is a disjoint union of Δ_{k-1} and the set $\{(L \setminus \{l\}, J) \mid l \in L\}$. Moreover, in the case $k = d$, we consider Δ_d to be equal to Δ_{d-1}, namely we define $\Delta_d := \Delta_{d-1}$. Then it is a routine to show, for each $i = 0, \ldots, k$, that $b^{L_i}_{J'} = 0$ for all $J' \subseteq J$ with $|J'| = i$, under suitable exchange of the representations c^P_Q's, where $(P, Q) \in \Delta_i$. This finishes the proof of Theorem 6.

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随伴次数環の正準加群への埋め込み

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1. はじめに

以下$(A,\mathfrak{m})$をNoether 局所環とし$A[t]$によりt を不定元とする環 A 上 多項式環を表す。A 内のイデアル I （$\neq A$）に対して

$$\mathcal{R}(I) := A[It] \subseteq A[t]$$

とおきイデアル I の Rees 環という。よく知られているように $\mathrm{Proj}\mathcal{R}(I)$ はイデアル I を中心とするAのblow upと呼ばれ、特異点の解析において重要な役割を果たす。ここでは$\mathcal{R}(I)$の環論的性質に注目したい。

本来、Rees 環はA内のイデアルのfiltrationに随伴するものである。$\mathcal{F}=\{F_i\}_{i\in\mathbb{Z}}$が$A$のイデアルのfiltrationであるとは、$F_i$はすべて$A$のイデアルであって (1) $F_i = A,\ \forall i \leq 0$. (2) $F_{i+1} \subseteq F_i,\ \forall i \in \mathbb{Z}$. (3) $F_iF_j \subseteq F_{i+j},\ \forall i, j \in \mathbb{Z}$. が成り立つことをいう。この様なイデアルの族 $\mathcal{F}$に対して

$$\mathcal{R}(\mathcal{F}) := \Sigma_{i\geq 0} F_i t^i \subseteq A[t],$$
$$\mathcal{R}'(\mathcal{F}) := \Sigma_{i\in\mathbb{Z}} F_i t^i \subseteq A[t, t^{-1}],$$
$$\mathcal{G}(\mathcal{F}) := \mathcal{R}'(\mathcal{F}) / t^{-1}\mathcal{R}'(\mathcal{F})$$

とおきそれぞれ $\mathcal{F}$ についてのRees 環、拡大Rees 環、随伴次数環と呼ぶ。一般に$\mathcal{R}(\mathcal{F})$はNoether環ではないが、この論文内ではAのイデアルのfiltration $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$を$\mathcal{R}(\mathcal{F})$がNoether環になるようにあたえ、しかも自明な場合を除くために$F_1 \neq A$とする。さて、$\mathcal{R}(\mathcal{F})$の環構造(特にCohen-Macaulay性、 Gorenstein性)は$\mathcal{G}(\mathcal{F})$の環構造とそのa-不変量によって特徴づけられることが知られている。従って$\mathcal{R}(\mathcal{F})$の環論的性質を解析するために$\mathcal{G}(\mathcal{F})$の環構造を判定する実際的な方法を見つけることは重要な問題だと思われる。ここでは、$\mathcal{G}(\mathcal{F})$のGorenstein性について考察してみたい。本稿の主定理は次の結果である。

定理(1.1) $G = \mathcal{G}(\mathcal{F})$, $a = a(G)$とおく。A はGorenstein局所環で、G はCohen-Macaulay環とする。そのとき$\mathcal{A} = \{P \in V(F_1) \mid ht_A P = \dim G_P/PG_P\}$とおけば$\mathcal{A}$は空でない有限集合であって、次の条件は同値である。

(1) G はGorenstein環である。

(2) 次数G-加群の単射$G(a) \to K_G$が存在する。

(3) すべての $P \in \mathcal{A}$ に対し G_P はGorenstein環であって、$a=a(G_P)$ が成り立つ。

但し、K_GはGの正準加群、a(G)はGのa-不変量を表す。即ち

$$a(G) := \sup \{ n \in \mathbb{Z} \mid [H^i_{\mathfrak{M}}(G)]_n \neq 0 \} \quad (\mathfrak{M} = \mathfrak{m}G + G_+)$$

である [GW]。この定理によると有名な次の結果がただちに導かれる。

系(1.2) ([HSV], Proposition (1.1)) AはGorenstein局所環とする。このとき、$\mathcal{G}(\mathcal{F})$がCohen-Macaulay整域ならば$\mathcal{G}(\mathcal{F})$はGorenstein環である。

系(1.3) ([GN], Part I, Corollary (5.8)) AはGorenstein局所環で、PをAの素イデアルとする。$P^{(n)}$によってPの記号的n乗を表し、$\mathcal{F}=\{P^{(n)}\}_{n \in \mathbb{Z}}$とおく。このとき$\mathcal{R}(\mathcal{F})$がNoether環であるならば、次の条件は同値である。

(1) $\mathcal{G}(\mathcal{F})$はGorenstein環である。

(2) $\mathcal{G}(\mathcal{F})$はCohen-Macaulay環であって、$\mathcal{G}(PA_P)$はGorenstein環である。

主定理の証明は第2節で行うが、それには、次の命題(1.4)で表示される$\mathcal{G}(\mathcal{F})$の正準加群を表わすAの正準加群K_Aの部分加群の族が重要な役割を果たす。

命題(1.4) AはGorenstein環の準同型像で、$\mathcal{G}(\mathcal{F})$はCohen-Macaulay環とすると、次の条件を満たすK_Aの部分加群の族$\omega=\{\omega_i\}_{i \in \mathbb{Z}}$が唯一つ存在する。

(1) $\omega_{i+1} \subseteq \omega_i$, $\forall i \in \mathbb{Z}$.

(2) $F_i \omega_j \subseteq \omega_{i+j}$, $\forall i, j \in \mathbb{Z}$.

(3) $\omega_i = K_A$, $\forall i \leq -a(G)-1$.

(4) $\oplus_{i \in \mathbb{Z}} \omega_i$は$\mathcal{R}'(\mathcal{F})$の正準加群である。

このとき、$\oplus_{i \geq -a(G)} \omega_{i-1}/\omega_i$は$\mathcal{G}(\mathcal{F})$の正準加群である。

そして第3節では正準加群の形を決める。つまり、AをGorenstein環の準同型像とし、$I\ (\neq A)$をAのイデアルで、そのあるreductionイデアル Qが$s=\mathrm{ht}_A I$ 個の元で生成されると仮定する。とくに$s=0$ときは$Q=(0)$と考える。例えばAの剰余体が無限体のとき、Iはequimultipleなイデアルにほかならない。rをQに関するIのreduction numberとおく。またイデアルの零又は負のべきはAとする。そして$\mathcal{G}(\mathcal{F})$はCohen-Macaulay環とすると、命題 (1.4)にある様なIの随伴次数環$\mathcal{G}(I)$の正準加群を表わす$\boldsymbol{\omega}$を、以下のようにイデアルの言葉で表わすことができた。

定理(1.5) $\mathcal{G}(I)$はCohen-Macaulay環とすると、すべての$i\in\mathbb{Z}$に対して次が正しい。

(1) $s\geq 1$のとき、$\boldsymbol{\omega}_i = Q^{i-s+r+1}K_A :_{K_A} I^r$ である。

(2) $s=0$のとき、$\boldsymbol{\omega}_i = (0):_{K_A} I^{-i}$ である。

この応用として次の結果が得られた。

系(1.6) [cf. [W] Proposition 9., [O] Theorem 1.5] $\mathcal{G}(I)$はCohen-Macaulay環とすると、次が正しい。

(1) $s\geq 1$のとき、次の条件は同値である。

(a) $\mathcal{G}(I)$はGorenstein環ある。

(b) AはGorenstein環あって、$I^i = Q^i :_A I^r$ $(1\leq \forall i \leq r)$ が成り立つ。

(2) $s=0$のとき、次の条件は同値である。

(a) $\mathcal{G}(I)$はGorenstein環ある。

(b) AはGorenstein環あって、$I^{r-i+1} = (0):_A I^i$ $(1\leq \forall i \leq r)$ が成り立つ。

2 .主定理の証明

この節ではAはGorenstein環の準同型像であると仮定する。$R = \mathcal{R}(\mathcal{F})$, $R' = \mathcal{R}'(\mathcal{F})$, $G = \mathcal{G}(\mathcal{F})$ とおき、K_A, K_R, $K_{R'}$, K_GによってそれぞれのA, R, R', Gの正準加群を 表すこととする。そして、$a = a(G)$ とおく。この節の目的は定理(1.1)を証明することである。まず次の命題を示すことから始める。

命題(2.1) GはCohen-Macaulay環とすると、次の条件を満たすの部分加群の族 $\omega=\{\omega_i\}_{i\in\mathbb{Z}}$が唯一つ存在する。

(1) $\omega_{i+1} \subseteq \omega_i$, $\forall i \in \mathbb{Z}$.

(2) $F_i\omega_j \subseteq \omega_{i+j}$, $\forall i, j \in \mathbb{Z}$.

(3) $\omega_i = K_A$, $\forall i \leq -a-1$.

(4)次数R'-加群として$K_{R'} \cong \Sigma_{i\in\mathbb{Z}}\, \omega_i \otimes_A A t^i$ ($\subseteq K_A \otimes_A A[t, t^{-1}]$)である。

このとき、次数G-加群の同型$K_G \cong \oplus_{i\geq -a}\, \omega_{i-1}/\omega_i$が存在する。

証明　自然に導かれる次数$R'_{t^{-1}}$-加群の同型$[K_{R'}]_{t^{-1}} \cong K_A \otimes_A A[t, t^{-1}]$より次数R'-加群の単射$\sigma: K_{R'} \to K_A \otimes_A A[t, t^{-1}]$が得られる。すべての$i \in \mathbb{Z}$に対して$\omega_i = \{ x \in K_A \mid x \otimes t^i \in \sigma([K_{R'}]_i) \}$とおく。このとき$\omega_i \otimes_A A t^i = \sigma([K_{R'}]_i)$ が成り立つので$\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ はK_Aの部分加群の族であり(4)をみたす。σは次数R'-線型写像なので ω が(1),(2)をみたすことは容易にわかる。さらに(3)は次数G-加群の同型 $K_G \cong \oplus_{i\in\mathbb{Z}}\omega_{i-1}/\omega_i$ が導かれることと、σが$K_{R'}$のt^{-1}による局所化であることに従う。また、(4)の同型を経由して得られる射 $\rho: K_{R'} \to K_A \otimes_A A[t, t^{-1}]$は、(3)の条件より$K_{R'}$の$t^{-1}$による局所化であり、さらに$K_A \otimes_A A[t, t^{-1}]$の次数付自己同型はAの単元のmultiplicationである。よってρとσの像が等しいので一意性が示される。□

これからはGはCohen-Macaulay環であると仮定し、命題(2.1)で定められたK_Aの部分加群の族を$\omega = \{\omega_i\}_{i\in\mathbb{Z}}$と書こう。定理(1.1)の証明の前に蛇足ではあるが次の注意を述べておく。

注意(2.2) RはCohen-Macaulay環であって、dimR = dimA +1 ならば、次数R-加群の同型

$$K_R \cong \Sigma_{i\geq 1}\, \omega_i \otimes_A A t^i$$

が存在する。

証明　RはCohen-Macaulay環であって、dimR = dimA +1 なので、**[TVZ]** によると次の条件(1)から(5)を満たすK_Aの部分加群の族$K=\{K_i\}_{i\in\mathbb{Z}}$が存在する；(1)$K_0 = K_A$

(2) $K_{i+1} \subseteq K_i$, $\forall i \in \mathbb{Z}$. (3)$F_i K_j \subseteq K_{i+j}$, $\forall i, j \in \mathbb{Z}$. (4)$K_R \cong \Sigma_{i\geq 1} K_i \otimes_A A t^i$ (5)$K_G \cong \oplus_{i\geq -a} K_{i-1}/K_i$。そして$C = \Sigma_{i\in\mathbb{Z}} K_i \otimes_A A t^i$とおくと$t^{-1}$はC-正則列であり、また$K_G \cong C/t^{-1}C$であるからCは極大Cohen-Macaulay R'-加群である。さらに$(0):_{R'} C = (0)$が容易に確かめられるので$C \cong K_{R'}$がわかる。よってωの一意性より$\omega = K$が導かれるので$K_R \cong \Sigma_{i\geq 1} \omega_i \otimes_A A t^i$を得る。□

定理(1.1)の証明 $\mathcal{A}$が空でない有限集合であることは$\mathcal{A} = \{ q\cap A \mid q\in \mathrm{Ass}G \}$を示せば十分である。$P = q\cap A$ ($q\in \mathrm{Ass}G$)とする。G_PはCohen-Macaulay環なので

$$\dim G_P = \dim G_P/qG_P \leqq \dim G_P/PG_P \leqq \dim G_P$$

が成り立つ。故に$\dim A_P = \dim G_P/PG_P$である。逆に$P\in\mathcal{A}$とするとPG_Pを含むようなG_Pの極小素イデアル$\mathfrak{p}$が存在する。このとき$q = \mathfrak{p}\cap G$はGの極小素イデアルであり、一方で$PA_P = \mathfrak{p}\cap A_P$なので$P = q\cap A$が成り立つ。故に$\mathcal{A}=\{q\cap A \mid q\in \mathrm{Ass}G\}$であることがわかった。これからは$A = K_A$, $K_G = \oplus_{i\geq -a}\omega_{i-1}/\omega_i$とする。(2)⇒(1):与えられた次数加群の単射を、$\psi: G(a)\to K_G$とおけば$\psi(1) = x^*$となるような$x\in A$が存在する。但しx^*はxを含むω_{-a}の類を表す。このとき、次数G-加群の射$\varphi: G(a)\to K_G$ ($\varphi(1) = 1^*$)とx^*によるK_Gのmultiplicationとの合成射は単射ψとなる。だからφは単射である。よってすべての$i\geq 0$に対して$\omega_{i-a-1}\cap F_{i-1} = F_i$が成り立つ。従ってiについての帰納法によって$\omega_{i-a-1} = F_i$を得る。(1)⇒(3)は明か。(3)⇒(2):次数G-加群の射$\varphi: G(a)\to K_G$ ($\varphi(1) = 1^*$)について$X = \ker\varphi \neq (0)$とする。このとき$\mathfrak{p}\in \mathrm{Ass}_G X$が存在し、そして$P = \mathfrak{p}\cap A$とおく。一般論によると$P\in \mathrm{Ass}_A X \subseteq \mathrm{Ass}_A G = \{q\cap A \mid q\in \mathrm{Ass}G\}$である。従って仮定より次数加群の同型 $G_P(a) \cong K_{G_P}$ が存在する。このとき$a=a(G_P)$であって、次数G_P-加群の射$\varphi\otimes A_P: G_P(a)\to K_{G_P}$の$-a$次は全射であるから、$\varphi\otimes A_P$は同型であることがわかる。故に$X_P = (0)$。これは矛盾である。□

3. ω について

ここでは、(A,m)をd次元Cohen-Macaulay局所環でGorenstein環の準同型像であると仮定する。I (≠A)をAのイデアルで $s=\mathrm{ht}_A I$ とおく。そしてIは、s個の元で生成

されるreductionイデアルQを持つと仮定せよ。とくにs=0ときは$Q=(0)$と考える。rをQに関するIのreduction numberとする。またイデアルの零又は負のべきはAとする。$R=\mathcal{R}(I)$, $R'=\mathcal{R}'(I)$, $G=\mathcal{G}(I)$ とおき、さらにK_A, K_R, $K_{R'}$, K_G, $K_{\mathcal{R}'(Q)}$によってそれぞれA, R, R', G, $\mathcal{R}'(Q)$の正準加群を表す。そしてa = a(G) とおく。以下 GはCohen-Macaulay環であると仮定し、命題(2.1)のK_G表わすK_Aの部分加群の族を$\omega=\{\omega_i\}_{i\in\mathbb{Z}}$とする。この節の目的は$\omega$を書き表すこと、すなわち、これらの正準加群の形を決めることである。

定理(3.1) すべての$i\in\mathbb{Z}$に対して次が正しい。

(1) $s\geq 1$のとき、$\omega_i=Q^{i-s+r+1}K_A:_{K_A}I^r$ である。

(2) s=0のとき、$\omega_i=(0):_{K_A}I^{-i}$ である。

証明 $s=\mathrm{ht}_A Q$でありAはCohen-Macaulay局所環なので、Qは長さsの正則列で生成される。よって$s\geq 1$のとき$\mathcal{G}(Q)$はs変数多項式環であるから、$\mathcal{G}(Q)$の正準加群$K_{\mathcal{G}(Q)}$の生成元は-s次にしか現われない。だから[HSV] によって、$K_{\mathcal{R}'(Q)}\cong\bigoplus_{i\in\mathbb{Z}}Q^{i-s+1}K_A$である。s=0のときは、$K_{\mathcal{R}'(Q)}\cong(K_A\otimes_A A[t^{-1}])(1)$となっている。これらのことより、どんな次元に対しても$K_{\mathcal{R}'(Q)}=\Sigma_{i\in\mathbb{Z}}Q^{i-s+1}K_A\otimes_A At^i$ と表記してよい。さて、$C=R'/\mathcal{R}'(Q)$として次の次数$\mathcal{R}'(Q)$-加群の完全列を考える。

$$0\to\mathcal{R}'(Q)\xrightarrow{f}R'\to C\to 0$$

ただしfは包含写像である。Cにt^{-1}の十分大きなべきをかければ消えるので、Cの次元はd+1ではない。よって$K_{\mathcal{R}'(Q)}$-dualをとれば、次数$\mathcal{R}'(Q)$-加群の単射

$$f^*:\mathrm{Hom}_{\mathcal{R}'(Q)}(R',K_{\mathcal{R}'(Q)})\to\mathrm{Hom}_{\mathcal{R}'(Q)}(\mathcal{R}'(Q),K_{\mathcal{R}'(Q)})$$

を得る。R'は$\mathcal{R}'(Q)$上有限生成加群なので$K_{R'}=\mathrm{Hom}_{\mathcal{R}'(Q)}(R',K_{\mathcal{R}'(Q)})$としてよい。そしてf*と標準的な同型$\mathrm{Hom}_{\mathcal{R}'(Q)}(\mathcal{R}'(Q),K_{\mathcal{R}'(Q)})\cong K_{\mathcal{R}'(Q)}$との合成射を、

$$\varphi:K_{R'}\to K_{\mathcal{R}'(Q)}\ \ (\subseteq K_A\otimes_A A[t,t^{-1}])$$

とおく。また、φのt^{-1}による局所化$\varphi_{t^{-1}}:K_A\otimes_A A[t,t^{-1}]\to K_A\otimes_A A[t,t^{-1}]$は同型であるので、$\varphi$を経由して得られる射$\varphi':K_{R'}\to K_A\otimes_A A[t,t^{-1}]$は$K_{R'}$の$t^{-1}$による局所化であることがわかる。だから$\mathrm{Im}\ \varphi=\Sigma_{i\in\mathbb{Z}}\omega_i\otimes_A At^i$が成り立つ。さらに、$\mathrm{Im}\ \varphi=\{g(1)\mid g\in\mathrm{Hom}_{\mathcal{R}'(Q)}(R',K_{\mathcal{R}'(Q)})\}$であるので、

$$\omega_i = \cap_{n\in\mathbb{Z}}(Q^{i-s+n+1}K_A :_{K_A} I^n)$$

が示される。実際、

$$I^n\omega_i \otimes_A A t^{i+n} \subseteq \omega_{i+n} \otimes_A A t^{i+n} = [\mathrm{Im}\varphi]_{i+n} \subseteq Q^{i-s+n+1}K_A \otimes_A A t^{i+n}$$

が成り立ち、逆に任意の右辺の元xに対して、

$$g : R' \to K_{\mathcal{R}'(Q)} ; h(\alpha) = \alpha(x\otimes_A t^i)$$

と定めることができ、gは次数がiの$\mathcal{R}'(Q)$-線型写像で$g(1) = x\otimes_A t^i$である。さて、次の補題を確かめることは標準的な方法でできる。

補題 $s\geq 1$とすれば次が正しい。

(1)$m, n\in\mathbb{Z}$ ならば $Q^m K_A :_{K_A} I^n \subseteq Q^{m-1}K_A :_{K_A} I^{n-1}$が成り立つ。

(2)$j\in\mathbb{Z}, n\geq r$ ならば $Q^{n+j}K_A :_{K_A} I^n = Q^{r+j}K_A :_{K_A} I^r$が成り立つ。

このことより$s\geq 1$のとき任意の$i\in\mathbb{Z}$に対して、$\omega_i = Q^{i-s+r+1}K_A :_{K_A} I^r$が示される。

$s=0$の場合は$Q=(0)$であるので、

$$\begin{aligned} Q^{i+n+1}K_A :_{K_A} I^n &= K_A :_{K_A} I^n = K_A && (n\leq -i-1)\\ &= (0) :_{K_A} I^n && (r\geq n\geq -i)\\ &= (0) :_{K_A} (0) = K_A && (n\geq r) \end{aligned}$$

となり、任意の$i\in\mathbb{Z}$に対して、$\omega_i = (0) :_{K_A} I^{-i}$が成り立つことがわかる。□

命題(3.2) $s\geq 1$のとき任意の$j\geq 0$に対して次が成り立つ。

$$Q^{r+j}K_A :_{K_A} I^r = Q^j(Q^r K_A :_{K_A} I^r)$$

証明 $a_1, \cdots, a_s$をQの生成元とすると、$a_1, \cdots, a_s$はA-正則列であり、$a_1t, \cdots, a_st$はG-正則列である。よって次数加群として

$$\begin{aligned} K_{G(I/Q)} &\cong K_{G/(a_1t, \cdots, a_st)G}\\ &\cong [K_G/(a_1t, \cdots, a_st)K_G](s)\\ &\cong [\oplus_{i\geq -a}\omega_{i-1}/Q\omega_{i-2}+\omega_i](s) \end{aligned}$$

であり、定理(3.1)(2)より$K_{G(I/Q)}$の1次以上は消えていることがわかるので、任意の$i\geq s+1$に対して$\omega_{i-1} = Q\omega_{i-2}+\omega_i$が成り立つ。よって$K_{R'}$は有限生成なので$\omega_{i-1} =$

$Q\omega_{i-2}$を得る。ここで定理(3.1)(1)を使えば主張を得る。□

μを極小生成系の個数とすると、Cohen-Macaulay型 $r(*)$ は次のように表わすことができる。

定理(3.3) 次が成り立つ。

(1) $r(G) = \Sigma_{0\leq i\leq r}\ \mu[Q^iK_A :_{K_A} I^r/(Q^{i+1}K_A :_{K_A} I^r + I(Q^{i-1}K_A :_{K_A} I^r))]$ $(s\geq 1)$

$\quad = \Sigma_{0\leq i\leq r}\ \mu[(0):_{K_A} I^{i+1}/((0):_{K_A} I^i + I((0):_{K_A} I^{i+2}))]$ $(s=0)$

(2) RはCohen-Macaulay環であって、$\dim R = \dim A + 1$ ならば、

$$r(R) = r(A)(-a-1) + \Sigma_{1\leq i\leq r}\ \mu[Q^iK_A :_{K_A} I^r/I(Q^{i-1}K_A :_{K_A} I^r)]$$

証明 $\mathfrak{M} = mR + R_+$ とおくと、

$K_G/\mathfrak{M}K_G \cong \oplus_{i\geq -a}\ \omega_{i-1}/(\omega_i + m\omega_{i-1} + I\omega_{i-2})$

であり、注意(2.2)より $K_R \cong \oplus_{i\geq 1}\ \omega_i$ なので

$$\begin{aligned} K_R/\mathfrak{M}K_R &\cong \omega_1/m\omega_1 \oplus \omega_2/(m\omega_2 + I\omega_1) \oplus \cdots \oplus \omega_{-a-1}/(m\omega_{-a-1} + I\omega_{-a-2}) \oplus \\ &\qquad (\oplus_{i\geq -a}\omega_i/(m\omega_i + I\omega_{i-1})) \\ &\cong (K_A/mK_A)^{(-a-1)} \oplus (\oplus_{i\geq -a}\omega_i/(m\omega_i + I\omega_{i-1})) \end{aligned}$$

となる。(2)の場合は$a\leq -1$であり$s\geq 1$となることに注意すれば、定理(3.1)と命題(3.2)に従う。□

系(3.4) 次が正しい。

(1) $s\geq 1$のとき、次の条件は同値である。

(a) GはGorenstein環ある。

(b) AはGorenstein環あって、$I^i = Q^i :_A I^r$ $(1\leq \forall i \leq r)$ が成り立つ。

(2) $s=0$のとき、次の条件は同値である。

(a) GはGorenstein環ある。

(b) AはGorenstein環あって、$I^{r-i+1} = (0):_A I^i$ $(1\leq \forall i \leq r)$ が成り立つ。

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(On local cohomological dimension and local connectedness),

in page 157—160 of this volume

to the public on the web, because of the request by the author.

Linear maximal Cohen-Macaulay modules over a one-dimensional Cohen-Macaulay homogeneous k-algebra

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1 Introduction.

The main purpose of this talk is to prove the following theorem.

Theorem 1.1 (cf. Yoshida [9]) Let k be an algebraically closed field and $A = k[A_1]$ a homogeneous Cohen-Macaulay k-algebra with $\dim A = 1$. Then A is reduced if and only if $\operatorname{gr}\mathcal{LC}(A)$ is of finite representation type, that is, there are only a finite number of isomorphism classes of indecomposable graded linear maximal Cohen-Macaulay A-modules up to degree shifting.

Moreover, in this case, any graded linear maximal Cohen-Macaulay A-module is isomorphic to a finite direct sum of copies of A/p, where p runs through minimal prime ideals of A.

Before proving the above theorem, we recall some basic terminology. Throughout this paper, let k be an infinite field and let $A = k[A_1]$ be an $\mathbb{N}$-graded Cohen-Macaulay k-algebra which is generated in degree one and we call such A a homogeneous Cohen-Macaulay k-algebra.

Let $\operatorname{gr}\mathcal{M}(A)$ (resp. $\operatorname{gr}\mathcal{C}(A)$) be the category of finite graded A-modules (resp. maximal Cohen-Macaulay (abbr. MCM) A-modules) and graded A-homomorphisms preserving degree.

In general, if M is a graded MCM A-module then the inequality

$$\mu_A(M) \leq e_A(M)$$

holds, where $\mu_A(M)$ denotes the minimal number of generators of M and $e_A(M)$ denotes the multiplicity of M with respect to the irrelevant maximal ideal $\mathfrak{m} := A_+$ of A. M is said to be *linear* MCM A-module if the equality holds in the above inequality. Then M satisfies $\mathfrak{m}M = \mathfrak{q}M$ for any minimal reduction $\mathfrak{q}$ of $\mathfrak{m}$, and the converse is also true. Note that any direct sum of linear MCM A-modules and any direct summand of a linear MCM A-module are also one. See e.g. [2],[5] for details.

Let $\operatorname{gr}\mathcal{LC}(A)$ denote the full subcategory of $\operatorname{gr}\mathcal{C}(A)$ which consists of graded linear MCM A-modules. Under the above notation, we give the definition of finite representation type.

Definition 1.2 (cf. [8]) We say that the category $\mathrm{gr}\,\mathcal{LC}(A)$ (resp. $\mathrm{gr}\,\mathcal{C}(A)$) is *of finite representation type* if there are only a finite number of isomorphism classes of indecomposable graded linear MCM (resp. MCM) A-modules up to degree shifting. Moreover, then we also say that A is *of finite linear CM (resp. finite CM) representation type.*

Now let k be an algebraically closed field of characteristic 0. Then the complete classification of homogeneous Cohen-Macaulay k-algebras of finite CM representation type has been given due to Eisenbud-Herzog [3]; see also Remark 3.

In connection with this result, we propose the following question.

Question 1.3 *When is* $\mathrm{gr}\,\mathcal{LC}(A)$ *of finite representation type ?*

However, there exists a still open conjecture saying that any Cohen-Macaulay k-algebra admits at least one linear MCM A-module. So we must add some superficial conditions to consider the above question. On the other hand, the existence of (graded) linear MCM A-modules are already known in the following cases:

(i) A is a strict complete intersection; see Herzog et.al. [5].

(ii) A has a maximal embedding dimension; see Brennan et.al. [2, Proposition (2.5)].

(iii) $\dim A \leq 1$.

(iv) A is a two-dimensional homogeneous Cohen-Macaulay domain over an infinite field; see brennan et.al. [2, Theorem (4.8)].

In this paper, we give an answer to the above Question (1.3) in the case of $\dim A = 1$ and $k = \overline{k}$.

2 Proof of Theorem (1.1).

In this section, we give a proof of Theorem (1.1). We first prove the only if part. In fact, the following proposition gives a structure theorem for linear MCM A-modules over a homogeneous reduced k-algebra with $\dim A = 1$.

Proposition 2.1 *Let k be an algebraically closed field of any characteristic and $A = k[A_1]$ a reduced homogeneous k-algebra with $\dim A = 1$. Then any graded linear MCM A-module M can be written as follows:*

$$M \cong F_1 \oplus F_2 \oplus \ldots \oplus F_e,$$

where $\mathrm{Min}(A) = \{p_1, \ldots, p_e\}$, the set of minimal prime ideals of A and each F_i is a graded finite free A/p_i-module.

In particular, $\mathrm{gr}\,\mathcal{LC}(A)$ is of finite representation type.

Before proving this proposition, we recall the following well-known lemma needed later.

Lemma 2.2 (cf. Goto and Watanabe [4, Proposition (2.2.11)]) *Let k be an algebraically closed field and $A = k[A_1]$ a homogeneous k-algebra with $\dim A = 1$. If A is an integral domain, then A is k-isomorphic to a polynomial ring over k.*

In particular, if p is a minimal prime ideal of A, then we can write as follows:

$$A = k[x_1, \ldots, x_{v-1}, x_v], \qquad p = (x_1, \ldots, x_{v-1}).$$

We now give a proof of the above proposition. By virtue of Lemma (2.2), for any minimal prime ideal p, we have that A/p is a polynomial ring over k and thus is a graded linear MCM A-module. Moreover, it goes without saying that it is indecomposable as an A-module.

We prove the converse, that is, any graded linear MCM A-module M can be written as

$$M \cong F_1 \oplus F_2 \oplus \ldots \oplus F_e,$$

where each F_i is a graded finite free A/p_i-module.

We first note that $\mathfrak{m}^{e-1}$ is a linear MCM A-module, where $e = e(A)$ is the multiplicity of A with respect to $\mathfrak{m}$. In order to see this, let uA a minimal reduction of $\mathfrak{m}$ and let r be a reduction exponent of $\mathfrak{m}$, the smallest non-negative integer r which satisfies the equality $\mathfrak{m}^{r+1} = u\mathfrak{m}^r$. Because A is a homogeneous Cohen-Macaulay k-algebra, we know that this integer r equals the integer s such that $\overline{A}_s \neq 0$, $\overline{A}_n = 0$ for all integer $n \geq s+1$, where $\overline{A} = A/uA$. Therefore we get $e = l(\overline{A}) \geq r+1$; hence $\mathfrak{m}\mathfrak{m}^{e-1} = u\mathfrak{m}^{e-1}$. Namely, $\mathfrak{m}^{e-1}$ is a linear MCM A-module; hence $\mu_A(\mathfrak{m}^{e-1}) = e_A(\mathfrak{m}^{e-1}) = e$.

On the other hand, by virtue of Lemma (2.2) again, we can easily obtain that $p_i + p_j = \mathfrak{m}$ for all $1 \leq i < j \leq e$. Therefore we get

$$\mathfrak{m}^{e-1} = \sum_{1 \leq i \leq e} p_1 \cdots \check{p}_i \cdots p_e.$$

In fact, since $p_1 \cdots \check{p}_i \cdots p_e \cap p_1 \cdots \check{p}_j \cdots p_e = 0$ for all $i \neq j$, it follows that

$$\mathfrak{m}^{e-1} = \bigoplus_{1 \leq i \leq e} p_1 \cdots \check{p}_i \cdots p_e.$$

Thus each $p_1 \cdots \check{p}_i \cdots p_e$ is also a non-zero linear MCM A-module; hence $\mu_A(p_1 \cdots \check{p}_i \cdots p_e) = 1$. So we put $p_1 \cdots \check{p}_i \cdots p_e = g_i A$ for every i.

Now let M be any graded linear MCM A-module. Then since $uM = \mathfrak{m}M$, we get

$$M \cong u^{e-1}M = \mathfrak{m}^{e-1}M = g_1 M + \ldots + g_e M.$$

As $g_i g_j = 0$ for $i \neq j$, we have

$$\mathfrak{m}^{e-1}\left[g_1 M \cap (g_2, \ldots, g_e)M\right] = 0.$$

It follows that $N := g_1 M \cap (g_2, \ldots, g_e)M$ is a module of finite length. On the other hand, as N is a submodule of M, it must be zero. Therefore

$$M \cong g_1 M \oplus \ldots \oplus g_e M.$$

Then because of $p_i g_i = 0$ we have that $g_i M$ is a graded linear MCM A/p_i-module and thus is a finite direct sum of A/p_i, for A/p_i is a polynomial ring over k. It follows that M itself can be written as the required form. **QED**

Example 1 *Let $A = k[X]$ be a polynomial ring in one variable. Then any graded MCM A-module is a graded finite free A-module.*

Example 2 *Let* $A = k[X, Y, Z]/(XY, XZ, YZ(Y+Z))$. *Then*

$$\mathrm{Min}(A) = \{(x, y), (x, z), (y, z), (x, y+z)\}$$

and we have

$$\mathfrak{m}^3 = (x^3, y^2(y+z), z^2(y+z), y^2z)A.$$

Example 3 *Let* $A = k[X_1, X_2, \ldots, X_v]/(X_iX_j \mid 1 \leq i < j \leq v)$ *and put* $u = x_1 + \ldots + x_v$. *Then* $e = v$ *and for any graded linear MCM* A*-module* M *we have*

$$M \cong \mathfrak{m}M = x_1M \oplus \ldots \oplus x_vM$$

and each x_iM *is a finite direct sum of* $A/(x_1, \ldots, \check{x}_i, \ldots, x_v)A$.

In order to complete the proof of Theorem (1.1), we next prove if part.

Proposition 2.3 *Let* k *be an algebraically closed field and* $A = k[A_1]$ *a homogeneous Cohen-Macaulay* k*-algebra with* $\dim A = 1$. *If* $\mathrm{gr}\,\mathcal{LC}(A)$ *is of finite representation type, then* A *is reduced.*

Proof. First of all, we may assume that A is irreducible. Actually, let $(0) = q_1 \cap \ldots \cap q_e$ be a homogeneous irredundant primary decomposition of (0) in A. Then since A/q_i also satisfies the assumption of this proposition, if the assertion is true in the irreducible case, we can obtain that A/q_i is an integral domain. It follows that A is reduced.

We assume that A is irreducible and $\mathrm{Min}(A) = \{p\}$. By virtue of Lemma (2.2), we can write as

$$S = k[X_1, \ldots, X_{v-1}, X_v], \quad P = (X_1, \ldots, X_{v-1})S, \quad A = S/I \quad \text{and } p = P/I,$$

where $X_1, \ldots, X_v$ are indeterminates over k. Moreover, we may assume that $t = X_v \bmod I$ generates a minimal reduction of $\mathfrak{m} = A_+$. For simplicity, we put $x_i = X_i \bmod I$ for $i = 1, \ldots, v-1$.

Suppose that A is not an integral domain (that is, not a polynomial ring over k). Then $\overline{A} := A/tA$ is a homogeneous Artin k-algebra and we have $\overline{A}_s \neq 0$, $\overline{A}_n = 0$ for some positive integer s and for all $n \geq s+1$. Putting $S' := k[X_1, \ldots, X_{v-1}] \subseteq S$, we can write as follows:

$$\overline{A} = S'/J = \overline{A}_0 \oplus \overline{A}_1 \oplus \ldots \oplus \overline{A}_s.$$

For each integer i with $1 \leq i \leq s$, we take elements $F_{\ell,1}, \ldots, F_{\ell,i_\ell} \in S'_\ell$ the images of which form a k-basis of $\overline{A}_\ell$, where $i_\ell = \dim_k \overline{A}_\ell$. Then we define the following ideal I_n of A for every integer $n \geq s+1$:

$$I_n := \left(t^n, \{f_{\ell,j}t^{s-\ell}\} \mid \ell = 1, \ldots, s\,; j = 1, \ldots, i_\ell\right) A,$$

where $f_{\ell,j}$ denotes the image of $F_{\ell,j}$ in A.

Then since I_n contains a non-zero divisor t^n, it is an $\mathfrak{m}$-primary ideal, and thus is a graded MCM A-module. Moreover, it follows that every I_n is an indecomposable A-module from the following remark.

Remark 1 Let A be a one-dimensional homogeneous Cohen-Macaulay irreducible k-algebra Then for any homogeneous $\mathfrak{m}$-primary ideal I, it is a graded indecomposable A-module.

Proof. Suppose that there exists a non-trivial decomposition $I = I_1 \oplus I_2$ of I. Then both I_1 and I_2 are homogeneous ideals of A and $I_1 I_2 = 0$. If at least one of them is an $\mathfrak{m}$-primary ideal (say I_1), then I_1 contains a non-zero divisor of A, which implies $I_2 = 0$. This contradicts the assumption. Therefore both I_1 and I_2 are contained in the unique minimal prime ideal p of A. Then I itself is contained in p; this is a contradiction. $\square$

We return to the proof of the above proposition. We show that the above I_n is a linear MCM A-module for every $n \geq s+1$.

Let $e = e(A)$ be the multiplicity of A with respect to $\mathfrak{m}$. Then we shall prove the following claim.

Claim: $\left\{t^n, \{f_{\ell,j}t^{s-\ell}\}_{\ell,j}\right\}$ form a minimal basis of I_n.

Suppose not. Noting $deg(t^n) = n > s$ and $\deg(f_{\ell,j}t^{s-\ell}) = s$, we get either one of the following equation:

$$t^n = \sum_{\substack{1\leq \ell \leq s \\ 1 \leq j \leq i_\ell}} f_{\ell,j} t^{s-\ell} g_{\ell,j} \qquad \text{for some } g_{\ell,j} \in A_{n-s} \qquad Eq.(1)$$

or

$$\sum_{\substack{1\leq \ell \leq s \\ 1 \leq j \leq i_\ell}} c_{\ell,j} f_{\ell,j} t^{s-\ell} = 0 \qquad \text{for some } c_{\ell,j} \in k \text{ with } \{c_{\ell,j}\} \neq \{0\}. \qquad Eq.(2)$$

Now first suppose that Eq.(1) holds. Then as $f_{\ell,j} \in p = (x_1, \ldots, x_{v-1})A$ we have $t^n \in p$; hence $t \in p$, which contradicts the choice of t. Next suppose that Eq.(2) holds. Then we have

$$\sum_{j=1}^{i_s} c_{s,j}\overline{f_{s,j}} = 0 \quad \text{in} \quad \overline{A}$$

Since $\{\overline{f_{s,j}}\}_{j=1,\ldots,i_s}$ form a k-basis of $\overline{A}_\ell$, we can get $c_{s,j} = 0$ for all j; then Eq.(2) can be replaced with

$$\sum_{\substack{1\leq \ell \leq s-1 \\ 1 \leq j \leq i_\ell}} c_{\ell,j} f_{\ell,j} t^{s-1-\ell} = 0,$$

because t is a non-zero divisor of A. Then

$$\sum_{j=1}^{i_{s-1}} c_{s-1,j}\overline{f_{s-1,j}} = 0 \quad \text{in} \quad \overline{A}.$$

This implies that $c_{s-1,j} = 0$ for all j. Repeating this argument, we can obtain that $c_{\ell,j} = 0$ for all $\ell = 1, \ldots, s$ and for all j; this is a contradiction. This completes the proof of the claim.

Therefore every I_n is a graded indecomposable linear MCM A-module and the minimal (resp. maximal) degree of its minimal system of generators equals s (resp. n). In particular, $\{I_n\}_{n\geq s+1}$ are infinite non-isomorphic graded indecomposable linear MCM A-modules. **QED**

Example 4 (cf. [8, Example (6.5)]) *Let* $A = k[X, Y]/(Y^2)$. *Put* $I_n = (x^n, y)A$ *for every positive integer* n. *Then* $\{I_n\}_{n\geq 1}$ *is the complete list of non-isomorphic graded indecomposable linear MCM A-modules.*

Any zero-dimensional homogeneous k-algebra is of finite linear CM representation type. In fact, in this case, the residue field k is a unique indecomposable linear MCM A-module. Therefore we have the following corollary.

Corollary 2.4 *Let k be an algebraically closed field and $A = k[A_1]$ a homogeneous Cohen-Macaulay k-algebra with* $\dim A \leq 1$. *Then A is of finite linear CM representation type if and only if it is of isolated singularity.*

Remark 2 Auslander [1] has proved that if A is of finite CM representation type then it is an isolated singularity.

In the final of this section, we quote one-dimensional case of the Classification Theorem for homogeneous Cohen-Macaulay k-algebras with finite CM representation type which is due to Eisenbud-Herzog [3].

Remark 3 (cf. [3],[8, Theorem (17.10)]) *Let k be an algebraically closed field of characteristic 0 and $A = k[A_1]$ be a homogeneous Cohen-Macaulay k-algebra with* $\dim A = 1$. *If A is of finite CM representation type, then A is k-isomorphic to one of the following rings:*

(1) $k[X]$.

(2) $k[X, Y]/(XY)$.

(3) $k[X, Y]/(XY(X+Y))$.

(4) $k[X, Y, Z]/(XY, YZ, ZX)$.

3 Two-dimensional cases.

In case of a certain two-dimenional Cohen-Macaulay homogeneous k-algebra, the author has proved an analogy of Theorem (1.1) in [10].

Theorem 3.1 (cf. Yoshida [10]) Let k be an algebraically closed field of characteristic $\neq 2$ and $A = k[A_1]$ a homogeneous Cohen-Macaulay k-algebra with $\dim A = 2$. Furthermore, we suppose that A has a maximal embedding dimension, that is, $\dim_k A_1 = \dim A + e(A) - 1$, where $e(A)$ is a multiplicity (degree) of A. Then the following conditions are equivalent.

(1) $\mathrm{gr}\,\mathcal{LC}(A)$ is of finite representation type, that is, there are only a finite number of isomorphism classes of indecomposable graded linear maximal Cohen-Macaulay A-modules up to degree shiting.

(2) $\mathrm{gr}\,\mathcal{LB}(A)$ is of finite representation type, that is, there are only a finite number of isomorphism classes of indecomposable graded linear maximal Buchsbaum A-modules up to degree shifting.

(3) A is a normal domain.

(4) A is k-isomorphic to the rational normal scroll of type (e), where $e = e(A)$;

$$A \cong k[X_0, X_1, \ldots, X_e] \Big/ I_2 \begin{pmatrix} X_0 & X_1 & \cdots & X_{e-1} \\ X_1 & X_2 & \cdots & X_e \end{pmatrix}, \qquad \text{where } \deg(X_i) = 1.$$

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REMARKS ON A DEPTH FORMULA, A GRADE INEQUALITY AND A CONJECTURE OF AUSLANDER

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1. A DEPTH FORMULA

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and let M and N be finitely generated R-modules. We define the integer $q^R(M, N)$ as follows;

$$q^R(M,N) = \sup\{i \mid \operatorname{Tor}_i^R(M,N) \neq 0\}.$$

Theorem 1. *Let M and N be finitely generated R-modules with $\operatorname{pd} N < \infty$. Putting $q = q^R(M,N)$, if $\operatorname{depth} \operatorname{Tor}_q^R(M,N) \leqq 1$ or if $q = 0$, then we have the equality*

$$\operatorname{depth} M + \operatorname{depth} N = \operatorname{depth} R + \operatorname{depth} \operatorname{Tor}_q^R(M,N) - q.$$

It is proved by Auslander. Huneke and Wiegand observe the case of $q = 0$ and define that M and N satisfy the *depth formula* provided the following equality is hold;

$$\operatorname{depth} M + \operatorname{depth} N = \operatorname{depth} R + \operatorname{depth} M \otimes_R N.$$

And they show the following;

Theorem 2. *Let R be a ring of complete intersection, and let M and N be finitely generated R-modules with $q^R(M,N) = 0$. Then M and N satisfy the depth formula.*

We show in the following a unified version of these two theorems using CI dimension whose definition is due to Avramov-Gasharov-Peeva.

Theorem 3. *Let M and N be finitely generated R-modules with $\text{CI-dim}_R N < \infty$ and assume that $q = q^R(M,N) < \infty$. If $\operatorname{depth} \operatorname{Tor}_q^R(M,N) \leqq 1$ or $q = 0$, then we have the equality*

$$\operatorname{depth} M + \operatorname{depth} N = \operatorname{depth} R + \operatorname{depth} \operatorname{Tor}_q^R(M,N) - q.$$

So M and N satisfy the depth formula if $\text{CI-dim}_R N < \infty$ and if $q = 0$.

2. A GRADE INEQUALITY

For finitely generated R-modules L and M, recall that $\operatorname{grade}(L, M)$ is defined as follows;

$$\operatorname{grade}(L,M) = \inf\{i \mid \operatorname{Ext}_R^i(L,M) \neq 0\}.$$

Note that $\operatorname{grade}(k, M) = \operatorname{depth} M$ for any finitely generated R-module M. Hence the depth formula can be described in the following form;

$$\operatorname{grade}(k,M) + \operatorname{grade}(k,N) = \operatorname{grade}(k,R) + \operatorname{grade}(k, M \otimes_R N).$$

And it is hold if $\text{CI-dim}_R N < \infty$ and $q^R(M, N) = 0$ by Theorem 3. We are interested in when the equality is hold for an R-module L instead of k.

Theorem 4. *Let M and N be finitely generated R-modules and suppose that $\operatorname{pd} N < \infty$ and $q^R(M, N) = 0$. Then, for any finitely generated R-module L, we have the inequality*

$$\operatorname{grade}(L, M \otimes_R N) \leqq \operatorname{grade}(L, M) \leqq \operatorname{grade}(L, M \otimes_R N) + \operatorname{pd} N$$

Example 5. For any non-negative integers $l \leqq m \leqq n$, there exists an example such that $\operatorname{grade}(L, M \otimes_R N) = l, \operatorname{grade}(L, M) = m$, and $\operatorname{grade}(L, M \otimes_R N) + \operatorname{pd} N = n$.

3. A CONJECTURE OF AUSLANDER

We define the integer $p^R(M, N) = \sup\{i | \operatorname{Ext}^i_R(M, N) \neq 0\}$. Then we can prove the followings;

Theorem 6. *Let M and N be finitely generated R-modules. Assume that $\text{CI-dim}_R M < \infty$ and $p^R(M, N) < \infty$. Then we have the equality*

$$p^R(M, N) = \operatorname{depth} R - \operatorname{depth} M.$$

In particular, $p^R(M, N)$ is independent of N in this case.

Auslander conjectured that for a finitery generated R-module M, the condition $\operatorname{Ext}^i_R(M \oplus R, M \oplus R) = 0$ for all $i > 0$ would imply that M is free. It is proved by Auslander for the cases where R is a complete intersection but it is still open for general cases. We prove it for the cases where M has finite CI dimension.

Theorem 7. *Let M be a finitely generated R-module with $\text{CI-dim}_R M < \infty$. If $p^R(M, M) = 0$, then M is free.*

Proof. Since $\text{CI-dim}_R M < \infty$, there exists a quasi-deformation $R \to R' \leftarrow S$ with $\operatorname{pd}_S M' < \infty$, and we may assume that R is complete local ring by [3] Prop1.13. Without loss of generality we may assume that $(S, \mathfrak{n})$ is a deformation of R of codimension r. Since $p^R(M, M) = 0$, we have $\operatorname{Ext}^2_R(M, M) = 0$, so M is liftable to S, i.e. there exists an S-module N such that $N \otimes_S R \cong M$ and $\operatorname{Tor}^S_i(N, R) = 0$ for $i > 0$. See [2] (1.7) or [8] (1.6) . Let $\mathbf{x}= x_1, x_2, \cdots, x_r \in \mathfrak{n}$ be an S-sequence with $S/(\mathbf{x}) \cong R$. Then $\mathbf{x}$ is also N-sequence. Since $\operatorname{pd}_S M < \infty$ and $N/(\mathbf{x})N \cong M$, we have $\operatorname{pd}_S N < \infty$ and hence $\operatorname{pd}_R M < \infty$. Thus we get from theorem 6 that $\operatorname{pd}_R M = \operatorname{depth} R - \operatorname{depth} M = p^R(M, M) = 0$, hence M is free. □

Corollary 8. *Let M be a finitely generated R-module with $\text{CI-dim}_R M < \infty$. Then we have the equality $\operatorname{pd}_R M = p^R(M, M)$.*

Example 9. Schulz showed that there is a counterexample if R is noncommutative.

Let L be a division ring and a is an element of L such that inner derivation $\partial_a : x \mapsto ax - xa$ is surjective. Such L exists according to [4]. Let $K = L((T))$ be the field of formal Laurent series in variable T which commutes with L. We put $R = K < X, Y > /(X^2, Y^2, XY + TYX)$ where X and Y are variables which commute with K. For each $i \geqq 0$, we define left R homomorphisms $d_i : R \ni \alpha \mapsto \alpha(X + aT^iY) \in R$. In this case, $\cdots \to R \xrightarrow{d_i} R \xrightarrow{d_{i-1}} R \to \cdots \xrightarrow{d_0} R$ is exact sequence. So we put $M = \operatorname{Coker} d_0$, then M is not projective but $\operatorname{Ext}^i_R(M, M) = 0$ for all $i > 0$.

The last of my talk, I said two problems.

Problem

(1) If $\text{G-dim}_R M < \infty$ and if $\operatorname{Ext}^i_R(M, M) = 0$ for all $i > 0$, Then M is free.

(2) If $\operatorname{Ext}^i_R(M, R) = 0$ for all $i > 0$, then it hold $\operatorname{Ext}^i_R(trM, R) = 0$ for all $i > 0$.

But Miyachi told me that there is a counterexample for (2) if R is non-commutative.

Example 10. Let k be a commutative field,$R = k < X, Y > /(X^2, Y^2, XY - \lambda YX)$ where X and Y are variables which commute with k and λ is non zero element of k such that $(-\lambda)^i \neq 1$ for all $i > 0$, and $M = R/R(X + Y)$.

We put $A = \begin{pmatrix} R & {}_RM_k \\ 0 & k \end{pmatrix}$, ${}_AX = \begin{pmatrix} \Omega^1_R M \\ 0 \end{pmatrix}$.

Then we have $\operatorname{Ext}^i_A(X, A) = 0$ for all $i > 0$.

On the other hand, by the following exact sequence;

$$0 \to \operatorname{Ext}^1_A(trX, A) \to X \to X^{**} \to \operatorname{Ext}^2_A(trX, A) \to 0,$$

and since X is not reflexive, we have $\operatorname{Ext}^1_A(trX, A) \neq 0$ or $\operatorname{Ext}^2_A(trX, A) \neq 0$.

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可換環上の単純拡大について（その2）

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この研究報告は昨年の可換環論 Symposium（第18回）（インテック大山研修センターで開催）の続きという事でその2とさせていただきました。

Notation

R: integral domain $\subset$ K: quotient field

α: K上代数的な元，拡大次数は d $(d \geq 1)$

$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$, $\eta_i \in K$, を α の最小多項式

$I_{\eta_i} = R :_R \eta_i$, $I_{[\alpha]} = \bigcap_{i=1}^{d} I_{\eta_i}$ を拡張された分母 ideal と呼ぶ.

$J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \cdots, \eta_d)$

α が R上 anti-integral であるとは

$$0 \longrightarrow \operatorname{Ker}\pi \longrightarrow R[X] \xrightarrow{\pi} R[\alpha] \longrightarrow 0$$

$\operatorname{Ker}\pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$ となるときであると定め

α が R上 super-primitive であるとは

$\operatorname{grade}(J_{[\alpha]}) > 1$ i.e., $J_{[\alpha]} \not\subset \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Spec} R$ and depth $R_{\mathfrak{p}} = 1$

super-primitive であれば anti-integral であることが知られている.

1. seminormal から integrally closed が出るための条件.

まず seminormal とは

$$\mathrm{Pic}(C) \cong \mathrm{Pic}(C[T])$$

が成り立つときで，一般的には injective は言える.

この判定法として有名なのは (2,3)-closed で.

まず C が normal ならば seminormal，そこで

C の normalization を $\overline{C}$ とするとき. $\lambda \in \overline{C}$ について

$\lambda^2, \lambda^3 \in C$ ならば $\lambda \in C$

そこで anti-integral element α による単純拡大 $A = R[\alpha]$

では,

Theorem

(1) $d=1$ のとき $\eta_1 \in I_{[\alpha]} : I_{[\alpha]}$

$d \geqq 2$ のときは $\eta_1, \eta_2 \in I_{[\alpha]} : I_{[\alpha]}$ という条件のもとで

$R[\alpha] \cap R[\alpha^{-1}]$ が seminormal ならば，この環は

normal (integrally closed) である。

上記の条件がない時には反例が存在する.

この証明等は

Note on extensions $R[\alpha]$ and $R[\alpha] \cap R[\alpha^{-1}]$ of an integral domain R, C. R. Math. Rep. Acad. Sci. Canada, vol 19, 1997, pp 21-23. (with T. Sugatani)

2. LCM-stable と Richman extension について

$R \subset A$ という ring extension について，これが LCM-stable であるとは，${}^\forall x, y \in R$ について $(xR \cap yR)A = xA \cap yA$ が成り立つことである．flat extension ならば LCM-stable (Richman さんがはじめだと思います)

A/R が Birational extension のときは LCM-stable と flat extension は同値であることが知られている．

birational extension でない場合は LCM-stable から flat extension は出てこない例を宮崎大の内田さんが与えた．その例を詳しくみてみることから次が与えられた．

Theorem

α は R 上 super-primitive とする．今 $J_{[\alpha]} \neq R$ (i.e, $R[\alpha]/R$ は flat extension ではない) とするとき，

$R[\alpha]/R$ が LCM-stable ならば $ht(J_{[\alpha]}) \geqq 3$

さて，LCM-stable は，$K \ni \alpha$ のとき，分母 ideal I_α について次の様に述べられる．

$R \subset A \subset K$ について，A/R が LCM-stable $\Leftrightarrow$ ${}^\forall \alpha \in A$ について $I_\alpha A = A$．

そこで LCM-stable は Birational extension については すばらしい情報，条件であるが High degree algebraic extension に対しては このままでは いけない，ではどうすればよいか，そこで，

Definition

$R \subset A$ が Richman ext とは, $^\forall \alpha \in A$ について

$I_{[\alpha]} A = A$ が成り立つ.

この Richman ext については, 今, 論文作成中ですので来年にでもお知らせしたいと思います. この拡張が本質的なものをもっているかどうかはまだわかりません.

$A = R[\alpha]$ については

Integrality and LCM-stableness of simple extension over Noetherian domains, Communication in Algebra, vol 24, 1996, 3229-3235. (with M. Kanemitsu, J. Sato)

Flatness and LCM-stability of simple extensions over Noetherian domains, to appear in Commutative in Algebra, (with M. Kanemitsu, T. Sugatani)

Anti-integral element and coefficients of its minimal polynomial, to appear in Math. J. Okayama Univ., (with S. Oda)

3. $R[\lambda\alpha+\mu] = R[\alpha]$ のとき，$\lambda \in R^\times$，$\mu \in R$ となること.

R の商体 K の中での integral closure を $\overline{R}$ とし，$\overline{R}$ は finite R-module とします.

α が ultra-primitive とは

$$\mathrm{grade}\,(I_{[\alpha]} + \mathfrak{C}(\overline{R}/R)) > 1$$

ultra-primitive であれば super-primitive，よって anti-integral.

Theorem

α が R 上 ultra-primitive で 拡大次数 $d \geq 3$，のとき，$R[\alpha] \cap K = R$ (i.e. exclusive とよんだ条件で，これは $\bigcap_{i=1}^{d-1} I_{\eta_i} \subsetneq I_{\eta_d}$ と同値，又 $\mathrm{grade}\,(I_{[\alpha]}(1, \eta_1, \cdots, \eta_{d-1})) > 1$ とも同値) であれば，$\lambda, \mu \in K$ について

$$R[\alpha] = R[\lambda\alpha+\mu] \iff \lambda \in R^\times,\ \mu \in R$$

この証明は

A linear generator of an ultra-primitive extension $R[\alpha]$, to appear in Far East J. Math. Sci (with S. Oda)

Far East J. は最近刊行された雑誌で手に入りにくいと思います. 関心のある方は私まですでにゲラ刷りがあります. 9ページと少々長い証明です.

4. 一見有限生成にみえて実は単純拡大になるもの

これも現在進行形で論文作成中ですが，大半が計算ばかりで，さてこういうもので論文にしてよいものやらと思って困っておりますが，私は気に入っています．

まずは結果から

Proposition

α：R上 anti-integral element of degree d.

$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_{d-1}X + \eta_d$, $\eta_i \in K$，をαの最小多項式，このとき

$\ell \geqq 1$の整数について

(i) $R[\alpha^2, \alpha^{2\ell+1}] = R[\alpha] \Leftrightarrow I_{[\alpha]} = I_{\eta_{d-1}} \cap I_{\eta_d}$で

$(\eta_{d-1}, \eta_d)R \supsetneqq R$かつ

$I_{[\alpha]}$は invertible ideal

i.e., $^\exists b_0, b_1 \in R$

$b_0\eta_{d-1} + b_1\eta_d = 1$

このとき $R[\alpha]/R$ は flat extension である．

(ii) この条件があれば $^\forall s, t \in R$ で

$$R[\alpha^2, \alpha^5 + s\alpha^3 + t\alpha] = R[\alpha]$$

逆は

$$\overset{4}{\bigoplus} I_{[\alpha]} \oplus R \xrightarrow{\psi} R^2$$

ψは $\begin{pmatrix} \eta_{d-1} & \eta_d & 0 & 0 & -s \\ \eta_{d-3} & \eta_{d-2} & \eta_{d-1} & \eta_d & -t \end{pmatrix}$ で与えられるもの

について $\mathrm{Im}\,\psi \ni \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

(iii) $R[\alpha^2, \alpha^3 - t\alpha] = R[\alpha]$ となるための条件は面白い事に拡大次数 d が odd と even では異なる条件が与えられる. 複雑になるので来年(今年)の Symposium で話しが出来ればと思っています.

以上のおきるときはすべて $R[\alpha]/R$ が flat extension のときに限られます. 従って not flat ならば単純拡大は一目見てそれと分かるといえるでしょう.

この参考資料はありません, その内論文が出来るでしょう

$R[\alpha^2] = R[\alpha]$ となる物もありまして,

Proposition

$R[\alpha^2] = R[\alpha]$ であれば, $I_{[\alpha]} = I_{\eta_{d-1}} \cap I_{\eta_d}$, $(\eta_{d-1}, \eta_d)R \supseteq R$

そして, $I_{[\alpha]}$ は invertible ideal. 逆に,

この条件があれば $R[\![\alpha^2]\!] = R[\![\alpha]\!]$ 形式的べき級数である.

逆については d が odd, even で変わります.

d: odd のとき, $d = 2t+1$

$$\begin{cases} b_1 F_1 + b_3 F_3 + \cdots + b_{2t+1} F_{2t+1} = 0 \\ b_1 G_1 + b_3 G_3 + \cdots + b_{2t+1} G_{2t+1} = -1 \end{cases}$$

をみたす $F_{2i+1}, G_{2i+1} \in \mathbb{Z}[\eta_1, \eta_2, \cdots, \eta_d]$ がある, ここで $\deg \eta_j = j$ と weight を与えれば, F, G は homogeneous となり $\deg F_{2i+1} = \deg G_{2i+1} = 2i + (d-1)$

この $F_1, F_3, \cdots, F_{2t+1}, G_1, G_3, \cdots, G_{2t+1}$ は t が与えれば
そのまま引き継がれます.

d: even のとき $d=2t$

$$\begin{cases} b_0 F_0 + b_2 F_2 + \cdots + b_{2t} F_{2t} = 0 \\ b_0 G_0 + b_2 G_2 + \cdots + b_{2t} G_{2t} = -1 \end{cases}$$

この F_{2i}, G_{2i} は d が odd のときと同様です.

この様な F, G が与えられれば, 正しくは F, G は
はじめから given ですから $\{b_i\}$ が与えられれば

$R[\alpha] = R[\alpha^2]$ となる.

これも $R[\alpha]/R$ が flat extension のときにしか
おきません.

∵ $A = R[\alpha]$ とおく. $J_{[\alpha]} \neq R$ (i.e., A/R は not flat)
とすれば $A/J_{[\alpha]}A \cong R/J_{[\alpha]}[T]$ T は変数.

よって polynomial ring では $k[t] = k[t^2]$ とか
$k[t^2, t^3] = k[t]$ etc はおこらないから.

7. 最後に計算ばかりだけどパソコンを使って（Maple とか Mathematica）面白い事をやりましょう.

$$\begin{array}{ccc} A = R[\alpha] & \hookrightarrow & M_d(K) \\ & & \Psi \\ \alpha & \longmapsto & M_\alpha \end{array}$$

$$M_\alpha = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \\ -\eta_d & -\eta_{d-1} & \cdots & & -\eta_1 \end{pmatrix}$$ コンパニオン行列

によって A の行列表現が与えられます.

$f(\alpha) \in A$, $M_{f(\alpha)} = f(M_\alpha)$ です.

この時 $G_A = \{ H \in GL_d(K) \mid HM_\alpha = M_\alpha H \}$

は A を不変にする group.

そこで $\forall f(\alpha) \in A$ について

$$G_{f(\alpha)} = \{ H \in M_d(K) \mid HM_{f(\alpha)} = M_{f(\alpha)}H \}$$

又は $$= \{ H \in GL_d(K) \mid HM_{f(\alpha)} = M_{f(\alpha)}H \}$$

を求めてみると面白い.

逆に ${}^\forall H \in M_d(K)$

$$B_H = \{ f(\alpha) \in A \mid H M_{f(\alpha)} = M_{f(\alpha)} H \}$$ は

A の subring を与える. これらの subring は一体どういう物か？

色々単純拡大と言っても, そう単純ではないのです.
私は一元生成と呼ぶべきだと考えています.
なかなか奥の深いものなのです.

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SIMPLE ALGEBRAIC EXTENSIONS OF RINGS AND KERNELS OF UNIVERSAL DERIVATIONS

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Introduction

Let R be a Noetherian ring containing the rational number field $\mathbb{Q}$ and let $A = R[X]/I$, where $R[X]$ is a polynomial ring in one variable over R and I is an ideal of $R[X]$. We denote by $\Omega_R(A)$ the module of differentials of A over R and by $d_{A/R}$ the universal R-derivation $A \longrightarrow \Omega_R(A)$. Then $C = \ker(d_{A/R})$ is an R-subalgebra of A, which is not Noetherian in general even if R is a regular ring. It is thus natural to ask what conditions imply that C is again Noetherian. The purpose of this note is to give some results on this problem under the assumption that $\mathfrak{c}(I)$ contains a regular element of A; here $\mathfrak{c}(I)$ is the ideal of R generated by the coefficients of the polynomials in I. Our main result is the following

Main Theorem. *Let R be a Noetherian ring containing the rational number field $\mathbb{Q}$ and let $A = R[X]/I$, where I is an ideal of $R[X]$. We set $C = \ker(d_{A/R})$. If $\mathfrak{c}(I)$ contains a regular element of A, then the following three conditions are equivalent to each other*:

(1) *C is finitely generated over $R/(I \cap R)$*;

(2) *C is Noetherian*;

(3) $\mathfrak{c}(I) = R$.

This note is a summarization of our paper [4].

1. Preliminary lemmas

We employ the same notation and assumption as in the introduction. For the proof of the main theorem stated above we need some preliminary lemmas.

Let $\pi\colon R[X] \longrightarrow A$ be the natural R-algebra map and let $\alpha = \pi(X)$. We set $I' = \delta(I)$, where $\delta\colon R[X] \longrightarrow R[X]$ stands for the canonical R-derivation of $R[X]$ defined by $\delta(f(X)) = f'(X)$ for $f(X) \in R[X]$. Then the map $\phi\colon A \longrightarrow A/\pi(I')$ defined by $\phi(g(\alpha)) = g'(\alpha) \bmod \pi(I')$ is a well-defined R-derivation, and there exists an A-module isomorphism $\rho\colon A/\pi(I') \longrightarrow \Omega_R(A)$ such that $\rho \cdot \phi = d_{A/R}$ (cf. [1, p. 195]). Hence we have $\ker(d_{A/R}) = \ker(\phi)$. It is then easy to check the following

Lemma 1.1. *We have $C = \pi(\delta^{-1}(I))$, and in particular $C \subseteq R + \mathfrak{c}(I)A$.*

The following two lemmas play important roles in the proof of the theorem. The proofs of the lemmas are omitted.

Lemma 1.2. *Let a be a non-zero element of $\mathfrak{c}(I)$. Then there exists a positive integer r such that $a\alpha^{r+i} \in C + C\alpha + \cdots + C\alpha^{r-1}$ for every $i \geq 0$.*

Lemma 1.3. *The element α is integral over C if and only if $\mathfrak{c}(I) = R$.*

2. Proof of the main theorem

By making use of the lemmas in the previous section, we can prove our main theorem as follows:

Since the implication (1) $\Longrightarrow$ (2) is obvious, it suffices to check (3) $\Longrightarrow$ (1) and (2) $\Longrightarrow$ (3). Let $S = R/(I \cap R)$. Then we have $\Omega_R(A) \cong \Omega_S(A)$, and $\ker(d_{A/R}) = \ker(d_{A/S})$. Moreover we have $I \cap R \subseteq \mathfrak{c}(I)$. Hence, replacing R by S if necessary, we may assume that $I \cap R = (0)$.

First, suppose that $\mathfrak{c}(I) = R$ and let $\alpha = X \bmod I$. Then, by Lemma 1.3, α is integral over C, and hence

$$\alpha^r + c_1\alpha^{r-1} + \cdots + c_r = 0$$

for some positive integer r and $c_1, c_2, \ldots, c_r \in C$. Let $B = R[c_1, c_2, \ldots, c_r]$. Then A is a finite B-module because $A = B[\alpha]$ and α is integral over B. Since $B \subseteq C \subseteq A$ and B is Noetherian, it then follows that C is also a finite B-module. Thus C is finitely generated over R.

Next, suppose that C is Noetherian and let $a \in \mathfrak{c}(I)$ be a regular element of A. Then, by Lemma 1.2, there exists a positive integer r such that

$$a\alpha^{r+i} \in M := C + C\alpha + \cdots + C\alpha^{r-1}$$

for every $i \geq 0$. Since a is a regular element of A, this implies that $C[\alpha^r] \subseteq (1/a)M$. Hence $C[\alpha^r]$ is a finite C-module because C is Noetherian and $(1/a)M$ is a finite C-module. Thus α^r is integral over C, and so is α over C. Then, by Lemma 1.3, we have $\mathfrak{c}(I) = R$. *Q.E.D.*

Recall that, in the case where $\mathfrak{c}(I)$ contains a regular element of R, if A is flat over R, then $\mathfrak{c}(I) = R$ (cf. [5, Corollary 1.3]). Hence, the main theorem implies the following

Corollary 2.1. *Let R be a Noetherian ring containing $\mathbb{Q}$ and let $A = R[X]/I$. Assume that $\mathfrak{c}(I)$ contains a regular element of R. If A is flat over R, then $C = \ker(d_{A/R})$ is finitely generated over $R/(I \cap R)$.*

Concerning the case where $\mathfrak{c}(I)$ contains no regular elements of A, we give the following two examples.

Example 2.2. Let R be a ring containing $\mathbb{Q}$ and let $A = R[X]/(af(X))$, where a is a regular element of R and $f(X)$ is a monic polynomial in $R[X]$ of positive degree d. Let $\beta_i = \int_0^{\alpha} aX^{i-1}f(X)dX$ for each $i \geq 1$ and let $\beta_0 = 1$. Then $C = R + R\beta_1 + \cdots + R\beta_{d-1}$. In fact, we have $C = R + R\beta_1 + R\beta_2 + \cdots$ by Lemma 1.1. Hence, to see the assertion, it suffices to check that $\beta_{d+i} \in C_0 := R + R\beta_1 + \cdots + R\beta_{d-1}$ for every $i \geq 0$. The proof is by induction on i; we omit details.

Example 2.3. Let k be a field of characteristic 0 and let $R = k[x, y]$, where x and y are indeterminates. Let $A = R[X]/(xX^2 - yX, xyX)$. Then we have $C = R + Ry\alpha^2 + Ry\alpha^3 + \cdots$ by Lemma 1.1. We will show that C is not Noetherian. Assume that C were Noetherian. Then the ideal $J = (y\alpha^2, y\alpha^3, \ldots)C$ would be finitely generated, say, $J = (y\alpha^2, y\alpha^3, \ldots, y\alpha^n)C$. Note that $y^2\alpha = 0$ because

$$y^2X = xyX^2 - y(xX^2 - yX) \in (xX^2 - yX,\ xyX).$$

Hence we have $J = Ry\alpha^2 + \cdots + Ry\alpha^n$. Since $y\alpha^{n+1} \in J$, it follows that

$$yX^{n+1} - (c_2yX^2 + \cdots + c_nyX^n) = (xX^2 - yX)F(X) + xyXG(X) \tag{1}$$

for some $F(X)$, $G(X) \in R[X]$ and $c_2, c_3, \ldots, c_n \in R$. From (3) we have $xX^2F(X) \in yR[X]$, which implies that $F(X) = yF_1(X)$ for some $F_1(X) \in R[X]$. We then have

$$yX^{n+1} - (c_2yX^2 + \cdots + c_nyX^n) = xy(X^2F_1(X) + XG(X)) - y^2XF_1(X).$$

Comparing the coefficient of X^{n+1} in the left hand side with that in the right hand side, we have $y \in (xy,\ y^2)R$, which is a contradiction.

Example 2.2 shows that the converse of Corollary 2.1 does not hold in general. However, the converse holds if the extension $R \subseteq A = R[X]/I$ is anti-integral; it is known that if $R \subseteq R[\alpha] \cong R[X]/I$ is an anti-integral extension of Noetherian domains, then $R[\alpha]$ is flat over R if and only if $\mathfrak{c}(I) = R$ (cf. [2, Proposition 2.6]). Hence, by the main theorem, we have the following

Theorem 2.4. *Let $R \subseteq A = R[\alpha]$ be an anti-integral extension of Noetherian domains and let $C = \ker(d_{A/R})$. If R contains $\mathbb{Q}$, then the following three conditions are eqivalent to each other:*
(1) *C is finitely generated over R;*
(2) *C is Noetherian;*
(3) *A is flat over R.*

Remark 2.5. We can generalize the main theorem to the case where C is the kernel of an R-derivation $D\colon A \longrightarrow M$ for some A-module M. (cf. [3]).

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Some results on Hilbert-Kunz multiplicity [1]

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Introduction.

The theory of Hilbert-Kunz multiplicity (HK multiplicity, in short) was developed by E. Kunz and P. Monsky. Yet there are many fundamental facts to become clear.

Also, since HK multiplicity is finer than the usual one, we can expect it will be a good invariant when we classify "singularities".

The other merit to consider HK multiplicity is that it will be a powerfull guide in the theory of tight closure of ideals and will help us to find important properties of tight closures.

Some parts of this article (including the most important ones) are either joint work with Ken-ichi Yoshida at Nagoya University or his results. I want to thank him for the collaboration. The detail of the proof not included here will appear in [WY].

Also, we want to thank the organizer of this conference for giving the chance of disscussion. In fact, Thorem 2.1 was found in the discussion during this conference.

I want to thank P. Monsky for giving me many informations and also to R.-O. Buchweitz and Q. Chen for sending me the TEX file of their paper [BCP].

§1. Fundamental Properties.

Definition (1.1) Let $(A, \mathfrak{m})$ be a Noetherian local ring of dim $A = d$

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of characteristic $p > 0$ and I be an $\mathfrak{m}$-primary ideal. We always agree to use the letter q for powers of p. Then

$$e_{HK}(I) := \lim_{q\to\infty} \frac{l_A(A/I^{[q]})}{q^d} \quad \text{and} \quad e_{HK}(A) := e_{HK}(\mathfrak{m}).$$

Also, for a finitely generated A module M, we define

$$e(I,M) = \lim_{n\to\infty} d!\, \frac{l_A(M/I^nM)}{n^d}, \quad e_{HK}(I,M) := \lim_{q\to\infty} \frac{l_A(M/I^{[q]}M)}{q^d}$$

We write $e(I) = e(I,A)$ for the usual multiplicity of $\mathfrak{m}$ primary ideal I and write $e(A) = e(\mathfrak{m})$.

From this definition, it is clear that $e_{HK}(I^{[q]}) = q^d e_{HK}(I)$ for every $\mathfrak{m}$-primary ideal I. This formula will be used in the proof of (2.1).

Hilbert-Kunz multiplicity characterizes tight closures in the same manner that usual multiplicity characterizes integral closures.

Theorem (1.2) [HH] Let $I \supset J$ be $\mathfrak{m}$-primary ideals. If $I \subset J^*$ then $e_{HK}(I) = e_{HK}(J)$. The converse holds if $\hat{A}$ is reduced and equidimensional.

The following lemma of Lech is fundamental in the theory of HK multiplicity.

Lech's Lemma (1.3) If $(x_1,\dots,x_d)$ is a s.o.p. of a finitely generated A module M, then

$$\lim_{n_1,\dots,n_d\to\infty} \frac{l_A(M/(x_1^{n_1},\dots,x^{n_d})M)}{n_1\cdots n_d} = e(M,(x_1,\dots,x_d)).$$

Corollary (1.4) For a parameter ideal I of A (resp. M), we have $e(I) = e_{HK}(I)$ (resp. $e(M,I) = e_{HK}(M,I)$).

Proposition (1.5) [M] (1) $e_{HK}(I)$ exists and

$$e(I) \geq e_{HK}(I) \geq \frac{e(I)}{d!}$$

(2) Let M_1, M_2 be A modules which satisfy $(M_1)_\mathfrak{p} \cong (M_2)_\mathfrak{p}$ for every minimal prime ideal $\mathfrak{p}$ of A, then we have $e_{HK}(I,M_1) = e_{HK}(I,M_2)$.

In particular, if A is an integral domain, $e_{HK}(I,M) = \text{rank } M . e_{HK}(A)$.

(3) If A is regular, then $e_{HK}(I) = l_A(A/I)$ (in particular, $e_{HK}(A) = 1$). In general, $e_{HK}(A) \geq 1$.

Remark (1.6) Let $A = k[[X_1, \ldots, X_d]]^{(r)}$ be the r-th Veronese subring of $k[[X_1, \ldots, X_d]]$. We can show that $e_{HK}(A) = (d + r - 1)!/((d!)(r!))$ by (1.8). If we fix d and let $r \to \infty$, then $\lim_{r \to \infty} e_{HK}(A)/e(A) = 1/d!$. Hence the inequalitis in (1.5)(1) are best possible. Does ther exist some A with $d = \dim A \geq 2$ with the equality $e(A) = d!.e_{HK}(A)$?

The following criterion of Kunz for regular local rings is essential to prove (2.1).

Theorem (1.7) (Kunz [K1]) Let A be a local ring of characteristic $p > 0$. Then the following conditions are equivalent.

(1) A is a regular local ring.

(2) A is reduced and it is flat over $A^p = (a^p \mid a \in A)$.

(3) $l_A(A/\mathfrak{m}^{[q]}) = q^d$ for all $q = p^e$, $e \geq 1$.

In general, HK multiplicity is very hard to calculate. (I hope the calculation will be much easier in the future.) But if A has a regular ring as a finite extension, then $e_{HK}(I)$ is rather easy to calculate. The follwoing formula is found by Buchweitz, Chen and Pardue independently ([BCP]).

Proposition (1.8) Let $A \subset B$ be a finite ring extension with rank ${}_A B = r$. Then for an $\mathfrak{m}$ primary ideal I of A, we have $e_{HK}(I) = \dfrac{e_{HK}(IB)}{r}$. If, in particular, B is regular, then we have

$$e_{HK}(I) = \frac{l_B(B/IB)}{r}.$$

The proof follows easily from (2) and (3) of (1.5).

§2. Colength and HK multiplicity. Characterization of regular local rings.

It is well known that A is regular if and only if A is unmixed and $e(A) = 1$ ([N] (40.6)). This also holds for HK multiplicity. Also, our proof seems to suggest a relation between colength and HK multiplicity of an $\mathfrak{m}$-primary ideal.

Theorem (2.1) (with Ken-ichi YOSHIDA) If A is unmixed with $e_{HK}(A) = 1$, then A is regular.

To prove this theorem, we begin with the "reverse" inequality for multiplicity and colength for the tight closure of parameter ideals.

First, we consider the following condition (#) for parameter ideals.

(2.1.1) $\mathfrak{q} = (a_1, \ldots, a_d)$ satisfies the condition (#) if

(#i) a_i is $A_{i-1}/\mathrm{H}^0_{\mathfrak{m}}(A_{i-1})$-regular, where $A_{i-1} = A/(a_1, \ldots, a_{i-1})A$ for each $i = 1, \ldots, d-1$.

(#ii) $0 :_{A_{d-1}} a_d = 0 :_{A_{d-1}} a_d^2$.

Theorem (2.2) (K. YOSHIDA) If $\mathfrak{q}$ satisfies (#), then $e(\mathfrak{q}) \geq l_A(A/\mathfrak{q}^*)$, where $e(\mathfrak{q})$ is the usual multiplicity of $\mathfrak{q}$ and $\mathfrak{q}^*$ is the tight closure of $\mathfrak{q}$.

To show this we study the ideal $\Sigma(\mathfrak{q})$ defined by Yamagishi–Goto as

$$\Sigma(\mathfrak{q}) = \sum_{i=1}^{d} (a_1, \ldots, \widehat{a_i}, \ldots, a_d) : a_i + (a_1, \ldots, a_d)A.$$

We can show that if $\mathfrak{q}$ satisfies (#), then $e(\mathfrak{q}) \geq l_A(A/\Sigma(\mathfrak{q}))$. Then we use the "colon capturing" property of the tight closure to show $\Sigma(\mathfrak{q}) \subset \mathfrak{q}^*$ and get (2.2).

Now, we ask, "What if the equality hols (2.2) ?" We don't know the general answer yet, but we have the following.

Theorem (2.3) (K. Yoshida) Let A be an excellent local ring with $d := \dim A \geq 2$. Suppose that the following two conditions hold:

(1) $A/\mathrm{H}^0_{\mathfrak{m}}(A)$ satisfies (S_2).

(2) $e(\mathfrak{q}) = l_A(A/\Sigma(\mathfrak{q}))$ for every system of parameters $\mathfrak{q}$ which satisfies (#).

Then $A/\mathrm{H}^0_{\mathfrak{m}}(A)$ is Cohen-Macaulay.

Now assume that $e_{HK}(A) = 1$. Take a parameter ideal $\mathfrak{q}$ which satisfy the condition (#) and take a composition series

$$A \supset I_1 = \mathfrak{m} \supset \ldots \supset I_s = \mathfrak{q}^*.$$

Since $I_j/I_{j+1} \cong A/\mathfrak{m}$ for every j, there is a surjection $A/\mathfrak{m}^{[q]} \to I_j^{[q]}/I_{j+1}^{[q]}$ for every j. Taking the length and dividing by q^d, we have

$$l_A(A/\mathfrak{q}^*).e_{HK}(A) \geq e_{HK}(\mathfrak{q}^*) = e_{HK}(\mathfrak{q}) = e(\mathfrak{q}).$$

Combinig this with (2.2) and putting $e_{HK}(A) = 1$, we have $e(\mathfrak{q}) \geq l_A(A/\mathfrak{q}^*).e_{HK}(A)$ and we must have equality everywhere. If we assume the condition (S_2) for A, then by (2.3), A is Cohen-Macaulay and $e(\mathfrak{q}) = l_A(A/\mathfrak{q}^*)$. Hence we have $\mathfrak{q} = \mathfrak{q}^*$, which implies that A is F-rational.

We assumed the condition (S_2) for A in the argument above. But we can show that to prove (2.1), we can reduce to the case A satisfies (S_2) after some discussion. The detail will appear in [WY].

Now, we will finally prove (2.1). The argument above shows that A is Cohen-Macaulay and it suffices to show the following.

Proposition (2.4) If A is Cohen-Macaulay with $e_{HK}(A) = 1$, then A is regular.

(Proof) Take arbitrary $\mathfrak{m}$-primary ideal I and take a parameter ideal $\mathfrak{q} \subset I$. Taking a composition series from A through I to $\mathfrak{q}$ and note 2 inequalities

$$l_A(A/I^{[q]}) \leq l_A(A/I).l_A(A/\mathfrak{m}^{[q]}), \qquad l_A(I^{[q]}/\mathfrak{q}^{[q]}) \leq l_A(I/\mathfrak{q}).l_A(A/\mathfrak{m}^{[q]}).$$

Since $e_{HK}(A) = 1$, we have $e_{HK}(I) \leq l_A(A/I)$ and $e_{HK}(\mathfrak{q}) - e_{HK}(I) \leq l_A(I/\mathfrak{q})$. On the other hand, we have $e_{HK}(\mathfrak{q}) = e(\mathfrak{q}) = l_A(A/\mathfrak{q})$ since A is Cohen-Macaulay. This shows that $l_A(A/I) = e_{HK}(I)$ for arbitrary I.

Now, take $I = \mathfrak{m}^{[q]}$. Then we have $l_A(A/\mathfrak{m}^{[q]}) = e_{HK}(\mathfrak{m}^{[q]}) = q^d.e_{HK}(A) = q^d$. By Kunz' criterion, A is regular.

Conjecture (2.5) Assume A is unmixed. If $l_A(A/\mathfrak{q}^*) = e(\mathfrak{q})$ for a parameter ideal $\mathfrak{q}$, is A F-rational ?

§3. Rings of dimension 2 with small HK multiplicity.

In this section, we seek the possibility to classify singularities in terms of HK-multiplicity. As the first step, we will classify the 2-dimensional singularities with small HK-multiplicity.

It turns out that HK-multuplicity classifies F-rational double points very sharply and gives very important invariant. Also, we have only one triple point with $e_{HK}(A) = 2$. The next value of $e_{HK}(A)$ after 2 seems to be $\frac{11}{5}$.

Theorem (3.1). Let A be a Cohen-Macauly local ring of dimension 2.

(1) $e_{HK}(A) < 2$ if and only if A is an F-rational double point. If A is not regular, then $e_{HK}(A) \geq \frac{3}{2}$.

(2) $e_{HK}(A) = 2$ if and only if either A is a non-F-rational double point or A is an "ordinary triple point" (that is, $G_{\mathfrak{m}}(A) \cong A/\mathfrak{m}[s^3, s^2t, st^2, t^3]$).

(Proof) First, let $e(A) = 2$. Then A has a parameter ideal $\mathfrak{q}$ with $l_A(A/\mathfrak{q}) = 2$. Namely, $\mathfrak{q} \subset \mathfrak{m} \subset A$ is the composition series. Then by (1.2) and (1.4), we have (1).

Rational double points are classified by Artin [A1,A2] and Lipman [L]. Among them, F-rational double points are those which can be expressed as a pure subring of a regular ring. Except the singularities of type (A_n), they are "quotient singularities" by a finite subgroup of $SL(2,k)$. Then their HK-multiplicity are given by the following.

Example (3.2) (1) If $A = k[[X,Y]]^G$ with G a finite subgroup of $SL(2,k)$, then $e_{HK}(A) = 2 - 1/|G|$

(2) If $A \cong k[[x,y,z]]/(xy - z^n)$, then $e_{HK}(A) = 2 - 1/n$.

To prove (3.1), we first exclude rings with $e(A) \geq 4$ and non-F-rational triple points.

Lemma (3.3) Let $\mathfrak{q}$ be a minimal reduction of $\mathfrak{m}$. If $\mathfrak{m}/\mathfrak{q}^*$ is minimally generated by r elements (note that $r \leq e(A) - 1$), then $e_{HK}(A) \geq \frac{r+2}{2(r+1)}e(A)$.

By this lemma, if A is not F-rational with $e(A) = 3$, then we have $e_{HK}(A) \geq \frac{9}{4}$ and if $e(A) \geq 4$, then we have $e(A) \geq \frac{5}{2}$.

Then it remains to show that for a F-rational triple point, $e_{HK}(A) = 2$ if and only if the associated graded ring $G_{\mathfrak{m}}(A)$ is an integral domain. Detail will appear in [WY].

Question (3.4) What are possible values of $e_{HK}(A)$ if A moves all 2-dimensional singularities ? Up to now, we have accumulation points from below (2, for example) but no accumulation points from above.

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Gorenstein 代数について

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この講演の目的は、可換なNoether環上の必ずしも可換ではない有限代数に、可換環論で著しい成功を収めたGorenstein環の理論[**B**]をなるべく忠実に拡張し、いかなる理論が展開されるかを調べることにあり、完全に一般の（非可換）Noether環[**Bj**, **GL**, **IS**]ではなく「Noether代数」と呼ばれる、可換に近いというか、ほんの少しだけ非可換とでもいうべき代数を研究対象としたものである。Noether代数には表現論からの精密で深い理論[**ARS**]があるが、入射次元の有限性という観点からNoether代数に的を絞って一般論を展開した論文は案外少いようであり([**BHa1**, **BHa2**, **BHM**, **GN2**, **N**, **R**, **U**])、またこの方面に精通した研究者も多数という訳ではないように見受けられる。私の講演も意を尽くせなかった点が多いが、幸いにも「第４２回代数学分科会シンポジウム」で同一の主題の下に講演を行う機会を与えられ、その記録は（ミスタイプが多いことを申し訳なく思うが）いわんとするところのほぼ全容を伝えているので、詳細は同シンポジウム報告集内の文章「On Gorenstein R-algebras」を参照されたい。

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