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ON H-EPIMORPHISMS AND CO-H-SEQUENCES IN TWO-SIDED HARADA RINGS

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Abstract. In [8] M. Harada studied a left artinian ring R such that every non-small left R-module contains a non-zero injective submodule. And in [13] K. Oshiro called the ring a left Harada ring (abbreviated left H-ring). We can see many results on left Harada rings in [6] and many equivalent conditions in [4, Theorem B]. In this paper, to characterize two-sided Harada rings, we intruduce new concepts "co-H-sequence" and "H-epimorphism" and study them.

In §1 we define a H-epimorphism, a co-H-sequence and a w-H-epimorphism. In §2 we characterize H-epimorphisms and left co-H-sequences of two-sided Harada rings using well-indexed set of left Harada ring. In Lemma 2.1 we show that a left (resp. right) H-epimorphism induces the inverse right (resp. left) H-epimorphism. In Theorem 2.2, using Lemma 2.1, we characterize left (right) H-epimorphisms. In Corollary 2.3 we characterize left co-H-sequences by a well-indexed set of left Harada ring. And in Example 2.4 we given a simple example of left co-H-sequences in a Nakayama ring. In §3 we consider a w-H-epimorphism. In Lemma 3.1 we consider a left H-epimorphism with the codomain the Jacobson radical of the first term of some left co-H-sequence. And in Lemma 3.2 we further consider the left (right) H-epimorphism in Lemma 3.1. In Lemma 3.3 we consider colocal pairs in two-sided Harada rings. And in Proposition 3.4 we consider w-H-epimorphisms.

1. Definitions

Let R be a basic artinian ring. A ring R is called a *left Harada ring* or a *left H-ring* if, for any primitive idempotent e of R, there exists a primitive idempotent f_e of R with $E(T({}_RRe)) \cong {}_RRf_e/S_{n_e}({}_RRf_e)$ for some $n_e \in \mathbb{N}$.

By, for instance, [4, Theorem B (5),(6),(14) and the proof of (6) $\Rightarrow$ (5)], the following are equivalent:

(a) R is a left Harada ring.

(b) There exist a basic set $\{ e_{i,j} \}_{i=1,j=1}^{m\ n(i)}$ of orthogonal primitive idempotents of R and a set $\{ f_i \}_{i=1}^{m}$ of primitive idempotents of R such

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that $E(T({}_RRe_{i,j})) \cong {}_RRf_i/S_{j-1}({}_RRf_i)$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n(i)$.

(c) There exists a basic set $\{ e_{i,j} \}_{i=1,j=1}^{m\ n(i)}$ of orthogonal primitive idempotents of R such that $e_{i,1}R_R$ is injective and $e_{i,j}R_R \cong e_{i,1}J_R^{j-1}$ for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n(i)$.

We may consider the sets

$$\{e_{i,j}\}_{i=1,j=1}^{m\ n(i)}$$

in $(b), (c)$ coincide and call it a *well-indexed set of left Harada ring* or a *left well-indexed set*.

Further, for primitive idempotents e, f of R, we call

$$(eR,\ Rf)$$

is an *i-pair* if both $S(eR_R) \cong T(fR_R)$ and $S({}_RRf) \cong T({}_RRe)$ hold. And, since $\{e_{i,1}R\}_{i=1}^m$ is a basic set of indecomposable projective injective right R-modules, for each $i = 1, 2, \ldots, m$, there exists $e_{\sigma(i),\rho(i)} \in \{e_{i,j}\}_{i=1,j=1}^{m\ n(i)}$ such that $(e_{i,1}R,\ Re_{\sigma(i),\rho(i)})$ is an i-pair by [7, Theorem 3.1], where $\sigma,\ \rho : \{1, 2, \ldots, m\} \to \mathbb{N}$ are mappings.

Unless otherwise stated, throughout this paper, we let R be a basic two-sided Harada ring, let $\{e_{i,j}\}_{i=1,j=1}^{m\ \ n(i)}$ be its well-indexed set of left Harada ring, let σ, ρ be mappings above, and, for each $i = 1, 2, \ldots, m$ and each $j = 2, 3, \ldots, n(i)$, let

$$\theta_{i,j} : e_{i,j}R_R \to e_{i,j-1}J_R$$

be an R-isomorphism.

Let R be an artinian ring, let $\{e_i\}_{i=1}^n$ be a complete set of orthogonal primitive idempotents of R and let $\{f_i\}_{i=1}^k \subseteq \{e_i\}_{i=1}^n$. A sequence $f_1R,\ f_2R,\ \ldots,\ f_kR$ is called a *right co-H-sequence* of R if the following (CHS1), (CHS2), (CHS3) hold.

(CHS1) For each $i = 1, 2, \ldots, k-1$, there exists an R-isomorphism $\xi_i : f_iR_R \to f_{i+1}J_R$.

(CHS2) The last term f_kR_R is injective.

(CHS3) $f_1R,\ f_2R,\ \ldots,\ f_kR$ is the longest sequence among the sequences which satisfy both (CHS1) and (CHS2), i.e., there does not exist an R-isomorphism: $fR_R \to f_1J_R$, where $f \in \{e_i\}_{i=1}^n$.

Similarly, we define a *left co-H-sequence* $Rf_1,\ Rf_2,\ \ldots,\ Rf_k$ of R.

Obviously, for each $i = 1, 2, \ldots, m$

$$e_{i,n(i)}R_R,\ e_{i,n(i)-1}R_R,\ \ldots,\ e_{i,1}R_R$$

is a right co-H-sequence of R. And, for an artinian ring R', it is a left Harada ring if and only if there exists a basic set $\{e_{i,j}\}_{i=1,j=1}^{m\ n(i)}$ of orthogonal primitive idempotents of R' such that $e_{i,n(i)}R'$, $e_{i,n(i)-1}R'$, $\ldots$, $e_{i,1}R'$ is a right co-H-sequence of R' for all $i = 1, 2, \ldots, m$.

From the definition of a left Harada ring, the following lemma holds:

Lemma 1.1. *For a left Harada ring R' and primitive idempotents $f_1, f_2, \ldots, f_k$ of R', the following are equivalent.*

(*a*) *$f_1R', f_2R', \ldots, f_kR'$ is a right co-H-sequence.*

(*b*) *$f_1R', f_2R', \ldots, f_kR'$ satisfies* (CHS1) *and the following* (CHS3$'$)*:*

(CHS3$'$) *$f_1R', f_2R', \ldots, f_kR'$ is the longest sequence among sequences which satisfy* (CHS1)*.*

In this paper, using a well-indexed set $\{e_{i,j}\}_{i=1,j=1}^{m\ n(i)}$ of left Harada ring, we characterize left co-H-sequences of R, i.e., we give the structure of R as a right Harada ring.

Let $\{e_i\}_{i=1}^n$ be a complete set of orthogonal primitive idempotents of R and let $\{f_i\}_{i=1}^{j+1} \subseteq \{e_i\}_{i=1}^n$, where $f_1, f_2, \ldots, f_{j+1}$ are mutually distinct. Then we call $\varphi : f_1R_R \to f_2J_R$ (resp. ${}_RRf_1 \to {}_RJf_2$) a *right* (resp. *left*) *H-epimorphism* if φ is a non-zero R-epimorphism with $J \cdot \operatorname{Ker}\varphi = 0$ (resp. $\operatorname{Ker}\varphi \cdot J = 0$). And we call $\varphi : f_1R_R \to f_{j+1}J_R^j$ (resp. ${}_RRf_1 \to {}_RJ^jf_{j+1}$) a *right* (resp. *left*) *weak H-epimorphism* (or simply a *right* (resp. *left*) *w-H-epimorphism*) if there exist right (resp. left) H-epimorphisms $\varphi_i : f_iR_R \to f_{i+1}J_R$ ($i = 1, 2, \ldots, j$) with $\varphi = \varphi_j\varphi_{j-1}\cdots\varphi_1$ (resp. $\varphi_i : {}_RRf_i \to {}_RJf_{i+1}$ ($i = 1, 2, \ldots, j$) with $\varphi = \varphi_1\varphi_2\cdots\varphi_j$).

For $a \in R$, we write the left (resp. right) multiplication map by a

$$(a)_L \ \ (\text{resp. } (a)_R)\,.$$

And, for primitive idempotents e, f and g, we use the following terminologies.

- If both $S({}_{eRe}eRf)$ and $S(eRf_{fRf})$ are simple, we call (eR, Rf) is a *colocal pair* following [12] and [11]. And then $S({}_{eRe}eRf) = S(eRf_{fRf})$ holds. We abbreviate it to
$$S(eRf)\,.$$
- We put
$$R(e) \overset{put}{:=} eRe\,.$$

2. H-epimorphisms and left co-H-sequences of two-sided Harada rings

First we give basic results.

Lemma A.

(I) For any $i = 1, 2, \ldots, m$ and any $j = 1, 2, \ldots, n(i)$, the following hold.

(1) ${}_RRe_{\sigma(i),\rho(i)}/S_{j-1}({}_RRe_{\sigma(i),\rho(i)}) \cong E(T({}_RRe_{i,j}))$. So $S_{n(i)}({}_RRe_{\sigma(i),\rho(i)})$ is uniserial.

(2) $S_j({}_RRe_{\sigma(i),\rho(i)}) = \oplus_{k=1}^{j} S(e_{i,k}R_R)$.

(3) $S(e_{ij}R_R) \cong T(e_{\sigma(i),\rho(i)}R_R)$ and $S(e_{i,j}R_R) = S(e_{i,j}R_R)\, e_{\sigma(i),\rho(i)} = S(e_{i,j}Re_{\sigma(i),\rho(i)})$.

(4) Suppose that $T(e_{i,1}J_R^{n(i)}) \oplus\!\!> T(e_{k,l}R_R)$. Then $l = 1$.

(5) $S(R_R) \cong \oplus_{i=1}^{m} T(e_{\sigma(i),\rho(i)}R_R)^{n(i)}$.

(II) Let
$$Rf_1,\ Rf_2,\ \ldots,\ Rf_{n'}$$
be a left co-H-sequence of R and let $(e_{i,1}R,\ Rf_{n'})$ be an i-pair, where $f_1, f_2, \ldots, f_{n'}$ are primitive idempotents of R. Then, for any $p = 1, 2, \ldots, n'$, the following hold.

(1) $e_{i,1}R_R/S_{p-1}(e_{i,1}R_R) \cong E(T(f_{n'-p+1}R_R))$. So $S_{n'}(e_{i,1}R_R)$ is uniserial.

(2) $S_p(e_{i,1}R_R) = \oplus_{k=n'-p+1}^{n'} S({}_RRf_k)$.

(3) $S({}_RRf_p) \cong T({}_RRe_{i,1})$ and $S({}_RRf_p) = e_{i,1}\, S({}_RRf_p) = S(e_{i,1}Rf_p)$.

So, in particular, if $S({}_RRe_{k,l}) \cong T({}_RRe_{s,t})$ for some $e_{k,l}$ and $e_{s,t}$, then $t = 1$ and $S({}_RRe_{k,l}) = e_{s,1}\, S({}_RRe_{k,l}) = S(e_{s,1}Re_{k,l})$.

(4) Suppose that $T({}_RJ^{n'}f_{n'}) \oplus\!\!> T({}_RRg)$, where g is a primitive idempotent of R. Then ${}_RRg$ is injective.

(5) For each $i = 1, 2, \ldots, m$, we let $Rf_{i,1}, Rf_{i,2}, \ldots, Rf_{i,n'_i} = Re_{\sigma(i),\rho(i)}$ be a left co-H-sequence with the last term $Re_{\sigma(i),\rho(i)}$. Then $S({}_RR) = \oplus_{i=1}^{m} \oplus_{j=1}^{n'_i} S({}_RRf_{i,j})$. So $S({}_RR) \cong \oplus_{i=1}^{m} T({}_RRe_{i,1})^{n'_i}$.

Proof.

(I) (1) By [6, Lemma 3.3.1].

(2) By [14, Proposition 3.5 (1)].

(3) By [3, Theorem 1], $S(e_{i,j}R_R) \cong T(e_{\sigma(i),\rho(i)}R_R)$ and $S({}_{R(e_{i,j})}e_{i,j}Re_{\sigma(i),\rho(i)})$ is simple. So the statements follow from, for instance, [3, Lemma 1 (2),(3)] since R is a basic ring.

(4) There exists $\varphi : e_{k,l}R_R \to e_{i,1}J_R^{n(i)}$ with $\operatorname{Im}\varphi \not\subseteq e_{i,1}J^{n(i)+1}$ by the assumption. Suppose that $l \neq 1$. There exists $\tilde{\varphi} : e_{k,l-1}R_R \to e_{i,1}R_R$ with $\tilde{\varphi}\,\theta_{k,l} = \varphi$ since $e_{i,1}R_R$ is injective. Then, because $\operatorname{Im}\varphi \subseteq e_{i,1}J^{n(i)}$ and $\operatorname{Im}\varphi \not\subseteq e_{i,1}J^{n(i)+1}$, $\operatorname{Im}\tilde{\varphi} \subseteq e_{i,1}J^{n(i)-1}$ and $\operatorname{Im}\tilde{\varphi} \not\subseteq e_{i,1}J^{n(i)}$. So $T(e_{k,l-1}R_R) \cong T(e_{i,1}J_R^{n(i)-1})$ since $e_{i,1}R_R/e_{i,1}J^{n(i)}$ is uniserial by the definition of $\{e_{i,j}\}_{i=1,j=1}^{m\ n(i)}$. On the other hand, $T(e_{i,1}J_R^{n(i)-1}) \cong T(e_{i,n(i)}R_R)$ by the definition of $\{e_{i,j}\}_{i=1,j=1}^{m\ n(i)}$. Therefore $(k, l) = (i, n(i)+1)$. But $e_{i,n(i)+1}$ does not exist, a contradiction.

(5) By the equivalent condition (c) given in the definition of a left Harada ring.

(II) By left-right symmetric argument of (I), we see that (1),(2),(4),(5) and the first half of (3) hold. We only show the second half of (3). We assume that $S({}_RRe_{k,l}) \cong T({}_RRe_{s,t})$ for some $e_{k,l}$ and $e_{s,t}$. Then $t = 1$, i.e., $e_{s,t}R_R$ is injective, by (II)(5). And $S({}_RRe_{k,l}) \cong S({}_RRe_{\sigma(s),\rho(s)})$. Now we consider a left co-H-sequence with the last term $Re_{\sigma(s),\rho(s)}$. And we may assume that $Re_{k,l}$ is a term of the left co-H-sequence. Hence $S({}_RRe_{k,l}) = e_{s,1}\,S({}_RRe_{k,l}) = S(e_{s,1}Re_{k,l})$ by the first half.

□

Now we give an important lemma which gives the relationshop between left H-epimorphisms and right H-epimorphisms.

Lemma 2.1.

(I) Suppose that $\zeta : {}_RRe_{i,j} \to {}_RJe_{k,l}$ is a left H-epimorphism. Then the left multiplication map

$$\xi \overset{put}{:=} (\,(e_{i,j})\zeta\,)_L : e_{k,l}R_R \to e_{i,j}J_R$$

by $(e_{i,j})\zeta$ is a right H-epimorphism.

Further, we put $I_k \overset{put}{:=} \{\ (p,q) \mid S({}_RRe_{p,q}) \cong T({}_RRe_{k,1})\ \}$ and let $Rf_1, Rf_2, \ldots, Rf_{n'} = Re_{\sigma(k),\rho(k)}$ be a left co-H-sequence with the last term $Re_{\sigma(k),\rho(k)}$, where $\{\,f_1, f_2, \ldots, f_{n'}\,\} \subseteq \{\,e_{i,j}\,\}_{i=1,j=1}^{m\ \ n(i)}$ (then, since R is a right Harada ring, $\{\,f_1, f_2, \ldots, f_{n'}\,\} = \{\,e_{p,q}\,\}_{(p,q)\in I_k}$). ξ satisfies the following conditions (1),(2),(3).

(1) $$\operatorname{Ker}\xi = e_{k,l}\,S({}_RR) = \begin{cases} \oplus_{(p,q)\in I_k} S({}_RRe_{p,q}) \neq 0 & (\text{if } l = 1) \\ 0 & (\text{if } l \neq 1) \end{cases}$$

(2) (i) Suppose that $e_{k,l}R_R$ is injective, i.e., $l = 1$. Then the following $(x), (y), (z)$ hold.

(x) For each $(p,q) \in I_k$,
$S({}_RRe_{p,q}) = e_{k,1}\, S({}_RRe_{p,q}) = S(e_{k,1}Re_{p,q})$.

(y) $\operatorname{Ker}\xi = \oplus_{i=1}^{n'} S({}_RRf_i) = S_{n'}(e_{k,1}R_R)$ and it is uniserial as a right R-module.

(z) $j = n(i)$, i.e., $\xi : e_{k,1}R_R \to e_{i,n(i)}J_R$.

(ii) Suppose that $e_{k,l}R_R$ is not injective, i.e., $l \neq 1$. Then $(k,l) = (i, j+1)$ and $j < n(i)$, i.e., $\xi : e_{i,j+1}R_R \to e_{i,j}J_R$.

(3) For any $e_{s,t}$, the restriction map of ξ

$$\xi_{s,t} \overset{put}{:=} \xi|_{e_{k,l}Re_{s,t}} : e_{k,l}Re_{s,t} \to e_{i,j}Je_{s,t}$$

is a right $R(e_{s,t})$-epimorphism with

$$\operatorname{Ker}\xi_{s,t} = \begin{cases} S(e_{k,1}Re_{s,t}) \begin{pmatrix} \text{if } S({}_RRe_{s,t}) \cong T({}_RRe_{k,l}), \\ \text{i.e., } l = 1 \text{ and } (e_{k,1}R,\, E({}_RRe_{s,t})) \text{ is an } i\text{-pair} \end{pmatrix} \\ 0 \qquad \begin{pmatrix} \text{if } S({}_RRe_{s,t}) \not\cong T({}_RRe_{k,l}), \\ \text{i.e., either } l \neq 1 \text{ or } (e_{k,1}R,\, E({}_RRe_{s,t})) \text{ is not an } i\text{-pair} \end{pmatrix} \end{cases}$$

(II) Suppose that $\xi : e_{i,j}R_R \to e_{k,l}J_R$ is a right H-epimorphism. Then the right multiplication map

$$\zeta \overset{put}{:=} (\, \xi(e_{i,j})\,)_R : {}_RRe_{k,l} \to {}_RJe_{i,j} \, ,$$

by $\xi(e_{i,j})$ is a left H-epimorphism. Further, if ${}_RRe_{k,l}$ is injective, we let $(e_{p,1}R,\, Re_{k,l})$ be an i-pair. ζ satisfies the following conditions (1),(2),(3),(4).

(1) $$\operatorname{Ker}\zeta = S(R_R)\, e_{k,l} = \begin{cases} \oplus_{q=1}^{n(p)} S(e_{p,q}R_R) \neq 0 & (\text{if } {}_RRe_{k,l} \text{ is injective}) \\ 0 & (\text{if } {}_RRe_{k,l} \text{ is not injective}) \end{cases}$$

(2) Suppose that ${}_RRe_{k,l}$ is injective. Then the following $(x), (y)$ hold.

(x) For each $q = 1, 2, \ldots, n(p)$,
$S(e_{p,q}R_R) = S(e_{p,q}R_R)\, e_{k,l} = S(e_{p,q}Re_{k,l})$.

(y) $\operatorname{Ker}\zeta = S_{n(p)}({}_RRe_{k,l})$ and it is uniserial as a left R-module.

(3) For any $e_{s,t}$, the restriction map of ζ

$$\zeta_{s,t} \overset{put}{:=} \zeta|_{e_{s,t}Re_{k,l}} : e_{s,t}Re_{k,l} \to e_{s,t}Je_{i,j}$$

is a left $R(e_{s,t})$-epimorphism with

$$\operatorname{Ker}\zeta_{s,t} = \begin{cases} S(e_{s,t}Re_{k,l}) & (\text{ if } S(e_{s,1}R_R) \cong T(e_{k,l}R_R), \text{ i.e., } (e_{s,1}R,\ Re_{k,l}) \\ & \text{ is an } i\text{-pair}) \\ 0 & (\text{ if } S(e_{s.1}R_R) \ncong T(e_{k,l}R_R)) \end{cases}$$

(4) For any $i = 1, 2, \ldots, m$, ${}_RRe_{i,n(i)}/J^{n(i)}e_{i,n(i)}$ is uniserial.

Proof. (I) First we show that ξ is surjective. Take any $e_{s,t}$. Since R is a right Harada ring, $E({}_RRe_{s,t}) \cong Re_{u,v}$ for some $e_{u,v}$, there exists an R-isomorphism $\iota : {}_RRe_{s,t} \to {}_RJ^ne_{u,v}$, where $n \in \mathbb{N}$, and ${}_RRe_{u,v}/J^{n+1}e_{u,v}$ is uniserial. Then, for any $\gamma \in e_{i,j}Je_{s,t}$, we can define the right multiplication map

$$(\gamma)_R : Re_{i,j}/\operatorname{Ker}\zeta \to Re_{s,t}$$

by γ since ζ is a left H-epimorphism. Therefore we consider the diagram

$$\begin{array}{ccccc} 0 & \to & Re_{i,j}/\operatorname{Ker}\zeta & \xrightarrow{\overline{\zeta}} & Re_{k,l}\ , \\ & & (\gamma)_R \downarrow & & \\ & & Re_{s,t} & & \\ & & \iota \downarrow \cong & & \\ & & J^ne_{u,v} & & \\ & & \cap & & \\ & & Re_{u,v} & & \end{array}$$

where $\overline{\zeta}$ is an R-monomorphism naturally induced from ζ. And we have $\psi : Re_{k,l} \to Re_{u,v}$ with $\overline{\zeta}\,\psi = (\gamma)_R\,\iota$ since ${}_RRe_{u,v}$ is injective. Then $J\operatorname{Im}\psi = (Je_{k,l})\psi = (\operatorname{Im}\overline{\zeta})\psi = \operatorname{Im}(\overline{\zeta}\psi) = \operatorname{Im}((\gamma)_R\,\iota) \subseteq J^{n+1}e_{u,v}$. So $\operatorname{Im}\psi \subseteq J^ne_{u,v}$ because ${}_RRe_{u,v}/J^{n+1}e_{u,v}$ is uniserial. Therefore there exists $\beta \in e_{k,l}Re_{s,t}$ with $\overline{\zeta}\,(\beta)_R = (\gamma)_R$. So

$$\xi(\beta) = ((e_{i,j})\zeta)_L(\beta) = (\overline{e_{i,j}})\overline{\zeta}\ (\beta)_R = (\overline{e_{i,j}})\ (\gamma)_R = \gamma\,.$$

Hence $\operatorname{Im}\xi \supseteq e_{i,j}Je_{s,t}$ for any $e_{s,t}$. We see that $\operatorname{Im}\xi = e_{i,j}J$.

Next we show that (1),(2),(3). From (1), we see that ξ is a right H-epimorphism.

(1) For any $a \in \operatorname{Ker}\xi$ ($\subset e_{k,l}R$), $0 = R(e_{i,j})\zeta\, a = Je_{k,l}a = Ja$ since ζ is surjective. So $a \in S({}_RR)$. Hence $\operatorname{Ker}\xi = e_{k,l}\,S({}_RR)$. And, by Lemma A (II)(5),

$$e_{k,l}\,S({}_RR) = \begin{cases} \oplus_{i=1}^{n'} S({}_RRf_i) & (\text{ if } l = 1) \\ 0 & (\text{ if } l \neq 1) \end{cases}$$

since $Rf_1, Rf_2, \dots, Rf_{n'} = Re_{\sigma(k),\rho(k)}$ is a left co-H-sequence with the last term $Re_{\sigma(k),\rho(k)}$. So the statement follows since $\{ f_1, f_2, \dots, f_{n'} \} = \{ e_{p,q} \}_{(p,q)\in I_k}$.

(2) (*i*) (*x*) Since $(p,q) \in I_k$, $S({}_RRe_{p,q}) \cong T({}_RRe_{k,1})$. So the statement holds by the second half of Lemma A (II)(3).

(*y*) $\operatorname{Ker}\xi = \oplus_{i=1}^{n'} S({}_RRf_i)$ by the proof of (1). The remainder follows from Lemma A (II)(1),(2).

(*z*) Assume that $j \in \{ 1, 2, \dots, n(i) - 1 \}$. Then $e_{i,j}J_R$ is projective by the definition of a well-indexed set $\{ e_{i,j} \}_{i=1,j=1}^{m \quad n(i)}$ of left Harada ring. So $e_{k,l} = e_{i,j+1}$ and ξ is an R-isomorphism because ξ is an R-epimorphism. But, since $l = 1$, $\operatorname{Ker}\xi \neq 0$ by (1), a contradiction. Hence $j = n(i)$.

(*ii*) Let $l \neq 1$. Then $\xi : e_{k,l}R_R \to e_{i,j}J_R$ is an R-isomorphism by (1). So $(k,l) = (i, j+1)$ and $j < n(i)$ by the definition of a well-indexed set $\{e_{i,j}\}_{i=1,j=1}^{m \quad n(i)}$ of left Harada ring.

(3) By (1), (2)(*i*)(*x*). (We only note that, if $S({}_RRe_{s,t}) \cong T({}_RRe_{k,l})$, then $l = 1$ by Lemma A (II)(5).)

(II) We see that ζ is a left H-epimorphism and (1),(2),(3) hold by the same way as in (I).

(4) For each $i = 1, 2, \dots, m$ and $j = 2, 3, \dots, n(i)$, we put $\xi \overset{put}{:=} \theta_{i,j} : e_{i,j}R_R \to e_{i,j-1}J_R$. And there exists a left R-epimorphism

$$\zeta \overset{put}{:=} (\, \xi(e_{i,j}) \,)_R : {}_RRe_{i,j-1} \to {}_RJe_{i,j} \,.$$

So ${}_RRe_{i,n(i)}/J^{n(i)}e_{i,n(i)}$ is uniserial.

□

Using Lemma 2.1, we characterize left (right) H-epimorphisms.

Theorem 2.2.

(I) Suppose that $\zeta : {}_RRe_{i,j} \to {}_RJe_{k,l}$ is a left H-epimorphism. And, if ${}_RRe_{i,j}$ is injective, we let $(e_{p,1}R,\, Re_{i,j})$ be an i-pair. Then the following hold.

(1) (*i*) Suppose that $e_{k,l}R_R$ is injective, i.e., $l = 1$. Then $j = n(i)$, i.e., $\zeta : {}_RRe_{i,n(i)} \to {}_RJe_{k,1}$.

(*ii*) Suppose that $e_{k,l}R_R$ is not injective, i.e., $l \neq 1$. Then $(k,l) = (i, j+1)$ $(\, j < n(i) \,)$, i.e., $\zeta : {}_RRe_{i,j} \to {}_RJe_{i,j+1}$.

(2) (i) $\operatorname{Ker}\zeta = S(R_R)\,e_{i,j} =$

$$\begin{cases} \oplus_{q=1}^{n(p)} S(e_{p,q}R_R) = S_{n(p)}({}_RRe_{i,j}) \neq 0 & (\text{if } {}_RRe_{i,j} \text{is injective}) \\ \text{and it is uniserial as a left } R\text{-module} & \\ 0 & (\text{if } {}_RRe_{i,j} \text{ is not injective}) \end{cases}$$

(ii) If ${}_RRe_{i,j}$ is injective, then, for each $q = 1, 2, \ldots, n(p)$, $S(e_{p,q}R_R) = S(e_{p,q}R_R)\,e_{i,j} = S(e_{p,q}Re_{i,j})\,.$

(II) Suppose that $\xi : e_{i,j}R_R \to e_{k,l}J_R$ is a right H-epimorphism. And, if $e_{i,j}R_R$ is injective, we put $I_i \overset{put}{:=} \{\, (p,q) \mid S({}_RRe_{p,q}) \cong T({}_RRe_{i,1})\,\}$ and let n' be the number of elements in I_i. Then the following hold.

(1) (i) Suppose that $e_{i,j}R_R$ is injective, i.e., $j = 1$. Then $l = n(k)$, i.e., $\xi : e_{i,1}R_R \to e_{k,n(k)}J_R\,.$

(ii) Suppose that $e_{i,j}R_R$ is not injective, i.e., $j \geq 2$. Then $(k,l) = (i, j-1)$ $(\, l < n(k)\,)$, i.e., $\xi : e_{i,j}R_R \to e_{i,j-1}J_R\,.$

(2) (i) $\operatorname{Ker}\xi = e_{i,j}\,S({}_RR) =$

$$\begin{cases} \oplus_{(p,q)\in I_i} e_{i,1}\,S({}_RRe_{p,q}) = S_{n'}(e_{i,1}R_R) \neq 0 & (\text{if } j = 1) \\ \text{and it is uniserial as a right } R\text{-module} & \\ 0 & (\text{if } j \neq 1) \end{cases}$$

(ii) If $e_{i,j}R_R$ is injective, i.e., $j = 1$, then, for each $(p,q) \in I_i$, $S({}_RRe_{p,q}) = e_{i,1}\,S({}_RRe_{p,q}) = S(e_{i,1}Re_{p,q})\,.$

Proof.

(I) (1) By Lemma 2.1 (I) (1), (2)$(i)(z)$, (ii), we can define a right H-epimorphism

$$\xi \overset{put}{:=} (\,(e_{i,j}\,)\zeta\,)_L : e_{k,l}R_R \to e_{i,j}J_R$$

which satisfies the following.

(i') Suppose that $e_{k,l}R_R$ is injective, i.e., $l = 1$. Then $j = n(i)$, i.e., $\xi : e_{k,1}R_R \to e_{i,n(i)}J_R\,.$

(ii') Suppose that $e_{k,l}R_R$ is not injective, i.e., $l \neq 1$. Then $(k,l) = (i, j+1)$ $(\, j < n(i)\,)$, i.e., $\xi : e_{i,j+1}R_R \to e_{i,j}J_R$ and it is a right R-isomorphism.

Using Lemma 2.1 (II) for the ξ, either the following (i) or (ii) holds.

(i) $e_{k,l}R_R$ is injective, i.e. $l = 1$, $j = n(i)$ and $\zeta = (\,(\,(e_{i,n(i)}\,)\zeta\,)_L(e_{k,1})\,)_R = (\,\xi(e_{k,1})\,)_R : {}_RRe_{i,n(i)} \to {}_RJe_{k,1}\,.$

(ii) $e_{k,l}R_R$ is not injective, i.e., $l \neq 1$, $(k,l) = (i, j+1)$ and $\zeta = (\,(\,(e_{i,j}\,)\zeta\,)_L(e_{i,j+1})\,)_R = (\,\xi(e_{i,j+1})\,)_R : {}_RRe_{i,j} \to {}_RJe_{i,j+1}$.

(2) Since $\zeta = (\,(\,(e_{i,j}\,)\zeta\,)_L(e_{k,l})\,)_R$, the statements follow from Lemma 2.1 (II) (1), (2).

(II) (1) By Lemma 2.1 (II), we can define a left H-epimorphism

$$\zeta \stackrel{put}{:=} (\,\xi(e_{i,j}\,)\,)_R : {}_RRe_{k,l} \to {}_RJe_{i,j}\,.$$

So, using Lemma 2.1 (I) (1), (2)$(i)(z)$, (ii) for the ζ, the statements hold.

(2) Since $\xi = (\,(e_{k,l})(\,\xi(e_{i,j}\,)\,)_R\,)_L$, the statements follow from Lemma 2.1 (I) (1), (2)$(i)(x)$, (y).

□

By the definition of a well-indexed set $\{\,e_{i,j}\,\}_{i=1,j=1}^{m\ n(i)}$ of left Harada ring,

$$e_{i,n(i)}R,\ e_{i,n(i)-1}R, \ldots, e_{i,1}R \quad (\,i = 1, 2, \ldots, m\,)$$

are right co-H-sequences of R. And, from Theorem 2.2, we obtain the following characterization left co-H-sequences of R using the same set $\{\,e_{i,j}\,\}_{i=1,j=1}^{m\ n(i)}$.

Corollary 2.3. Every left co-H-sequence of R is of the form

$Re_{i_1,s},\ Re_{i_1,s+1}, \ldots,\ Re_{i_1,n(i_1)},\ Re_{i_2,1},\ Re_{i_2,2},\ \ldots,\ Re_{i_2,n(i_2)},\ Re_{i_3,1},\ \ldots\ ,Re_{i_u,t}$,

where $1 \le i_1, i_2, \ldots, i_u \le m$, $1 \le s \le n(i_1)$ and $1 \le t \le n(i_u)$.

Proof. By Theorem 2.2 (I) (1). □

Example 2.4. *Let R be a basic indecomposable Nakayama ring with a complete set $\{\,g_i\,\}_{i=1}^{7}$ of orthogonal primitive idempotents which satisfies*

(i) $T(g_iJ_R) \cong T(g_{i+1}R_R)$ *for any* $i = 1, 2, \ldots, 6$, *and*

(ii) $c(\,g_1R_R\,) = 10$, $c(\,g_2R_R\,) = 9$,
$c(\,g_3R_R\,) = 10$, $c(\,g_4R_R\,) = 9$,
$c(\,g_5R_R\,) = 11$, $c(\,g_6R_R\,) = 10$, $c(\,g_7R_R\,) = 9$,
where $c(\,M\,)$ means the composition length of an R-module M.

We put

$$e_{1,1} \stackrel{put}{:=} g_1,\ e_{1,2} \stackrel{put}{:=} g_2,\ e_{2,1} \stackrel{put}{:=} g_3,\ e_{2,2} \stackrel{put}{:=} g_4,\ e_{3,1} \stackrel{put}{:=} g_5,\ e_{3,2} \stackrel{put}{:=} g_6,\ e_{3,3} \stackrel{put}{:=} g_7.$$

And $\{\,e_{1,1},\ e_{1,2},\ e_{2,1},\ e_{2,2},\ e_{3,1},\ e_{3,2},\ e_{3,3}\,\}$ is a left well-indexed set of R and

$$(\,e_{1,1}R,\ Re_{2,1}\,),\quad (\,e_{2,1}R,\ Re_{3,1}\,),\quad (\,e_{3,1}R,\ Re_{1,1}\,)$$

are i-pairs and

$$Re_{1,2},\ Re_{2,1}$$
$$Re_{2,2},\ Re_{3,1}$$
$$Re_{3,2},\ Re_{3,3},\ Re_{1,1}$$

are left co-H-sequences.

3. w-H-EPIMORPHISMS

In section 2, we characterize H-epimorphisms in Theorem 2.2 and left co-H-sequences by a well-indexed set $\{e_{i,j}\}_{i=1,j=1}^{m\ \ n(i)}$ of left Harada ring as a corollary (Corollary 2.3). In this section, we characterize w-H-epimorphisms.

Lemma 3.1. *Let Rf_1, Rf_2, ..., $Rf_{n'}$ be a left co-H-sequence and let $f_1 = e_{k,l}$.*

(1) *Suppose that $l \geq 2$. Then we have a left H-epimorphism ζ' : ${}_RRe_{k,l-1} \to {}_RJf_1$.*

(2) *Suppose that $l = 1$. If there exists a left H-epimorphism ζ' : ${}_RRe_{i,j} \to {}_RJe_{k,1} = {}_RJf_1$, then $j = n(i)$, i.e., $\zeta' : {}_RRe_{i,n(i)} \to {}_RJf_1$.*

Proof.

(1) By Lemma 2.1 (II), $(\,\theta_{k,l}(e_{k,l})\,)_R$: ${}_RRe_{k,l-1} \to {}_RJe_{k,l}$ is a left H-epimorphism.

(2) By Theorem 2.2 (I)(1).

□

Now we further consider a left H-epimorphism the codomain of which is the Jacobson radical of the first term of some left co-H-sequence.

Lemma 3.2.

(I) Let Rf_1, Rf_2, ... , $Rf_{n'}$ be a left co-H-sequence. And suppose that there exists a left H-epimorphism ζ' : ${}_RRf_0 \to {}_RJf_1$. Then the following hold.

(0) Then ${}_RRf_0$ is injective.

So we let $(e_{k,1}R,\ Rf_0)$ be an i-pair. Then the following hold.

(1) $\operatorname{Ker}\zeta' = S_{n(k)}({}_RRf_0)$.

(2) ${}_RRf_0/S_{j-1}({}_RRf_0) \cong E(T({}_RRe_{k,j}))$ for any $j = 1, 2, \ldots, n(k)$.

(3) $S_{n(k)+1}({}_RRf_0)$ is uniserial as a left R-module.

Further we let $(e_{l,1}R,\ Rf_{n'})$ be an i-pair. Then the following hold.

(4) $S\big({}_RRf_0/S_{n(k)+j-1}({}_RRf_0)\big) \cong T({}_RRe_{l,j})$ for any $j = 1, 2, \dots, n(l)$.

(II) Suppose that $\xi' : e_{l,1}R_R \to e_{k,n(k)}J_R$ is a right H-epimorphism and we let both $Rg_1, Rg_2, \dots, Rg_{n_l} = Re_{\sigma(l),\rho(l)}$ and $Rh_1, Rh_2, \dots, Rh_{n_k} = Re_{\sigma(k),\rho(k)}$ be left co-H-sequences. Then the following hold.

(1) $\operatorname{Ker}\xi' = S_{n_l}(e_{l,1}R_R)$.

(2) $e_{l,1}R/S_{j-1}(e_{l,1}R_R) \cong E(T(g_{n_l-j+1}R_R))$ for any $j = 1, 2, \dots, n_l$.

(3) $S_{n_l+1}(e_{l,1}R_R)$ is uniserial as a right R-module.

(4) $S(e_{l,1}R_R/S_{n_l+j-1}(e_{l,1}R_R)) \cong T(h_{n_k-j+1}R_R)$ for any $j = 1, 2, \dots, n_k$.

Proof.

(I) (0) By (CHS3) in the definition of a left co-H-sequence and Theorem 2.2 (I)(2)(i).

(1) By Theorem 2.2 (I)(2)(i).

(2) By Lemma A (I)(1).

(3) $S_{n(k)}({}_RRf_0)$ is uniserial by Lemma A (I)(1). And $S_{n(k)+1}({}_RRf_0)/S_{n(k)}({}_RRf_0)$ is simple by (i) since ${}_RRf_1$ is colocal.

(4) By (1), ${}_RRf_0/S_{n(k)}({}_RRf_0) \cong {}_RJf_1 \cong {}_RJ^{n'}f_{n'}$. And, for any $j = 1, 2, \dots, n(l)$, $S\big({}_RRf_{n'}/S_{j-1}({}_RRf_{n'})\big) \cong T({}_RRe_{l,j})$ by Lemma A (I)(1) since $(e_{l,1}R,\ Rf_{n'})$ is an i-pair. So the statement holds.

(II) We see by the same way as in (I).

□

Next we consider colocal pairs in two-sided Harada rings.

Lemma 3.3. *Let $Rf_1,\ Rf_2,\ \dots,\ Rf_{n'}$ be a left co-H-sequence and let $(e_{l,1}R,\ Rf_{n'})$ be an i-pair. Then the following hold.*

(I) (0) *$S(\,e_{l,j}Rf_s\,)$ is defined for any $j = 1, 2, \dots, n(l)$ and any $s = 1, 2, \dots, n'$.*

(1) *Suppose that there exists a left H-epimorphism $\zeta' : {}_RRg_{n''} \to {}_RJf_1$ and we let $Rg_1,\ Rg_2,\ \dots,\ Rg_{n''}$ be a left co-H-sequence. Then $S(\,e_{l,j}Rg_t\,)$ is defined for any $j = 1, 2, \dots, n(l)$ and any $t = 1, 2, \dots, n''$.*

(2) *We further suppose that there exists a left H-epimorphism ζ'' : ${}_RRh_{n'''} \to {}_RJg_1$ and we let Rh_1, Rh_2, $\ldots$, $Rh_{n'''}$ be a left co-H-sequence. Then $S(\, e_{l,j}Rh_u\,)$ is defined for any $j = 1, 2, \ldots, n(l)$ and any $u = 1, 2, \ldots, n'''$.*

(II) (1) *Suppose that there exists a right H-epimorphism ξ' : $e_{l',1}R_R \to e_{l,n(l)}J_R$. Then $S(\, e_{l',t}Rf_j\,)$ is defined for any $t = 1, 2, \ldots, n(l')$ and any $j = 1, 2, \ldots, n'$.*

(2) *We further suppose that there exists a right H-epimorphism ξ'' : $e_{l'',1}R_R \to e_{l',n(l')}J_R$. Then $S(\, e_{l'',u}Rf_j\,)$ is defined for any $u = 1, 2, \ldots, n(l'')$ and any $j = 1, 2, \ldots, n'$.*

Proof.

(I) (0) $S(\, e_{l,1}Rf_s\,)$ is defined by Lemma A (II)(3). So $S(\, e_{l,j}Rf_s\,{}_{R(f_s)}\,)$ is simple since $e_{l,j}R_R \cong e_{l,1}J_R^{j-1}$. On the other hand, $S({}_{R(e_{l,j})}\, e_{l,j}Rf_{n'})$ is defined by Lemma A (I)(3). So $S({}_{R(e_{l,j})}e_{l,j}Rf_s)$ is simple since ${}_RRf_s \cong {}_RJ^{n'-s}f_{n'}$. Hence $S(\, e_{l,j}Rf_s\,)$ is defined.

(1) First we consider the case ${}_RRg_{n''} \cong {}_RRf_{n'}$. Then $g_s = f_s$. So the statement holds from (0).

Next we consider the case ${}_RRg_{n''} \ncong {}_RRf_{n'}$. Let $(e_{k,1}R,\, Rg_{n''})$ be an i-pair. Then

$$r_{Rg_{n''}}(e_{l,j}R) = S_{n(k)+j-1}(Rg_{n''})$$

and

$$S\big(\, Rg_{n''}/r_{Rg_{n''}}(e_{l,j}R)\big) \cong T({}_RRe_{l,j})$$

by Lemma 3.2 (I)(2),(4). So

$${}_RRg_{n''}/r_{Rg_{n''}}(e_{l,j}R) \;\cong\; {}_RJ^{n'}f_{n'}/S_{j-1}({}_RRf_{n'})$$

and

$$E(\, {}_RRg_{n''}/r_{Rg_{n''}}(e_{l,j}R)\,) \;\cong\; {}_RRf_{n'}/S_{j-1}({}_RRf_{n'})$$

by Lemma 3.2 (I)(1) and Lemma A (I)(1) since $Rf_1, Rf_2, \ldots, Rf_{n'}$ is a left co-H-sequence. Therefore

$${}_RRg_t/r_{Rg_t}(e_{l,j}R) \;\cong\; {}_RJ^{n'+(n''-t)}f_{n'}/S_{j-1}({}_RRf_{n'})$$

and

$$E(\, {}_RRg_t/r_{Rg_t}(e_{l,j}R)\,) \;\cong\; {}_RRf_{n'}/S_{j-1}({}_RRf_{n'})$$

since $Rg_1, Rg_2, \ldots, Rg_{n''}$ is a left co-H-sequence. So ${}_RRg_t/r_{Rg_t}(e_{l,j}R)$ is quasi-injective. Hence $S(\, e_{l,j}Rg_t\,)$ is defined by [5, Corollary 1.6].

(2) If ${}_RRh_{n'''} \cong {}_RRf_{n'}$, the statement holds from (0). So we assume that ${}_RRh_{n'''} \ncong {}_RRf_{n'}$. Let $(e_{k',1}R,\, Rh_{n'''})$ be an i-pair. Then

the follong $(i), (ii), (iii), (iv), (v), (vi)$ hold by Lemma 3.2 (I)(1),(2),(4) and Lemma A (I)(1).

(i) $r_{Rh_{n'''}}(e_{l,j}R) = S_{n(k')+n(k)+j-1}({}_RRh_{n'''})$

(ii) $S({}_RRh_{n'''}/r_{Rh_{n'''}}(e_{l,j}R)) \cong T({}_RRe_{l,j})$

(iii) ${}_RRh_{n'''}/r_{Rh_{n'''}}(e_{l,j}R) \cong {}_RJ^{n'+n''}f_{n'}/S_{j-1}({}_RRf_{n'})$

(iv) $E({}_RRh_{n'''}/r_{Rh_{n'''}}(e_{l,j}R)) \cong {}_RRf_{n'}/S_{j-1}({}_RRf_{n'})$

(v) ${}_RRh_u/r_{Rh_u}(e_{l,j}R) \cong {}_RJ^{n'+n''+(n'''-u)}f_{n'}/S_{j-1}({}_RRf_{n'})$

(vi) $E({}_RRh_u/r_{Rh_u}(e_{l,j}R)) \cong {}_RRf_{n'}/S_{j-1}({}_RRf_{n'})$

So ${}_RRh_u/r_{Rh_u}(e_{l,j}R)$ is quasi-injective and $S(e_{l,j}Rh_u)$ is defined by [5, Corollary 1.6].

(II) We see by the left-right symmetrical argument of (I).

□

Using Lemma 3.3, last we generalize Lemma 2.1 to w-H-epimorphisms.

Proposition 3.4. Let $f_1, f_2, \ldots, f_{u+1}$ be distinct elements in $\{e_{i,j}\}_{i=1,j=1}^{m\ n(i)}$. Suppose that

$$\zeta \overset{put}{:=} \zeta_1\zeta_2\cdots\zeta_u : {}_RRf_1 \to {}_RJ^u f_{u+1}$$

is a left w-H-epimorphism, where $\zeta_i : {}_RRf_i \to {}_RJf_{i+1}$ is a left H-epimorphism for $i = 1, 2, \ldots, u$. For each $i = 1, 2, \ldots, u$, we consider a right H-epimorphism

$$\xi_i \overset{put}{:=} ((f_i)\zeta_i)_L : f_{i+1}R \to f_iJ$$

given in Lemma 2.1 (I). And we put

$$\xi \overset{put}{:=} ((f_1)\zeta)_L : f_{u+1}R \to f_1J^u\,.$$

Further we put

$$X \overset{put}{:=} \{\, i \in \{2, 3, \ldots, u+1\} \mid f_iR_R \text{ is injective}\,\}\,.$$

And, for each $i \in X$, put $I_i \overset{put}{:=} \{(p,q) \mid S({}_RRe_{p,q}) \cong T({}_RRf_i)\}$, let $(f_iR,\, Rg_i)$ be an i-pair, let n'_i be the length of a left co-H-sequence with the last term Rg_i and put $n' \overset{put}{:=} \sum_{i\in X} n'_i$.

Then the following hold.

(1) ξ is a right w-H-epimorphism with $\xi = \xi_1\xi_2\cdots\xi_u$.

(2) $\operatorname{Ker}\xi = \oplus_{i\in X}\oplus_{(p,q)\in I_i} S(f_{u+1}Re_{p,q}) = S_{n'}(f_{u+1}R_R)$ and it is uniserial as a right R-module.

We note that the left-right symmetric statement of (I) also holds for a right w-H-epimorphism $\xi \overset{put}{:=} \xi_1\xi_2\cdots\xi_u : f_{u+1}R_R \to f_1J_R^u$, where $\xi_i : f_{i+1}R_R \to f_iJ_R$ is a right H-epimorphism for $i = 1, 2, \ldots, u$.

Proof.

(1)
$$\begin{aligned}
\xi(f_{u+1}) &= ((f_1)\zeta)_L(f_{u+1}) \\
&= ((f_1)\zeta_1\zeta_2\cdots\zeta_u)_L(f_{u+1}) \\
&= ((f_1)((f_1)\zeta_1)_R((f_2)\zeta_2)_R\cdots((f_u)\zeta_u)_R)_L(f_{u+1}) \\
&= f_1\cdot(f_1)\zeta_1\cdot(f_2)\zeta_2\cdot\cdots\cdot(f_u)\zeta_u\cdot f_{u+1} \\
&= ((f_1)\zeta_1)_L((f_2)\zeta_2)_L\cdots((f_u)\zeta_u)_L(f_{u+1}) \\
&= \xi_1\xi_2\cdots\xi_u(f_{u+1}).
\end{aligned}$$

So $\xi = \xi_1\xi_2\cdots\xi_u$. Therefore ξ is a right w-H-epimorphism.

(2) If f_2R_R is not injective, i.e., $2 \notin X$, then $\operatorname{Ker}\xi_1 = 0$ by Lemma 2.1 (I)(1). If f_2R_R is injective, i.e., $2 \in X$, then
$$\operatorname{Ker}\xi_1 = \oplus_{(p,q)\in I_2}S(f_2Re_{p,q}) = S_{n'_2}(f_2R_R)$$
and it is uniserial as a right R-module by Lemma 2.1 (I)(1), (2)(i)(x), (y).

Next, with respect f_2 and f_3, we consider the following four cases.

(i) If both f_2R_R and f_3R_R are not injective, i.e., $2, 3 \notin X$, then $\xi_2^{-1}(\operatorname{Ker}\xi_1) = 0$ by Lemma 2.1 (I)(1).

(ii) If f_2R_R is not injective and f_3R_R is injective, i.e., $2 \notin X$ and $3 \in X$, then
$$\begin{aligned}
\xi_2^{-1}(\operatorname{Ker}\xi_1) &= \xi_2^{-1}(0) \\
&= \oplus_{(p,q)\in I_3}S(f_3Re_{p,q}) \\
&= S_{n'_3}(f_3R_R)
\end{aligned}$$
and it is uniserial as a right R-module by Lemma 2.1 (I)(1), (2)(i)(x), (y).

(iii) If f_2R_R is injective but f_3R_R is not injective, i.e., $2 \in X$ and $3 \notin X$, then
$$\begin{aligned}
\xi_2^{-1}(\operatorname{Ker}\xi_1) &= \xi_2^{-1}(\oplus_{(p,q)\in I_2}S(f_2Re_{p,q})) \\
&= \oplus_{(p,q)\in I_2}S(f_3Re_{p,q\ R(e_{p,q})}) \\
&= S_{n'_2}(f_3R_R)
\end{aligned}$$
and it is uniserial as a right R-module by Lemma 2.1 (I)(1), (2)(i)(x), (y). Further $S(f_3Re_{p,q\ R(e_{p,q})}) = S(f_3Re_{p,q})$ for any $(p, q) \in I_2$ by Lemma 3.3 (II)(1).

(*iv*) If both f_2R_R and f_3R_R are injective, i.e., $2, 3 \in X$, then

$$\begin{aligned} &\xi_2^{-1}(\operatorname{Ker}\xi_1) \\ &= \xi_2^{-1}(\oplus_{(p,q)\in I_2} S(f_2Re_{p,q})) \\ &= (\oplus_{(p,q)\in I_2} S(f_3Re_{p,q\ R(e_{p,q})})) \oplus (\oplus_{(p',q')\in I_3} S(f_3Re_{p',q'})) \\ &= S_{n_2'+n_3'}(f_3R_R) \end{aligned}$$

and it is uniserial as a right R-module by Lemma 2.1 (I)(1), (2)(i)(x), (y). Further $S(f_3Re_{p,q\ R(e_{p,q})}) = S(f_3Re_{p,q})$ for any $(p,q) \in I_2$ by Lemma 3.3 (II)(1).

Inductively we obtain

$$\operatorname{Ker}\xi = \oplus_{i\in X} \oplus_{(p,q)\in I_i} S(f_{u+1}Re_{p,q}) = S_{n'}(f_{u+1}R_R)$$

and it is uniserial as a right R-module.

$\square$

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