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ON SOME FAMILIES OF INVARIANT POLYNOMIALS DIVISIBLE BY THREE AND THEIR ZETA FUNCTIONS

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Abstract. In this note, we establish an analog of the Mallows-Sloane bound for Type III formal weight enumerators. This completes the bounds for all types (Types I through IV) in synthesis of our previous results. Next we show by using the binomial moments that there exists a family of polynomials divisible by three, which are not related to linear codes but are invariant under the MacWilliams transform for the value 3/2. We also discuss some properties of the zeta functions for such polynomials.

1. Introduction

This article, as a sequel of [3]-[5], investigates some polynomials of the form

$$W(x,y) = x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i \in \mathbf{C}[x,y] \quad (A_d \neq 0) \tag{1.1}$$

that satisfy certain transformation rules: for a linear transformation $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the action of σ on $W(x,y)$ is defined by

$$W^{\sigma}(x,y) = W(ax+by, cx+dy)$$

and we are interested in $W(x,y)$ of the form (1.1) with the property

$$W^{\sigma_q}(x,y) = \pm W(x,y),$$

where

$$\sigma_q = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & q-1 \\ 1 & -1 \end{pmatrix} \quad \text{(the MacWilliams transform).}$$

We call $W(x,y)$ with $W^{\sigma_q}(x,y) = W(x,y)$ a "σ_q-invariant polynomial" and $W(x,y)$ with $W^{\sigma_q}(x,y) = -W(x,y)$ a formal weight enumerator. We sometimes say a "q-formal weight enumerator" when we specify the value q. Moreover, $W(x,y)$ is called "divisible by c" ($c>1$) if "$A_i \neq 0 \Rightarrow c|i$". In this article, we are interested in the case $c=3$.

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The earliest example of the divisible formal weight enumerator in the literature is the case $(q, c) = (2, 4)$, which is given in Ozeki [13] (the denomination "formal weight enumerator" is also due to him). Ozeki's formal weight enumerators are members of the polynomial ring

$$R_{\mathrm{II}}^{-} := \mathbf{C}[W_{\mathcal{H}_8}(x, y), W_{12}(x, y)],$$

where

$$\begin{aligned} W_{\mathcal{H}_8}(x, y) &= x^8 + 14x^4y^4 + y^8, \\ W_{12}(x, y) &= x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12} \end{aligned}$$

($W_{12}(x, y)$ satisfies ${W_{12}}^{\sigma_2}(x, y) = -W_{12}(x, y)$). Note that $W_{\mathcal{H}_8}(x, y)$ is the weight enumerator of the extended Hamming code. We will call formal weight enumerators in R_{II}^{-} "Type II formal weight enumerators", since they resemble weight enumerators of Type II codes, which are divisible by four and σ_2-invariant. We also have rings of formal weight enumerators for the cases $(q, c) = (2, 2), (3, 3)$ and $(4, 2)$ which we shall call Types I, III and IV, respectively:

$$\begin{aligned} &(\text{Type I}) && R_{\mathrm{I}}^{-} := \mathbf{C}[W_{2,2}(x, y), \varphi_4(x, y)], \\ &(\text{Type III}) && R_{\mathrm{III}}^{-} := \mathbf{C}[W_4(x, y), \psi_6(x, y)], \\ &(\text{Type IV}) && R_{\mathrm{IV}}^{-} := \mathbf{C}[W_{2,4}(x, y), \varphi_3(x, y)], \end{aligned}$$

where,

$$\begin{aligned} W_{2,q}(x, y) &= x^2 + (q-1)y^2, \\ \varphi_4(x, y) &= x^4 - 6x^2y^2 + y^4, \\ W_4(x, y) &= x^4 + 8xy^3, \\ \psi_6(x, y) &= x^6 - 20x^3y^3 - 8y^6, \\ \varphi_3(x, y) &= x^3 - 9xy^2. \end{aligned}$$

The ring R_{III}^{-} is introduced by Ozeki [14], R_{I}^{-} and R_{IV}^{-} are dealt with in [5].

Our first goal in this article is to complete the following theorem by proving the case of Type III (the cases Types I and IV are already proved in [5] and the case Type II is proved in [1]):

Theorem 1.1. *For all formal weight enumerators of Types I through IV of the form (1.1), we have the following:*

$$\begin{aligned} &(\text{Type I}) && d \le 2\left[\frac{n-4}{8}\right] + 2, \\ &(\text{Type II}) && d \le 4\left[\frac{n-12}{24}\right] + 4, \end{aligned}$$

$$(1.2) \qquad \text{(Type III)} \qquad d \le 3\left[\frac{n-6}{12}\right]+3,$$
$$\text{(Type IV)} \qquad d \le 2\left[\frac{n-3}{6}\right]+2,$$

where $[x]$ means the greatest integer not exceeding x for $x \in \mathbf{R}$.

This is an analog of the famous Mallows-Sloane bound for weight enumerators of divisible self-dual codes ([11]). Similarly to the case of codes, we can define the extremal formal weight enumerator:

Definition 1. A formal weight enumerator of Types I through IV is called extremal if an equality holds in Theorem 1.1.

Our interest in divisible formal weight enumerators arose from the consideration of their zeta functions. Zeta functions of this type were defined in Duursma [6] for weight enumerators of linear codes (see also [7]-[9]) and some generalization was made by the present author ([1], [2]):

Definition 2. For any homogeneous polynomial of the form (1.1) and $q \in \mathbf{R}$ ($q > 0, q \neq 1$), there exists a unique polynomial $P(T) \in \mathbf{C}[T]$ of degree at most $n-d$ such that

$$(1.3) \quad \frac{P(T)}{(1-T)(1-qT)}(y(1-T)+xT)^n = \cdots + \frac{W(x,y)-x^n}{q-1}T^{n-d} + \cdots .$$

We call $P(T)$ and $Z(T) = P(T)/(1-T)(1-qT)$ the zeta polynomial and the zeta function of $W(x,y)$, respectively.

We must assume $d, d^\perp \ge 2$ where $d^\perp$ is defined by

$$W^{\sigma_q}(x,y) = \pm x^n + A_{d^\perp}x^{n-d^\perp}y^{d^\perp} + \cdots \quad (A_{d^\perp} \neq 0)$$

when considering zeta functions ([7, p.57]). The Riemann hypothesis is formulated as follows:

Definition 3 (Riemann hypothesis)**.** A polynomial of the form (1.1) with $W^{\sigma_q}(x,y) = \pm W(x,y)$ satisfies the Riemann hypothesis if all the zeros of $P(T)$ have the same absolute value $1/\sqrt{q}$.

Our second result is the following theorem, which is an analog of Okuda's theorem ([12, Theorem 5.1]), of which proof will be given briefly in Section 2 :

Theorem 1.2. *Let $W(x,y)$ be the Type III extremal formal weight enumerator of degree $n = 12k+6$ ($k \ge 1$). Then*

$$W^*(x,y) := \frac{1}{(n-3)_4}\frac{\partial}{\partial x}\left(\frac{\partial^3}{\partial x^3}+\frac{\partial^3}{\partial y^3}\right)W(x,y)$$

is the extremal formal weight enumerator of degree $n-4$. Moreover, the zeta polynomial $P(T)$ of $W(x,y)$ and $P^*(T)$, that of $W^*(x,y)$ are related by $P^*(T)=(3T^2-3T+1)P(T)$. The Riemann hypothesis of $W(x,y)$ and that of $W^*(x,y)$ are equivalent.

These results, together with the ones in [1] and [5] suggest that formal weight enumerators of Types I through IV have similar properties to the weight enumerators of corresponding Types.

The last feature of this article is the discovery of $\sigma_{3/2}$-invariant polynomials. They are also divisible by three:

$$R_{3/2} := \mathbf{C}[\eta_6(x,y), \eta_{24}(x,y)], \tag{1.4}$$

where

$$\eta_6(x,y) = x^6+\frac{5}{2}x^3y^3-\frac{1}{8}y^6, \tag{1.5}$$

$$\begin{aligned}\eta_{24}(x,y) &= x^{24}+\frac{253}{4}x^{18}y^6+\frac{1265}{32}x^{15}y^9+\frac{7659}{256}x^{12}y^{12}\\ &\quad -\frac{1265}{256}x^9y^{15}+\frac{253}{256}x^6y^{18}+\frac{1}{4096}y^{24}.\end{aligned} \tag{1.6}$$

We can also construct the ring of 3/2-formal weight enumerators:

$$R^-_{3/2} := \mathbf{C}[\eta_6(x,y), \eta_{12}(x,y)], \tag{1.7}$$

where

$$\eta_{12}(x,y) = x^{12}-11x^9y^3-\frac{11}{8}x^3y^9-\frac{1}{64}y^{12}. \tag{1.8}$$

These families were discovered by the use of the binomial moments. We will explain it and observe their Riemann hypothesis in Section 3.

In what follows, we put $\tau = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$ $(\omega = (-1+\sqrt{-3})/2)$. The Pochhammer symbol $(a)_n$ means $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \geq 1$ and $(a)_0 = 1$.

2. Type III formal weight enumerators

First we give an outline of the proof of Theorem 1.1 (Type III). For a homogeneous polynomial $p(x,y) \in \mathbf{C}[x,y]$, $p(x,y)(D)$ means a differential operator obtained by replacing x by $\partial/\partial x$ and y by $\partial/\partial y$. Here we use $p(x,y) = y(y^3-8x^3)$. In a similar manner to Duursma [9, Lemma 2], we can prove the following (see also [5, Proposition 3.1]):

Proposition 4. Let $W(x,y)$ be a Type III formal weight enumerator with $d \geq 6$. Then we have

$$\{y(x^3-y^3)\}^{d-4} | p(x,y)(D)W(x,y). \tag{2.1}$$

Using this, we can prove (1.2). Proof is similar to that of [5, Theorem 3.3] and the notation follows it:

(Proof of Theorem 1.1 (Type III))

Let $a(x,y) = \{y(x^3 - y^3)\}^{d-4}$ and we put

$$p(x,y)(D)W(x,y) = a(x,y)\tilde{a}(x,y).$$

Note that

$$\begin{aligned} p^{{}^t\sigma_3}(x,y) &= p(x,y), \\ W^{\sigma_3}(x,y) &= -W(x,y), \\ a^{\sigma_3}(x,y) &= a(x,y). \end{aligned}$$

So we have

$$\tilde{a}^{\sigma_3}(x,y) = -\tilde{a}(x,y).$$

Similarly, the transformation rules

$$\begin{aligned} p^{{}^t\tau}(x,y) &= \omega p(x,y), \\ W^{\tau}(x,y) &= W(x,y), \\ a^{\tau}(x,y) &= \omega^2 a(x,y) \end{aligned}$$

imply

$$\tilde{a}^{\tau}(x,y) = \tilde{a}(x,y).$$

Considering the terms of $p(x,y)(D)W(x,y)$ and $a(x,y)$, we can verify that $\tilde{a}(x,y)$ has a term of a power of x only (it is the term of $x^{n-4d+12}$). Therefore, we can see that $\tilde{a}(x,y)$ is a constant times a certain formal weight enumerator of the form (1.1) and that $\psi_6(x,y)|\tilde{a}(x,y)$. Since $(a(x,y), \psi_6(x,y)) = 1$, we can conclude

$$a(x,y)\psi_6(x,y)|p(x,y)(D)W(x,y).$$

Comparing the degrees on both sides, we have $4(d-4)+6 \le n-4$. Putting $d = 3d'$, we get $d' \le (n-6)/12 + 1$. Since $d' \in \mathbf{Z}$, it is equivalent to $d' \le [(n-6)/12] + 1$. The conclusion follows immediately. $\square$

Remark. Some numerical examples of zeta polynomials for Type III formal weight enumerators are given and the extremal property is mentioned up to degree 18 in [1, Section 4].

(Proof of Theorem 1.2)

We follow the method of Okuda [12] (see also [5, Theorem 3.9]). Here we use $p(x,y) = x(x^3 + y^3)$. Let $W(x,y)$ be the extremal formal weight enumerator of degree $n = 12k + 6$, that is,

$$W(x,y) = x^{12k+6} + A_{3k+3}x^{9k+3}y^{3k+3} + \cdots .$$

Then from the rules

$$p^{{}^t\sigma_3}(x,y) = p^{{}^t\tau}(x,y) = p(x,y),$$
$$W^{\sigma_3}(x,y) = -W(x,y),\ W^{\tau}(x,y) = W(x,y),$$

$W^*(x,y)$ is also a Type III formal weight enumerator. It is of the form

$$W^*(x,y) = x^{12k+2} + A'_{3k}x^{9k+2}y^{3k} + \cdots .$$

By the uniqueness of the extremal formal weight enumerator at each degree, we can see that $W^*(x,y)$ is extremal at the degree $12k+2$ (note that $3[(n-6)/12]+3 = 3k$ if $n = 12k+2$). Next we use the MDS weight enumerators for $q = 3$. Let $M_{n,d} = M_{n,d}(x,y)$ be the $[n, k = n-d+1, d]$ MDS weight enumerator. If the genus of $W(x,y)$ is $n/2 - d + 1$, then the zeta polynomial $P(T)$ of $W(x,y)$ satisfies $\deg P(T) = n-2d+2$. Let $P(T) = \sum_{i=0}^{n-2d+2} a_i T^i$. Then $P(T)$ and $W(x,y)$ are related by

$$W(x,y) = a_0 M_{n,d} + a_1 M_{n,d+1} + \cdots + a_{n-2d+2} M_{n,n-d+2} \tag{2.2}$$

(see [7, formula (5)]). By the use of the "puncturing and averaging operator" and the "shortening and averaging operator" in [7, Section 3], we have

$$\begin{aligned} x(D)M_{n,i}(x,y) &= nM_{n-1,i}(x,y), \\ y(D)M_{n,i}(x,y) &= n(M_{n-1,i-1}(x,y) - M_{n-1,i}(x,y)). \end{aligned}$$

Applying these rules repeatedly to the both sides of (2.2), we can verify that the zeta polynomial of $x^4(D)W(x,y)/(n-3)_4$ is $P(T)$, that of $xy^3(D)W(x,y)/(n-3)_4$ is $(1-T)^3P(T)$. Adjusting the degrees, we can conclude that $P^*(T) = (3T^2 - 3T + 1)P(T)$. The equivalence of the Riemann hypothesis is straightforward. □

3. Polynomials for $q = 3/2$

Our construction of $\eta_6(x,y)$ (see (1.5)) uses the binomial moments. We give an outline (see also [3]). We search a σ_q-invariant polynomial $W(x,y)$ of the form

$$W(x,y) = \sum_{i=0}^{[2n/3]} A_i x^{2n-3i} y^{3i} \quad (A_0 = 1). \tag{3.1}$$

The formula of the binomial moments for (3.1) becomes

$$\sum_{i=0}^{[(2n-\nu)/3]} \binom{2n-3i}{\nu} A_i - q^{n-\nu} \sum_{i=0}^{[\nu/3]} \binom{2n-3i}{2n-\nu} A_i = 0 \quad (\nu = 0, 1, \cdots, 2n) \tag{3.2}$$

(it is obtained from [10, p.131, Problem (6)]). In (3.2), the values ν and $2n-\nu$ give essentially the same formula, so it suffices to consider the cases

$\nu = 0, 1, \cdots, n$. Moreover, (3.2) is trivial when $\nu = n$. Thus (3.2) gives n linear equations of $[2n/3]+1$ unknowns $A_0, A_1, \cdots, A_{[2n/3]}$. The number of equations and unknowns coincide when $n = 3$, in which case the system of equations becomes

$$\begin{cases} (1-q^3)A_0 + A_1 + A_2 = 0, \\ 6(1-q^2)A_0 + 3A_1 = 0, \\ 15(1-q)A_0 + 3A_1 = 0. \end{cases}$$

Since $A_0 = 1$, we have $2q^2 - 5q + 3 = 0$. We get a non-trivial value $q = 3/2$. We can determine other coefficients $A_1 = 5/2$, $A_2 = -1/8$ and get $\eta_6(x,y)$. We can verify it is indeed $\sigma_{3/2}$-invariant. We can also verify (with some computer algebra system) that there is no $\sigma_{3/2}$-invariant polynomial $W(x,y)$ of even degrees in the range $8 \le \deg W(x,y) \le 22$ except for $\eta_6(x,y)^2$ and $\eta_6(x,y)^3$, but we can find $\eta_{24}(x,y)$ in (1.6) at degree 24 ($\eta_6(x,y)$ and $\eta_{24}(x,y)$ are algebraically independent). We can furthermore find $\eta_{12}(x,y)$ from the condition that it is invariant under $\sigma_{3/2}\tau\sigma_{3/2}$. The ring $R_{3/2}$ is the invariant polynomial ring of the group $\langle \sigma_{3/2}, \tau \rangle$, and $R^-_{3/2}$ is that of $\langle \sigma_{3/2}\tau\sigma_{3/2}, \tau \rangle$. Clearly, we have $R^-_{3/2} \supset R_{3/2}$.

For the members of $R^-_{3/2}$ (including $R_{3/2}$), there seems to be bounds similar to Theorem 1.1 (proof seems to be difficult):

Conjecture 5. (i) All $\sigma_{3/2}$-invariant polynomials of the form (1.1) in $R_{3/2}$ satisfy

$$d \le 3\left[\frac{n}{24}\right] + 3.$$

(ii) All 3/2-formal weight enumerators of the form (1.1) in $R^-_{3/2}$ satisfy

$$d \le 3\left[\frac{n-12}{24}\right] + 3.$$

Here are some examples of zeta polynomials for the members of $R^-_{3/2}$. The zeta polynomial of $\eta_6(x,y)$ is $P_6(T) = (3T^2+3T+2)/8$, that of $\eta_{12}(x,y)$ is $P_{12}(T) = (3T^2-2)(27T^6+27T^5+36T^4+26T^3+24T^2+12T+8)/160$ (the zeta polynomial of $\eta_{24}(x,y)$ is a polynomial of degree 14). From numerical experiments we can conjecture that extremal $\sigma_{3/2}$-invariant polynomials and extremal formal weight enumerators in $R^-_{3/2}$ satisfy the Riemann hypothesis.

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