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A REPRESENTATION FOR ALGEBRAIC K-THEORY OF QUASI-COHERENT MODULES OVER AFFINE SPECTRAL SCHEMES

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Abstract. In this paper, we study K-theory of spectral schemes by using locally free sheaves. Let us regard the K-theory as a functor K on affine spectral schemes. Then, we prove that the group completion $\Omega B^{\mathcal{G}}(B^{\mathcal{G}}GL)$ represents the sheafification of K with respect to Zariski (resp. Nisnevich) topology $\mathcal{G}$, where $B^{\mathcal{G}}GL$ is a classifying space of a colimit of affine spectral schemes GL_n.

1. Introduction

An ∞-category is a notion of categories up to coherent homotopy. The spectral algebraic geometry in terms of ∞-category, introduced by Lurie [15] [16], is a generalization of algebraic geometry.

In this paper, we study the K-theory on spectral schemes. For the K-theory of a category of projective modules of finite rank in the sense of spectral algebraic geometry, we construct an object in the ∞-category which represents the K-theory. The main theorem (Theorem 1.1) is a generalization of the representation theorem of the K-theory in the classical algebraic geometry proved by Morel and Voevodsky [20, Proposition 3.9].

1.1. **Statement of the main theorem.** By an ∞-category with w^∞-cofibrations we mean a pointed ∞-category with a class of morphisms (called w^∞-cofibrations) satisfying certain conditions (see Definition 5.1 for details). Let $\mathcal{C}$ be an ∞-category with w^∞-cofibrations. We define the algebraic K-theory space of $\mathcal{C}$ by

$$K(\mathcal{C}) = \Omega|S_\bullet(\mathcal{C})|,$$

where $S_\bullet$ is the S-construction (cf. [6]) and $|-|$ is the geometric realization.

We say that a spectrum E is connective if $\pi_n E \simeq 0$ for $n < 0$. Let R be a connective $\mathbb{E}_\infty$-ring, and CAlg^{cn} the ∞-category of connective $\mathbb{E}_\infty$-rings. Let $\mathrm{Mod}_R^{\infty proj}$ be an ∞-category of projective R-modules of finite rank which we recall in Definition 2.1. It becomes an ∞-category with w^∞-cofibrations by Definition 5.3.

We denote by $\mathrm{CAlg}^{\mathcal{G}}$ an opposite ∞-category $(\mathrm{CAlg}^{cn})^{op}$ equipped with either the Zariski topology or the Nisnevich topology $\mathcal{G}$, and by $\mathrm{Spec}^{\mathcal{G}} R$ an object in the essential image of Yoneda functor $\mathrm{CAlg}^{\mathcal{G}} \to \mathcal{S}hv(\mathrm{CAlg}^{\mathcal{G}})$,

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where $\mathcal{S}hv(\mathrm{CAlg}^{\mathcal{G}})$ is the ∞-category of sheaves on $\mathrm{CAlg}^{\mathcal{G}}$ given in Section 2. Here, we denote by $(-)^{op}$ the opposite ∞-category.

Let $\widehat{\mathcal{S}}$ be the ∞-category of not-necessary small spaces. We define a functor

$$K : (\mathrm{CAlg}^{\mathcal{G}})^{op} \to \widehat{\mathcal{S}} \tag{1.1}$$

which carries a spectral scheme $\mathrm{Spec}^{\mathcal{G}} R$ to the K-theory $K(\mathrm{Mod}_R^{\infty proj})$.

We denote by $\widetilde{(-)}^{\mathcal{G}}$ the sheafification from the ∞-category of functors on $(\mathrm{CAlg}^{\mathcal{G}})^{op}$ to $\mathcal{S}hv(\mathrm{CAlg}^{\mathcal{G}})$ which we recall in Definition 2.4.

Let GL_n be the affine group spectral scheme of general linear group in $\mathcal{S}hv(\mathrm{CAlg}^{\mathcal{G}})$. Let $B^{\mathcal{G}}GL = \coprod_{n\in\mathbb{N}} B^{\mathcal{G}}GL_n$ be the coproduct of the classifying sheaf $B^{\mathcal{G}}GL_n$ of GL_n, where $B^{\mathcal{G}}$ is a functor given by taking classifying sheaf which we recall in Section 4. Let $\Omega B^{\mathcal{G}}$ be a functor on $\mathcal{S}hv(\mathrm{CAlg}^{\mathcal{G}})$ defined in Definition 3.15. We denote by $\Omega B^{\mathcal{G}}(B^{\mathcal{G}}GL)$ the group completion (cf. Definition 4.15).

Theorem 1.1 (cf. Theorem 6.2). *Let* $\mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(-,-)$ *denote the mapping space of the ∞-category* $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$ *which we recall in Definition 3.8. There is an equivalence*

$$\mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(\mathrm{Spec}^{\mathcal{G}} R, \Omega B^{\mathcal{G}}(B^{\mathcal{G}}GL)) \simeq \widetilde{K}^{\mathcal{G}}(\mathrm{Mod}_R^{\infty proj}).$$

Remarks for Theorem 1.1. It is known that the sheaf $\widetilde{K}^{\mathcal{G}}$ is representable by an object in $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$ by the brown representability theorem in the sense of ∞-category [11, Proposition 5.5.2.7]. The importance is that we give a concrete object which represents $\widetilde{K}^{\mathcal{G}}$.

Morel and Voevodsky used the cofinality theorem in Quillen's K-theory in the proof of [20, Proposition 3.9]. Since the cofinality theorem is not established in the K-theory of ∞-categories, their proof cannot be applied to our case directly. To avoid this problem, we treat an ∞-category $\mathrm{Mod}_R^{\infty proj}$ instead of the finitely generated projective modules.

1.2. **Outline of this paper.** This paper is organized as follows. In Section 2, we introduce the terminology of an ∞-category endowed with Grothendieck topology and sheaves on it. In Section 3, we characterize the group completion functor on $\mathcal{S}hv(\mathrm{CAlg}^{\mathcal{G}})$ explicitly in Proposition 3.16, which is one of the important lemmas in this paper. In Section 4, we show the correspondence between the value of the affine group scheme GL_n on S and the automorphisms of S^n, where S is an $\mathbb{E}_\infty$-ring in Proposition 4.8. We also demonstrate that the classifying sheaf $B^{\mathcal{G}}GL_n$ is equivalent to the sheaf of projective modules of finite rank in Proposition 4.21. We also define the notion of Zariski connectedness and relate a certain mapping space of

$B^{\mathcal{G}}GL_n$ is equivalent to the sheaf of projective modules of finite rank in Proposition 4.26. In Section 5, we recall the several notion and proposition with respect to K-theory. In Section 6, by using these results, we prove Theorem 6.2.

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2. Preliminary

We fix the universe $\mathbb{U}$ such that $\mathbb{N} \in \mathbb{U}$. We define the Cat_∞ by $\mathbb{U}$-small ∞-category, which is locally $\mathbb{U}$-small.

Although there are a lot of languages of higher category theory, we use the same notation in Lurie's book [11] and papers [12], [15] and [13].

Let $\mathrm{Fun}(-,-)$ be an ∞-category of functors. For a pair of functors $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ between ∞-categories, according to [11, Proposition 5.2.2.8], we say that the functor f is a left adjoint to g (resp. g is a right adjoint to f) if there exists a unit map $u : id_{\mathcal{C}} \to g \circ f$ given in [11, Defnition 5.2.2.7].

For an $\mathbb{E}_\infty$-ring R, we have the ∞-category Mod_R of R-modules [12, Section 4.2]. Since the tensor product on the ∞-category of spectra is compatible with the geometric realizations [12, Corollary 4.8.2.19], Mod_R becomes the symmetric monoidal ∞-category by [12, Theorem 4.5.2.1]. We denote by $\otimes_R$ the tensor product on Mod_R.

Let R be an $\mathbb{E}_\infty$-ring and $a \in \pi_0 R$ an element. The localization of R with respect to a, which is denoted by $R[a^{-1}]$, is an $\mathbb{E}_\infty$-ring (see [16, Remark 2.9] and [12, 7.2.4]).

Definition 2.1. Let R be a connective $\mathbb{E}_\infty$-ring and M an R-module.

(i) We say that M is free of rank n if $M \simeq R^{\oplus n}$.

(ii) We say that M is finitely generated projective if there exists $n \in \mathbb{N}$ such that M is a retract free R-module of rank n.

(iii) We say that M is projective of rank n if it is finitely generated projective and we can choose elements $x_1, \cdots, x_m \in \pi_0 R$ such that they generate the unit ideal and all of the localization $M[x_i^{-1}]$ are free modules of rank n over $R[x_i^{-1}]$.

(iv) We say that M is projective of finite rank if there exists $n \in \mathbb{N}$ such that M is projective of rank n.

We denote by Mod_R^{nfree} (resp. Mod_R^{nproj}) an ∞-category of free (resp. projective) R-modules of rank n. We denote by Mod_R^{proj} (resp. $\mathrm{Mod}_R^{\infty proj}$) the ∞-category of finitely generated projective R-modules (resp. projective R-modules of finite rank).

Remark 2.2. In general, $\mathrm{Mod}_R^{\infty proj}$ and Mod_R^{proj} is not equivalent.

As in [11], [12] and [4], to treat certain size of limits and colimits, we need to enlarge the universe and fix the size of universes properly.

We adopt the axiom of universes which allows us to consider that every cardinal can be strictly upper bounded by a strongly inaccessible cardinal. Then, there exists a bijection between strongly inaccessible cardinals and universes, and thus we can take a suitable enlargement of universe which we need.

By the axiom of universe, there exists an enlargement of universes $\mathbb{U} \in \mathbb{V}$ such that every $\mathbb{U}$-small object is also $\mathbb{V}$-small, so that Cat_∞ becomes a $\mathbb{V}$-small category.

In this paper, sometimes we need to treat the ∞-category CAlg as small, and to regard the maximal ∞-groupoid $(\mathrm{Mod}_R)^{\simeq}$ of Mod_R as a space. We also need to treat the size of limits and colimits. In these cases, we enlarge the universe as the following definition and proceed the arguments.

Definition 2.3. Let $\mathcal{S}$ be the ∞-category of $\mathbb{U}$-small spaces. Throughout this paper, we enlarge the universe $\mathbb{U} \in \mathbb{V}$ such that Mod_R is a $\mathbb{V}$-small ∞-category. We use the notation $\widehat{\mathcal{S}}$ for $\mathcal{S}$ and $\widehat{\mathrm{Cat}_\infty}$ for Cat_∞ after changing the universe from $\mathbb{U}$ to $\mathbb{V}$.

2.1. **Sheaves and spectral affine schemes.** Let us take $\mathcal{X}$ as an ∞-category equipped with a Grothendieck topology as in [11, Definition 6.2.2.1] and $\mathcal{C}$ an ∞-category which admits limits. We denote by $\mathcal{S}hv_{\mathcal{C}}(\mathcal{X})$ an ∞-category of $\mathcal{C}$-valued sheaves on $\mathcal{X}$ in [11, Definition 6.2.2.6].

Definition 2.4. Let $\widetilde{(-)} : \mathrm{Fun}(\mathcal{X}^{op}, \mathcal{S}) \to \mathcal{S}hv_{\mathcal{S}}(\mathcal{X})$ be an localization defined in [11, Definition 6.2.2.6], which is a left adjoint of the inclusion. For an object F of $\mathrm{Fun}(\mathcal{X}, \mathcal{S})$, we say that $\widetilde{F}$ is a sheafification of F.

Remark 2.5 ([11] Construction 6.2.2.9, Remark 6.2.2.12)**.** For an ∞-category $\mathcal{C}$ equipped with a Grothendieck topology, and a presheaf F on $\mathcal{C}$, the sheafification $\widetilde{F}$ of F is given by the following formula: for any $C \in \mathcal{C}$,

$$\widetilde{F}(C) = \mathrm{colim}_{\mathcal{C}^{(0)}_{/C}} \lim_{C' \in \mathcal{C}^{(0)}_{/C}} F(C'),$$

where the first limit is the limit of the simplicial diagram associated to $C' \to C$ [11, Corollary 6.2.3.5] and the second colimit is taken over the collection of covering sieves on C.

Here is a list of morphisms of $\mathbb{E}_\infty$-rings.

- Let $f : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings. Recall that $f : A \to B$ is said to be flat if $\pi_0 B$ is flat $\pi_0 A$-module and the underlying map of commutative rings $\pi_0 A \to \pi_0 B$ induces the isomorphism $(\pi_i A) \otimes_{\pi_0 A} (\pi_0 B) \simeq \pi_i B$ for every integer i.
- Let $f : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings. $f : A \to B$ is said to be étale if it is flat and the underlying map of commutative rings $\pi_0 A \to \pi_0 B$ is étale. The class of étale morphisms on $(\mathrm{CAlg}^{cn})^{op}$ satisfies the axiom of admissible morphisms by [15, Proposition 2.4.17].
- We say that a morphism $f : A \to B$ of $\mathbb{E}_\infty$-rings is faithfully flat if it is flat morphism and the underlying map of commutative rings $\pi_0 A \to \pi_0 B$ is faithfully flat. By a class of faithfully flat morphisms, Grothendieck topology is defined on CAlg^{op}, which is called flat topology [16, Definition 5.2, Proposition 5.4].

Let $\mathbb{S}$ be a sphere spectrum. Recall the Grothendieck topology given by the admissibility on covering sieves from [15, Definition 1.2.1]. Let $\mathcal{G}^{Sp}_{Zar}(\mathbb{S})$ be an ∞-category with Grothendieck topology given as [16, Definition2.10]. We denote by CAlg^{Zar} an ∞-category $\mathrm{Ind}(\mathcal{G}^{Sp}_{Zar}(\mathbb{S})^{op})^{op}$ equipped with Zariski topology [15, Notation 2.2.6]. Moreover, this Zariski geometry is finitary [15, Remark 2.2.8] by the definition. (For the base change assumption, see [15, Remark 1.2.4]).

Let A be a connective $\mathbb{E}_\infty$-ring. Let CAlg^{et}_A be an ∞-category spanned by connective étale algebras over A. Let $\mathcal{C} \subset (\mathrm{CAlg}^{et}_A)^{op}$ be a sieve containing A. We say that $\mathcal{C}$ is a Nisnevich covering sieve on A if it contains a collection of morphisms $A \to A_a$ such that their underlying maps of commutative rings $\pi_0 A \to \pi_0 A_a$ determine a Nisnevich covering defined in [14, Definition 1.1]. Let A be a connective $\mathbb{E}_\infty$-ring. We define the admissible morphisms in $(\mathrm{CAlg}^{cn})^{op}$ by the morphisms corresponding to étale morphisms in CAlg^{cn}, and the collection of admissible morphisms $A \to A_a$ generates a covering sieve on A if and only if it corresponds to a Nisnevich covering sieve. Then, it generates a Grothendieck topology on $(\mathrm{CAlg}^{cn})^{op}$, which is called the Nisnevich topology. We denote by CAlg^{Nis} an opposite ∞-category $(\mathrm{CAlg}^{cn})^{op}$ endowed with the Nisnevich topology.

Let $\mathrm{CAlg}^{\mathcal{G}}$ denote either CAlg^{Zar} or CAlg^{Nis}. Let us denote an object in the essential image of Yoneda functor $\mathrm{CAlg}^{\mathcal{G}} \to \mathcal{S}hv_{\mathcal{S}}(\mathrm{CAlg}^{\mathcal{G}})$ by $\mathrm{Spec}^{\mathcal{G}} R$, and we call the object an affine spectral scheme.

The following proposition is a special case of [16, Proposition 5.7].

Proposition 2.6. *Let us consider* CAlg^{op} *endowed with the flat topology. Then, in the case of* ∞*-topos* CAlg^{op} *with flat topology, a functor* F *is a sheaf if, it preserves finite products and for any covering* $X \to Y$ *in* $(\mathrm{CAlg})^{op}$*,*

$$F(Y) \to \lim_{\Delta} F(X_{\bullet})$$

is an equivalence. Here, $X_{\bullet} \to Y$ *is the simplicial object associated to* $X \to Y$*, and the notation* $\lim_{\Delta}$ *in the right hand side is a limit taken over the simplicial diagram* $X_{\bullet}$*.*

Lemma 2.7. *(i) A Zariski covering sieve is a Nisnevich covering sieve on* $(\mathrm{CAlg}^{cn})^{op}$*.*

(ii) A Zariski covering sieve and a Nisnevich covering sieve are covering sieves on the flat topology on $(\mathrm{CAlg}^{cn})^{op}$*.*

Proof. Let R be a connective $\mathbb{E}_{\infty}$-ring and $f_1, \cdots, f_n$ elements in $\pi_0 R$ which generate the unit ideal of $\pi_0 R$. To prove (i), we show that each $R \to R[f_i^{-1}]$ is flat for all n and $\{\pi_0 R \to \pi_0 R[f_i^{-1}]\}$ is an ordinary Nisnevich covering of $\pi_0 R$. Since each $R \to R[f_i^{-1}]$ is flat by [16, Remark 2.9] and $\{\pi_0 R \to \pi_0 R[f_i^{-1}]\}$ is an ordinary Zariski covering of $\pi_0 R$, so that it is a Nisnevich covering of $\pi_0 R$ [14, Remark 1.13].

To show (ii), we show that, for any collection of morphisms $\{R \to R_a\}$ such that their underlying maps of commutative rings $\{\pi_0 R \to \pi_0 R_a\}$ form a Nisnevich covering, $R \to \prod_{\alpha} R_{\alpha}$ is faithfully flat. Note that we can assume that this Nisnevich covering is a finite collection of morphisms [14, Remark 1.6].

Since each $R \to R_{\alpha}$ is étale, it is flat. It follows that $R \to \prod_{\alpha} R_{\alpha}$ is also flat. By [14, Remark 1.12], $\pi_0 R \to \pi_0(\prod_{\alpha} R_{\alpha}) \cong \prod_{\alpha} \pi_0 R_{\alpha}$ is faithfully flat. $\square$

3. Group completion in an ∞-topos

Definition 3.1. Let $\mathcal{C}$ be an ∞-category with finite products, and $\mathcal{O}^{\otimes}$ be either the ∞-operad $\mathrm{N}_{\Delta}(\mathcal{A}ssoc^{\otimes})$ given in [12, Definition 4.1.1.3] or $\mathrm{N}_{\Delta}(\mathcal{F}\mathrm{in}_*)$. Note that we have a natural map $\rho : \mathrm{N}_{\Delta}(\mathcal{A}ssoc^{\otimes}) \to \mathrm{N}_{\Delta}(\mathcal{F}\mathrm{in}_*)$ by forgetting linear orderings (cf. [12, Remark 4.1.1.4]).

(i) An ∞-monoid in $\mathcal{C}$ is a functor $M : \mathrm{N}_{\Delta}(\mathcal{A}ssoc^{\otimes}) \to \mathcal{C}$ such that the morphism $M(\langle n \rangle) \to M(\langle 1 \rangle)$ induced by the inert maps $\sigma^i : \langle n \rangle \to \langle 1 \rangle$ with linear orderings in each $1 \leq i \leq n$ induces an equivalence $M(\langle n \rangle) \simeq M(\langle 1 \rangle)^n$ for all $n \in \mathbb{N}$.

(ii) A commutative ∞-monoid in $\mathcal{C}$ is a functor $M : \mathrm{N}_{\Delta}(\mathcal{F}\mathrm{in}_*) \to \mathcal{C}$ such that the morphism $M(\langle n \rangle) \to M(\langle 1 \rangle)$ induced by the inert maps $\sigma^i : \langle n \rangle \to \langle 1 \rangle$ induces an equivalence $M(\langle n \rangle) \simeq M(\langle 1 \rangle)^n$ for all $n \in \mathbb{N}$.

(iii) An ∞-monoid is an ∞-group if its image in $h\mathcal{C}$ is a group object. Also, a commutative ∞-monoid is a commutative ∞-group if its image in $h\mathcal{C}$ is a commutative group object. We denote by $Mon(\mathcal{C})$ and $nMon(\mathcal{C})$ an ∞-category of commutative ∞-monoids in $\mathcal{C}$ and not-neccesary commutative ∞-monoids in $\mathcal{C}$ respectively. We also denote by $Gp(\mathcal{C})$ and $nGp(\mathcal{C})$ an ∞-category of commutative ∞-groups in $\mathcal{C}$ and not-neccesary commutative ∞-groups in $\mathcal{C}$ respectively (cf. [9, Proposition 1.1]).

Note that we have a diagram

$$\begin{tikzcd} Gp(\mathcal{C}) \arrow[r, "\rho^*"] \arrow[d] & nGp(\mathcal{C}) \arrow[d] \\ Mon(\mathcal{C}) \arrow[r, "\rho^*"] & nMon(\mathcal{C}), \end{tikzcd}$$

where the horizontal morphisms are induced from $\rho : \mathrm{N}_\Delta(\mathcal{A}ssoc^{\otimes}) \to \mathrm{N}_\Delta(\mathcal{F}\mathrm{in}_*)$ and the vertical morphisms are obtained by regarding ∞-groups as ∞-monoids.

Remark 3.2. In terminology of [9], a commutative ∞-monoid is called an $\mathbb{E}_\infty$-monoid, and a commutative ∞-group is called an $\mathbb{E}_\infty$-group by [9, Proposition 1.1]. We adopt the terminology, and also describe the not-necessary commutative case in the terminology since we apply the not-necessary commutative case to the group completion on the ∞-category $\mathcal{S}$ of spaces in the following Lemma 3.9.

3.1. **The group completion functor.** We consider morphisms which go to the other direction to the vertical morphisms in the diagram in Definition 3.1.

Definition 3.3. Let $\mathcal{C}$ be an ∞-category with finite products as in Definition 3.1. We say that a functor $Mon(\mathcal{C}) \to Gp(\mathcal{C})$ is group completion on $\mathcal{C}$ if it is the left adjoint to the inclusion functor $Gp(\mathcal{C}) \to Mon(\mathcal{C})$ given in the diagram in Definition 3.1. We also say that a functor $nMon(\mathcal{C}) \to nGp(\mathcal{C})$ is group completion on $\mathcal{C}$ if it is the left adjoint to the inclusion functor $nGp(\mathcal{C}) \to nMon(\mathcal{C})$. By the universal property of adjoint functors, the group completion functor is uniquely determined up to equivalence [11, Remark 5.2.2.2].

Definition 3.4. (i) Let $\mathrm{Fun}^{\prod}(\mathcal{C}, \mathcal{D})$ denote an ∞-category of functors which preserve the finite products.

(ii) We define the product on $\mathrm{Fun}^{\prod}(\mathcal{C}, \mathcal{D})$ by taking the objectwise product in $\mathcal{D}$.

Lemma 3.5 ([9] Lemma 1.6)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories with finite products. Let us take $\mathrm{Fun}^{\Pi}(\mathcal{C},\, \mathcal{D})$ as an ∞-category of functors which preserve the finite products. Then, we have*

$$Mon(\mathrm{Fun}^{\Pi}(\mathcal{C},\, \mathcal{D})) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C},\, Mon(\mathcal{D})),$$

and

$$Gp(\mathrm{Fun}^{\Pi}(\mathcal{C},\, \mathcal{D})) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C},\, Gp(\mathcal{D})).$$

$\square$

Definition 3.6 (cf. [11] 6.1.2.7)**.** Let $\mathcal{C}$ be an ∞-topos. For an ∞-monoid object G, there exists a colimit of the simplicial homotopy diagram by [12, Lemma 5.2.2.6]

$$\cdots \rightrightarrows G \times G \times G \rightrightarrows G \times G \rightrightarrows G \rightrightarrows 1.$$

where the face map is given by the multiplication on G and the degeneracy map is given by the unit morphism. We define BG is a colimit of the simplicial homotopy diagram.

Remark 3.7. Let $\mathcal{S}$ be the symmetric cartesian monoidal ∞-category of spaces. In terminology of topology, an object M in $\mathcal{S}$ is grouplike if $\pi_0 M$ is a group object in $h\mathcal{S}$ [12, Example 5.2.6.4]. A grouplike object in $\mathcal{S}$ is just an ∞-group in $\mathcal{S}$ by Definition 3.1. If $\mathcal{C}$ is a symmetric cartesian monoidal model category, an ∞-monoid object can be regarded as an associative algebra object by [12, Proposition 2.4.2.5].

3.2. **Characterization of the group completion on $\mathcal{S}$.** For a simplicial set S and its vertexes x and y, recall that a simplicial set $\mathrm{Map}_S(x,\, y)$ is defined as follows. This construction is due to Joyal.

Definition 3.8. [cf. [11] 1.2.2.2, Corollary 4.2.1.8] Let Set_Δ be the category of simplicial sets. Let S be a simplicial set and S^{Δ^1} a simplicial set which sends $[n]$ to $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^1 \times \Delta^n,\, S)$, where Δ^n is the n-simplex for $n \geq 0$ and $\mathrm{Hom}_{\mathrm{Set}_\Delta}(-,\, -)$ is the hom-set of Set_Δ. Let $s, t : [0] \to [1]$ be maps defined by $s(0) = 0$ and $t(0) = 1$.

(i) Take vertexes $x, y \in S$. The mapping space $\mathrm{Map}_S(x,\, y)$ from x to y is defined by the pullback

$$\begin{tikzcd} \mathrm{Map}_S(x,\, y) \arrow[r] \arrow[d] & S^{\Delta^1} \arrow[d, "(s,t)"] \\ * \arrow[r, "(x,y)"] & S \times S, \end{tikzcd}$$

where the morphism (s, t) is induced by s and t, e.g., it sends $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\{0\} \times \Delta^n,\, S)$ to the n-simplices S_n of the first factor and $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\{1\} \times$

Δ^n, S) to the n-simplices S_n of the second factor, and the morphism (x, y) sends $*$ to $(x, y) \in S \times S$.

(ii) If S is an ∞-category, $\mathrm{Map}_S(x, y)$ becomes a Kan complex by [11, Proposition 1.2.2.3]. For an ∞-category S and objects $x, y \in S$, we say that $\mathrm{Map}_S(x, y)$ is the mapping space between x and y.

Let $\mathcal{S}$ be the ∞-category of spaces, and $\mathcal{S}_*$ the pointed ∞-category of spaces (cf. [12, Notation 5.2.6.11]).

For a space X, we define the loop space of X by $\mathrm{Map}(S^1, X)$, where S^1 is a simplicial circle Δ^1/Δ^0.

There is the group completion theorem on simplicial monoids due to Quillen, which is given by taking classifying space and loop space. By taking the simplicial localization [7] of categories, we have an equivalence from the ∞-category of connected spaces to the ∞-category of not-neccesary commutative ∞-groups in $\mathcal{S}$ obtained by restricting the adjoint functors $B : nMon(\mathcal{S}) \rightleftarrows \mathcal{S}_* : \Omega$, which includes the case arising from commutative topological monoids [19]. There is also an equivalence from the ∞-category Sp^{cn} to $Gp(\mathcal{S})$ obtained by B and Ω. Moreover, we have the following lemma.

Lemma 3.9. *Let $\mathcal{S}$ be an ∞-category of spaces. Let $i : nGp(\mathcal{S}) \to nMon(\mathcal{S})$ be the forgetfull functor. With the previous notations, we have the adjunction*

$$\Omega B : nMon(\mathcal{S}) \rightleftarrows nGp(\mathcal{S}) : i,$$

where i is the forgetfull functor. In other words, ΩB is the group completion on $\mathcal{S}$.

□

We can also take the left adjoint of the inclusion $i : Gp(\mathcal{S}) \to Mon(\mathcal{S})$ restricted in commutative case, so that it is also the restriction of the left adjoint in Lemma 3.9. Note that, since sifted colimits commute with finite products, ΩB commutes with finite products by the definition of B.

3.3. Characterization of the group completion on $\mathrm{Fun}^{\Pi}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})$.

We characterize the group completion functor on $\mathrm{Fun}^{\Pi}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})$ by using Ω and B.

Definition 3.10. (i) In this paper, we will also denote by ΩB the left adjoint functor given by $\Omega B : Mon(\mathcal{S}) \rightleftarrows Gp(\mathcal{S})$ after Lemma 3.9.

(ii) We also denote by ΩB the functor

$$\mathrm{Fun}((\mathrm{CAlg}^{\mathcal{G}})^{op}, Mon(\widehat{\mathcal{S}})) \to \mathrm{Fun}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})$$

which sends a presheaf F to a presheaf given by $R \mapsto \Omega B(F(R))$.

We will give an explicit description of the group completion functor for the ∞-category of presheaves which preserves finite products.

Definition 3.11. We define a functor

$$\Omega B^{\prod} : Mon(\mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \to Gp(\mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}}))$$

by the restriction of ΩB, i.e., $\Omega B^{\prod}$ is the functor which satisfies the following commutative diagram

$$\begin{tikzcd} Mon(\mathrm{Fun}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \arrow[r, "\Omega B"] & Gp(\mathrm{Fun}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \\ Mon(\mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \arrow[r, "\Omega B^{\prod}"] \arrow[u] & Gp(\mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \arrow[u] \end{tikzcd} ,$$

where the vertical morphisms are inclusions.

Lemma 3.12. *The functor $\Omega B^{\prod}$ gives the group completion on the ∞-category $\mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})$ up to equivalences.*

Proof. Let $\mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})$ be an ∞-category of presheaves on $\mathrm{CAlg}^{\mathcal{G}}$ which preserve the finite products. This ∞-category has the pointwise finite products. By Lemma 3.5, we have the equivalences

$$Mon(\mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \simeq \mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, Mon(\widehat{\mathcal{S}}))$$

and

$$Gp(\mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \simeq \mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, Gp(\widehat{\mathcal{S}})).$$

Therefore, the group completion functor on $\mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})$ is determined by the point-wise values in $\mathcal{S}$. The problem is reduced to the group completion functor $nMon(\mathcal{S}) \to nGp(\mathcal{S})$ for $\mathcal{S}$, which is ΩB by Lemma 3.9. $\square$

3.4. Characterization of the group completion on $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$.

Definition 3.13 (cf. [16] Proposition 1.15)**.** We define the product on $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$ by the pointwise product, i.e., the product induced from the formation of product in $\mathcal{S}$ under the sheafification.

Since a finite limit of sheaves are again a sheaf, the pointwise product on $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$ is the restriction of the product on $\mathrm{Fun}^{\prod}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})$ in Definition 3.4.

Lemma 3.14. *We have $\mathcal{S}hv_{Gp(\widehat{\mathcal{S}})}(\mathrm{CAlg}^{\mathcal{G}}) \subset Gp(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}}))$.*

Proof. By definition, the objectwise products of presheaves becomes pointwise products of sheaves after sheafification. Since the sheafification is left exact [11, Definition 5.3.2.1], it commutes with the finite products. Therefore, we have $\mathcal{S}hv_{Gp(\widehat{\mathcal{S}})}(\mathrm{CAlg}^{\mathcal{G}}) \subset Gp(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}}))$. $\square$

Definition 3.15. We define the functor

$$\Omega B^{\mathcal{G}} : Mon(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})) \to Gp(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$$

by the composition of the inclusion

$$i' : Mon(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})) \to Mon(\mathrm{Fun}^{\Pi}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}}))$$

with

$$Mon(\mathrm{Fun}^{\Pi}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \to Gp(\mathrm{Fun}^{\Pi}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \to Shv_{Gp(\widehat{\mathcal{S}})}(\mathrm{CAlg}^{\mathcal{G}}),$$

where the first functor is the functor which is induced from the pointwise group completion $\Omega B^{\Pi} : Mon(\mathrm{Fun}^{\Pi}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}})) \to Gp(\mathrm{Fun}^{\Pi}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}}))$, and the second functor is obtained by the equivalence in Lemma 3.5 and the sheafification $(\tilde{-}) : \mathrm{Fun}^{\Pi}((\mathrm{CAlg}^{\mathcal{G}})^{op}, Gp(\widehat{\mathcal{S}})) \to Shv_{Gp(\widehat{\mathcal{S}})}(\mathrm{CAlg}^{\mathcal{G}})$. Note that, we have $\mathcal{S}hv_{Gp(\widehat{\mathcal{S}})}(\mathrm{CAlg}^{\mathcal{G}}) \subset Gp(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}}))$ by Lemma 3.14.

Proposition 3.16. *Let* $\mathrm{CAlg}^{\mathcal{G}}$ *be the* ∞*-category equipped with the Grothendieck topology which is defined in Section 2. Then,* $\Omega B^{\mathcal{G}}$ *is the group completion on* $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$*.*

Proof. Let us take F and G from $Mon(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}}))$ and $Gp(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}}))$ respectively. By the adjunction in Lemma 3.9 and the definition of $\Omega B^{\mathcal{G}}$, we have

$$\mathrm{Map}(\Omega B^{\mathcal{G}} F,\, G) \simeq \mathrm{Map}((-)^{\simeq} \circ \Omega B^{\Pi} \circ i'(F),\, G) \simeq \mathrm{Map}(\Omega B iF,\, iG) \simeq \mathrm{Map}(F,\, iG),$$

where the left two mapping spaces are taken in $Gp(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}}))$, the third and the last are in $Gp(\mathrm{Fun}^{\Pi}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}}))$ and $Mon(\mathrm{Fun}^{\Pi}((\mathrm{CAlg}^{\mathcal{G}})^{op}, \widehat{\mathcal{S}}))$ respectively. Since F and G are already sheaves and the sheafification commutes with finite products, we can regard the last mapping space as that taken in $Mon(\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}}))$. □

4. Classifying sheaf of GL and projective modules of finite rank

4.1. The affine spectral scheme GL_n.

Definition 4.1 ([12] Notation 3.1.3.8)**.** Let R be an $\mathbb{E}_\infty$-ring, and CAlg_R the ∞-category of R-algebras.

(i) We define a functor $\mathrm{Sym}_R : \mathrm{Mod}_R \to \mathrm{CAlg}_R$ the left adjoint of the forgetful functor $\mathrm{CAlg}_R \to \mathrm{Mod}_R$ which sends an R-algebra S to an R-module S.

(ii) For a free R-module $R^{\oplus n^2}$ of rank n^2, $\pi_0\mathrm{Sym}_R R^{\oplus n^2}$ is isomorphic to the polynomial ring $(\pi_0 R)[x_{11}, \cdots, x_{nn}]$ over $\pi_0 R$. Let us denote $\mathrm{Sym}_R R^{\oplus n^2}$ by $R\{x_{11}, \cdots, x_{nn}\}$, where each x_{ij} is the representative of the indeterminants of the polynomial ring over $\pi_0 R$.

Definition 4.2. (i) For an object X in a symmetric monoidal ∞-category, an endomorphism object $\mathrm{End}(X)$ is an object equipped with the evaluation morphism $e : \mathrm{End}(X) \otimes X \to X$ which induces a weak homotopy equivalence $\mathrm{Map}(Y \otimes X,\, X) \simeq \mathrm{Map}(Y,\, \mathrm{End}(X))$ for every Y.
(ii) Let R be an $\mathbb{E}_\infty$-ring. We have the endomorphism object of $R^{\oplus n}$ in Mod_R. Let us denote it by $\mathrm{End}_R(R^n)$.

The following lemma is explained in [12, Remark 7.1.2.2].

Lemma 4.3. *We have an isomorphism*

$$\pi_*\mathrm{End}_R(R^{\oplus n}) \cong \pi_*\mathrm{Map}_{\mathrm{Mod}_R}(R^{\oplus n},\, R^{\oplus n})$$

□

The object $R^{\oplus n^2}$ of Mod_R satisfies the universal property of endomorphism, we have an equivalence $R^{\oplus n^2} \simeq \mathrm{End}_R(R^{\oplus n})$. Since we have $R^{\oplus n^2} \simeq \mathrm{End}_R(R^{\oplus n})$, we have an equivalence $\mathrm{Sym}_R\mathrm{End}_R(R^{\oplus n}) \simeq R\{x_{11}, \cdots, x_{nn}\}$ by Definition 4.1 (ii).

Definition 4.4. Let R be an $\mathbb{E}_\infty$-ring. Let us denote $\mathrm{Spec}^{Zar}\mathrm{Sym}_R\mathrm{End}_R(R^{\oplus n})$ by $M_{n,R}$ in $\mathcal{S}hv_{\mathcal{S}}(\mathrm{CAlg}^{Zar})$.

(i) Under the equivalence $\mathrm{Sym}_R\mathrm{End}_R(R^{\oplus n}) \simeq R\{x_{11}, \cdots, x_{nn}\}$, we define the element $(det) \in \pi_0 M_{n,R}$ by the determinant relation $\Sigma_{\tau\in S_n} sgn(\tau) \prod x_{i,\tau(i)}$ of $x_{ij}s$.
(ii) We define an affine scheme $GL_{n,R}$ by inverting the determinant element of $M_{n,R}$.
(iii) In the case that the base scheme R is the sphere spectrum $\mathbb{S}$, we denote $M_{n,R}$ and $GL_{n,\mathbb{S}}$ by M_n and GL_n.

Let us take S as an R-algebra. We denote by $M_{n,R}(S)$ and $GL_{n,R}(S)$ the mapping spaces

$$\mathrm{Map}_{\mathrm{CAlg}_R}(\mathrm{Sym}_R\mathrm{End}_R(R^{\oplus n}), S)$$

and

$$\mathrm{Map}_{\mathrm{CAlg}_R}(\mathrm{Sym}_R\mathrm{End}_R(R^{\oplus n})[(det)^{-1}], S)$$

respectively.

Remark 4.5. Since GL_n is corepresented by an $\mathbb{E}_\infty$-ring, it is flat sheaf by [16, VII, Theorem 5.15], so that it is already a Nisnevich sheaf. We also use the notation GL_n for the image in $\mathcal{S}hv_{\mathcal{S}}(\mathrm{CAlg}^{Nis})$ under the sheafification.

4.2. The equivalence $GL_n(R) \simeq Aut(R^n)$ as ∞-groups.

Definition 4.6. We define the space $Aut_R(S^n)$ by the following pullback of simplicial sets

$$\begin{tikzcd} Aut_R(S^n) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{Mod}_R}(S^n, S^n) \arrow[d, "\pi_0"] \\ (\pi_0\mathrm{Map}_{\mathrm{Mod}_R}(S^n, S^n))^\times \arrow[r] & \pi_0\mathrm{Map}_{\mathrm{Mod}_R}(S^n, S^n), \end{tikzcd}$$

where we regard $(\pi_0\mathrm{Map}_{\mathrm{Mod}_R}(R^n, R^n))^\times$ and $\pi_0\mathrm{Map}_{\mathrm{Mod}_R}(R^n, R^n)$ as constant simplicial sets and $(\pi_0\mathrm{Map}_{\mathrm{Mod}_R}(R^n, R^n))^\times$ is the invertible objects in $\pi_0\mathrm{Map}_{\mathrm{Mod}_R}(R^n, R^n)$.

Remark 4.7. Note that the constant simplicial sets which appear in the diagram of Definition 4.6 are also Kan complexes since the homotopy set of the mapping space is made from the same mapping space by replacing 1-simplices with isomorphism. (Note that any 1-simplex in a Kan complex is invertible.)

Proposition 4.8. *For an R-algebra S, we have an equivalence $GL_{n,R}(S) \simeq Aut_R(S^n)$ as ∞-groups in $\mathcal{S}$, which is functorial with respect to R-algebra S.*

Proof. Note that the right vertical morphism π_0 in the diagram in Definition 4.6 is a Kan fibration.

Since $GL_{n,R}(S)$ is formulated by the following pullback of simplicial sets

(4.1)
$$\begin{tikzcd} GL_{n,R}(S) \arrow[r] \arrow[d] & M_{n,R}(S) \arrow[d, "\pi_0"] \\ (\pi_0 M_{n,R}(S))^\times \arrow[r] & \pi_0 M_{n,R}(S), \end{tikzcd}$$

where we regard $\pi_0 M_{n,R}(S)$ as a constant simplicial set and $(\pi_0 M_{n,R}(S))^\times$ is the invertible objects in $\pi_0 M_{n,R}(S)$, and by the coglueing lemma (cf. [11] A.2.4.3), it suffices to construct the two morphisms $M_{n,R}(S) \to \mathrm{Map}_{\mathrm{Mod}_R}(S^n, S^n)$ and $\pi_0 M_{n,R}(S) \to \pi_0\mathrm{Map}_{\mathrm{Mod}_R}(S^n, S^n)$ which preserve the multiplication and show that the following diagram is commutative:

(4.2)
$$\begin{tikzcd} (\pi_0 M_{n,R}(S))^\times \arrow[r] \arrow[d] & \pi_0 M_{n,R}(S) \arrow[d] & M_{n,R}(S) \arrow[l, "\pi_0"'] \arrow[d] \\ (\pi_0\mathrm{Map}_{\mathrm{Mod}_R}(S^n, S^n))^\times \arrow[r] & \pi_0\mathrm{Map}_{\mathrm{Mod}_R}(S^n, S^n) & \mathrm{Map}_{\mathrm{Mod}_R}(S^n, S^n). \arrow[l, "\pi_0"] \end{tikzcd}$$

Note that objects in the diagram (4.1) are Kan complexes by the same reason of Remark 4.7

An element of $M_{n,R}(S)$ is a morphism $\mathrm{Sym}_R\mathrm{End}_R(R^{\oplus n}) \to S$ of R-algebras for an R-algebra S. By the adjointness of Sym_R, the morphism of R-algebras is corresponding to a morphism of R-modules

$$R^{\oplus n^2} \simeq \mathrm{End}_R(R^{\oplus n}) \to S.$$

On the other hand, we have an equivalence $R^{\oplus n^2} \simeq \mathrm{End}_R(R^{\oplus n})$ obtained by evaluating the each factor of R^n. Therefore, if we regard $S \otimes_R R^{\oplus n^2}$ as an S-module, the above morphism corresponds to a morphism $S \otimes_R R^{\oplus n^2} \to S$ of S-modules. This gives an identification $M_{n,R}(S) \simeq \mathrm{End}_S(S^{\oplus n})$ as S-modules.

To show that $GL_{n,R}(S) \to Aut_R(S^n)$ is a morphism of ∞-groups, we fix the choice of the second equivalence : $\mathrm{Map}_{\mathrm{Mod}_S}(S^{\oplus n^2}, S) \simeq \mathrm{Map}_{\mathrm{Mod}_S}(S, S)^{\oplus n^2} \simeq \mathrm{Map}_{\mathrm{Mod}_S}(S^n, S^n)$. Then, by the composition of S-module endomorphisms on $S^{\oplus n}$, $\mathrm{End}_S(S^{\oplus n})$ has a canonical ∞-monoid structure for each S defined in Definition 3.1, the spectral scheme $M_{n,R}$ is an ∞-monoid.

We can identify the discrete group $\pi_0 M_{n,R}(S)^\times$ with the class of $\pi_0 S$-algebra morphisms $\pi_0\mathrm{Sym}_R\mathrm{End}_R(R^{\oplus n})[det^{-1}] \to \pi_0 S$. Here, det is the determinant element given by the determinant relation of $x_{ij}s$ in $\pi_0 R[x_{11}, \cdots, x_{nn}]$. By ordinary theory of affine group schemes, these morphisms correspond to $(\pi_0\mathrm{End}_S(S^{\oplus n}))^\times$ by a similar choice $\mathrm{Hom}(\pi_0 S, \pi_0 S)^{\oplus n^2} \cong \mathrm{Hom}(\pi_0 S^n, \pi_0 S^n)$ of isomorphism. Since 2-morphisms are invertible, this induces an equivalence between $Aut_R(S^n)$ and $\mathrm{Sym}_R\mathrm{End}_R(R^{\oplus n})[det^{-1}] \to S$ by the above pullback.

Since we fix the choice of equivalences in the proof, $GL_{n,R}$ is an ∞-group scheme with respect to the monoid structure of $\mathrm{Map}_{\mathrm{Mod}_R}(R^n, R^n)$. $\square$

4.3. **The $\widehat{\mathrm{Cat}_\infty}$-valued functor** $(nProj)$**.** In this subsection, we show that the functor $(nProj) : \mathrm{CAlg}^{cn} \to \widehat{\mathrm{Cat}_\infty}$ defined by sending an $\mathbb{E}_\infty$-ring R to the maximal Kan complex of ∞-category Mod_R^{nproj} of projective modules of rank n is a sheaf.

Definition 4.9 ([13] Definition 2.6.14)**.** Let P be a property for objects (A, M) in an ∞-category $\mathrm{CAlg}^{cn} \times_{\mathrm{CAlg}} \mathrm{Mod}$. We say that P is local for the flat topology if the following conditions are satisfied:

(i) Let $f : A \to B$ be a flat morphism of connective $\mathbb{E}_\infty$-rings, and M an A-module. If (A, M) has the property P, $(B, B \otimes_A M)$ has the property P. If f is faithfully flat, the converse holds.
(ii) For any finite collection (A_i, M_i) of the objects in $\mathrm{CAlg}^{cn} \times_{\mathrm{CAlg}} \mathrm{Mod}$ such that each (A_i, M_i) has the property P, the product $(\prod A_i, \prod M_i)$ has the property P.

Theorem 4.10 ([16] Corollary 6.13, Lemma 6.17)**.** *A functor* $\mathrm{CAlg}^{cn} \to \widehat{\mathrm{Cat}_\infty}$ *given by* $R \mapsto \mathrm{Mod}_R$ *is a sheaf.*

Since $(nProj)$ is a subfunctor of the sheaf $R \mapsto \mathrm{Mod}_R$, if Mod_R^{nproj} consisting of objects which satisfy the conditions in Definition 4.9 for each connective $\mathbb{E}_\infty$-ring, we note that $(nProj)$ is also a sheaf by the similar argument in [13, cf. Remark 1.5.3].

Lemma 4.11. *Let* $(nProj) : \mathrm{CAlg}^{cn} \to \widehat{\mathrm{Cat}_\infty}$ *be a functor given by* $R \mapsto \mathrm{Mod}_R^{nproj}$*. Then, the functor* $(nProj)$ *is a sheaf with respect to flat topology.*

Proof. Since a functor $\mathrm{CAlg}^{cn} \to \widehat{\mathrm{Cat}_\infty}$ given by $R \mapsto \mathrm{Mod}_R$ is already a sheaf By Theorem 4.10, it suffices to check that the projective modules of rank n satisfy the condition of Definition 4.9 (i) and (ii).

By [13, Proposition 2.6.15 (1), (6), (9)], the condition of finitely generated projective is flat local property. Since the tensor product preserves rank, the projective modules of rank n satisfy the condition of Definition 4.9 (i).

We will check the condition (ii) of Definition 4.9. Assume that $(A_i,\, M_i)$ is a pair such that A_i is connective $\mathbb{E}_\infty$-ring and M_i is a projective A_i-module of rank n for $1 \leq i \leq m$. To show that $\prod_i M_i$ is a projective $\prod_i A_i$-module of rank n, it suffices to show that there exists a finite set $\{g_a\}_a$ of objects in $\prod_i A_i$ such that each $\prod_i M_i[g_a^{-1}]$ is a free $\prod_i A_i[g_a^{-1}]$-module of rank n. We choose such $\{g_a\}_a$ as follows.

For each i, we take $f_{i1}, \cdots, f_{ik} \in \pi_0 A_i$ such that $(M_i)[f_{il}^{-1}]$ is a free $A_i[f_{il}^{-1}]$-module of rank n for $1 \leq l \leq k$. Since $\prod_i A_i \to A_i$ is flat and the essential image is generated by the form $\prod (A_i \otimes_A A_j) \otimes_{A_j} M$ and $A_i \otimes_A A_j \simeq 0$, $A_j \otimes_{\prod_i A_i} \prod_i M_i \simeq M_j$.

From this, we have the equivalence $\prod_i A_i[f_{il}^{-1}] \otimes_{\prod_i A_i} \prod_i M_i \simeq \prod_i M_i[f_{il}^{-1}]$, where $\prod_i M_i[f_{il}^{-1}]$ is regarded as a $\prod_i A_i[f_{il}^{-1}]$-module and is free of rank n. □

Corollary 4.12. $(nProj)$ is a sheaf in Nisnevich topology and Zariski topology.

Proof. It follows from Lemma 2.7 and Lemma 4.11. □

Definition 4.13 ([11] Theorem 3.1.5.1)**.** Let $\widehat{\mathcal{S}}$ be the ∞-category of spaces and $\widehat{\mathrm{Cat}_\infty}$ the ∞-category of ∞-categories defined in Definition 2.3 after enlarging the universe respectively. We regard the Kan complexes as ∞-categories, so that we have an inclusion functor $i : \widehat{\mathcal{S}} \to \widehat{\mathrm{Cat}_\infty}$.

Since the inclusion preserves small colimits, by [11, Corollary 5.5.2.9], there is an adjunction

$$i : \widehat{\mathcal{S}} \rightleftarrows \widehat{\mathrm{Cat}_\infty} : (-)^{\simeq}. \tag{4.3}$$

Note that the right adjoint $(-)^{\simeq}$ is given by taking the maximal ∞-groupoid.

Lemma 4.14. *The right adjoint in (4.3) induces a functor from $\widehat{\mathrm{Cat}_\infty}$-valued sheaves to $\widehat{\mathcal{S}}$-valued sheaves.*

Proof. Since $(-)^{\simeq}$ is a right adjoint, it preserves limits. Especially, it preserves the limits of simplicial diagrams in Proposition 2.6. Therefore, $(-)^{\simeq}$ sends $\widehat{\mathrm{Cat}_\infty}$-valued sheaves to $\widehat{\mathcal{S}}$-valued sheaves. □

4.4. **Comparison between $B(GL_n(R))$ to $(\mathrm{Mod}_R^{nfree})^{\simeq}$.** Recall that the notion of classifying object from Definition 3.6. We denote by $B^{\mathcal{G}}$ the classifying functor on $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$. Acording to Remark 4.5, we have GL_n in $\mathcal{S}hv_{\mathcal{S}}(\mathrm{CAlg}^{\mathcal{G}})$.

Definition 4.15. (i) We use the notation $B(GL_n(R))$ for the classifying space of the value $GL_n(R)$ in $\mathcal{S}$, and $(B^{\mathcal{G}}GL_n)(R)$ for the value at R of the classifying sheaf of GL_n in $\mathcal{S}hv_{\mathcal{S}}(\mathrm{CAlg}^{\mathcal{G}})$.
(ii) We let $(\mathrm{Mod}_{(-)}^{nfree})^{\simeq} : \mathrm{CAlg}^{cn} \to \widehat{\mathcal{S}}$ be a functor which sends R to the ∞-groupoid $(\mathrm{Mod}_R^{nfree})^{\simeq}$ of free R-modules of rank n.

Note that $GL_n(R) \simeq Aut_R(R^{\oplus n})$, and $B(GL_n(R))$ is not equal to $(B^{\mathcal{G}}GL_n)(R)$.

Remark 4.16. Apparently, the classifying sheaf $B^{\mathcal{G}}GL_n$ of GL_n in $\mathcal{S}hv_{\mathcal{S}}(\mathrm{CAlg}^{\mathcal{G}})$ in Definition 4.15 depends the Grothendieck topology on $\mathrm{CAlg}^{\mathcal{G}}$, but consequently, we show that it is equivalent to the flat sheaf $(nProj)$ in Proposition 4.21 below, $B^{\mathcal{G}}GL_n$ in Zariski topology is equivalent to that in Nisnevich topology.

Lemma 4.17 (cf. [1] Section B.3)**.** *Let us regard $(\mathrm{Mod}_R^{nfree})^{\simeq}$ as a simplicial set, and take a vertex $x \in (\mathrm{Mod}_R^{nfree})^{\simeq}$. Then, we have a morphism $B(Aut_R(x)) \to (\mathrm{Mod}_R^{nfree})^{\simeq}$ of simplicial sets.*

Proof. Since we have an equivalence

$$Aut_R(x) \to \mathrm{Map}_{\mathrm{Mod}_R^{nfree}}(x,\, x)^{\simeq},$$

which is induced from the inclusion in Definition 4.6. We note that this morphism factors through the equivalence $\mathrm{Map}_{(\mathrm{Mod}_R^{nfree})^{\simeq}}(x,\, x) \simeq \mathrm{Map}_{\mathrm{Mod}_R^{nfree}}(x,\, x)^{\simeq}$ and $\mathrm{Map}_{(\mathrm{Mod}_R^{nfree})^{\simeq}}(x,\, x)$ is 1-simplices of $(\mathrm{Mod}_R^{nfree})^{\simeq}$ by taking $x = y$ and identifying the image in $S \times S$ with S in Definition 3.8. Since the morphism $Aut_R(x) \to \mathrm{Map}_{(\mathrm{Mod}_R^{nfree})^{\simeq}}(x,\, x)$ is obtained by the inclusion, it preserves ∞-groupoid structure, so that it preserves the monoid structure. Since both $B(Aut_R(x))$ and $(\mathrm{Mod}_R^{nfree})^{\simeq}$ are connected, the induced morphism $B(Aut_R(x)) \to (\mathrm{Mod}_R^{nfree})^{\simeq}$ is an equivalence. □

Let $B(GL_n(-)) : \mathrm{CAlg}^{cn} \to \widehat{\mathcal{S}}$ be a functor which sends R to the space $B(GL_n(R))$ defined in Definition 4.15. We will compare the functor $B(GL_n(-))$ with a functor given in Definition 4.15(ii). The following proposition is obtained by the similar arguments in the case [1, Section b.3] of rank 1.

Proposition 4.18. *The functor $B(GL_n(-))$ is equivalent to the functor $(\mathrm{Mod}_{(-)}^{nfree})^{\simeq}$ defined in Definition 4.15(ii).*

Proof. By taking $x = y = R^n$ in Lemma 4.17, we have a morphism from $B(GL_n(R)) \to (\mathrm{Mod}_R^{nfree})^{\simeq}$ for each $R \in \mathrm{CAlg}^{cn}$.

By composing a morphism $B(GL_n(R)) \simeq BAut_{\mathbb{S}}(R^n)$ which we obtained by Proposition 4.8 with $B(Aut_{\mathbb{S}}(R)) \to (\mathrm{Mod}_R^{nfree})^{\simeq}$ obtained by Lemma 4.17, we have a morphism $B(GL_n(R)) \to (\mathrm{Mod}_R^{nfree})^{\simeq}$. It suffices to show that a morphism $B(GL_n(R)) \to (\mathrm{Mod}_R^{nfree})^{\simeq}$ is an equivalence and functorial with respect to R.

By construction, we identify $B(GL_n(R))$ with a full ∞-subgroupoid of $(\mathrm{Mod}_R^{nfree})^{\simeq}$ such that the object is only $R^{\oplus n}$ and the class of morphisms is identified with $GL_n(R) \simeq Aut_{\mathbb{S}}(R^n)$ by Proposition 4.8, which is fully faithfull functorial assignment with respect to R.

We show that it is essentially surjective. It suffices to show that $(\mathrm{Mod}_R^{nfree})^{\simeq}$ is connected as a simplicial set. This is obvious since, for $M \in \mathrm{Mod}_R^{nfree}$, we have $M \simeq R^{\oplus n}$. $\square$

4.5. **A natural transformation from $B(GL_n(-))$ to $(B^{\mathcal{G}}GL_n)(-)$.** We construct a natural transformation from $B(GL_n(-))$ to $(B^{\mathcal{G}}GL_n)(-)$ by using the comparison between the sheaf of projective modules of rank n and the sheafification of the functor of free modules of rank n.

For each R, we have the following colimit in $\widehat{\mathcal{S}}$:

$$GL_n(R) \rightrightarrows * \longrightarrow B(GL_n(R)).$$

Note that the functor corresponds to this cofiber is not a sheaf since the second condition of [13, Definition 2.6.14 (1)] fails. According to the notation in Definition 2.4, we write $B(\widetilde{GL_n(-)})$ for the sheafification of $B(GL_n(-))$.

Remark 4.19. In the stable ∞-category of spectra, taking B is equivalent to taking Σ by stability. So, although the notation Σ of the suspention can be used, we use the notation B in the colimit of the above coequalizer since objects appearing in the coequalizer are grouplike and can be regarded as connective spectra via the equivalence from $Gp(\mathcal{S})$ to Sp^{cn} which preserves sifted colimits.

Lemma 4.20. *We have the following cofiber sequence*

$$GL_n \rightrightarrows * \longrightarrow B(\widetilde{GL_n(-)})$$

in $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$.

Proof. Since GL_n is corepresented by an $\mathbb{E}_\infty$-ring, it is flat sheaf by [16, VII, Theorem 5.15], so that it is a Zariski (resp. Nisnevich) sheaf. Since sheafification is left adjoint, it commutes with the cofiber sequence. ☐

For each $A \to R$ in CAlg^{cn}, the natural inclusions induce the commutative diagram

$$\begin{tikzcd} \mathrm{Mod}_A^{nfree} \arrow[r] \arrow[d] & \mathrm{Mod}_A^{nproj} \arrow[d] \\ \mathrm{Mod}_R^{nfree} \arrow[r] & \mathrm{Mod}_R^{nproj}, \end{tikzcd}$$

in $\widehat{\mathrm{Cat}_\infty}$, so that we have a natural transformation

$$\mathrm{Mod}_{(-)}^{nfree} \to \mathrm{Mod}_{(-)}^{nproj}. \tag{4.4}$$

By the adjointness of sheafification, we have a morphism $B(\widetilde{GL_n(-)}) \to (nProj)^{\simeq}$ of $\widehat{\mathcal{S}}$-valued sheaves.

4.6. **An equivalence $B^{\mathcal{G}}GL_n \simeq (nProj)^{\simeq}$.** The weak homotopy eqivalence in the following statement is written in [2, Proposition 5.1]. However, we show it by a fortiori argument using morphisms induced by previous subsection.

Proposition 4.21. *The morphism (4.4) in the previous subsection gives an equivalence $B^{\mathcal{G}}GL_n \simeq (nProj)^{\simeq}$ in $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$.*

Proof. From adjointness of a morphism $f_{(-)} : B(\widetilde{GL_n(-)}) \to (nProj)^{\simeq}$ of $\widehat{\mathcal{S}}$-valued sheaves, we have the following homotopy commutative diagram of $\widehat{\mathcal{S}}$-valued presheaves:

$$\begin{tikzcd} & B(\widetilde{GL_n(-)}) \arrow[dr, "f"] & \\ B(GL_n(-)) \arrow[ur, "L"] \arrow[rr, "i"] & & (nProj)^{\simeq}, \end{tikzcd}$$

where L is the morphism associated to the sheafification and i is induced from the inclusion.

Let us denote the limits of the following simplicial diagrams by $B(GL_n(A))_R$:

$$B(GL_n(A)) \rightrightarrows B(GL_n(A \otimes_R A)) \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \cdots ,$$

where $R \to A$ is a faithfully flat morphism. By Remark 2.5, the sheafification $B(\widetilde{GL_n(-)})$ is described by the term of $B(GL_n(-))$, i.e., it is the colimit of those $B(GL_n(A))_R$ taken over the every covering sieve $R \to A$.

For $f_R : B(\widetilde{GL_n(R)}) \to (nProj)^{\simeq}(R)$ and an object Q, i.e., a vertex in $(nProj)^{\simeq}(R)$, let $(Q)^{\simeq}$ denote the full ∞-subgroupoid spanned by Q. Since $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$ is locally presentable, there exists a corresponding combinatorial simplicial model category up to Quillen equivalences. We take it under the consideration, and construct $h_{R,Q} : (Q)^{\simeq} \to B(\widetilde{GL_n(R)})$ as follows.

We take a Zariski local A of R such that an object $Q \in (nProj)^{\simeq}(R)$ is trivialized on A. Then, we naturally identified the ∞-groupoid spanned by $Q \otimes A$ with $B(GL_n(A))$, so that we have an object in $B(\widetilde{GL_n(R)})$. This is the assignment under the morphisms of ∞-groupoids. By $f \circ L \simeq i$ in the above diagram, we have $f_R \circ h_{R,Q}$ is homotopic to identity.

Conversely, for an object P in $B(\widetilde{GL_n(R)})$, we take its value on the covering sieve $R \to R$ under the morphisms associated to the colimit, and denote by P'. Then, by an extension of the coefficients of P' gives an object in $B(\widetilde{GL_n(R)})$, which is equivalent to P by the construction. This shows that $h_{R,f(P)} \circ f_R$ is homotopic to identity.

Since f_R is obviously a Kan fibration and, by the above arguments, any fiber of f_R is contractible. By [11, Lemma 4.1.3.2, Corollary 4.1.2.6], it is a weak homotopy equivalence. □

4.7. **Zariski connected $\mathbb{E}_\infty$-rings.** Next, we consider the condition on a connective $\mathbb{E}_\infty$-ring R such that any finitely generated projective R-module has finite constant rank.

Definition 4.22. (i) We say that an $\mathbb{E}_\infty$-ring is Zariski non-connected if there exists those objects $f, g \in \pi_0 R$ such that $R[f^{-1}] \otimes_R R[g^{-1}] \simeq 0$ and $R \simeq R[f^{-1}] \times R[g^{-1}]$.

(ii) We say that an $\mathbb{E}_\infty$-ring is Zariski connected if it is not Zariski non-connected.

Lemma 4.23. *The following conditions are equivalent:*

(i) R be an $\mathbb{E}_\infty$-ring such that $\pi_0 R$ has no non-trivial idempotent element.
(ii) R is Zariski connected.
(iii) Any $P \in \mathrm{Mod}_R^{proj}$ has finite constant rank.

Proof. For proving (iii) from (i), let R be an $\mathbb{E}_\infty$-ring such that $\pi_0 R$ has no non-trivial idempotent, and $P \in \mathrm{Mod}_R^{proj}$. Note that $\pi_0 P$ is a finitely generated projective $\pi_0 R$-module [12, Remark 7.2.2.20]. Then, there exists $f_1, \cdots, f_m$ which generate the unit ideal of $\pi_0 R$ such that each $\pi_0 P[f_i^{-1}]$ is free of finite rank n_i over $\pi_0 R[f_i^{-1}]$. For an ordinary commutative ring $\pi_0 R$, in this the condition (i) is equivalent to that n_is are constant, and let us denote it by n. Since finitely generated free $\pi_0 R[f_i^{-1}]$-modules $\pi_0 P[f_i^{-1}]$ can be lifted by finitely generated $R[f_i^{-1}]$-modules $P[f_i^{-1}]$ and a morphism between flat R-modules is an equivalence if and only if it induces an isomorphism on π_0 [12, Lemma 7.2.2.17], locally P has a constant rank.

Conversely, if $\pi_0 R$ has a non-trivial idempotent element, we have that the rank n_i are not constant. In this case, the above argument shows that (iii) implies (i).

We show that (i) implies (ii). Since π_0 preserves finite products, if R is Zariski non-connected, by passing to π_0 and applying the ordinary commutative ring theory, $\pi_0 R$ has a non-trivial idempotent. Conversely, if $\pi_0 R$ has a non-trivial idempotent e, it also a non-trivial idempotent of $\pi_* R$. Since the localization of an $\mathbb{E}_\infty$-ring with one element commutes with π_*, we have $\pi_*(R[e^{-1}]) \cong (\pi_* R)[e^{-1}]$ and $\pi_*(R[(1-e)^{-1}]) \cong (\pi_* R)[(1-e)^{-1}]$. Since π_* commutes with finite products, by taking π_* of $R[e^{-1}] \times R[(1-e)^{-1}]$, we conclude that R is equivalent to $R[e^{-1}] \times R[(1-e)^{-1}]$, so that R is Zariski non-connected. Thus, (ii) implies (i). $\square$

4.8. **Comparison between $B^{\mathcal{G}}GL$ and $\mathrm{Mod}_R^{\infty proj}$.** Now, we will prove the main proposition in this section by applying the following proposition.

Proposition 4.24 ([16] Lemma 3.21)**.** *Let $\mathcal{C}$ be an ∞-topos, I an index set and $\{X_i\}_{i\in I}$ a collection of objects in $\mathcal{C}$. For every subset $J \subset I$, let $X_J \simeq \coprod_{i\in J} X_i$. Let $C \in \mathcal{C}$ be an object such that every covering of C has a finite subcovering* [16, Definition 3.1]*. Then, the canonical morphism*

$$\operatorname*{colim}_{J\subset I} \mathrm{Map}_{\mathcal{C}}(C, X_J) \to \mathrm{Map}_{\mathcal{C}}(C, X_I)$$

induces a homotopy equivalence. Here, the colimit in the left hand side is run through the all finite subsets $J \subset I$.

$\square$

Lemma 4.25. *Let $I \subset \mathbb{N}$ be a finite index set. If R is Zariski connected, the canonical morphism*

$$\coprod_{i\in I} \mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(\mathrm{Spec}^{\mathcal{G}} R, B^{\mathcal{G}}GL_i) \to \mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(\mathrm{Spec}^{\mathcal{G}} R, \coprod_{i\in I} B^{\mathcal{G}}GL_i),$$

is an equivalence.

Proof. To check that the canonical morphism is an equivalence, it suffices to show that, for any morphism ϕ in $\mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(\mathrm{Spec}^{\mathcal{G}} R, \coprod_{i\in I} B^{\mathcal{G}}GL_i)$, there exists an index $j \in \mathbb{N}$ which is uniquely determined by ϕ, the morphism ϕ has the unique factorization

$$\mathrm{Spec}^{\mathcal{G}} R \to B^{\mathcal{G}}GL_j \to \coprod_{i\in I} B^{\mathcal{G}}GL_i.$$

Since the sheafification commutes with coproducts, $\coprod_{i\in I} B^{\mathcal{G}}GL_i$ is equivalent to the sheafification of the presheaf given by $R \mapsto \coprod_{i\in I}(B^{\mathcal{G}}GL_i(R))$. (Note that the coproduct of sheaves are the sheafification of the objectwise coproduct as presheaves, and an functor obtained by objectwise coproduct of sheaves is not always a sheaf.) Therefore, an object of $(\coprod_{i\in I} B^{\mathcal{G}}GL_i)(R)$ can be represented by a finitely generated projective R-module whose local ranks are contained in I.

Let $I\mathrm{Mod}_R^{proj}$ be an ∞-category of finitely generated projective modules whose local rank is contained in I. Since R is Zariski connected, by Lemma 4.23(iii), we deduce that $I\mathrm{Mod}_R^{proj} \simeq \coprod_{\{i\in I\}} \mathrm{Mod}_R^{iProj}$. Therefore, for any object of $(\coprod_{i\in I} B^{\mathcal{G}}GL_i)(R)$, there exists a unique $j \in I$ such that the object can be represented by a finitely generated projective R-module of rank j.

By applying the Yoneda embedding [11, Section 5.1.3], we have that an object of $(\coprod_{i\in I} B^{\mathcal{G}}GL_i)(R)$ represented by a finitely generated projective R-module of rank j corresponds to the morphism $\mathrm{Spec}^{\mathcal{G}} R \to \coprod_{i\in I} B^{\mathcal{G}}GL_i$ which has the unique factorization

$$\mathrm{Spec}^{\mathcal{G}} R \to B^{\mathcal{G}}GL_j \to \coprod_{i\in I} B^{\mathcal{G}}GL_i.$$

□

We consider a decomposition of an $\mathbb{E}_\infty$-ring R by using the idempotent element in $\pi_0 R$.

Proposition 4.26. *Let* $\mathrm{Spec}^{\mathcal{G}} R : (\mathrm{CAlg}^{\mathcal{G}})^{op} \to \widehat{\mathcal{S}}$ *be a spectral scheme and* $\mathrm{Spec}^{\mathcal{G}} R \to B^{\mathcal{G}}GL_n$ *a morphism in* $\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})$. *Let* $B^{\mathcal{G}}GL$ *be a sheaf* $\coprod_{n\in\mathbb{N}} B^{\mathcal{G}}GL_n$. *For* $I \subset J \subset \mathbb{N}$, *we have the system* $\coprod_{i\in I} B^{\mathcal{G}}GL_i \to \coprod_{j\in J} B^{\mathcal{G}}GL_j$ *given by inclusions. Then there is an equivalence of* ∞*-groupoids;*

$$\mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(\mathrm{Spec}^{\mathcal{G}} R,\, B^{\mathcal{G}}GL) \simeq (\mathrm{Mod}_R^{\infty proj})^{\simeq}.$$

Proof. By Proposition 4.24, we have

$$\operatorname*{colim}_{I\subset\mathbb{N}} \mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(\mathrm{Spec}^{\mathcal{G}} R,\, \coprod_{i\in I} B^{\mathcal{G}}GL_i) \simeq \mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(\mathrm{Spec}^{\mathcal{G}} R,\, B^{\mathcal{G}}GL).$$

By decomposing R with the Zariski connected $\mathbb{E}_\infty$-rings (given by the corresponding irreducible decomposition on π_0) and applying Lemma 4.25, for $I = \{1, \cdots, n\}$, we have the equivalence

$$\operatorname*{colim}_{I\subset\mathbb{N}} \coprod_{i\in I} \mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(\mathrm{Spec}^{\mathcal{G}} R,\, B^{\mathcal{G}}GL_i) \simeq \operatorname*{colim}_{I\subset\mathbb{N}} \mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(\mathrm{Spec}^{\mathcal{G}} R,\, \coprod_{i\in I} B^{\mathcal{G}}GL_i)$$

By Proposition 4.21, we identify the left hand side with

$$\operatorname*{colim}_{i\in\mathbb{N}}(\mathrm{Mod}_R^{iproj})^{\simeq} \simeq (\mathrm{Mod}_R^{\infty Proj})^{\simeq}.$$

□

5. K-theory of $\mathrm{Mod}_R^{\infty proj}$

Definition 5.1 (cf. [4] 1.2, [6] Definition 2.1)**.** Let $\mathcal{C}$ be a pointed ∞-category. A class of w^∞-cofibrations is a class of morphisms in $\mathcal{C}$ which satisfies the following conditions:

(i) $* \to X$ is a w^∞-cofibration for any object X,
(ii) The class of w^∞-cofibrations includes weak equivalences,
(iii) Any composition of w^∞-cofibrations is a w^∞-cofibration,
(iv) For a w^∞-cofibration $X \to Y$ and a morphism $X \to Z$, there exists a pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, "f"'] & W, \end{tikzcd}$$

in which the morphism f is a w^∞-cofibration.

We call such a pair of $\mathcal{C}$ and a w^∞-cofibrations an ∞-category with w^∞-cofibrations.

In terminology of [5], a w^∞-cofibration is called a cofibration. In terminology of [3], a w^∞-cofibration is called an igressive morphism.

Remark 5.2. We say that a w^∞-cofibration is a split w^∞-cofibration if it is split as a morphism.

Definition 5.3. We make into $\mathrm{Mod}_R^{\infty proj}$ to be an ∞-category with w^∞-cofibrations as follows.

Declare a morphism $f : P_1 \to P_2$ in $\mathrm{Mod}_R^{\infty proj}$ to be a w^∞-cofibration if it is a morphism in Mod_R and the cofiber of f is an object in $\mathrm{Mod}_R^{\infty proj}$.

Then, Mod_R^{proj} (resp. $\mathrm{Mod}_R^{\infty proj}$) is an ∞-category with w^∞-cofibrations.

Lemma 5.4. *(i) A w^∞-cofibration in $\mathrm{Mod}_R^{\infty proj}$ is always split, e.g., it is a split w^∞-cofibration in Remark 5.2.*
(ii) Let $R_1 \to R_2$ be a morphism of connective $\mathbb{E}_\infty$-rings. Then, a functor $\mathrm{Mod}_{R_1}^{\infty proj} \to \mathrm{Mod}_{R_2}^{\infty proj}$ given by the extension of coefficients is an exact functor in the sence of K-theory.

Proof. By [12, Proposition 7.2.2.6 (5)], a cofiber sequence of finitely generated projective modules is always split up to homotopy. Thus, (i) holds.

Since the tensor product commutes with the cofiber sequences and preserves rank, the extension of coefficients is exact. □

Remark 5.5. In this paper, we consider the connective K-theory. We can regard the K-theory spectrum as a K-theory space.

Let $\mathcal{C}$ be a pointed additive ∞-category.

Recall that the maximal ∞-groupoid $\mathcal{C}^{\simeq}$ of the additive ∞-category $\mathcal{C}$ inherits the structure of ∞-monoid space by applying [9, Corollary 6.6] to the functor $\mathrm{Cat}_\infty \to \mathcal{S}$ given by taking maximal ∞-groupoid. Especially, the additive ∞-category $(\mathrm{Mod}_R^{\infty proj})^{\simeq}$ is a commutative ∞-monoid.

Note that the composition of the two lax symmetric monoidal functors $Mon(Cat_\infty) \to Mon(\mathcal{S})$ and the group completion on the ∞-category of spaces $Mon(\mathcal{S}) \to Gp(\mathcal{S})$ in Lemma 3.9 is factored by the group completion of Cat_∞ with the natural inclusion $Gp(\mathcal{S}) \to Gp(\mathrm{Cat}_\infty)$ by [9, Proposition 8.14].

There is an ∞-version of the additivity theorem [21, Theorem 1.8.7] given by [8]. We state the theorem as in Lurie's unpublished note [17].

Recall the notion of a split w^∞-cofibration from Remark 5.2. We will apply the following theorem for $\mathrm{Mod}_R^{\infty proj}$ whose w^∞-cofibrations defined in Definition 5.3 are split w^∞-cofibrations by Lemma 5.4.

Theorem 5.6 ([8], cf. [17] Theorem 10)**.** *Let $\mathcal{C}$ be a pointed additive ∞-category with split w^∞-cofibrations, and $\mathcal{C}^{\simeq}$ be the maximal ∞-groupoid of $\mathcal{C}$ as in (4.3). Let $K(\mathcal{C})$ be the algebraic K-theory and $\Omega B(\mathcal{C}^{\simeq})$ the group completion of $\mathcal{C}^{\simeq}$.*

Then, there is an equivalence

$$\Omega B(\mathcal{C}^{\simeq}) \to K(\mathcal{C}) \tag{5.1}$$

of spases. Here, Ω and B are defined in Section 4.

Corollary 5.7. We have $\Omega B((\mathrm{Mod}_R^{\infty proj})^{\simeq}) \simeq K(\mathrm{Mod}_R^{\infty proj})$.

Proof. The ∞-category $\mathrm{Mod}_R^{\infty proj}$ (resp. Mod_R^{proj}) is additive, pointed by 0, with split w^∞-cofibrations. Thus, Theorem 5.6 can be applied. □

6. Proof of Theorem 1.1

Let us keep the notation explained in the previous sections. Now we prove Theorem 1.1.

Let $B^{\mathcal{G}}GL = \coprod_{n\in\mathbb{N}} B^{\mathcal{G}}GL_n$. Note that, the sheafification preserves coproducts and finite products. By Proposition 4.26, we can regard $B^{\mathcal{G}}GL$ as the sheaf $(\mathrm{Mod}_{(-)}^{\infty proj})^{\simeq}$. Since, for an $\mathbb{E}_\infty$-ring R, $\mathrm{Mod}_R^{\infty proj}$ is a symmetric monoidal ∞-category with direct sum and with tensor product which preserves direct sum in each variable separately. Moreover, since the direct sum commutes with the sheafification, the juxtaposition of $B^{\mathcal{G}}GL$ arises from that of $B(GL(-))$ which is obtained in objectwise manner. So, we obtain the following proposition.

Proposition 6.1. *Let $B^{\mathcal{G}}GL = \coprod_{n\in\mathbb{N}} B^{\mathcal{G}}GL_n$. Then, $B^{\mathcal{G}}GL$ becomes a commutative ∞-monoid.*

Theorem 6.2. *Let $\mathrm{CAlg}^{\mathcal{G}}$ be the ∞-category $(\mathrm{CAlg}^{cn})^{op}$ equipped with Zariski or Nisnevich topology defined in Section 2. Let $B^{\mathcal{G}}GL = \coprod_{n\in\mathbb{N}} B^{\mathcal{G}}GL_n$ and $\Omega B^{\mathcal{G}}(B^{\mathcal{G}}GL)$ the group completion as a sheaf on $\mathrm{CAlg}^{\mathcal{G}}$. Note that $B^{\mathcal{G}}GL$ becomes a commutative ∞-monoid as in Proposition 6.1.*

There is an equivalence of ∞-groupoids:

$$\mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(\mathrm{Spec}^{\mathcal{G}} R,\ \Omega B^{\mathcal{G}}(B^{\mathcal{G}}GL)) \simeq \widetilde{K}^{\mathcal{G}}(\mathrm{Mod}_R^{\infty proj}),$$

where $\widetilde{K}^{\mathcal{G}}$ is the sheafification of K defined by (1.1).

Proof. By Proposition 4.21, we have an equivalence

$$\Omega B(B^{\mathcal{G}}GL(R)) \simeq \Omega B((\mathrm{Mod}_R^{\infty proj})^{\simeq}),$$

where $(-)^{\simeq}$ denotes the maximal ∞-groupoid (4.3). Since all w^{∞}-cofibrations in $\mathrm{Mod}_R^{\infty proj}$ are split and the homotopy category of Mod_R^{proj} is additive, by Corollary 5.7, we obtain that $\Omega B((\mathrm{Mod}_R^{\infty proj})^{\simeq})$ is equivalent to the algebraic K-theory $K(\mathrm{Mod}_R^{\infty proj})$ given by $S_\bullet$ construction.

On the other hand, we have an equivalence induced by Yoneda embedding

$$\mathrm{Map}_{\mathcal{S}hv_{\widehat{\mathcal{S}}}(\mathrm{CAlg}^{\mathcal{G}})}(\mathrm{Spec}^{\mathcal{G}} R,\ \Omega B^{\mathcal{G}}(B^{\mathcal{G}}GL)) \simeq (\Omega B^{\mathcal{G}} B^{\mathcal{G}}GL)(R).$$

By virtue of Proposition 6.1, we can apply Proposition 3.16 to the commutative ∞-monoid sheaf $B^{\mathcal{G}}GL$. The sheafification of the objectwise group completion functor $R \mapsto \Omega B(B^{\mathcal{G}}GL(R))$ is equivalent to the group completion of the sheaf given by the assignment $R \mapsto (\Omega B^{\mathcal{G}} B^{\mathcal{G}}GL)(R)$. Thus, we have an equivalence $(\Omega B^{\mathcal{G}} B^{\mathcal{G}}GL)(R) \simeq \Omega B(B^{\mathcal{G}}GL(R))$ after the sheafification. $\square$

Remark 6.3. Since Yoneda embedding and the sheafification preserve finite products, and since the functor $(-)^{\simeq}$ taking maximal ∞-groupoid is lax monoidal [9, Remark 8.7], the equivalence in Theorem 6.2 preserves commutative ∞-group structure.

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