PASSAGE OF PROPERTY $(Bw)$ FROM TWO OPERATORS
TO THEIR TENSOR PRODUCT

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Abstract. A Banach space operator satisfies property $(Bw)$ if the complement of its B-Weyl spectrum in its the spectrum is the set of finite multiplicity isolated eigenvalues of the operator. Property $(Bw)$ does not transfer from operators $T$ and $S$ to their tensor product $T \otimes S$. We give necessary and/or sufficient conditions ensuring the passage of property $(Bw)$ from $T$ and $S$ to $T \otimes S$. Perturbations by Riesz operators are considered.

1. Introduction

Given Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{X} \otimes \mathcal{Y}$ denote the completion (in some reasonable uniform cross norm) of the tensor product of $\mathcal{X}$ and $\mathcal{Y}$. For Banach space operators $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$, let $T \otimes S \in \mathcal{B}(\mathcal{X} \otimes \mathcal{Y})$ denote the tensor product of $T$ and $S$.

For a bounded linear operator $S \in \mathcal{B}(\mathcal{X})$, let $\ker(S), \mathcal{R}(S), \sigma(S)$ and $\sigma_a(S)$ denote, respectively, the kernel, the range, the spectrum and the approximate point spectrum of $S$. Let $\alpha(S)$ and $\beta(S)$ denote the nullity and the deficiency of $S$, defined by $\alpha(S) = \dim \ker(S)$ and $\beta(S) = \dim \mathcal{R}(S)$.

If the range $\mathcal{R}(S)$ of $S$ is closed and $\alpha(S) < \infty$ (resp. $\beta(S) < \infty$), then $S$ is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator. If $S \in \mathcal{B}(\mathcal{X})$ is either upper or lower semi-Fredholm, then $S$ is called a semi-Fredholm operator, and ind$(S)$, the index of $S$, is then defined by ind$(S) = \alpha(S) - \beta(S)$. If both $\alpha(S)$ and $\beta(S)$ are finite, then $S$ is a Fredholm operator. The ascent, denoted $a(S)$, and the descent, denoted $d(S)$, of $S$ are given by $a(S) = \inf \{ n \in \mathbb{N} : \ker(S^n) = \ker(S^{n+1}) \}$, $d(S) = \inf \{ n \in \mathbb{N} : \mathcal{R}(S^n) = \mathcal{R}(S^{n+1}) \}$ (where the infimum is taken over the set of non-negative integers); if no such integer $n$ exists, then $a(S) = \infty$, respectively $d(S) = \infty$.

Let $T \in \mathcal{B}(\mathcal{X})$. Define

\[ \sigma_p(T) = \{ \lambda \in \mathbb{C} : \ker(\lambda - T) \neq \{0\} \}; \]
\[ \sigma_c(T) = \{ \lambda \in \mathbb{C} : \ker(\lambda - T) \neq \{0\}, \overline{\mathcal{R}(\lambda - T)} = \mathcal{X} \text{ but } \mathcal{R}(\lambda - T) \neq \mathcal{X} \}; \]
\[ \sigma_e(T) = \{ \lambda \in \mathbb{C} : \ker(\lambda - T) = \{0\} \text{ but } \overline{\mathcal{R}(\lambda - T)} \neq \mathcal{X} \}. \]
\( \sigma_p(T), \sigma_c(T) \) and \( \sigma_r(T) \) are called respectively the point spectrum, the continuous spectrum and the residual spectrum of \( T \). Clearly, \( \sigma_p(T), \sigma_c(T) \) and \( \sigma_r(T) \) are disjoint and \( \sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) \). Let \( \sigma_w(T) = \{ \lambda \in \sigma(T) : \lambda - T \text{ is not a Fredholm operator of index 0} \} \) be the Weyl spectrum of \( T \), which is a subset of the whole spectrum \( \sigma(T) \). The set \( \sigma_0(T) = \{ \lambda \in \sigma_p(T) : \Re(\lambda - T) \text{ is closed and } \alpha(\lambda - T) = \alpha(\lambda - T^*) < \infty \} \) is precisely the complement of the Weyl spectrum \( \sigma_w(T) \) in the whole spectrum \( \sigma(T) \). Hence

\[
\sigma_w(T) = \sigma(T) \setminus \sigma_0(T),
\]

and so \( \{ \sigma_w(T), \sigma_0(T) \} \) forms another partition of the spectrum of \( \sigma(T) \). Set \( \sigma_{PF}(T) = \{ \lambda \in \sigma_p(T) : \alpha(\lambda - T) < \infty \} \); the set of all eigenvalues of finite multiplicity, so that \( \sigma_0(T) \subseteq \sigma_{PF}(T) \) and \( \sigma_r(T) \cup \sigma_c(T) \cup (\sigma_p(T) \setminus \sigma_{PF}(T)) \subseteq \sigma_w(T) \). Set

\[
E^0(T) = \text{iso } \sigma(T) \cap \sigma_{PF}(T) = \{ \lambda \in \text{iso } \sigma(T) : 0 < \alpha(\lambda - T) < \infty \}.
\]

According to Coburn [6], Weyl’s theorem holds for \( T \) if \( \Delta(T) = \sigma(T) \setminus \sigma_{w}(T) \) is closed and \( \sigma_0(T) = E^0(T) \) and that Browder’s theorem holds for \( T \) if \( \Delta(T) = \sigma(T) \setminus \sigma_w(T) = \pi^0(T) \), or equivalently \( \sigma_0(T) \subseteq E^0(T) \). In this paper we prove that if \( T \) and \( S \) are isoloid, obey property \( (Bw) \), and the generalized Weyl identity holds, then \( T \otimes S \) obeys property \( (Bw) \).

2. Preliminaries

For \( S \in \mathcal{B}(X) \) and a nonnegative integer \( n \) define \( S_{[n]} \) to be the restriction of \( S \) to \( \Re(S^n) \) viewed as a map from \( \Re(S^n) \) into \( \Re(S^n) \) (in particular, \( S_{[0]} = S \)). If for some integer \( n \) the range space \( \Re(S^n) \) is closed and \( S_{[n]} \) is an upper (a lower) semi-Fredholm operator, then \( S \) is called an upper (a lower) semi-\( B \)-Fredholm operator. In this case the index of \( S \) is defined as the index of the semi-\( B \)-Fredholm operator \( S_{[n]} \); see [4]. Moreover, if \( S_{[n]} \) is a Fredholm operator, then \( S \) is called a \( B \)-Fredholm operator. A semi-\( B \)-Fredholm operator is an upper or a lower semi-\( B \)-Fredholm operator. An operator \( S \) is said to be a \( B \)-Weyl operator [3, Definition 1.1] if it is a \( B \)-Fredholm operator of index zero. The \( B \)-Weyl spectrum \( \sigma_{BW}(S) \) of \( S \) is defined by \( \sigma_{BW}(S) = \{ \lambda \in \mathbb{C} : S - \lambda I \text{ is not a } B \)-Weyl operator \}. An operator \( S \in \mathcal{B}(X) \) is called Drazin invertible if it has a finite ascent and descent. The Drazin spectrum \( \sigma_D(S) \) of an operator \( S \) is defined by \( \sigma_D(S) = \{ \lambda \in \mathbb{C} : S - \lambda I \text{ is not Drazin invertible} \}. \) Define also the set \( LD(X) \) by \( LD(X) = \{ S \in \mathcal{B}(X) : a(S) < \infty \text{ and } \Re(T^{a(S)+1}) \text{ is closed} \} \) and \( \sigma_{LD}(S) = \{ \lambda \in \mathbb{C} : S - \lambda \notin LD(X) \}. \) Following [2], an operator \( S \in \mathcal{B}(X) \) is said to be left Drazin invertible if \( S \in LD(X) \). We say that \( \lambda \in \sigma_a(T) \) is a left pole of \( S \) if \( S - \lambda I \in LD(X) \), and that \( \lambda \in \sigma_a(S) \) is a left
pole of $S$ of finite rank if $\lambda$ is a left pole of $T$ and $\alpha(S - \lambda I) < \infty$. Let $\pi_2(S)$ denotes the set of all left poles of $S$ and let $\pi_2^0(S)$ denotes the set of all left poles of $S$ of finite rank. From [2, Theorem 2.8] it follows that if $S \in \mathcal{B}(\mathcal{X})$ is left Drazin invertible, then $S$ is an upper semi-B-Fredholm operator of index less than or equal to 0. Note that $\pi_2(S) = \sigma_2(S) \setminus \sigma_{LD}(S)$ and hence $\lambda \in \pi_2(S)$ if and only if $\lambda \notin \sigma_{LD}(S)$.

According to [13], $T \in \mathcal{B}(\mathcal{X})$ satisfies Property (Bw) if $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$. We say that $T$ satisfies Property (Bb) if $\sigma(T) \setminus \sigma_{BW}(T) = \pi^0(T)$ [14]. Property (Bw) implies Weyl’s theorem but converse is not true also Property (Bb) implies Property (Bw) but converse is not true [14]. Let $\mathcal{SBF}^{-\infty}(\mathcal{X})$ denote the class of all is upper B-Fredholm operators such that $\text{ind}(T) \leq 0$. The upper B-Weyl spectrum $\sigma_{\mathcal{SBF}^{-\infty}}(T)$ of $T$ is defined by

$$\sigma_{\mathcal{SBF}^{-\infty}}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \notin \mathcal{SBF}^{-\infty}(\mathcal{X}) \}.$$ 

The operator $T \in \mathcal{B}(\mathcal{X})$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at $\lambda_0$) if for every open disc $\mathbb{D}$ centred at $\lambda_0$, the only analytic function $f : \mathbb{D} \to \mathbb{C}$ which satisfies the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to have SVEP if $T$ has SVEP at every point $\lambda \in \mathbb{C}$. Obviously, every $T \in \mathcal{B}(\mathcal{X})$ has SVEP at the points of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic function, it easily follows that $T \in \mathcal{B}(\mathcal{X})$, as well as its dual $T^*$, has SVEP at every point of the boundary $\partial \sigma(T) = \partial \sigma(T^*)$ of the spectrum $\sigma(T)$. In particular, both $T$ and $T^*$ have SVEP at every isolated point of the spectrum, see [1]. Let $T \in \mathcal{B}(\mathcal{X})$ and let $s \in \mathbb{N}$ then $T$ has uniform descent for $n \geq s$ if $\Re(T) + \ker(T^n) = \Re(T) + \ker(T^s)$ for all $n \geq s$. If in addition if $\Re(T) + \ker(T^s)$ is closed then $T$ is said to have topological descent for $n \geq s$ [7]. Recall that an operator $T$ is said to be isloid if $\lambda \in \text{iso}(\sigma(T))$ implies $\lambda \in \sigma_p(T)$ and that $T \in \mathcal{B}(\mathcal{X})$ is said to be $a$-isloid if $\lambda \in \text{iso}_a(\sigma(T))$ implies $\lambda \in \sigma_p(T)$. It is well-known that if $T$ is $a$-isloid, then $T$ is isloid but not conversely.

**Lemma 2.1.** ([8]) Let $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$. If $T$ and $S$ are isloid, then $T \otimes S$ is isloid.

**Lemma 2.2.** ([11]) If $T$ and $S$ are isloid operators on infinite-dimensional space, then

$$E^0(T \otimes S) \subseteq E^0(T)E^0(S).$$

### 3. Property (Bw) and Tensor Product

The problem of transferring property (Bb), property (Sw), generalized Weyl’s theorem and Property (b) from operators $T$ and $S$ to their tensor product $T \otimes S$ was considered in [16], [15], [17] and [18]. The main objective
of this section is to study the transfer of property \((Bw)\) from a bounded linear operator \(T\) acting on a Banach space \(\mathcal{X}\) and a bounded linear operator \(S\) acting on a Banach space \(\mathcal{Y}\) to their tensor product \(T \otimes S\).

Let \(BF_+\) denote the set of upper semi B-Fredholm operators and let 
\[
\sigma_{SBF_+} = \{ \lambda \in \mathbb{C} : \lambda \notin BF_+(X) \}. 
\]
We write \(\sigma_{BW}(T) = \{ \lambda \in \mathbb{C} : \lambda \in \sigma_{SBF_+}, \text{ or } \text{ind}(T - \lambda) > 0 \}\).

The quasinilpotent part \(H_0(T - \lambda)\) and the analytic core \(K(T - \lambda)\) of \(T - \lambda I\) are defined by

\[
H_0(T - \lambda) := \{ x \in X : \lim_{n \to \infty} \| (T - \lambda)^n x \|^{\frac{1}{n}} = 0 \}. 
\]

and

\[
K(T - \lambda) = \{ x \in X : \text{there exists a sequence } \{ x_n \} \subset X \text{ and } \delta > 0 \text{ for which } x = x_0, (T - \lambda)x_{n+1} = x_n \text{ and } \| x_n \| \leq \delta^n \| x \| \text{ for all } n = 1, 2, \cdots \}. 
\]

We note that \(H_0(T - \lambda)\) and \(K(T - \lambda)\) are generally non-closed hyperinvariant subspaces of \(T - \lambda\) such that \((T - \lambda)^{-p}(0) \subset H_0(T - \lambda)\) for all \(p = 0, 1, \cdots\) and \((T - \lambda)K(T - \lambda) = K(T - \lambda)\). Recall that if \(\lambda \in \text{iso}(\sigma(T))\), then \(H_0(T - \lambda) = \chi_T(\{ \lambda \})\), where \(\chi_T(\{ \lambda \})\) is the glacial spectral subspace consisting of all \(x \in X\) for which there exists an analytic function \(f : \mathbb{C} \setminus \{ \lambda \} \to X\) that satisfies \((T - \mu)f(\mu) = x\) for all \(\mu \in \mathbb{C} \setminus \{ \lambda \}\).

**Theorem 3.1.** Let \(T \in B(\mathcal{X})\). If \(T\) obeys property \((Bb)\). Then the following statements are equivalent.

(i) \(T\) obeys property \((Bw)\);

(ii) \(\sigma_{BW}(T) \cap E^0(T) = \emptyset\);

(iii) \(E^0(T) = \pi^0(T)\).

**Proof.** (i)\(\Rightarrow\) (ii). Let \(\lambda \in \sigma(T) \setminus \sigma_{BW}(T)\). Since \(T\) satisfies \((Bb)\), \(\lambda \in \pi^0(T)\).

Thus \(\lambda \in \sigma(T) \setminus \sigma_b(T)\) and hence \(\sigma_b(T) \subset \sigma_{BW}(T)\). Since the reverse inclusion is always true, we have \(\sigma_b(T) = \sigma_{BW}(T)\).

(ii)\(\Rightarrow\) (i). Assume that \(\sigma_b(T) = \sigma_{BW}(T)\) and we will establish that \(\Delta^q(T) = \pi^q(T)\). Suppose \(\lambda \in \Delta^q(T)\). Then \(\lambda \in \sigma(T) \setminus \sigma_b(T)\). Hence \(\lambda \in \pi^0(T)\). Conversely suppose \(\lambda \in \pi^0(T)\). Since \(\sigma_{BW}(T) = \sigma_b(T), \lambda \in \Delta^q(T)\).

(ii)\(\Rightarrow\) (iii). Let \(\lambda \in \Delta^q(T)\). Since \(\sigma_{BW}(T) = \sigma_b(T), \lambda \in \sigma(T) \setminus \sigma_b(T)\), i.e., \(\lambda \in \pi^0(T)\) which implies that \(\lambda \in E^0(T)\). Thus \(\sigma_{BW}(T) \cup E^0(T) \supseteq \sigma(T)\). Since \(\sigma_{BW}(T) \cup E^0(T) \subseteq \sigma(T)\), always we must have \(\sigma_{BW}(T) \cup E^0(T) = \sigma(T)\).

(iii)\(\Rightarrow\) (ii). Suppose that \(E^0(T) = \pi^0(T)\). As \(T\) obeys property \((Bb)\) then \(\sigma(T) \setminus \sigma_{BW}(T) = \pi^0(T)\) and so \(\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)\). That is, \(E^0(T) \cap \pi^0(T) = \emptyset\). \(\blacksquare\)
The following result may be found in [16], we give the proof for completeness.

**Theorem 3.2.** Let \( T \in \mathcal{B}(\mathcal{X}) \). Then the following statements are equivalent.

(i) \( T \) satisfies property \((Bb)\);
(ii) \( \sigma_{BW}(T) = \sigma_b(T) \);
(iii) \( \sigma_{BW}(T) \cup E^0(T) = \sigma(T) \).

**Proof.** (i)\( \Rightarrow \) (ii) Since the opposite inclusion \( \sigma_{BW}(T) \subseteq \sigma_b(T) \) is always true, we have to show that \( \sigma_b(T) \subseteq \sigma_{BW}(T) \). Let \( \lambda \notin \sigma_{BW}(T) \). Since \( T \) satisfies property \((Bb)\), \( \lambda \in \pi^0(T) \). Hence, \( \lambda \notin \sigma_b(T) \).

(ii)\( \Rightarrow \) (i) Assume that \( \sigma_b(T) = \sigma_{BW}(T) \) and we will establish that \( \sigma(T) \setminus \sigma_{BW}(T) = \pi^0(T) \). Suppose \( \lambda \in \sigma(T) \setminus \sigma_{BW}(T) \). The hypothesis \( \sigma_b(T) = \sigma_{BW}(T) \) implies that \( \lambda \in \sigma(T) \setminus \sigma_b(T) \). Hence \( \lambda \in \pi^0(T) \) and so \( \sigma(T) \setminus \sigma_{BW}(T) \subseteq \pi^0(T) \). Conversely suppose \( \lambda \in \pi^0(T) \). Since \( \sigma_{BW}(T) = \sigma_b(T) \), \( \lambda \in \sigma(T) \setminus \sigma_{BW}(T) \).

(iii)\( \Rightarrow \) (ii) Let \( \lambda \in \sigma(T) \setminus \sigma_{BW}(T) \). Since \( \sigma_{BW}(T) = \sigma_b(T) \), \( \lambda \in \sigma(T) \setminus \sigma_b(T) \), that is, \( \lambda \in \pi^0(T) \) which implies that \( \lambda \notin E^0(T) \). Thus \( \sigma_{BW}(T) \cup E^0(T) \supseteq \sigma(T) \). Since \( \sigma_{BW}(T) \cup E^0(T) \subseteq \sigma(T) \), always we must have \( \sigma_{BW}(T) \cup E^0(T) = \sigma(T) \).

Example 3.3. Let \( T \) be a non-zero nilpotent operator and let \( S \) be a quasinilpotent which is not nilpotent. Then it easy to see that

\[
\sigma(T) = \{0\}, \sigma_{BW}(T) = \emptyset \text{ and } \sigma(S) = \sigma_{BW}(S) = \{0\}.
\]

Hence \( T \) and \( S \) satisfy property \((Bw)\). Since \( T \otimes S \) is nilpotent, we have \( \sigma_{BW}(T \otimes S) = \emptyset \). Hence \( T \otimes S \) satisfies property \((Bw)\). However,

\[
\sigma_{BW}(T)\sigma(S) \cup \sigma_{BW}(S)\sigma(T) = \{0\} \neq \sigma_{BW}(T \otimes S).
\]

Here \( 0 \in \text{iso} \sigma(T \otimes S) \) and \( 0 \) is a pole. Moreover, we note that \( T, S \) and \( T \otimes S \) satisfies generalized \( a \)-Browder’s theorem

**Lemma 3.4.** [16, Lemma 3.1] Let \( A \in \mathcal{B}(\mathcal{X}) \) and \( B \in \mathcal{B}(\mathcal{Y}) \). Then

\[
\sigma_{BW}(A \otimes B) \subseteq \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A) \subseteq \sigma_{w}(A)\sigma(B) \cup \sigma_{w}(B)\sigma(A) \subseteq \sigma_b(A)\sigma(B) \cup \sigma_b(B)\sigma(A) = \sigma_b(A \otimes B).
\]
Lemma 3.5. [16, Lemma 3.2] Let $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$ obey property (Bb). Then $T \otimes S$ obeys property (Bb) if and only if $\sigma_{BW}(T \otimes S) = \sigma_{BW}(T) \sigma(S) \cup \sigma_{BW}(S) \sigma(T)$.

In [11], Kubrusly and Duggal studied the stability of Weyl’s theorem under tensor product in the infinite dimensional space setting. Rashid [15] studied the stability of generalized Weyl’s theorem under tensor product in the infinite dimensional space setting. Rashid [15] studied the stability of Weyl’s theorem under tensor product in the infinite dimensional space setting. Let

\[ \sigma_{PF}(T) = \{ \lambda \in \sigma_{p}(T) : \alpha(T - \lambda) < \infty \} = \{ \lambda \in \mathbb{C} : 0 < \alpha(T - \lambda) < \infty \}. \]

**Definition 1.** An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be finitely isoloid if all the isolated points of its spectrum are eigenvalues of finite multiplicity i.e. $\text{iso}\sigma(T) \subseteq E_{0}(T)$. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be finitely polaroid (resp., polaroid) if all the isolated points of its spectrum are poles of finite rank i.e. $\text{iso}\sigma(T) \subseteq \pi_{0}(T)$, (resp., $\text{iso}\sigma(T) \subseteq \pi(T)$).

**Theorem 3.6.** Let $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$ such that $T$ and $S$ are finite-isoloid and $0 \notin \text{iso}\sigma(T \otimes S)$. If property (Bw) holds for $T$ and $S$, then the following statements are equivalent.

(a) $T \otimes S$ satisfies property (Bw).

(b) $\sigma_{BW}(T \otimes S) = \sigma_{BW}(T) \sigma(S) \cup \sigma(T) \sigma_{BW}(S)$.

**Proof.** (a) $\implies$ (b): Assume that $T \otimes S$ satisfies property (Bw). Let

\[ \lambda \in E_{0}(T \otimes S) = \sigma(T) \sigma_{BW}(S) \cup \sigma_{BW}(T) \sigma(S). \]

Since $0 \notin \text{iso}\sigma(T \otimes S)$, then $\lambda \neq 0$. Hence $\lambda \in \text{iso}(T \otimes S) = \text{iso}(T) \text{iso}(S)$. That is, $\lambda = \mu \nu$ with $\mu \in \text{iso}(T)$ and $\nu \in \text{iso}(S)$. Since $T$ and $S$ are finite-isoloid, then $\mu \in E_{0}(T) = \sigma(T) \backslash \sigma_{BW}(T)$ and $\nu \in E_{0}(S) = \sigma(S) \backslash \sigma_{BW}(S)$, and hence $\lambda = \mu \nu \notin \sigma_{BW}(T) \sigma(S) \cup \sigma(T) \sigma_{BW}(S)$. Thus

\[ \sigma_{BW}(T) \sigma(S) \cup \sigma(T) \sigma_{BW}(S) \subseteq \sigma_{BW}(T \otimes S). \]

Conversely, let $\lambda \in \sigma(T \otimes S) \backslash (\sigma_{BW}(T) \sigma(S) \cup \sigma(T) \sigma_{BW}(S))$, then for $\lambda = \mu \nu$ we have that $\mu \in \sigma(T)$ and $\nu \in \sigma(S)$, hence $\mu \in E_{0}(T)$ and $\nu \in E_{0}(S)$. Thus $\lambda = \mu \nu \in E_{0}(T \otimes S) = \sigma(T \otimes S) \backslash \sigma_{BW}(T \otimes S)$. Therefore,

\[ \sigma_{BW}(T \otimes S) = \sigma_{BW}(T) \sigma(S) \cup \sigma(T) \sigma_{BW}(S). \]

(b) $\implies$ (a): Since $T$ and $S$ obey property (Bw), then

\[ \sigma(T) \backslash \sigma_{BW}(T) = E_{0}(T) \text{ and } \sigma(S) \backslash \sigma_{BW}(S) = E_{0}(S). \]

Assume that

\[ \sigma_{BW}(T \otimes S) = \sigma_{BW}(T) \sigma(S) \cup \sigma(T) \sigma_{BW}(S). \]
Let $\lambda \in E^0(T \otimes S)$. Then there exists $\mu \in \text{iso}(T)$ and $\nu \in \text{iso}(S)$ such that $\lambda = \mu \nu$. Since $T$ and $S$ are finite-isoloid, then $\mu \in E^0(T)$ and $\nu \in E^0(S)$. Hence $\mu \notin \sigma_{BW}(T)$ and $\nu \notin \sigma_{BW}(S)$. Then $\lambda \notin \sigma_{BW}(T \otimes S)$. Thus

$$E^0(T \otimes S) \subseteq \sigma(T \otimes S) \setminus \sigma_{BW}(T \otimes S).$$

Conversely, assume that $\lambda \notin \sigma(T \otimes S) \setminus \sigma_{BW}(T \otimes S)$, then there exists $\mu \in \sigma(T) \setminus \sigma_{BW}(T)$ and $\nu \in \sigma(S) \setminus \sigma_{BW}(S)$ such that $\lambda = \mu \nu$. Since

$$T \otimes S = (T - \mu) \otimes S + \mu I \otimes (S - \nu),$$

then we can see that $\lambda \in E^0(T \otimes S)$. Hence $T \otimes S$ obeys property $(Bw)$. ■

**Lemma 3.7.** If $T \in B(\mathcal{X})$ and $S \in B(\mathcal{Y})$ are finitely polaroid, then so is $T \otimes S$.

**Proof.** If $\text{iso}(T) = \text{iso}(S) = \emptyset$, then $\text{iso}(T \otimes S) = \emptyset$. Observe also that if either of $\text{iso}(T)$ or $\text{iso}(S)$ is the empty set, say $\text{iso}(T) = \emptyset$, then it follows from [10, Proposition 3] that $\text{iso}(T \otimes S) \subseteq \{0\}$ and $0 \in \text{iso}(S)$. But then $0 \in \pi^0(S)$, which implies that $0 \in \pi^0(T \otimes S)$. Let $\lambda \in \text{iso}(T \otimes S)$ be such that $\lambda = \mu \nu$, $\mu \in \text{iso}(T)$ and $\nu \in \text{iso}(S)$. Then $\mu \in \pi^0(T)$ and $\nu \in \pi^0(S)$. Hence, we have $\lambda \in \pi^0(T \otimes S)$. ■

$T \in B(\mathcal{X})$ polaroid implies $T^*$ polaroid. It is known that if $T$ or $T^*$ has SVEP and $T$ is polaroid, then $T$ and $T^*$ satisfy generalized Weyl’s theorem. Note as well known is the fact, [13, Theorem 2.15] that if $T$ or $T^*$ has SVEP and $T$ is finitely polaroid, then $T$ obeys property $(Bw)$. The following theorem is the tensor product analogue of this result.

**Theorem 3.8.** Suppose that $T \in B(\mathcal{X})$ and $S \in B(\mathcal{Y})$ are finitely polaroid. If $T$ and $S$ have SVEP (or $T^*$ and $S^*$ have SVEP), then $T \otimes S$ satisfies property $(Bw)$.

**Proof.** The hypotheses by [13, Theorem 2.15] imply that $T$ and $S$ obey property $(Bw)$ and it then follows from Theorem 2.5 of [13] that $T$ and $S$ satisfy generalized Browder’s theorem and $\pi(T) = E^0(T)$ and $\pi(S) = E^0(S)$. Hence $T \otimes S$ satisfies generalized Browder’s theorem. Thus generalized Browder’s theorem transfer from $T$ and $S$ to $T \otimes S$. Hence

$$\sigma_{BW}(T \otimes S) = \sigma(T)\sigma_{BW}(S) \cup \sigma(S)\sigma_{BW}(T).$$

Evidently, $T \otimes S$ is finitely polaroid (3.7); combining with $T \otimes S$ satisfies generalized Browder’s theorem, it follows that

$$\sigma(T \otimes S) \setminus \sigma_{BW}(T \otimes S) = \pi(T \otimes S) = E^0(T \otimes S),$$

that is, $T \otimes S$ obeys property $(Bw)$. ■
Example 3.9. Let $T, S \in \mathcal{B}(\ell^2)$ be defined by

$$T = 3 \oplus I = \begin{pmatrix} 3 \\ I \end{pmatrix} \text{ and } S = \text{diag}\{1,2\} \oplus 6I = \begin{pmatrix} 1 \\ 2 \\ 6I \end{pmatrix}$$

so that

$$T \otimes S = 3S \oplus I \otimes S = \begin{pmatrix} 3S \\ I \otimes S \end{pmatrix} = \begin{pmatrix} 3S \\ S \\ \cdots \end{pmatrix}.$$ 

Hence

$$\sigma(T) = \text{iso}\sigma(T) = \{1,3\}, \quad \sigma(S) = \text{iso}\sigma(S) = \{1,2,6\},$$

$$E^0(T) = \{3\}, \quad \sigma_{BW}(T) = \sigma_w(T) = \{1\}, \quad E^0(S) = \{1,2\}, \quad \sigma_{BW}(S) = \sigma_w(S) = \{6\},$$

$$\sigma(T \otimes S) = \text{iso}(T \otimes S) = \{1,2,3,6,18\} \text{ and } E^0(T \otimes S) = \{3\}.$$

Since $T, S$ and $T \otimes S$ are self-adjoint, they all satisfy property $(Bw)$. We now give an example to show that property $(Bw)$ does not transfer from operators $T$ and $S$ to the tensor product $T \otimes S$.

Example 3.10. Let $R$ be the backward shift operator on $\ell^2$,

$$R : \ell^2 \to \ell^2 \text{ defined by } R(x_1, x_2, \cdots) = (x_2, x_3, \cdots).$$

It is known that $R$ satisfies property $(Bw)$. In fact $\text{ind}(R - \lambda) = 1$ for $|\lambda| < 1$ and so

$$\sigma(R) = \sigma_w(R) = \sigma_{BW}(R) = \mathbb{D}, \quad \text{iso}(R) = E^0(R) = \emptyset.$$ 

Let $P$ be a finite rank projection on $\ell^2$. Then $P$ satisfies property $(Bw)$ and

$$\sigma(P) = \{0,1\}, \quad \sigma_w(P) = \sigma_{BW}(P) = \{0\}.$$ 

Consider operators

$$T = P \oplus \left( \frac{1}{2}R - 1 \right) \text{ and } S = (-P) \oplus \left( \frac{1}{2}R^* + 1 \right)$$

acting on the Hilbert space $\mathcal{H} = \ell^2 \oplus \ell^2$. We have

$$\sigma(T) = \{0,1\} \cup \left( \frac{1}{2}\mathbb{D} - 1 \right) \quad \sigma(S) = \{0,-1\} \cup \left( \frac{1}{2}\mathbb{D} + 1 \right)$$

$$\sigma_w(T) = \sigma_{BW}(T) = \{0\} \cup \left( \frac{1}{2}\mathbb{D} - 1 \right) \quad \sigma_w(S) = \sigma_{BW}(S) = \{0\} \cup \left( \frac{1}{2}\mathbb{D} + 1 \right),$$

where $\mathbb{D}$ is the closed unit disc in the complex plane $\mathbb{C}$. So, $T$ and $S^*$ have SVEP. Note that $T$ and $S$ both satisfy property $(Bw)$. In particular $T$
and $S$ satisfy generalized Browder’s theorem. Furthermore, $1 \in \sigma(T \otimes S) \setminus \sigma_{BW}(T \otimes S)$. However, since
\[
\sigma(T \otimes S) = \left\{ \{0, 1\} \cup \left\{ \frac{1}{2} \mathbb{D} - 1 \right\} \right\} \cap \left\{ \{0, -1\} \cup \left\{ \frac{1}{2} \mathbb{D} + 1 \right\} \right\}.
\]
1 $\in \text{acc} \sigma(T \otimes S) \implies 1 \in \sigma_b(T \otimes S)$.

Then $T \otimes S$ does not satisfy Browder’s theorem, and hence property $(Bw)$.

4. Perturbations

Let $[T, S] = TS - ST$ denote the commutator of the operators $T$ and $S$. If $Q_1 \in \mathcal{B}(\mathcal{X})$ and $Q_2 \in \mathcal{B}(\mathcal{Y})$ are quasinilpotent operators such that $[Q_1, T] = [Q_2, S] = 0$ for some operators $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$, then
\[
(T + Q_1) \otimes (S + Q_2) = (T \otimes S) + Q,
\]
where $Q = Q_1 \otimes S + T \otimes Q_2 + Q_1 \otimes Q_2 \in \mathcal{B}(\mathcal{X} \otimes \mathcal{Y})$ is quasinilpotent operator.

**Theorem 4.1.** Let $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$ having SVEP and let $Q_1 \in \mathcal{B}(\mathcal{X})$ and $Q_2 \in \mathcal{B}(\mathcal{Y})$ be quasinilpotent operators such that $[Q_1, T] = [Q_2, S] = 0$. If $T \otimes S$ is finitely isoloid, then $T \otimes S$ satisfies property $(Bw)$ implies $(T + Q_1) \otimes (S + Q_2)$ satisfies property $(Sw)$. 

**Proof.** Recall that $\sigma((T + Q_1) \otimes (S + Q_2)) = \sigma(T \otimes S), \sigma_{BW}((T + Q_1) \otimes (S + Q_2)) = \sigma_{BW}(T \otimes S)$ and that the perturbation of an operator by a commuting quasinilpotent has SVEP if and only if the operator has SVEP. If $T \otimes S$ satisfies property $(Bw)$, then
\[
E^0(T \otimes S) = \sigma(T \otimes S) \setminus \sigma_{BW}(T \otimes S) = \sigma((T + Q_1) \otimes (S + Q_2)) \setminus \sigma_{BW}((T + Q_1) \otimes (S + Q_2)).
\]
We prove that $E^0(T \otimes S) = E^0((T + Q_1) \otimes (S + Q_2))$. Observe that if $\lambda \in \text{iso}(T \otimes S)$, then $T^* \otimes S^*$ has SVEP at $\lambda$, equivalently, $(T^* + Q_1^*) \otimes (S^* + Q_2^*)$ has SVEP at $\lambda$. Let $\lambda \in E^0(T \otimes S)$; then $\lambda \in \sigma((T + Q_1) \otimes (S + Q_2)) \setminus \sigma_{BW}((T + Q_1) \otimes (S + Q_2))$. Since $(T + Q_1, S + Q_2)^*$ has SVEP at $\lambda$, it follows that $\lambda \notin \sigma_{sw}(T + Q_1) \otimes (S + Q_2)$ and $\lambda \in \text{iso}(T + Q_1) \otimes (S + Q_2)$. Thus $\lambda \in E^0((T + Q_1) \otimes (S + Q_2))$. Hence $E^0(T \otimes S) \subseteq E^0((T + Q_1) \otimes (S + Q_2))$. Conversely, if $\lambda \in E^0((T + Q_1) \otimes (S + Q_2))$, then $\lambda \in \text{iso}(T \otimes S)$, and this, since $T \otimes S$ is finitely isoloid, implies that $\lambda \in E^0(T \otimes S)$. Hence $E^0((T + Q_1) \otimes (S + Q_2)) \subseteq E^0(T \otimes S)$. 

From [5], we recall that an operator $R \in \mathcal{B}(\mathcal{X})$ is said to be Riesz if $R - \lambda I$ is Fredholm for every non-zero complex number $\lambda$. For a bounded operator $T$ on $\mathcal{X}$, we denote by $E_{0f}(T)$ the set of isolated points $\lambda$ of $\sigma(T)$ such that $\ker(T - \lambda I)$ is finite-dimensional. Evidently, $E_0(T) \subseteq E_0f(T)$. 


**Lemma 4.2.** Let $T$ be a bounded operator on $X$. If $R$ is a Riesz operator that commutes with $T$, then

$$E_0(T + R) \cap \sigma(T) \subseteq \text{iso}\sigma(T).$$

**Proof.** Clearly,

$$E_0(T + R) \cap \sigma(T) \subseteq E_{0f}(T + R) \cap \sigma(T),$$

and by Lemma 2.3 of [12] the last set contained in $\text{iso}\sigma(T)$. □

Now we consider the perturbations by commuting Riesz operators. Let $T, R \in \mathcal{B}(X)$ be such that $R$ is Riesz and $[T, R] = 0$; the tensor product $T \otimes R$ is not a Riesz operator (the Fredholm spectrum $\sigma_F(T \otimes R) = \sigma(T)\sigma_F(R) \cup \sigma_F(T)\sigma(R) = \{0\}$ for a particular choice of $T$ only). However, $\sigma_w$ (also, $\sigma_b$) is stable under perturbation by commuting Riesz operators [19], and so $T$ satisfies Browder’s theorem if and only if $T + R$ satisfies Browder’s theorem. Thus, if $\sigma(T) = \sigma(T + R)$ for a certain choice of operators $T, R \in \mathcal{B}(X)$ (such that $R$ is Riesz and $[T, R] = 0$), then

$$\pi^0(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T + R) \setminus \sigma_w(T + R) = \pi^0(T + R),$$

where $\pi^0(T)$ is the set of $\lambda \in \text{iso}\sigma(T)$ which are finite rank poles of the resolvent of $T$. If we now suppose additionally that $T$ satisfies property $(Bw)$, then

$$(4.1) \quad E^0(T) = \sigma(T) \setminus \sigma_{BW}(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T + R) \setminus \sigma_w(T + R),$$

and a necessary and sufficient condition for $T + R$ to satisfy property $(Bw)$ is that $E^0(T + R) = E^0(T)$. One such condition, namely $T$ is finitely isoloid.

**Proposition 4.3.** Let $T, R \in \mathcal{B}(X)$, where $R$ is Riesz, $[T, R] = 0$ and $T$ is finitely isoloid. Then $T$ satisfies property $(Bw)$ implies $T + R$ satisfies property $(Bw)$.

**Proof.** Observe that if $T$ obeys property $(Bw)$, then identity $(4.1)$ holds. Let $\lambda \in E^0(T)$. Then it follows from Lemma 4.2 that $\lambda \in E^0(T) \cap \sigma(T) = E^0(T + R - R) \subseteq \text{iso}(T + R)$ and so $T^* + R^*$ has SVEP at $\lambda$. Since $\lambda \in \sigma(T + R) \setminus \sigma_w(T + R)$, $T^* + R^*$ has SVEP at $\lambda$ implies $T + R - \lambda$ is Fredholm of index 0 and so $\lambda \in E^0(T + R)$. Thus $E^0(T) \subseteq E^0(T + R)$. Now let $\lambda \in E^0(T + R)$. Then $\lambda \in E^0(T + R) \cap \sigma(T + R) = E^0(T + R) \cap \sigma(T) \subseteq \text{iso}(T)$, which by the finite isoloid property of $T$ implies $\lambda \in E^0(T)$. Hence $E^0(T + R) \subseteq E^0(T)$. □

**Theorem 4.4.** Let $T \in \mathcal{B}(X)$ and $S \in \mathcal{B}(X)$ be finitely isoloid operators which satisfy property $(Bw)$. If $R_1 \in \mathcal{B}(X)$ and $R_2 \in \mathcal{B}(Y)$ are Riesz operators such that $[T, R_1] = [S, R_2] = 0$, $\sigma(T + R_1) = \sigma(T)$ and $\sigma(S + R_2) = \sigma(S)$, then $T \otimes S$ satisfies property $(Bw)$ implies $(T + R_1) \otimes (S + R_2)$ satisfies
property \((Bw)\) if and only if Browder’s theorem transforms from \(T + R_1\) and \(S + R_2\) to their tensor product.

**Proof.** The hypotheses imply (by Proposition 4.3) that both \(T + R_1\) and \(S + R_2\) satisfy property \((Bw)\). Suppose that \(T \otimes S\) satisfies property \((Bw)\). Then \(\sigma(T \otimes B) \setminus \sigma_{BW}(T \otimes S) = E^0(T \otimes S)\). Evidently \(T \otimes S\) satisfies Browder’s theorem, and so the hypothesis \(T\) and \(S\) satisfy property \((Bw)\) implies that Browder’s theorem transfers from \(T\) and \(S\) to \(T \otimes S\). Furthermore, since \(\sigma(T + R_1) = \sigma(T)\), \(\sigma(S + R_2) = \sigma(S)\), and \(\sigma_w\) is stable under perturbations by commuting Riesz operators,

\[
\sigma_{BW}(T \otimes S) = \sigma_w(T \otimes S) = \sigma(T)\sigma_w(S) \cup \sigma_w(T)\sigma(S) = \sigma(T + R_1)\sigma_w(S + R_2) \cup \sigma_w(T + R_1)\sigma(S + R_2) = \sigma(T + R_1)\sigma_{BW}(S + R_2) \cup \sigma_{BW}(T + R_1)\sigma(S + R_2)
\]

Suppose now that Browder’s theorem transfers from \(T + R_1\) and \(S + R_2\) to \((T + R_1) \otimes (S + R_2)\). Then

\[
\sigma_w(T \otimes S) = \sigma_w((T + R_1) \otimes (S + R_2))
\]

and

\[
E^0(T \otimes S) = \sigma((T + R_1) \otimes (S + R_2)) \setminus \sigma_w((T + R_1) \otimes (S + R_2)).
\]

Let \(\lambda \in E^0(T \otimes S)\). Then \(\lambda \neq 0\), and hence there exist \(\mu \in \sigma(T + R_1) \setminus \sigma_w(T + R_1)\) and \(\nu \in \sigma(S + R_2) \setminus \sigma_w(S + R_2)\) such that \(\lambda = \mu \nu\). As observed above, both \(T + R_1\) and \(S + R_2\) satisfy property \((Bw)\); hence \(\mu \in E^0(S + R_2)\) and \(\nu \in E^0(S + R_2)\). This, since \(\lambda \in \sigma(T \otimes S) = \sigma((T + R_1) \otimes (S + R_2))\), implies \(\lambda \in E^0((T + R_1) \otimes (S + R_2))\). Conversely, if \(\lambda \in E^0((T + R_1) \otimes (S + R_2))\), then \(\lambda \neq 0\) and there exist \(\mu \in E^0(T + R_1) \subseteq \text{iso}(T)\) and \(\nu \in E^0(S + R_2) \subseteq \text{iso}(S)\) such that \(\lambda = \mu \nu\). Recall that \(E^0((T + R_1) \otimes (S + R_2)) \subseteq E^0(T + R_1)E^0(S + R_2)\). Since \(T\) and \(S\) are finite isoloid, \(\mu \in E^0(T)\) and \(\nu \in E^0(S)\). Hence, since \(\sigma((T + R_1) \otimes (S + R_2)) = \sigma(T \otimes S)\), \(\lambda = \mu \nu \in E^0(T \otimes S)\). To complete the proof, we observe that the implication of the statement of the theorem holds, then (necessarily) \((T + R_1) \otimes (S + R_2)\) satisfies Browder’s theorem. This, since \(T + R_1\) and \(S + R_2\) satisfy Browder’s theorem, implies Browder’s theorem transfers from \(T + R_1\) and \(S + R_2\) to \((T + R_1) \otimes (S + R_2)\)

**References**


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