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PASSAGE OF PROPERTY (Bw) FROM TWO OPERATORS TO THEIR TENSOR PRODUCT

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Abstract. A Banach space operator satisfies property (Bw) if the complement of its B-Weyl spectrum in its the spectrum is the set of finite multiplicity isolated eigenvalues of the operator. Property (Bw) does not transfer from operators T and S to their tensor product $T \otimes S$. We give necessary and /or sufficient conditions ensuring the passage of property (Bw) from T and S to $T \otimes S$. Perturbations by Riesz operators are considered.

1. Introduction

Given Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{X} \otimes \mathcal{Y}$ denote the completion (in some reasonable uniform cross norm) of the tensor product of $\mathcal{X}$ and $\mathcal{Y}$. For Banach space operators $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$, let $T \otimes S \in \mathcal{B}(\mathcal{X} \otimes \mathcal{Y})$ denote the tensor product of T and S.

For a bounded linear operator $S \in \mathcal{B}(\mathcal{X})$, let $\ker(S), \Re(S), \sigma(S)$ and $\sigma_a(S)$ denote, respectively, the kernel, the range, the spectrum and the approximate point spectrum of S and if $G \subseteq \mathbb{C}$, then $\operatorname{iso} G$ denote the isolated points of G. Let $\alpha(S)$ and $\beta(S)$ denote the nullity and the deficiency of S, defined by $\alpha(S) = \dim\ker(S)$ and $\beta(S) = \operatorname{co}\dim \Re(S)$.
If the range $\Re(S)$ of S is closed and $\alpha(S) < \infty$ (resp. $\beta(S) < \infty$), then S is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator. If $S \in \mathcal{B}(\mathcal{X})$ is either upper or lower semi-Fredholm, then S is called a semi-Fredholm operator, and $\operatorname{ind}(S)$, the index of S, is then defined by $\operatorname{ind}(S) = \alpha(S) - \beta(S)$. If both $\alpha(S)$ and $\beta(S)$ are finite, then S is a Fredholm operator. The ascent, denoted $a(S)$, and the descent, denoted $d(S)$, of S are given by $a(S) = \inf\left\{n \in \mathbb{N} : \ker(S^n) = \ker(S^{n+1}\right\}$, $d(S) = \inf\left\{n \in \mathbb{N} : \Re(S^n) = \Re(S^{n+1}\right\}$ (where the infimum is taken over the set of non-negative integers); if no such integer n exists, then $a(S) = \infty$, respectively $d(S) = \infty$.)
Let $T \in \mathcal{B}(\mathcal{X})$. Define

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(\lambda - T) \neq \{0\}\};$$

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : \ker(\lambda - T) \neq \{0\}, \overline{\Re(\lambda - T)} = \mathcal{X} \text{ but } \Re(\lambda - T) \neq \mathcal{X}\};$$

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : \ker(\lambda - T) = \{0\} \text{ but } \overline{\Re(\lambda - T)} \neq \mathcal{X}\}.$$

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$\sigma_p(T), \sigma_c(T)$ and $\sigma_r(T)$ are called respectively the point spectrum, the continuous spectrum and the residual spectrum of T. Clearly, $\sigma_p(T), \sigma_c(T)$ and $\sigma_r(T)$ are disjoint and $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$. Let $\sigma_w(T) = \{\lambda \in \sigma(T) : \lambda - T \text{ is not a Fredholm operator of index } 0\}$ be the Weyl spectrum of T, which is a subset of the whole spectrum $\sigma(T)$. The set $\sigma_0(T) = \{\lambda \in \sigma_p(T) : \Re(\lambda - T) \text{ is closed and } \alpha(\lambda - T) = \alpha(\overline{\lambda} - T^*) < \infty\}$ is precisely the complement of the Weyl spectrum $\sigma_w(T)$ in the whole spectrum $\sigma(T)$. Hence

$$\sigma_w(T) = \sigma(T) \setminus \sigma_0(T),$$

and so $\{\sigma_w(T), \sigma_0(T)\}$ forms another partition of the spectrum of $\sigma(T)$. Set $\sigma_{PF}(T) = \{\lambda \in \sigma_p(T) : \alpha(\lambda - T) < \infty\}$; the set of all eigenvalues of finite multiplicity, so that $\sigma_0(T) \subseteq \sigma_{PF}(T)$ and $\sigma_r(T) \cup \sigma_c(T) \cup (\sigma_p(T) \setminus \sigma_{PF}(T)) \subseteq \sigma_w(T)$. Set

$$E^0(T) = \operatorname{iso}\sigma(T) \cap \sigma_{PF}(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda - T) < \infty\}.$$

According to Coburn [6], *Weyl's theorem* holds for T if $\Delta(T) = \sigma(T) \setminus \sigma_w(T) = E^0(T)$, or equivalently $\sigma_0(T) = E^0(T)$ and that *Browder's theorem* holds for T if $\Delta(T) = \sigma(T) \setminus \sigma_w(T) = \pi^0(T)$, or equivalently $\sigma_0(T) \subseteq E^0(T)$. In this paper we prove that if T and S are isoloid, obey property (Bw), and the generalized Weyl identity holds, then $T \otimes S$ obeys property (Bw).

2. Preliminaries

For $S \in \mathcal{B}(\mathcal{X})$ and a nonnegative integer n define $S_{[n]}$ to be the restriction of S to $\Re(S^n)$ viewed as a map from $\Re(S^n)$ into $\Re(S^n)$ (in particular, $S_{[0]} = S$). If for some integer n the range space $\Re(S^n)$ is closed and $S_{[n]}$ is an upper (a lower) semi-Fredholm operator, then S is called an *upper* (a *lower*) *semi-B-Fredholm* operator. In this case the index of S is defined as the index of the semi-B-Fredholm operator $S_{[n]}$, see [4] Moreover, if $S_{[n]}$ is a Fredholm operator, then S is called a *B-Fredholm* operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator S is said to be a *B-Weyl operator* [3, Definition 1.1] if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(S)$ of S is defined by $\sigma_{BW}(S) = \{\lambda \in \mathbb{C} : S - \lambda I \quad \text{is not a B-Weyl operator}\}$. An operator $S \in \mathcal{B}(\mathcal{X})$ is called *Drazin invertible* if it has a finite ascent and descent. The *Drazin spectrum* $\sigma_D(S)$ of an operator S is defined by $\sigma_D(S) = \{\lambda \in \mathbb{C} : S - \lambda I \quad \text{is not Drazin invertible}\}$. Define also the set $LD(\mathcal{X})$ by $LD(\mathcal{X}) = \left\{S \in \mathcal{B}(\mathcal{X}) : a(S) < \infty \quad \text{and}\, \Re(T^{a(S)+1}) \text{ is closed}\right\}$ and $\sigma_{LD}(S) = \{\lambda \in \mathbb{C} : S - \lambda \notin LD(\mathcal{X})\}$. Following [2], an operator $S \in \mathcal{B}(\mathcal{X})$ is said to be left Drazin invertible if $S \in LD(\mathcal{X})$. We say that $\lambda \in \sigma_a(T)$ is a left pole of S if $S - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(S)$ is a left

pole of S of finite rank if λ is a left pole of T and $\alpha(S-\lambda I)<\infty$. Let $\pi_a(S)$ denotes the set of all left poles of S and let $\pi_a^0(S)$ denotes the set of all left poles of S of finite rank. From [2, Theorem 2.8] it follows that if $S\in\mathcal{B}(\mathcal{X})$ is left Drazin invertible, then S is an upper semi-B-Fredholm operator of index less than or equal to 0. Note that $\pi_a(S)=\sigma_a(S)\setminus\sigma_{LD}(S)$ and hence $\lambda\in\pi_a(S)$ if and only if $\lambda\notin\sigma_{LD}(S)$.
According to [13], $T\in\mathcal{B}(\mathcal{X})$ satisfies Property (Bw) if $\sigma(T)\setminus\sigma_{BW}(T)=E^0(T)$. We say that T satisfies Property (Bb) if $\sigma(T)\setminus\sigma_{BW}(T)=\pi^0(T)$ [14]. Property (Bw) implies Weyl's theorem but converse is not true also Property (Bw) implies Property (Bb) but converse is not true [14]. Let $\mathcal{SBF}_+^-(\mathcal{X})$ denote the class of all is upper B-Fredholm operators such that $\operatorname{ind}(T)\leq 0$. The upper B-Weyl spectrum $\sigma_{SBF_+^-}(T)$ of T is defined by

$$\sigma_{SBF_+^-}(T)=\{\lambda\in\mathbb{C}: T-\lambda\notin\mathcal{SBF}_+^-(\mathcal{X})\}.$$

The operator $T\in\mathcal{B}(\mathcal{X})$ is said to have the *single valued extension property* at $\lambda_0\in\mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc $\mathbb{D}$ centred at λ_0, the only analytic function $f:\mathbb{D}\to$ which satisfies the equation $(T-\lambda)f(\lambda)=0$ for all $\lambda\in\mathbb{D}$ is the function $f\equiv 0$. An operator $T\in\mathcal{B}(\mathcal{X})$ is said to have SVEP if T has SVEP at every point $\lambda\in\mathbb{C}$. Obviously, every $T\in\mathcal{B}(\mathcal{X})$ has SVEP at the points of the resolvent $\rho(T):=\mathbb{C}\setminus\sigma(T)$. Moreover, from the identity theorem for analytic function, it easily follows that $T\in\mathcal{B}(\mathcal{X})$, as well as its dual T^*, has SVEP at every point of the boundary $\partial\sigma(T)=\partial\sigma(T^*)$ of the spectrum $\sigma(T)$. In particular, both T and T^* have SVEP at every isolated point of the spectrum, see [1]. Let $T\in\mathcal{B}(\mathcal{X})$ and let $s\in\mathbb{N}$ then T has uniform descent for $n\geq s$ if $\Re(T)+\ker(T^n)=\Re(T)+\ker(T^s)$ for all $n\geq s$. If in addition if $\Re(T)+\ker(T^s)$ is closed then T is said to have topological descent for $n\geq s$ [7]. Recall that an operator T is said to be isoloid if $\lambda\in\operatorname{iso}\sigma(T)$ implies $\lambda\in\sigma_p(T)$ and that $T\in\mathcal{B}(\mathcal{X})$ is said to be a-isoloid if $\lambda\in\operatorname{iso}\sigma_a(T)$ implies $\lambda\in\sigma_p(T)$. It is well-known that if T is a-isoloid, then T is isoloid but not conversely.

Lemma 2.1. *([8]) Let $T\in\mathcal{B}(\mathcal{X})$ and $S\in\mathcal{B}(\mathcal{Y})$. If T and S are isoloid, then $T\otimes S$ is isoloid.*

Lemma 2.2. ([11]) *If T and S are isoloid operators on infinite-dimensional space, then*

$$E^0(T\otimes S)\subseteq E^0(T)E^0(S).$$

3. Property (Bw) and tensor product

The problem of transferring property (Bb), property (Sw), generalized Weyl's theorem and Property (b) from operators T and S to their tensor product $T\otimes S$ was considered in [16], [15], [17] and [18]. The main objective of this section is to study the transfer of property (Bw) from a bounded linear operator T acting on a Banach space $\mathcal{X}$ and a bounded linear operator S acting on a Banach space $\mathcal{Y}$ to their tensor product $T \otimes S$.

Let BF_+ denote the set of upper semi B-Fredholm operators and let $\sigma_{SBF_+} = \{\lambda \in \mathbb{C} : \lambda \notin BF_+(X)\}$. We write $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SBF_+}$ or $\text{ind}(T-\lambda) > 0\}$.

The quasinilpotent part $H_0(T-\lambda)$ and the analytic core $K(T-\lambda I)$ of $T-\lambda I$ are defined by

$$H_0(T-\lambda) := \{x \in X : \lim_{n \longrightarrow \infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\}.$$

and

$$\begin{aligned} &K(T-\lambda) = \{x \in X : \text{there exists a sequence } \{x_n\} \subset X \text{ and } \quad \delta > 0 \\ &\text{for which} \quad x = x_0, (T-\lambda)x_{n+1} = x_n \text{and} \quad \|x_n\| \leq \delta^n \|x\| \text{for all} \quad n = 1, 2, \cdots\}. \end{aligned}$$

We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are generally non-closed hyper-invariant subspaces of $T-\lambda$ such that $(T-\lambda)^{-p}(0) \subseteq H_0(T-\lambda)$ for all $p = 0, 1, \cdots$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$. Recall that if $\lambda \in iso(\sigma(T))$, then $H_0(T-\lambda) = \chi_T(\{\lambda\})$, where $\chi_T(\{\lambda\})$ is the glocal spectral subspace consisting of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus \{\lambda\} \longrightarrow X$ that satisfies $(T-\mu)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus \{\lambda\}$.

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{X})$. If T obeys property (Bb). Then the following statements are equivalent.*

(i) *T obeys property (Bw);*
(ii) *$\sigma_{BW}(T) \cap E^o(T) = \emptyset$;*
(iii) *$E^0(T) = \pi^0(T)$.*

Proof. (i)$\Longrightarrow$ (ii). Let $\lambda \in \sigma(T) \backslash \sigma_{BW}(T)$. Since T satisfies (Bb), $\lambda \in \pi^0(T)$. Thus $\lambda \in \sigma(T) \setminus \sigma_b(T)$ and hence $\sigma_b(T) \subseteq \sigma_{BW}(T)$. Since the reverse inclusion is always true, we have $\sigma_b(T) = \sigma_{BW}(T)$.
(ii)$\Longrightarrow$ (i). Assume that $\sigma_b(T) = \sigma_{BW}(T)$ and we will establish that $\Delta^g(T) = \pi^0(T)$. Suppose $\lambda \in \Delta^g(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_b(T)$. Hence $\lambda \in \pi^0(T)$. Conversely suppose $\lambda \in \pi^0(T)$. Since $\sigma_{BW}(T) = \sigma_b(T)$, $\lambda \in \Delta^g(T)$.
(ii)$\Longrightarrow$ (iii). Let $\lambda \in \Delta^g(T)$. Since $\sigma_{BW}(T) = \sigma_b(T)$, $\lambda \in \sigma(T) \setminus \sigma_b(T)$, i.e., $\lambda \in \pi^0(T)$ which implies that $\lambda \in E^0(T)$. Thus $\sigma_{BW}(T) \cup E^o(T) \supseteq \sigma(T)$. Since $\sigma_{BW}(T) \cup E^0(T) \subseteq \sigma(T)$, always we must have $\sigma_{BW}(T) \cup E^0(T) = \sigma(T)$.
(iii)$\Longrightarrow$ (ii). Suppose that $E^0(T) = \pi^0(T)$. As T obeys property (Bb) then $\sigma(T) \setminus \sigma_{BW}(T) = \pi^0(T)$ and so $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$. That is, $E^0(T) \cap \pi^0(T) = \emptyset$. ∎

The following result may be found in [16], we give the proof for completeness.

Theorem 3.2. *Let $T \in \mathcal{B}(\mathcal{X})$. Then the following statements are equivalent.*

(i) *T satisfies property (Bb);*
(ii) $\sigma_{BW}(T) = \sigma_b(T)$;
(iii) $\sigma_{BW}(T) \cup E^0(T) = \sigma(T)$.

Proof. (i)$\Longrightarrow$ (ii) Since the opposite inclusion $\sigma_{BW}(T) \subseteq \sigma_b(T)$ is always true, we have to show that $\sigma_b(T) \subseteq \sigma_{BW}(T))$. Let $\lambda \notin \sigma_{BW}(T)$. Since T satisfies property(Bb), $\lambda \in \pi^0(T)$ .Hence, $\lambda \notin \sigma_b(T)$
(ii)$\Longrightarrow$ (i) Assume that $\sigma_b(T) = \sigma_{BW}(T)$ and we will establish that $\sigma(T) \setminus \sigma_{BW}(T) = \pi^0(T)$. Suppose $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. The hypothesis $\sigma_b(T) = \sigma_{BW}(T)$ implies that $\lambda \in \sigma(T) \setminus \sigma_b(T)$. Hence $\lambda \in \pi^0(T)$ and so $\sigma(T) \setminus \sigma_{BW}(T) \subseteq \pi^0(T)$ Conversely suppose $\lambda \in \pi^0(T)$. Since $\sigma_{BW}(T) = \sigma_b(T)$, $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$.
(ii)$\Longrightarrow$ (iii) Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since $\sigma_{BW}(T) = \sigma_b(T)$, $\lambda \in \sigma(T) \setminus \sigma_b(T)$, that is, $\lambda \in \pi^0(T)$ which implies that $\lambda \in E^0(T)$. Thus $\sigma_{BW}(T) \cup E^0(T) \supseteq \sigma(T)$. Since $\sigma_{BW}(T) \cup E^0(T) \subseteq \sigma(T)$, always we must have $\sigma_{BW}(T) \cup E^0(T) = \sigma(T)$.
(iii)$\Longrightarrow$(ii) Suppose $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since $\sigma_{BW}(T) \cup E^0(T) = \sigma(T)$, $\lambda \in E^0(T)$. In particular λ is an isolated point of $\sigma(T)$. Then by [3, Theorem 4.2] that $\lambda \notin \sigma_D(T)$ and this implies that $\lambda \in \pi(T)$ and so $a(T-\lambda) = d(T-\lambda) < \infty$. So, it follows from [1, Theorem 3.4] that $\beta(T-\lambda) = \alpha(T-\lambda) < \infty$. Hence $\lambda \in \pi^0(T)$. Therefore, $\lambda \notin \sigma_b(T)$. Since the other inclusion is always verified, we have $\sigma_{SBF_+^-}(T) = \sigma_b(T)$. This completes the proof. ■

Example 3.3. Let T be a non-zero nilpotent operator and let S be a quasinilpotent which is not nilpotent. Then it easy to see that

$$\sigma(T) = \{0\}, \sigma_{BW}(T) = \emptyset \text{ and } \sigma(S) = \sigma_{BW}(S) = \{0\}.$$

Hence T and S satisfy property (Bw). Since $T \otimes S$ is nilpotent, we have $\sigma_{BW}(T \otimes S) = \emptyset$. Hence $T \otimes S$ satisfies property (Bw). However,

$$\sigma_{BW}(T)\sigma(S) \cup \sigma_{BW}(S)\sigma(T) = \{0\} \neq \sigma_{BW}(T \otimes S).$$

Here $0 \in \operatorname{iso} \sigma(T \otimes S)$ and 0 is a pole. Moreover, we note that T, S and $T \otimes S$ satisfies generalized a-Browder's theorem

Lemma 3.4. [16, Lemma 3.1] *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$. Then*

$$\begin{aligned}\sigma_{BW}(A \otimes B) &\subseteq \sigma_{BW}(A)\sigma(B) \cup \sigma_{BW}(B)\sigma(A) \subseteq \sigma_w(A)\sigma(B) \cup \sigma_w(B)\sigma(A)\\ &\subseteq \sigma_b(A)\sigma(B) \cup \sigma_b(B)\sigma(A) = \sigma_b(A \otimes B).\end{aligned}$$

Lemma 3.5. [16, Lemma 3.2] *Let $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$ obey property (Bb). Then $T \otimes S$ obeys property (Bb) if and only if $\sigma_{BW}(T \otimes S) = \sigma_{BW}(T)\sigma(S) \cup \sigma_{BW}(S)\sigma(T)$.*

In [11], Kubrusly and Duggal studied the stability of Weyl's theorem under tensor product in the infinite dimensional space setting. Rashid [15] studied the stability of generalized Weyl's theorem under tensor product in the infinite dimensional Banach space. The following main theorem shows if isoloid operators T and S satisfies property (Bw) and the equality $\sigma_{BW}(T \otimes S) = \sigma_{BW}(T)\sigma(S) \cup \sigma_{BW}(S)\sigma(T)$, then $T \otimes S$ satisfies property (Bw) in the infinite dimensional space setting. Let $\sigma_{PF}(T) = \{\lambda \in \sigma_p(T) : \alpha(T-\lambda) < \infty\} = \{\lambda \in \mathbb{C} : 0 < \alpha(T-\lambda) < \infty\}$.

Definition 1. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be finitely isoloid if all the isolated points of its spectrum are eigenvalues of finite multiplicity i.e. $\text{iso}\sigma(T) \subseteq E^0(T)$. An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be finitely polaroid (resp., polaroid) if all the isolated points of its spectrum are poles of finite rank i.e. $\text{iso}\sigma(T) \subseteq \pi^0(T)$, (resp., $\text{iso}\sigma(T) \subseteq \pi(T)$).

Theorem 3.6. *Let $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$ such that T and S are finite-isoloid and $0 \notin \text{iso}\sigma(T \otimes S)$. If property (Bw) holds for T and S, then the following statements are equivalent.*

(a) *$T \otimes S$ satisfies property (Bw).*

(b) *$\sigma_{BW}(T \otimes S) = \sigma_{BW}(T)\sigma(S) \cup \sigma(T)\sigma_{BW}(S)$.*

Proof. (a)$\Longrightarrow$ (b): Assume that $T \otimes S$ satisfies property (Bw). Let

$$\lambda \in E^0(T \otimes S) = \sigma(T)\sigma_{BW}(S) \cup \sigma_{BW}(T)\sigma(S).$$

Since $0 \notin \text{iso}\sigma(T \otimes S)$, then $\lambda \neq 0$. Hence $\lambda \in \text{iso}(T \otimes S) = \text{iso}(T)\text{iso}(S)$. That is, $\lambda = \mu\nu$ with $\mu \in \text{iso}(T)$ and $\nu \in \text{iso}(S)$. Since T and S are finite-isoloid, then $\mu \in E^0(T) = \sigma(T) \setminus \sigma_{BW}(T)$ and $\nu \in E^0(S) = \sigma(S) \setminus \sigma_{BW}(S)$, and hence $\lambda = \mu\nu \notin \sigma_{BW}(T)\sigma(S) \cup \sigma(T)\sigma_{BW}(S)$. Thus

$$\sigma_{BW}(T)\sigma(S) \cup \sigma(T)\sigma_{BW}(S) \subseteq \sigma_{BW}(T \otimes S).$$

Conversely, let $\lambda \in \sigma(T \otimes S) \setminus (\sigma_{BW}(T)\sigma(S) \cup \sigma(T)\sigma_{BW}(S))$, then for $\lambda = \mu\nu$ we have that $\mu \in \sigma(T)$ and $\nu \in \sigma(S)$, hence $\mu \in E^0(T)$ and $\nu \in E^0(S)$. Thus $\lambda = \mu\nu \in E^0(T \otimes S) = \sigma(T \otimes S) \setminus \sigma_{BW}(T \otimes S)$. Therefore,

$$\sigma_{BW}(T \otimes S) = \sigma_{BW}(T)\sigma(S) \cup \sigma(T)\sigma_{BW}(S).$$

(b)$\Longrightarrow$ (a): Since T and S obey property (Bw), then

$$\sigma(T) \setminus \sigma_{BW}(T) = E^0(T) \text{ and } \sigma(S) \setminus \sigma_{BW}(S) = E^0(S).$$

Assume that

$$\sigma_{BW}(T \otimes S) = \sigma_{BW}(T)\sigma(S) \cup \sigma(T)\sigma_{BW}(S).$$

Let $\lambda \in E^0(T \otimes S)$. Then there exists $\mu \in \text{iso}(T)$ and $\nu \in \text{iso}(S)$ such that $\lambda = \mu\nu$. Since T and S are finite-isoloid, then $\mu \in E^0(T)$ and $\nu \in E^0(S)$. Hence $\mu \notin \sigma_{BW}(T)$ and $\nu \notin \sigma_{BW}(S)$. Then $\lambda \notin \sigma_{BW}(T \otimes S)$. Thus

$$E^0(T \otimes S) \subseteq \sigma(T \otimes S) \setminus \sigma_{BW}(T \otimes S).$$

Conversely, assume that $\lambda \notin \sigma(T \otimes S) \setminus \sigma_{BW}(T \otimes S)$, then there exists $\mu \in \sigma(T) \setminus \sigma_{BW}(T)$ and $\nu \in \sigma(S) \setminus \sigma_{BW}(S)$ such that $\lambda = \mu\nu$. Since

$$T \otimes S = (T - \mu) \otimes S + \mu I \otimes (S - \nu),$$

then we can see that $\lambda \in E^0(T \otimes S)$. Hence $T \otimes S$ obeys property (Bw). ■

Lemma 3.7. *If $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$ are finitely polaroid, then so is $T \otimes S$.*

Proof. If $\text{iso}\sigma(T) = \text{iso}\sigma(S) = \emptyset$, then $\text{iso}\sigma(T \otimes S) = \emptyset$. Observe also that if either of $\text{iso}\sigma(T)$ or $\text{iso}\sigma(S)$ is the empty set, say $\text{iso}\sigma(T) = \emptyset$, then it follows from [10, Proposition 3] that $\text{iso}\sigma(T \otimes S) \subseteq \{0\}$ and $0 \in \text{iso}\sigma(S)$. But then $0 \in \pi^0(S)$, which implies that $0 \in \pi^0(T \otimes S)$. Let $\lambda \in \text{iso}\sigma(T \otimes S)$ be such that $\lambda = \mu\nu$, $\mu \in \text{iso}\sigma(T)$ and $\nu \in \text{iso}\sigma(S)$. Then $\mu \in \pi^0(T)$ and $\nu \in \pi^0(S)$. Hence, we have $\lambda \in \pi^0(T \otimes S)$. ■

$T \in \mathcal{B}(\mathcal{X})$ polaroid implies T^* polaroid. It is known that if T or T^* has SVEP and T is polaroid, then T and T^* satisfy generalized Weyl's theorem. Note as well known is the fact, [13, Theorem 2.15] that if T or T^* has SVEP and T is finitely polaroid, then T obeys property (Bw). The following theorem is the tensor product analogue of this result.

Theorem 3.8. *Suppose that $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$ are finitely polaroid. If T and S have SVEP (or T^* and S^* have SVEP), then $T \otimes S$ satisfies property (Bw).*

Proof. The hypotheses by [13, Theorem 2.15] imply that T and S obey property (Bw) and it then follows from Theorem 2.5 of [13] that T and S satisfy generalized Browder's theorem and $\pi(T) = E^0(T)$ and $\pi(S) = E^0(S)$. Hence $T \otimes S$ satisfies generalized Browder's theorem. Thus generalized Browder's theorem transfer from T and S to $T \otimes S$. Hence

$$\sigma_{BW}(T \otimes S) = \sigma(T)\sigma_{BW}(S) \cup \sigma(S)\sigma_{BW}(T).$$

Evidently, $T \otimes S$ is finitely polaroid (3.7); combining with $T \otimes S$ satisfies generalized Browder's theorem, it follows that

$$\sigma(T \otimes S) \setminus \sigma_{BW}(T \otimes S) = \pi(T \otimes S) = E^0(T \otimes S),$$

that is, $T \otimes S$ obeys property (Bw). ■

Example 3.9. Let $T, S \in \mathcal{B}(\ell^2)$ be defined by

$$T = 3 \oplus I = \begin{pmatrix} 3 & \\ & I \end{pmatrix} \text{ and } S = diag\{1,2\} \oplus 6I = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 6I \end{pmatrix}$$

so that

$$T \otimes S = 3S \oplus I \otimes S = \begin{pmatrix} 3S & \\ & I \otimes S \end{pmatrix} = \begin{pmatrix} 3S & & & \\ & S & & \\ & & S & \\ & & & \ddots \end{pmatrix}.$$

Hence

$$\sigma(T) = \text{iso}\sigma(T) = \{1,3\},\ \sigma(S) = \text{iso}\sigma(S) = \{1,2,6\},$$
$$E^0(T) = \{3\}, \sigma_{BW}(T) = \sigma_w(T) = \{1\},\ E^0(S) = \{1,2\},\ \sigma_{BW}(S) = \sigma_w(S) = \{6\},$$
$$\sigma(T \otimes S) = \text{iso}(T \otimes S) = \{1,2,3,6,18\} \text{ and } E^0(T \otimes S) = \{3\}.$$

Since T, S and $T \otimes S$ are self-adjoint, they all satisfy property (Bw).

We now give an example to show that property (Bw) does not transfer from operators T and S to the tensor product $T \otimes S$.

Example 3.10. Let R be the backward shift operator on ℓ^2,

$$R : \ell^2 \to \ell^2 \text{ defined by } R(x_1, x_2, \cdots) = (x_2, x_3, \cdots).$$

It is known that R satisfies property (Bw). In fact $\text{ind}(R - \lambda) = 1$ for $|\lambda| < 1$ and so

$$\sigma(R) = \sigma_w(R) = \sigma_{BW}(R) = \mathbb{D},\ \text{iso}\sigma(R) = E^0(R) = \emptyset.$$

Let P be a finite rank projection on ℓ^2. Then P satisfies property (Bw) and

$$\sigma(P) = \{0,1\},\ \sigma_w(P) = \sigma_{BW}(P) = \{0\}.$$

Consider operators

$$T = P \oplus (\frac{1}{2}R - 1) \text{ and } S = (-P) \oplus (\frac{1}{2}R^* + 1)$$

acting on the Hilbert space $\mathcal{H} = \ell^2 \oplus \ell^2$. We have

$$\sigma(T) = \{0,1\} \cup (\frac{1}{2}\mathbb{D} - 1)\ \sigma(S) = \{0,-1\} \cup (\frac{1}{2}\mathbb{D} + 1)$$
$$\sigma_w(T) = \sigma_{BW}(T) = \{0\} \cup (\frac{1}{2}\mathbb{D} - 1)\ \sigma_w(S) = \sigma_{BW}(S) = \{0\} \cup (\frac{1}{2}\mathbb{D} + 1),$$

where $\mathbb{D}$ is the closed unit disc in the complex plane $\mathbb{C}$. So, T and S^* have SVEP. Note that T and S both satisfy property (Bw). In particular T

and S satisfy generalized Browder's theorem. Furthermore, $1 \in \sigma(T \otimes S) \setminus \sigma_{BW}(T \otimes S)$. However, since

$$\sigma(T \otimes S) = \left\{ \{0, 1\} \cup \{\frac{1}{2}\mathbb{D} - 1\} \right\} . \left\{ \{0, -1\} \cup \{\frac{1}{2}\mathbb{D} + 1\} \right\} .$$

$$1 \in \operatorname{acc}\sigma(T \otimes S) \Longrightarrow 1 \in \sigma_b(T \otimes S).$$

Then $T \otimes S$ does not satisfy Browder's theorem, and hence property (Bw).

4. Perturbations

Let $[T, S] = TS - ST$ denote the commutator of the operators T and S. If $Q_1 \in \mathcal{B}(\mathcal{X})$ and $Q_2 \in \mathcal{B}(\mathcal{Y})$ are quasinilpotent operators such that $[Q_1, T] = [Q_2, S] = 0$ for some operators $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$, then

$$(T + Q_1) \otimes (S + Q_2) = (T \otimes S) + Q,$$

where $Q = Q_1 \otimes S + T \otimes Q_2 + Q_1 \otimes Q_2 \in \mathcal{B}(\mathcal{X} \otimes \mathcal{Y})$ is quasinilpotent operator.

Theorem 4.1. *Let $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{Y})$ having SVEP and let $Q_1 \in \mathcal{B}(\mathcal{X})$ and $Q_2 \in \mathcal{B}(\mathcal{Y})$ be quasinilpotent operators such that $[Q_1, T] = [Q_2, S] = 0$. If $T \otimes S$ is finitely isoloid, then $T \otimes S$ satisfies property (Bw) implies $(T + Q_1) \otimes (S + Q_2)$ satisfies property (Sw).*

Proof. Recall that $\sigma((T + Q_1) \otimes (S + Q_2)) = \sigma(T \otimes S)$, $\sigma_{BW}((T + Q_1) \otimes (S + Q_2)) = \sigma_{BW}(T \otimes S)$ and that the perturbation of an operator by a commuting quasinilpotent has SVEP if and only if the operator has SVEP. If $T \otimes S$ satisfies property (Bw), then

$$\begin{aligned} E^0(T \otimes S) &= \sigma(T \otimes S) \setminus \sigma_{BW}(T \otimes S) \\ &= \sigma((T + Q_1) \otimes (S + Q_2)) \setminus \sigma_{BW}((T + Q_1) \otimes (S + Q_2)). \end{aligned}$$

We prove that $E^0(T \otimes S) = E^0((T + Q_1) \otimes (S + Q_2))$. Observe that if $\lambda \in \operatorname{iso}\sigma(T \otimes S)$, then $T^* \otimes S^*$ has SVEP at λ; equivalently, $(T^* + Q_1^*) \otimes (S^* + Q_2^*)$ has SVEP at λ. Let $\lambda \in E^0(T \otimes S)$; then $\lambda \in \sigma((T + Q_1) \otimes (S + Q_2)) \setminus \sigma_{BW}((T + Q_1) \otimes (S + Q_2))$. Since $(T + Q_1)^* \otimes (S + Q_2)^*$ has SVEP at λ, it follows that $\lambda \notin \sigma_w((T + Q_1) \otimes (S + Q_2))$ and $\lambda \in \operatorname{iso}((T + Q_1) \otimes (S + Q_2))$. Thus $\lambda \in E^0((T + Q_1) \otimes (S + Q_2))$. Hence $E^0(T \otimes S) \subseteq E^0((T + Q_1) \otimes (S + Q_2))$. Conversely, if $\lambda \in E^0((T + Q_1) \otimes (S + Q_2))$, then $\lambda \in \operatorname{iso}(T \otimes S)$, and this, since $T \otimes S$ is finitely isoloid, implies that $\lambda \in E^0(T \otimes S)$. Hence $E^0((T + Q_1) \otimes (S + Q_2)) \subseteq E^0(T \otimes S)$. ■

From [5], we recall that an operator $R \in \mathcal{B}(\mathcal{X})$ is said to be Riesz if $R - \lambda I$ is Fredholm for every non-zero complex number λ. For a bounded operator T on $\mathcal{X}$, we denote by $E_{0f}(T)$ the set of isolated points λ of $\sigma(T)$ such that $\ker(T - \lambda I)$ is finite-dimensional. Evidently, $E_0(T) \subseteq E_{0f}(T)$.

Lemma 4.2. *Let T be a bounded operator on $\mathcal{X}$. If R is a Riesz operator that commutes with T, then*

$$E_0(T+R) \cap \sigma(T) \subseteq iso\sigma(T).$$

Proof. Clearly,

$$E_0(T+R) \cap \sigma(T) \subseteq E_{0f}(T+R) \cap \sigma(T).$$

and by Lemma 2.3 of [12] the last set contained in $iso\sigma(T)$. ■

Now we consider the perturbations by commuting Riesz operators. Let $T, R \in \mathcal{B}(\mathcal{X})$ be such that R is Riesz and $[T, R] = 0$; the tensor product $T \otimes R$ is not a Riesz operator (the Fredholm spectrum $\sigma_F(T \otimes R) = \sigma(T)\sigma_F(R) \cup \sigma_F(T)\sigma(R) = \sigma_F(T)\sigma(R) = \{0\}$ for a particular choice of T only). However, σ_w (also, σ_b) is stable under perturbation by commuting Riesz operators [19], and so T satisfies Browder's theorem if and only if $T + R$ satisfies Browder's theorem. Thus, if $\sigma(T) = \sigma(T + R)$ for a certain choice of operators $T, R \in \mathcal{B}(\mathcal{X})$ (such that R is Riesz and $[T, R] = 0$), then

$$\pi^0(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T+R) \setminus \sigma_w(T+R) = \pi^0(T+R),$$

where $\pi^0(T)$ is the set of $\lambda \in \mathrm{iso}\sigma(T)$ which are finite rank poles of the resolvent of T. If we now suppose additionally that T satisfies property (Bw), then

$$E^0(T) = \sigma(T) \setminus \sigma_{BW}(T) = \sigma(T) \setminus \sigma_w(T) = \sigma(T+R) \setminus \sigma_w(T+R), \tag{4.1}$$

and a necessary and sufficient condition for $T + R$ to satisfy property (Bw) is that $E^0(T+R) = E^0(T)$. One such condition, namely T is finitely isoloid.

Proposition 4.3. *Let $T, R \in \mathcal{B}(\mathcal{X})$, where R is Riesz, $[T, R] = 0$ and T is finitely isoloid. Then T satisfies property (Bw) implies $T + R$ satisfies property (Bw).*

Proof. Observe that if T obeys property (Bw), then identity (4.1) holds. Let $\lambda \in E^0(T)$. Then it follows from Lemma 4.2 that $\lambda \in E^0(T) \cap \sigma(T) = E^0(T + R - R) \subseteq \mathrm{iso}\sigma(T + R)$ and so $T^* + R^*$ has SVEP at λ. Since $\lambda \in \sigma(T+R) \setminus \sigma_w(T+R)$, $T^* + R^*$ has SVEP at λ implies $T + R - \lambda$ is Fredholm of index 0 and so $\lambda \in E^0(T+R)$. Thus $E^0(T) \subseteq E^0(T+R)$. Now let $\lambda \in E^0(T+R)$. Then $\lambda \in E^0(T+R) \cap \sigma(T+R) = E^0(T+R) \cap \sigma(T) \subseteq \mathrm{iso}\sigma(T)$, which by the finite isoloid property of T implies $\lambda \in E^0(T)$. Hence $E^0(T+R) \subseteq E^0(T)$. ■

Theorem 4.4. *Let $T \in \mathcal{B}(\mathcal{X})$ and $S \in \mathcal{B}(\mathcal{X})$ be finitely isoloid operators which satisfy property (Bw). If $R_1 \in \mathcal{B}(\mathcal{X})$ and $R_2 \in \mathcal{B}(\mathcal{Y})$ are Riesz operators such that $[T, R_1] = [S, R_2] = 0$, $\sigma(T+R_1) = \sigma(T)$ and $\sigma(S+R_2) = \sigma(S)$, then $T \otimes S$ satisfies property (Bw) implies $(T + R_1) \otimes (S + R_2)$ satisfies* *property (Bw) if and only if Browder's theorem transforms from $T+R_1$ and $S+R_2$ to their tensor product.*

Proof. The hypotheses imply (by Proposition 4.3) that both $T+R_1$ and $S+R_2$ satisfy property (Bw). Suppose that $T\otimes S$ satisfies property (Bw). Then $\sigma(T\otimes B)\backslash\sigma_{BW}(T\otimes S)=E^0(T\otimes S)$. Evidently $T\otimes S$ satisfies Browder's theorem, and so the hypothesis T and S satisfy property (Bw) implies that Browder's theorem transfers from T and S to $T\otimes S$. Furthermore, since , $\sigma(T+R_1)=\sigma(T)$, $\sigma(S+R_2)=\sigma(S)$, and σ_w is stable under perturbations by commuting Riesz operators,

$$\begin{aligned}\sigma_{BW}(T\otimes S)&=\sigma_w(T\otimes S)=\sigma(T)\sigma_w(S)\cup\sigma_w(T)\sigma(S)\\&=\sigma(T+R_1)\sigma_w(S+R_2)\cup\sigma_w(T+R_1)\sigma(S+R_2)\\&=\sigma(T+R_1)\sigma_{BW}(S+R_2)\cup\sigma_{BW}(T+R_1)\sigma(S+R_2)\end{aligned}$$

Suppose now that Browder's theorem transfers from $T+R_1$ and $S+R_2$ to $(T+R_1)\otimes(S+R_2)$. Then

$$\sigma_w(T\otimes S)=\sigma_w((T+R_1)\otimes(S+R_2))$$

and

$$E^0(T\otimes S)=\sigma((T+R_1)\otimes(S+R_2))\setminus\sigma_w((T+R_1)\otimes(S+R_2)).$$

Let $\lambda\in E^0(T\otimes S)$. Then $\lambda\neq 0$, and hence there exist $\mu\in\sigma(T+R_1)\backslash\sigma_w(T+R_1)$ and $\nu\in\sigma(S+R_2)\setminus\sigma_w(S+R_2)$ such that $\lambda=\mu\nu$. As observed above, both $T+R_1$ and $S+R_2$ satisfy property (Bw); hence $\mu\in E^0(S+R_1)$ and $\nu\in E^0(S+R_2)$. This, since $\lambda\in\sigma(T\otimes S)=\sigma((T+R_1)\otimes(S+R_2))$, implies $\lambda\in E^0((T+R_1)\otimes(S+R_2))$. Conversely, if $\lambda\in E^0((T+R_1)\otimes(S+R_2))$, then $\lambda\neq 0$ and there exist $\mu\in E^0(T+R_1)\subseteq\mathrm{iso}\sigma(T)$ and $\nu\in E^0(S+R_2)\subseteq\mathrm{iso}\sigma(S)$ such that $\lambda=\mu\nu$. Recall that $E^0((T+R_1)\otimes(S+R_2))\subseteq E^0(T+R_1)E^0(S+R_2)$. Since T and S are finite isoloid, $\mu\in E^0(T)$ and $\nu\in E^0(S)$. Hence, since $\sigma((T+R_1)\otimes(S+R_2))=\sigma(T\otimes S)$, $\lambda=\mu\nu\in E^0(T\otimes S)$. To complete the proof, we observe that if the implication of the statement of the theorem holds, then (necessarily) $(T+R_1)\otimes(S+R_2)$ satisfies Browder's theorem. This, since $T+R_1$ and $S+R_2$ satisfy Browder's theorem, implies Browder's theorem transfers from $T+R_1$ and $S+R_2$ to $(T+R_1)\otimes(S+R_2)$. ■

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