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THE NUMBER OF SIMPLE MODULES IN A BLOCK WITH KLEIN FOUR HYPERFOCAL SUBGROUP

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Abstract. A 2-block of a finite group having a Klein four hyperfocal subgroup has the same number of irreducible Brauer characters as the corresponding 2-block of the normalizer of the hyperfocal subgroup.

1. Introduction

Let p be a prime and k an algebraically closed field of characteristic p. Let G be a finite group. Denote by $G_{p'}$ the set of p-regular elements of G. Denote by C_n the cyclic group of order n. Let b be a (p-)block (idempotent) of kG. Denote by $l(b)$ the number of isomorphism classes of simple kGb-modules. Let (P, b_P) be a maximal b-Brauer pair and let (S, b_S) be the unique b-Brauer pair contained in (P, b_P) for $S \leq P$. Denote by b_S^H the block of H associated with b_S where the group H is such that $C_G(S) \leq H \leq N_G(S)$, see [11, V, section 3]. Let Q be the hyperfocal subgroup of b with respect to (P, b_P), that is, $Q = \langle\, [S, N_G(S, b_S)_{p'}] \,|\, S \leq P \rangle = \langle\, [S, N_G(S, b_S)_{p'}] \,|\, S \leq P,\ (S, b_S)$ is maximal or essential$\rangle$, see [12].

Rouquier raised a question on a derived equivalence between b and $b_Q^{N_G(Q)}$ (see [13, A.2] for a precise statement). In this context, Watanabe showed that if b has a cyclic hyperfocal subgroup Q, then $l(b) = l(b_Q^{N_G(Q)})$ ([16, Theorem 1(i)]). In this article, we show the following:

Theorem 1.1. *If b has a hyperfocal subgroup Q isomorphic to $C_2 \times C_2$, then $l(b) = l(b_Q^{N_G(Q)})$.*

Above Rouquier's problem is verified affirmatively in some concrete cases, see for example [7]. Its character version, that is, existance of a perfect isometry between the corresponding blocks, is proved in some situations, see [8], [15].

2. Lower defect group of a block

In this section, we collect needed facts concerning lower defect groups of a block. For basic facts on lower defect groups of a block as stated in the

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next two paragraphs, see for example [5, V section 10], [11, V section 11] and [16, section 4].

Let I be the k-subspace of $Z(kG)$ with a basis $\{\hat{C}\,|\,C{\in}\mathrm{Cl}(G_{p'})\}$ where $\mathrm{Cl}(G_{p'})$ is the set of p-regular conjugacy classes of G and $\hat{C} = \sum_{x\in C} x$. Then denoting by $\mathrm{Bl}(G)$ the set of blocks of G, $I = \oplus_{a\in\mathrm{Bl}(G)} Ia$ and there exists a "block partition" $\mathrm{Cl}(G_{p'}) = \cup_{a\in\mathrm{Bl}(G)} X(a)$ (disjoint union) of $\mathrm{Cl}(G_{p'})$ so that $\{\hat{C}a\,|\,C{\in}X(a)\}$ is a k-basis of Ia.

For a p-subgroup S of G, set $m(b,S) = |\{C \in X(b)\,|\,C \text{ has a defect group } S\}|$. (We call S a lower defect group of b if $m(b,S) \neq 0$.) The multiplicity of p^n in elementary divisors of the Cartan matrix of b is equal to $\sum_S m(b,S)$ where S ranges over a set of G-conjugacy classes of p-subgroups of G of order p^n, and $m(b,S) = \sum_e m(e^{N_G(S,e)}, S)$ where e ranges over a set of $N_G(S)$-conjugacy classes of blocks of $C_G(S)$ such that (S,e) is a b-Brauer pair. In particular, choosing a set $\mathcal{T}$ of subgroups of P such that $\{(T,b_T)\,|\,T \in \mathcal{T}\}$ is a set of representatives of G-conjugacy classes of not maximal b-Brauer pairs, we have $l(b) = \sum_{S\in\mathcal{T}\cup\{P\}} m(b_S^{N_G(b_S,S)}, S)$. Here, we may take $\mathcal{T}$ so that (T,b_T) is extremal in (P,b_P) ([1, Corollary 4.5, Remark 4.9]), that is, $N_P(T)$ is a defect group of $b_T^{N_G(T,b_T)}$. Since $m(b_P^{N_G(P,b_P)}, P) = m(b,P) = 1$, for $l(b)$ it suffices to know $m(b_T^{N_G(b_T,T)}, T)$ for $T \in \mathcal{T}$.

Below let $T \in \mathcal{T}$ and denote $P' = N_P(T)$.

Lemma 2.1. *If $b_T^{N_G(T,b_T)}$ is nilpotent, then $m(b_T^{N_G(T,b_T)}, T) = 0$.*

Proof. Since $l\big(b_T^{N_G(T,b_T)}\big) = 1 = m(b_T^{N_G(T,b_T)}, P')$, we have $m(b_T^{N_G(T,b_T)}, T) = 0$. $\square$

For a normal p-subgroup Z of G, denote by μ_Z the canonical epimorphism $kG \to k[G/Z]$. When $|G : C_G(Z)|$ is a p-power, we see $m(b,S) = m(\mu_Z(b), S/Z)$ by [11, Theorem V.8.11, Lemma V.8.9].

Lemma 2.2. *If $T \cap Q = 1$, then $m(b_T^{N_G(T,b_T)}, T) = m(\mu_T(b_T^{N_G(T,b_T)}), 1)$.*

Proof. For $x \in N_G(T,b_T)_{p'}$, we have $[T,\langle x\rangle] = T \cap Q = 1$ and so $x \in C_{N_G(T,b_T)}(T)$. Hence $|N_G(T,b_T) : C_{N_G(T,b_T)}(T)|$ is a p-power. $\square$

The following is proved in the proof of [16, Theorem 4], in which hyperfocal subalgebra of the block is used. Note that $m(b,1)$ is equal to the multiplicity of 1 in the set of elementary divisors of the Cartan matrix of b.

Theorem 2.3. *If no simple kGb-module is relatively Q-projective, then any Cartan integer of b is divisible by p and so $m(b,1) = 0$.*

From this we have:

Lemma 2.4. *If Q is abelian, $Q < P$ and $|Q| \le |Z(P)|$, then any Cartan integer of b is divisible by p and so $m(b,1) = 0$.*

Proof. Assume there exists a simple kGb-module M having a vertex V such that $V \le Q$. Then there exists a self-centralizing b-Brauer pair (V, e) by [10, Corollary 3.7] (see [14, Section 41]). There exists $g \in G$ such that $(V,e)^g \le (P, b_P)$. Then $C_P(V^g) \le V^g < P$. If $V^g \le Z(P)$, then $P = C_P(V^g) \le V^g < P^g$, a contradiction. If $V^g \not\le Z(P)$, then $Z(P) < V^g Z(P) \le C_P(V^g) \le V^g \le Q^g$, a contradiction. Hence, by Theorem 2.3, the assertion follows. □

3. Hyperfocal subgroup of a block

In this section, we collect needed facts concerning hyperfocal subgroup of a block.

Lemma 3.1. *Let K be such that $TC_G(T) \trianglelefteq K \trianglelefteq N_G(T, b_T)$. Then the hyperfocal subgroup Q' of b_T^K with respect to $(P' \cap K, b_{P' \cap K})$ is contained in Q.*

Proof. See the proof of [16, Lemma 6]. □

Lemma 3.2. *If Z is a normal p-subgroup of G such that $|G : C_G(Z)|$ is a p-power, then $\mu_Z(b)$ has a hyperfocal subgroup QZ/Z.*

Proof. We use $^-$ for μ_Z. Let S be such that $Z \trianglelefteq S \le P$. Denote by $\hat{C}_G(S)$ the inverse image in G of $C_{\overline{G}}(\overline{S})$. Then we see that b_S is covered by a unique block $\hat{b_S}$ of $\hat{C}_G(S)$, $\overline{\hat{b_S}}$ is a block of $C_{\overline{G}}(\overline{S})$, $(\overline{P}, \overline{\hat{b_P}})$ is a maximal $\overline{b}$-Brauer pair, $(\overline{S}, \overline{\hat{b_S}}) \le (\overline{P}, \overline{\hat{b_P}})$, and $\hat{N}_G(S, b_S) = N_G(S, b_S)\hat{C}_G(S)$ where $\hat{N}_G(S, b_S)$ is the inverse image in G of $N_{\overline{G}}(\overline{S}, \overline{\hat{b_S}})$, see the proof of [16, Lemma 8] for details.

Let $Q_{\overline{b}}$ be the hyperfocal subgroup of $\overline{b}$ with respect to $(\overline{P}, \overline{\hat{b_P}})$. Then $Q_{\overline{b}} = \langle\, [\overline{S}, N_{\overline{G}}(\overline{S}, \overline{\hat{b_S}})_{p'}] \,|\, \overline{S} \le \overline{P} \rangle = \langle\, \overline{[S, N_G(S, b_S)_{p'}]} \,|\, Z \le S \le P \rangle$. On the other hand, $Q = \langle\, [S, N_G(S, b_S)_{p'}] \,|\, S \le P,\ (S, b_S) \text{ is maximal or essential} \rangle = \langle\, [S, N_G(S, b_S)_{p'}] \,|\, Z \le S \le P \rangle$ since $Z \trianglelefteq G$, see [16, Lemma 2]. Hence, $Q_{\overline{b}} = \overline{Q}$. □

The canonical epimorphism $\pi : N_G(P, b_P)/C_G(P) \to N_G(P, b_P)/PC_G(P)$ splits since $p \nmid |N_G(P, b_P)/PC_G(P)|$. Let $\sigma : N_G(P, b_P)/PC_G(P) \to N_G(P, b_P)/C_G(P)$ be a monomorphism such that $\pi\sigma = Id_{N_G(P, b_P)/PC_G(P)}$. Let $E(b) = \sigma\big(N_G(P, b_P)/PC_G(P)\big)$ and $\hat{E}(b)$ be the inverse image of $E(b)$ in $N_G(P, b_P)$. Note that σ and $E(b)$ are determined up to conjugation. We may view $E(b) \le \mathrm{Aut}(P)$.

Let $C = C_G(Q)$, and note $N_G(P, b_P) \le N_G(Q, b_Q)$.

Lemma 3.3. *$\hat{E}(b) \cap C = C_G(P)$ and $E(b) \le \mathrm{Aut}(Q)$.*

Proof. See the proof of [16, Lemma 3]. □

Lemma 3.4. *If $E(b) \neq 1$ and $E(b)$ acts regularly on $Q - \{1\}$, then $P = Q \rtimes C_P\big(E(b)\big)$.*

Proof. See the proof of [16, Lemma 4(i)]. □

Let $\mathcal{F}_{(P,b_P)}(G,b)$ be the Brauer category of b whose objects are b-Brauer pairs contained in (P, b_P).

Lemma 3.5. *If $Q \trianglelefteq G$ and G/C is abelian, then there is no essential b-Brauer pair and so $N_G(P, b_P)$ controls fusion of $\mathcal{F}_{(P,b_P)}(G,b)$.*

Proof. See the proof of [16, Theorem 3]. □

Let $N = N_G(Q, b_Q)$ and $c = b_Q^{N_G(Q,b_Q)}$.

As is well-known, $\uparrow_N^{N_G(Q)}$ gives a Morita equivalence between kNc and $kN_G(Q)b_Q^{N_G(Q)}$, so $l(c) = l(b_Q^{N_G(Q)})$ ([11, Theorem V.5.10]). Hence, we will show $l(b) = l(c)$.

The Brauer pair (P, b_P) of G can be viewed as a Brauer pair of N and is a maximal c-Brauer pair.

Theorem 3.6. ([16, Theorem 2]) *If Q is abelian, then $\mathcal{F}_{(P,b_P)}(G,b) \simeq \mathcal{F}_{(P,b_P)}(N,c)$. In particular, c has a hyperfocal subgroup Q.*

Lemma 3.7. *If Q is abelian, then $Q = \langle [Q, N_{p'}] \rangle$. In particular, C_2 cannot be a hyperfocal subgroup of a block.*

Proof. Clearly, $Q \geq \langle [Q, N_{p'}] \rangle$. We also have $Q \leq \langle [Q, N_{p'}] \rangle$. In fact, for $S \leq P$ and $x \in N_G(S, b_S)_{p'} = (N_N(S, b_S)C_G(S))_{p'}$, $[S, \langle x \rangle] = \big[[S, \langle x \rangle], \langle x \rangle\big] \leq [Q, N_{p'}]$ using [6, Theorem 5.3.6]. □

4. Proof of the main result

Below, we assume $p = 2$ and $Q \simeq C_2 \times C_2$. Note $\mathrm{Aut}(Q) \simeq GL(2,2) \simeq S_3$.

A block is nilpotent if and only if its hyperfocal subgroup is trivial. Hence, from Lemma 2.1, Lemma 3.1 and Lemma 3.7, if $m(b_T^{N_G(T,b_T)}, T) \neq 0$, then Q is a hyperfocal subgroup of $b_T^{N_G(T,b_T)}$ with respect to $(P', b_{P'})$.

Let $F = N/C$. We may view $F \leq \mathrm{Aut}(Q)$.

Since b_Q is nilpotent ([12, Proposition 4.2]) and c is not nilpotent, F is not a p-group by [4, Theorem 2] and so

$$F \simeq C_3 \text{ (Case(i)) or } F \simeq S_3 \text{ (Case(ii))}.$$

(Principal 2-blocks of A_4 and S_4 give Case(i) and Case(ii) respectively.) Then there exists a unique subgroup H such that $C \lhd H \trianglelefteq N$, and $H/C \simeq C_3$. The subgroup H is P-invariant since C and N are so. Let $U = C_P(Q)$. Note that Case(i) means $H = N$, $Q \leq Z(P)$ and $U = P$, and Case(ii) means $H < N$, $Q \not\leq Z(P)$ and $U < P$.

Let $f = b_Q^H$. Then f and b_Q have a defect group $P \cap H = P \cap C = U$. Since $l(f) = 3$, f is not nilpotent and so has a hyperfocal subgroup Q.

Lemma 4.1. $l(c) = \begin{cases} 3 & (Case(i)) \\ 2 & (Case(ii)). \end{cases}$

Proof. Case(ii): Since $|N : H| = 2$ and $l(f) = 3$, there exists an N-invariant simple kHf-module. The other two simple kHf-modules are permuted by conjugation by N, and the assertion follows. □

For a maximal f-Brauer pair (U, b_U), $E(f)$ is such that $\mathrm{Aut}(U) \geq N_H(U, b_U)/C_H(U) = UC_H(U)/C_H(U) \rtimes E(f)$, and $\hat{E}(f)$ is the inverse image of $E(f)$ in $N_H(U, b_U)$.

Lemma 4.2. $E(f) \simeq C_3$.

Proof. By the Frattini argument, we have $H = N_H(U, b_U)C$ and so $H = \hat{E}(f)C$. Then $E(f) = \hat{E}(f)/C_H(U) = \hat{E}(f)/\hat{E}(f) \cap C \simeq \hat{E}(f)C/C = H/C \simeq C_3$ using Lemma 3.3 for f. □

Since $(U, b_U) \trianglelefteq (P, b_P)$, P normalizes (U, b_U) and so $N_H(U, b_U)$. The conjugation action of P on $N_H(U, b_U)$ induces the action of P on $N_H(U, b_U)/C_H(U)$. By the uniqueness of the p-complement up to conjugation, for $u \in P$ there exists $w \in U$ such that $E(f)^u = E(f)^w$.

Let $R = C_U\big(E(f)\big)$. Note that (R, b_R) is extremal in (P, b_P) by Lemma 4.3(ii) below, and so we will assume $R \in \mathcal{T}$.

Lemma 4.3. *(i) $U = Q \times R$.* *(ii) $R \lhd P$.*

Proof. (i) We can apply Lemma 3.4 for f and U.

(ii) For $u \in P$, there exists $w \in U$ so that $R^u = C_{U^u}\big(E(f)^u\big) = C_{U^w}\big(E(f)^w\big) = R^w = R$. □

Note that R does not depend on the choice of $E(f)$ since $R \lhd U$.

Proposition 4.4. *Let $T \leq R$.*

(i) If $T = R$, then $m(b_T^{N_G(T,b_T)}, T) = \begin{cases} 2 & (Case(i)) \\ 1 & (Case(ii)). \end{cases}$

(ii) If $T < R$, then $m(b_T^{N_G(T,b_T)}, T) = 0$.

Proof. The pair (U, b_U) can be viewed as a $b_R^{N_G(R,b_R)}$-Brauer pair, and we have $\hat{E}(f) \leq N_{N_G(R,b_R)}(U, b_U)$ and $\hat{E}(f) \not\leq C_{N_G(R,b_R)}(U)$. Hence, $b_R^{N_G(R,b_R)}$ is not nilpotent, and for the statement we may assume $b_T^{N_G(T,b_T)}$ has a hyperfocal subgroup Q. Then $\mu_T\big(b_T^{N_G(T,b_T)}\big)$ is a block with a defect group P'/T and a hyperfocal subgroup $QT/T \simeq Q$, see Lemma 3.2.

Let $m = m(b_T^{N_G(T,b_T)}, T) = m(\mu_T(b_T^{N_G(T,b_T)}), 1)$, see Lemma 2.2.

(i) An elementary divisor of the Cartan matrix of a block with dihedral defect group D_{2^n} ($n \geq 2$) is 2^n or 1 ([3, Proposition 4G]). In Case(i), $P/R \simeq Q$, and since a block having Klein four as a defect group and as a hyperfocal subgroup has three irreducible Brauer characters ([2, Proposition 7D]), we have $m = 2$. In Case(ii), $P/R \simeq D_8$, and since a block having D_8 as a defect group and having Klein four as a hyperfocal subgroup has two irreducible Brauer characters ([3, Theorem 2]), we have $m = 1$.

(ii) We have $C_{P'}(Q) = Q \times (P' \cap R)$ and $P' \cap R > T$. Then $C_{P'}(Q)/T = QT/T \times (P' \cap R)/T$, and QT/T and $(P' \cap R)/T$ are non-trivial normal subgroup of P'/T. We have $P'/T > QT/T$ and $|Z(P'/T)| \geq 4 = |QT/T|$, and so $m = 0$ by Lemma 2.4. $\square$

Lemma 4.5. *$l(b) = m(b, P) + m(b, R) = m(b_P^{N_G(P,b_P)}, P) + m(b_R^{N_G(R,b_R)}, R)$ when $Q \trianglelefteq G$.*

Proof. From Lemma 4.1 and Proposition 4.4(i), we have $l(b) = m(b_P^{N_G(P,b_P)}, P) + m(b_R^{N_G(R,b_R)}, R)$ and so $l(b) = m(b, P) + m(b, R)$. $\square$

Lemma 4.6. *If $T \cap Q = Q$, then $m(b_T^{N_G(T,b_T)}, T) = 0$.*

Proof. Let $G' = N_G(T, b_T)$ and $b' = b_T^{G'}$. We may assume b' has a hyperfocal subgroup Q. Then we have a normal subgroup R' of P' for b' as R for b. Since $G' = N_N(T, b_T)C_G(T) \leq N$, we have $l(b') = m(b', P') + m(b', R')$ by Lemma 4.5 for b', and so $m(b', T) = 0$. Note T and R' are not G'-conjugate since $Q \leq T$ and $Q \not\leq R'$. $\square$

Lemma 4.7. *If $T \cap Q \simeq C_2$, then $b_T^{N_G(T,b_T)}$ is nilpotent and so $m(b_T^{N_G(T,b_T)}, T) = 0$.*

Proof. Let $Q_1 = T \cap Q$. Since $N_G(T, b_T) \cap N \leq N_G(Q_1, b_{Q_1}) = C_G(Q_1)$, we have $N_G(T, b_T) = N_N(T, b_T)C_G(T) \leq C_G(Q_1)$ and Q_1 is a central p-subgroup of $N_G(T, b_T)$. If $b_T^{N_G(T,b_T)}$ is not nilpotent, then $\mu_{Q_1}(b_T^{N_G(T,b_T)})$ would have a hyperfocal subgroup isomorphic to C_2. $\square$

Proposition 4.8. *If $m(e^{N_G(S,e)}, S) \neq 0$ for a b-Brauer pair (S, e), then (S, e) is G-conjugate to (P, b_P) or (R, b_R).*

Proof. By Proposition 4.4 it suffices to show that if $m(b_T^{N_G(T,b_T)}, T) \neq 0$, then $T \leq R$. The condition $m(b_T^{N_G(T,b_T)}, T) \neq 0$ implies that $b_T^{N_G(T,b_T)}$ has a hyperfocal subgroup Q and that $T \cap Q = 1$ by Lemma 4.6 and Lemma 4.7.

Firstly, assume $N_G(T, b_T) = G$. Then $T \triangleleft G$. QT is a direct product, since Q normalizes T, T normalizes Q and $T \cap Q = 1$. In particular $T < U$ and so $\hat{E}(f)$ acts on T through $\hat{E}(f)/C_H(U) = E(f) \simeq C_3$. Then $[T, \hat{E}(f)] \leq [T, N_G(T, b_T)_{p'}] \leq T \cap Q = 1$. Hence $T \leq R$.

Next, assume $N_G(T, b_T) < G$. We will show by the induction on $|G|$.

When $|G|$ is sufficiently small, then $Q \trianglelefteq G$ and the assertion holds by Lemma 4.5.

Let $G' = N_G(T, b_T)$ and $b' = b_T^{G'}$. Let $(T', b'_{T'})$ be the b'-Brauer pair contained in $(P', b_{P'})$ for $T' \leq P'$. Note $(T, b'_T) = (T, b_T)$.

Let $N' = N_{G'}(Q, b'_Q)$ and $C' = C_{G'}(Q)$. Then there exists unique H' such that $C' \triangleleft H' \trianglelefteq N'$, which satisfies $H'/C' \simeq C_3$. Let $f' = {b'}_Q^{H'}$ and $U' = C_{P'}(Q)$. For a maximal f'-Brauer pair $(U', b'_{U'})$, $E(f')$ is such that $\mathrm{Aut}(U') \geq N_{H'}(U', b'_{U'})/C_{H'}(U') = U'C_{H'}(U')/C_{H'}(U') \rtimes E(f')$. Then $E(f') \simeq C_3$ by Lemma 4.2 for b', and let $R' = C_{U'}\big(E(f')\big)$. Then $U' = Q \times R'$ by Lemma 4.3 for b'. Note that R' does not depend on the choice of $E(f')$.

We can consider the statement of this proposition for b'. Since $G' < G$, by the induction hypothesis, if $m({e'}^{N_{G'}(S', e')}, S') \neq 0$ for a b'-Brauer pair (S', e'), then (S', e') is G'-conjugate to $(P', b_{P'})$ or $(R', b'_{R'})$. Since the condition $m(b_T^{N_G(T, b_T)}, T) \neq 0$ can be viewed as a condition $m({b'}_T^{N_{G'}(T, b'_T)}, T) \neq 0$ of b'-Brauer pair, the assumption $m(b_T^{N_G(T, b_T)}, T) \neq 0$ and $T < P'$ implies (T, b'_T) is G'-conjugate to $(R', b'_{R'})$ and so $T = R'$.

Then we see $(U', b_{U'}) = (U', b'_{U'})$ and $H' = N_{H'}(U')$. Hence we have $N_{H'}(U, b_U) \leq N_{H'}(U', b'_{U'})$. On the other hand, since $N_H(U, b_U)$ controls fusion of $\mathcal{F}_{(U, b_U)}(H, f)$ by Lemma 3.5 for f and $C_H(U') = C_{H'}(U')$, we have $N_{H'}(U', b'_{U'}) \leq N_{H'}(U, b_U)C_{H'}(U')$. Therefore we have $N_{H'}(U', b'_{U'}) = N_{H'}(U, b_U)C_{H'}(U')$.

The quotient group $N_{H'}(U, b_U)/C_{H'}(U)$ is a subgroup of $N_H(U, b_U)/C_H(U)$ and acts on U' through $N_{H'}(U, b_U)/N_{H'}(U, b_U) \cap C_{H'}(U') \simeq N_{H'}(U, b_U)C_{H'}(U')/C_{H'}(U') = N_{H'}(U', b'_{U'})/C_{H'}(U')$. Then we can take $E(f)$ and $E(f')$ so that $E(f) \leq N_{H'}(U, b_U)/C_{H'}(U)$ and $E(f)$ acts on U' as $E(f')$.

Then we have $T = R' = C_{U'}\big(E(f')\big) = C_U\big(E(f)\big) \cap U' \leq R$. $\square$

From Proposition 4.8 and Proposition 4.4(i), we have

Theorem 4.9. $l(b) = m(b, P) + m(b, R) = m(b_P^{N_G(P, b_P)}, P) + m(b_R^{N_G(R, b_R)}, R)$
$$= \begin{cases} 3 & (Case(i)) \\ 2 & (Case(ii)). \end{cases}$$

Then Theorem 1.1 follows from Theorem 4.9.

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Remark After a presence of our results at The Mathematical Society of Japan Autumn Meeting 2017 (September 13th), the article [9] by Hu and Zhou which treats more general situation appears.

References

[1] J. Alperin and M. Broué, *Local methods in block theory*, Ann. of Math. (2) **110** (1979), no. 1, 143–157.

[2] R. Brauer, *Some applications of the theory of blocks of characters of finite groups IV*, J. Algebra **17** (1971), 489–521.

[3] R. Brauer, *On 2-blocks with dihedral defect groups*, Symposia Mathematica, Vol. XIII, 367–393, Academic Press, London, 1974.

[4] M. Cabanes, *Extensions of p-groups and construction of characters,* Comm. Algebra **15** (1987), no. 6, 1297–1311.

[5] W. Feit, The Representation Theory of Finite Groups, North-Holland, New York, 1982.

[6] D. Gorenstein, Finite groups, Second edition, Chelsea Publishing Co., New York, 1980.

[7] M. Holloway, S. Koshitani and N. Kunugi, *Blocks with nonabelian defect groups which have cyclic subgroups of index p,* Arch. Math. (Basel) **94** (2010), no. 2, 101–116.

[8] H. Horimoto and A. Watanabe, *On a perfect isometry between principal p-blocks of finite groups with cyclic p-hyperfocal subgroups,* arXiv:1611.02486, 2016.

[9] X. Hu and Y. Zhou, *2-blocks with hyperfocal subgroup* $C_{2^n} \times C_{2^n}$, arXiv:1709.05983, 2017.

[10] R. Knörr, *On the vertices of irreducible modules,* Ann. of Math. (2) **110** (1979), no. 3, 487–499.

[11] H. Nagao and Y. Tsushima, Representations of Finite Groups, Academic Press, Boston, 1989.

[12] L. Puig, *The hyperfocal subalgebra of a block*, Invent. Math. **141** (2000), no. 2, 365–397.

[13] R. Rouquier, *Block theory via stable and Rickard equivalences*, Modular representation theory of finite groups (Charlottesville, VA, 1998), 101–146, de Gruyter, Berlin, 2001.

[14] J. Thévenaz, G-Algebras and Modular Representation Theory, Oxford Univ. Press, New York, 1995.

[15] A. Watanabe, *On p-power extensions of cyclic defect blocks of finite groups,* preprint, 2012.

[16] A. Watanabe, *The number of irreducible Brauer characters in a p-block of a finite group with cyclic hyperfocal subgroup*, J. Algebra **416** (2014), 167–183.

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