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ON THE DIOPHANTINE EQUATION IN THE FORM THAT A SUM OF CUBES EQUALS A SUM OF QUINTICS

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Abstract. In this paper, theory of elliptic curves is used for solving the Diophantine equation $a({X'_1}^5+{X'_2}^5)+\sum_{i=0}^{n}a_iX_i{}^5=b({Y'_1}^3+{Y'_2}^3)+\sum_{i=0}^{m}b_iY_i{}^3$, where $n,m\in\mathbb{N}\cup\{0\}$, and, $a,b\neq 0$, a_i, b_i, are fixed arbitrary rational numbers. This equation shows that how sums of some quintics can be written as sums of some cubes. We transform this Diophantine equation to a quartic or cubic elliptic curve. If the corresponding elliptic curve has positive rank, then we get infinitely many solutions for the aforementioned equation. We solve the Diophantine equation for some values of n, m, a, b, a_i, b_i, and obtain infinitely many nontrivial integer solutions for each case.

1. Introduction

Euler conjectured that the Diophantine equation $A^4+B^4+C^4=D^4$, or more generally $A_1^N+A_2^N+\cdots+A_{N-1}^N=A_N^N$, $(N\geq 4)$, has no solution in positive integers (see [1]). Nearly two centuries later, a computer search (see [6]) found the first counterexample to the general conjecture (for $N=5$):

$$27^5+84^5+110^5+133^5=144^5.$$

In 1986 Noam Elkies, by elliptic curves, found counterexamples for the $N=4$ case (see [2]). His smallest counterexample was:

$$2682440^4+15365639^4+18796760^4=20615673^4.$$

In [3], [4] and [5], we used elliptic curves to solve the following three types of Diophantine equations:

$$\sum_{i=1}^{n}a_ix_i^4=\sum_{j=1}^{n}a_jy_j^4, \tag{1.1}$$

where $a_1,\cdots,a_n$ $(n\geq 3)$ are arbitrary fixed integers,

$$X^4+Y^4=2U^4+\sum_{i=1}^{n}T_iU_i^{\alpha_i}, \tag{1.2}$$

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where $n, \alpha_i \in \mathbb{N}$, and T_i, are appropriate fixed rational numbers, and,

$$\sum_{i=1}^{n} a_i x_i^6 + \sum_{i=1}^{m} b_i y_i^3 = \sum_{i=1}^{n} a_i X_i^6 \pm \sum_{i=1}^{m} b_i Y_i^3, \tag{1.3}$$

where n, $m \geq 1$ and a_i, b_i, are arbitrary fixed nonzero integers.

In this paper, we are interested in the study of the Diophantine equation:

$$a({X'_1}^5 + {X'_2}^5) + \sum_{i=0}^{n} a_i {X_i}^5 = b({Y'_1}^3 + {Y'_2}^3) + \sum_{i=0}^{m} b_i {Y_i}^3, \tag{1.4}$$

where $n, m \in \mathbb{N} \cup \{0\}$, and, $a, b \neq 0$, a_i, b_i, are fixed arbitrary rational numbers.

We note that if $(X'_1, X'_2, X_0, \cdots, X_n, Y'_1, Y'_2, Y_0, \cdots, Y_m)$ is a rational solution for the equation (1.4), then for every arbitrary rational number μ, $(\mu^3 X'_1, \mu^3 X'_2, \mu^3 X_0, \cdots, \mu^3 X_n, \mu^5 Y'_1, \mu^5 Y'_2, \mu^5 Y_0, \cdots, \mu^5 Y_m)$ is also a solution. Therefore a rational solution for (1.4) produces an integer solution for (1.4). We try by some auxiliary variables to transform (1.4) to a cubic or quartic elliptic curve of positive rank and get infinitely many integer solutions for (1.4).

2. Main results

Our first main result is the following

Theorem 2.1. *Given the Diophantine equation* (1.4), *define the quartic elliptic curve in t, v by*

$$E_{\{\alpha_i\}_{i=0}^{n}, \{\beta_i\}_{i=0}^{m}, x_1} : \tag{2.1}$$
$$v^2 = \left(\frac{2a + \sum_{i=0}^{n} a_i \alpha_i^5}{6b}\right)t^4 + \left(\frac{20ax_1^2 - 2b - \sum_{i=0}^{m} b_i \beta_i^3}{6b}\right)t^2 + \left(\frac{5a}{3b}\right)x_1^4,$$

where $\{\alpha_i\}_{i=0}^{n}$, $\{\beta_i\}_{i=0}^{m}$ and x_1 are rational parameters. Then any solution (t_0, v_0) to this elliptic curve $E_{\{\alpha_i\}_{i=0}^{n}, \{\beta_i\}_{i=0}^{m}, x_1}$ with given parameters $\{\alpha_i\}_{i=0}^{n}$, $\{\beta_i\}_{i=0}^{m}$ and x_1 produces a solution to (1.4) *by*

$$X'_1 = t + x_1,\ X'_2 = t - x_1,\ Y'_1 = t + v,\ Y'_2 = t - v,$$
$$X_i = \alpha_i t\ (0 \leq i \leq n),\ Y_i = \beta_i t\ (0 \leq i \leq m).$$

Proof. Let us assume that

$$X'_1 = t + x_1, X'_2 = t - x_1, Y'_1 = t + v, Y'_2 = t - v, X_i = \alpha_i t, Y_i = \beta_i t, \tag{2.2}$$

where all the variables are rational numbers. By substituting these variables in (1.4) we get

$$a(2t^5+10x_1^4t+20x_1^2t^3)+\sum_{i=0}^{n}a_i\alpha_i^5t^5=b(2t^3+6tv^2)+\sum_{i=0}^{m}b_i\beta_i^3t^3. \tag{2.3}$$

Then after some simplifications and sorting the equation by degrees in t, we obtain

$$v^2=(\frac{2a+\sum_{i=0}^{n}a_i\alpha_i^5}{6b})t^4+(\frac{20ax_1^2-2b-\sum_{i=0}^{m}b_i\beta_i^3}{6b})t^2+(\frac{5a}{3b})x_1^4. \tag{2.4}$$

Now it is clear that any solution (t_0, v_0) of (2.4) (this solution can be obtained by choosing appropriate values for α_i $(0\le i\le n)$, β_i $(0\le i\le m)$ and x_1) produces a solution to (1.4). This completes the proof. □

Corollary 2.2. If in theorem 2.1 the rank of the quartic (2.4) is positive after choosing appropriate values for α_i $(0\le i\le n)$, β_i $(0\le i\le m)$ and x_1, we obtain infinitely many integer solutions for (1.4). □

Remark 2.3. It is interesting to see that by choosing appropriate values for the variables α_i and β_i so that the relation

$$2a+\sum_{i=0}^{n}a_i\alpha_i=2b+\sum_{i=0}^{m}b_i\beta_i$$

holds, for the obtained solutions we will have

$$a(X_1'+X_2')+\sum_{i=0}^{n}a_iX_i=b(Y_1'+Y_2')+\sum_{i=0}^{m}b_iY_i,$$

too. See the example 3.1.

If the constant term of (2.4), $Q=:(\frac{5a}{3b})x_1^4$, to be square (this is possible for some appropriate values of a and b), we may use the following lemma for transforming the quartic (2.4) to a cubic elliptic curve of the form $y^2+a_1xy+a_3y=x^3+a_2x^2+a_4x+a_6$.

Lemma 2.4 (see [8], page 37, theorem 2.17)**.** *Let K be a field of characteristic not equal to 2. Consider the equation*

$$v^2=au^4+bu^3+cu^2+du+q^2,$$

where all of the coefficients a, b, c, d, q are in K. Let

$$x=\frac{2q(v+q)+du}{u^2},\ \ y=\frac{4q^2(v+q)+2q(du+cu^2)-(\frac{d^2u^2}{2q})}{u^3},$$

and define

$$a_1 = \frac{d}{q},\ a_2 = c - (\frac{d^2}{4q^2}),\ a_3 = 2qb,\ a_4 = -4q^2a,\ a_6 = a_2a_4.$$

Then $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. *The inverse transformation is given by* $u = \frac{2q(x+c)-(\frac{d^2}{2q})}{y}$, $v = -q + \frac{u(ux-d)}{2q}$. *The point* $(u,v) = (0,q)$ *corresponds to the point* $(x,y) = \infty$, *and* $(u,v) = (0,-q)$ *corresponds to* $(x,y) = (-a_2, a_1a_2 - a_3)$. $\square$

Generally, it is not necessary for Q to be square, but we may transform the quartic (2.4) to a new quartic in which the constant number is square if the rank of the quartic (2.4) is positive. The only important thing is that the rank of the quartic elliptic curve (2.4) to be positive for getting infinitely many solutions for (1.4), see the example 3.1.

Before stating our 2nd theorem note that if in the equation (1.4), n and m to be odd, $a_{2k} = a_{2k+1}$ $(0 \leq k \leq \frac{n-1}{2})$, and $b_{2k} = b_{2k+1}$ $(0 \leq k \leq \frac{m-1}{2})$, then the equation (1.4) transforms to the special case

$$\begin{aligned} &a({X'_1}^5 + {X'_2}^5) + a_0(X_0^5 + X_1^5) + a_2(X_2^5 + X_3^5) + \cdots + a_{n-1}(X_{n-1}^5 + X_n^5) \\ &= b({Y'_1}^3 + {Y'_2}^3) + b_0(Y_0^3 + Y_1^3) + b_2(Y_2^3 + Y_3^3) + \cdots + b_{m-1}(Y_{m-1}^3 + Y_m^3). \end{aligned}$$

After renaming the coefficients, the variables and the number of terms, the above equation is in the form

$$\sum_{i=0}^{N} A_i({Z_i}^5 + {z_i}^5) = \sum_{i=0}^{M} B_i({W_i}^3 + {w_i}^3). \tag{2.5}$$

Note that in this case any two consecutive variables in both sides of the equation have the same coefficients. In the following 2nd theorem, we transform (2.5) to a quartic elliptic curve by another method.

Theorem 2.5. *Given the Diophantine equation* (2.5), *define the quartic elliptic curve in* t, v *by*

$$E'_{\{x_i\}_{i=0}^N,\{y_i\}_{i=1}^M} : v^2 = (\frac{\sum_{i=0}^N A_i}{3B_0})t^4 + (\frac{10\sum_{i=0}^N A_i x_i^2 - \sum_{i=0}^M B_i}{3B_0})t^2 + (\frac{5\sum_{i=0}^N A_i x_i^4 - 3\sum_{i=1}^M B_i y_i^2}{3B_0}) \tag{2.6}$$

where $\{x_i\}_{i=0}^N$, *and* $\{y_i\}_{i=1}^M$ *are rational parameters. Then any solution* (t_0, v_0) *to this elliptic curve* $E'_{\{x_i\}_{i=0}^N,\{y_i\}_{i=1}^M}$ *with given parameters* $\{x_i\}_{i=0}^N$, *and* $\{y_i\}_{i=1}^M$ *produces a solution to* (2.5) *by*

$$Z_i = t + x_i,\ z_i = t - x_i,\ W_i = t + y_i,\ w_i = t - y_i \quad (i \geq 0).$$

Proof. Let $Z_i = t + x_i$, $z_i = t - x_i$, $W_i = t + y_i$, $w_i = t - y_i$ $(i \geq 0)$. By substituting these variables in (2.5) we get

$$\sum_{i=0}^{N} A_i(2t^5 + 10x_i^4 t + 20x_i^2 t^3) = \sum_{i=0}^{M} B_i(2t^3 + 6ty_i^2). \tag{2.7}$$

Then after some simplifications, sorting the equation by degrees in t, and letting $y_0 = v$, we obtain

$$\begin{aligned} v^2 = (\frac{\sum_{i=0}^{N} A_i}{3B_0})t^4 + (\frac{10\sum_{i=0}^{N} A_i x_i^2 - \sum_{i=0}^{M} B_i}{3B_0})t^2 \\ + (\frac{5\sum_{i=0}^{N} A_i x_i^4 - 3\sum_{i=1}^{M} B_i y_i^2}{3B_0}). \end{aligned} \tag{2.8}$$

Then, similar to previous theorem, any solution (t_0, v_0) of (2.8) produces a solution to the main equation (2.5). The proof is completed. □

Corollary 2.6. If the rank of (2.6) is positive after choosing appropriate values for x_i $(0 \leq i \leq N)$, y_i $(1 \leq i \leq M)$, then we obtain infinitely many solutions for (2.5). □

3. Application to Examples

3.1. **Example:** $X_1^5 + X_2^5 + X_3^5 = Y_1^3 + Y_2^3 + Y_3^3$**.**
Let: $X_1 = t + x_1$, $X_2 = t - x_1$, $X_3 = \alpha t$, $Y_1 = t + v$, $Y_2 = t - v$, $Y_3 = \beta t$. Then we get

$$v^2 = \frac{2 + \alpha^5}{6}t^4 + \frac{20x_1^2 - 2 - \beta^3}{6}t^2 + (\frac{5}{3})x_1^4. \tag{3.1}$$

Since $(\frac{5}{3})x_1^4$, is not a square, the lemma 2.4 can not be applied. Let us take $x_1 = 1$, $\alpha = \beta = 2$. Then the quartic (3.1) becomes

$$v^2 = \frac{17}{3}t^4 + \frac{5}{3}t^2 + \frac{5}{3}. \tag{3.2}$$

By quick computer search (this can be done by choosing the values $1, ..., 100$ for the variable t, and considering whether the number $\sqrt{\frac{17}{3}t^4 + \frac{5}{3}t^2 + \frac{5}{3}}$ is rational or not), we see that the above quartic has two rational points $P_1 = (1, 3)$, and $P_2 = (7, 117)$ among the other points. Let us put $T = t - 1$. Then we get

$$v^2 = \frac{17}{3}T^4 + \frac{68}{3}T^3 + \frac{107}{3}T^2 + 26T + 9. \tag{3.3}$$

Now with the inverse transformation

$$T = \frac{6(X + \frac{107}{3}) - \frac{26^2}{6}}{Y}, \quad v = -3 + \frac{T(XT - 26)}{6}, \tag{3.4}$$

the quartic (3.3) maps to the cubic elliptic curve

$$Y^2 + \frac{26}{3}XY + 136Y = X^3 + \frac{152}{9}X^2 - 204X - \frac{10336}{3}. \tag{3.5}$$

This elliptic curve is of rank 2 and $P_1 = (X', Y') = (\frac{-152}{9}, \frac{280}{27})$, and $P_2 = (X'', Y'') = (\frac{-44}{3}, \frac{20}{9})$ are two generator of $E(\mathbb{Q})$. (Here, $E(\mathbb{Q})$ denotes the Mordell-Weil group of E.) Letting $M = Y + \frac{13}{3}X + 68$, we get the Weierstrass form

$$M^2 = X^3 + \frac{107}{3}X^2 + \frac{1156}{3}X + \frac{3536}{3}. \tag{3.6}$$

The generators for this elliptic curve are the two points $G_1 = (X', M') = (\frac{-44}{3}, \frac{20}{3})$ and $G_2 = (X'', M'') = (\frac{-152}{9}, \frac{140}{27})$. Thus we transformed the main quartic (3.2) to the cubic elliptic curve (3.6) of rank 2. For G_1, we get the point $(t, v) = (7, -117)$ on (3.2), and finally a solution to X_i, Y_i as follows:

$$8^5 + 6^5 + 14^5 = (-110)^3 + 124^3 + 14^3.$$

It is interesting to see that

$$8 + 6 + 14 = -110 + 124 + 14,$$

too. Also $2G_2$ gives rise to the point $(t, v) = (\frac{11}{47}, \frac{2943}{2209})$ on (3.2) and again a solution for X_i, Y_i, as follows:

$$128122^5 + (-79524)^5 + 48598^5$$
$$= 359227580^3 + (-251874598)^3 + 107352982^3.$$

Since these points are of infinite order, there are infinity solutions for the Diophantine equation.

Remark 3.1. Similarly, by letting $\alpha = 0$ or $\beta = 0$ in (3.1) and choosing appropriate values for the other variables such that the rank of the quartic (3.1) to be positive, we may obtain infinitely many integer solutions for the Diophantine equations $X_1^5+X_2^5 = Y_1^3+Y_2^3+Y_3^3$ and $X_1^5+X_2^5+X_3^5 = Y_1^3+Y_2^3$, respectively. In the following example, we solve one of these equations by another method.

3.2. **Example:** $X_1'^5 + X_2'^5 + X_0{}^5 = Y_1'^3 + Y_2'^3$**.**
Let us first solve the Diophantine equation $3^5(x_1'^5+x_2'^5)+x_0{}^5 = 5^3(y_1'^3+y_2'^3)$, then we can easily get a solution for the main equation. This work is done because we wish the constant term of (2.4) to be suqare. This trick can always be applied for squaring the constant term of (2.4). By letting $\alpha_0 = 1$, and $x_1 = \frac{1}{2}$, the quartic (2.4) becomes

$$v^2 = \frac{487}{750}t^4 + \frac{193}{150}t^2 + (\frac{9}{20})^2. \tag{3.7}$$

With the inverse transformation

$$t = \frac{\frac{9}{10}(X + \frac{193}{150})}{Y} \qquad v = \frac{-9}{20} + \frac{t^2 X}{\frac{9}{10}} \tag{3.8}$$

the quartic (3.7) maps to the cubic elliptic curve

$$Y^2 = X^3 + \frac{193}{150}X^2 - \frac{13149}{25000}X - \frac{845919}{1250000}. \tag{3.9}$$

This elliptic curve is of rank 1 and

$$P = (X, Y) = (\frac{86566439079589977}{6406966694860900}, \frac{266164579103117452813038 63}{512836230849099294977000})$$

is a generator of $E(\mathbb{Q})$. Then we get

$$(t, v) = (\frac{9545509085509985}{37198561615950929}, \frac{1490430349241988789626953913910028 1}{2767465972591395580443769871926082 0}),$$

and a rational solution for the title equation is as follows

$$X_1' = 3.(\frac{56289579786970899}{74397123231901858}), \quad X_2' = 3.(\frac{-18107543444930959}{74397123231901858}),$$
$$X_0 = \frac{9545509085509985}{37198561615950929},$$
$$Y_1' = 5.(\frac{22005887649879139538857553129621581}{27674659725913955804437698719260820}),$$
$$Y_2' = 5.(\frac{-7802719334960636253681525148578981}{27674659725913955804437698719260820}).$$

Remark 3.2. It is a well-known theorem that every integer number can be written as a sum of four cubics. Two above examples show that how (infinitely many) sums of 3 quintics can be written as sums of 2 or 3 cubics.

3.3. **Example:** $5.({X_1'}^5 + {X_2'}^5) = 3.({Y_1'}^3 + {Y_2'}^3)$.

For this equation, (2.8) becomes

$$v^2 = \frac{5}{9}t^4 + \frac{50x_1^2 - 3}{9}t^2 + (\frac{25}{9})x_1^4, \tag{3.10}$$

with square constant term on the right side. Letting $x_1 = 1$, the quartic elliptic curve (3.10) becomes

$$v^2 = \frac{5}{9}t^4 + \frac{47}{9}t^2 + (\frac{25}{9}). \tag{3.11}$$

With the inverse transformation

$$t = \frac{\frac{10}{3}(X + \frac{47}{9})}{Y} \qquad v = \frac{-5}{3} + \frac{t^2 X}{\frac{10}{3}} \tag{3.12}$$

the quartic (3.11) maps to the cubic elliptic curve

$$Y^2 = X^3 + \frac{47}{9}X^2 - \frac{500}{81}X - \frac{23500}{729}. \tag{3.13}$$

This elliptic curve is of rank 1 and $P = (X, Y) = (\frac{-609566}{164025}, \frac{-225298052}{66430125})$ is a generator of $E(\mathbb{Q})$. Then we get $(t, v) = (\frac{-335475}{226658}, \frac{-633289965055}{154121546892})$, and

$$X_1' = -150939399313320876,$$
$$X_2' = -779731267671365724,$$
$$Y_1' = -812465974773035511128800250760,$$
$$Y_2' = 382156766244967634925328109160.$$

3.4. **Example:** $n.(X_1^5 + X_2^5 + X_3^5 + X_4^5) = m.(Y_1^3 + Y_2^3 + Y_3^3 + Y_4^3)$.

From this equation, (2.8) becomes

$$v^2 = (\frac{2n}{3m})t^4 + (\frac{10nx_1^2 + 10nx_2^2 - 2m}{3m})t^2 + (\frac{5nx_1^4 + 5nx_2^4 - 3my_2^2}{3m}). \tag{3.14}$$

By the lemma 2.5 we need the constant term

$$(\frac{5nx_1^4 + 5nx_2^4 - 3my_2^2}{3m}), \tag{3.15}$$

to be a square, say q^2. This can be done if for values of n, m, there exist values of x_1, x_2, y_2 to make the expression square. For instance, if $m = 85$, $n = 6$, we may take $x_1 = 1$, $x_2 = 2$, $y_2 = 1$, $(q = 1)$. Then (3.15) becomes

$$v^2 = \frac{4}{85}t^4 + \frac{26}{51}t^2 + 1. \tag{3.16}$$

With the inverse transformation

$$t = \frac{2(X + \frac{26}{51})}{Y} \qquad v = -1 + \frac{t^2X}{2} \tag{3.17}$$

the corresponding cubic elliptic curve is

$$Y^2 = X^3 + \frac{26}{51}X^2 - \frac{16}{85}X - \frac{416}{4335}. \tag{3.18}$$

This elliptic curve is of rank 2 and the points $P_1 = (X', Y') = (\frac{2}{3}, \frac{28}{51})$, $P_2 = (X'', Y'') = (\frac{20777}{21675}, \frac{-1908281}{1842375})$ are two generators of $E(\mathbb{Q})$. For the point P_1, the point $(t, v) = (\frac{30}{7}, \frac{251}{49})$ is on (3.16) and finally we get a solution for the Diophantine equation as

$$6.(1813^5 + 1127^5 + 2156^5 + 784^5)$$
$$= 85.(158123^3 + (-14063)^3 + 88837^3 + 55223^3).$$

If $m = 17$, $n = 3$, in (3.14), we may choose appropriate values $x_1 = 1$, $x_2 = 2$, $y_2 = 1$, $(q = 2)$. Then (3.14) becomes

$$v^2 = \frac{2}{17}t^4 + \frac{116}{51}t^2 + 4. \tag{3.19}$$

With the inverse transformation

$$t = \frac{4(X + \frac{116}{51})}{Y} \qquad v = -2 + \frac{t^2 X}{4} \tag{3.20}$$

and the corresponding cubic elliptic curve

$$Y^2 = X^3 + \frac{116}{51}X^2 - \frac{32}{17}X - \frac{3712}{867}. \tag{3.21}$$

This elliptic curve is of rank 1 and $P = (X, Y) = (4, \frac{160}{17})$ is a generator of $E(\mathbb{Q})$. From the point P we get $(t, v) = (\frac{8}{3}, \frac{46}{9})$ and the corresponding solution for the Diophantine equation as

$$3.(99^5 + 45^5 + 126^5 + 18^5) = 17.(1890^3 + (-594)^3 + 891^3 + 405^3).$$

The Sage software has been used for calculating the rank of the elliptic curves (see [7]).

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References

[1] L. E. Dickson, History of the Theory of Numbers, Vol. II: Diophantine Analysis, G. E. Stechert, 1934

[2] N. Elkies, *On $A^4 + B^4 + C^4 = D^4$*. Math. Comp. **51** (1987) no. 184, 825-835.

[3] F. Izadi and M. Baghalaghdam, *On the Diophantine equation $\sum_{i=1}^n a_i x_i^4 = \sum_{j=1}^n a_j y_j^4$*, submitted, 2016.

[4] F. Izadi and M. Baghalaghdam, *On the Diophantine equation $X^4 + Y^4 = 2U^4 + \sum_{i=1}^n T_i U_i^{\alpha_i}$*, submitted, 2016.

[5] F. Izadi and M. Baghalaghdam, *On the Diophantine equations $\sum_{i=1}^n a_i x_i^6 + \sum_{i=1}^m b_i y_i^3 = \sum_{i=1}^n a_i X_i^6 \pm \sum_{i=1}^m b_i Y_i^3$*, submitted, 2016.

[6] L. J. Lander, T. R. Parkin *Counterexamples to Euler's conjecture on sums of like powers*, Bull. Amer. Math. Soc, **72** (1966), p.1079.

[7] Sage software, available from http://sagemath.org.

[8] L. C. Washington, Elliptic Curves: Number Theory and Cryptography, Chapman-Hall, 2008.

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