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A BINOMIAL-COEFFICIENT IDENTITY ARISING FROM THE MIDDLE DISCRETE SERIES OF SU(2, 2)

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Abstract. The aim of this paper is to answer the question in Remark 8.2 of Takahiro Hayata, Harutaka Koseki, and Takayuki Oda, *Matrix coefficients of the middle discrete series of* SU(2, 2), J. Funct. Anal. **185** (2001), 297–341, by giving an elementary proof of certain identities on binomials.

1. Introduction

The aim of this paper is to show an elementary proof of certain identities on binomials and state an answer to [4, Remark 8.2].

Since the identity which we prove in this paper stems from the representation theory of real semi-simple Lie groups, which admit discrete series, we begin with describing the representation theoretical aspect of the identity. We borrow the terminology about the Lie groups and their representations from [4] only in this section.

Let G be a real semi-simple Lie group with finite center and K be its maximal compact group. Take an irreducible unitary representation π of G. Consider the map

$$\phi : \pi \to C^\infty(K\backslash G; \tau)$$

We identify the map ϕ with the $\tau \otimes \tau^*$-valued function $\phi(g)$ of G in $C^\infty(K\backslash G/K; \tau \otimes \tau^*)$. For a K-finite vector v, $\phi(g) \in C^\infty(K\backslash G/K; \tau \otimes \tau^*)$ satisfies, by definition, $\phi(kgk') = \tau(k^{-1})\tau^*(k')\phi(g)$ $(k, k' \in K, g \in G)$ We say $\phi(g)$ a matrix coefficient of π with respect to τ. Because G has the Cartan decomposition $G = KAK$ where A is a maximal split torus in G, the radial part, *i.e.*, the restriction of matrix coefficients to A, is regarded as a $\tau \otimes \tau^*$-valued function on Euclidean domain.

Now we assume $\mathrm{rank}(G) = \mathrm{rank}(K)$ for G to admit a discrete series representation also denoted by π. If G is of hermitian type and π is holomorphic, then $\phi(g)$ is described by the Laurent polynomials of certain hyperbolic functions if we take its radial part $\phi|_A(a)$ for $a \in A$, known in the theory of

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Bergman kernels on the symmetric domain. If the Gel'fand-Kirillov dimension of π is high enough, the radial part can be also highly transcendental. But the dimension is relatively low, the radial part function is expected to be tractable. In fact, it is turned out to be feasible when G is a unitary group of degree 4 defined by the hermitian form of type $(2,2)$, say, $|z_1|^2+|z_2|^2-|z_3|^2-|z_4|^2$, and π is in the second lowest discrete series (in [2], [4], it is called *a middle discrete series*).

Since the situation that the unitary group G with respect to the hermitian form of type $(2,2)$ is the origin of the binomial relations in this paper, we briefly review the ingredients of [4]. We specify the situation as follows.

We take the representation π from the middle discrete series. We fix the K-type τ as the minimal one (cf. [5, Chapter VIII]). In this case $\dim A = 2$ which means the radial part of matrix coefficient $\phi|_A$ is essentially a function on $\mathbb{R}^2_{>0}$, *i.e.*, the function can be characterized by two-variable functions. Since $\phi(g)$ admits an action of differential operators [4, Section 3], we find the set of differential equations satisfied by $\phi|_A(a)$, which forcibly leads us to the function into separation of variables [4, Section 4]; one side is described by polynomials and the other side is essentially a Gaussian hypergeometric function ${}_2F_1$ as the reflection of the Gel'fand-Kirillov dimension of the middle discrete series representation of G. The binomials $\beta_m(r,s,k,l)$ we treat here in Equation 1.2 are nothing but the coefficients appearing on the polynomial side.

Besides, the structure inside the polynomial side itself has its own remarkable interest. We explain the direct connection of the binomial relation in this paper and the result in [4].

The coefficients appearing on the polynomial side of $\phi|_A(a)$ are represented by the sum of products of binomials with respect to the parameters from the representation. The proof of Theorem 8.1 in [4] uses generating functions to show that the polynomial side can be represented by such product of the binomial coefficients. Since the matrix coefficients at the group unit $e \in G$ reduces to the unit matrix $I_{r\times r}$ of the size of the dimension $r = \dim\tau$ of K-type τ, we have an identity $\phi(e) = cI_{r\times r}$ if we suitably choose the constant multiple c (cf. [4, Remark 8.1]). We then have an identity of the sum of products of binomials ([4, Remark 8.2], referred as *the binomial-coefficient identities* (Theorem 1.1), which we discuss in this paper.

However we can say that these binomial-coefficient identities hold as a corollary of Theorem 8.1 in [4] using the definition of the matrix coefficients as also pointed in [4, Remark 8.2], we attempted the different way in this paper, motivated by the question in [4, Remark 8.2]. We prove the binomial-coefficient identities in a direct manner using the elementary combinatorics of the binomials in Theorem 1.1. The proof is in §2. We remark that the proof in [4] is hardly specialized to an elementary proof as we did in Theorem 1.1 because of the method using generating functions.

Because we believe this kind of computation against matrix coefficients should work in somewhat broader contexts (for instance [3]) and likely to produce similar identities containing involved binomial coefficients, we hope our computation helps those who try to prove them. Even apart from the theory of the representations the real semi-simple Lie groups, the binomial coefficients have the vast extension in the combinatorics. We hope not only the identity but also the elementary proof could contribute to a generalization or a similar topic of other fields.

Next, some terminology is defined before stating the main theorem. Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_{>0}$ denote the set of integers, non-negative integers and positive integers, respectively. If k is a positive integer and $r_1, \ldots, r_k$ are integers such that $n = r_1 + \cdots + r_k$ is non-negative integer, the *multinomial coefficient* $\binom{n}{r_1,\ldots,r_k}$ is, by definition,

$$\begin{cases} \frac{n!}{r_1!\cdots r_k!} & \text{if all } r_i \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

In particular, when $k = 2$, $\binom{n}{r} = \binom{n}{r,n-r}$ is called the *binomial coefficient*. (For many interesting identities these famous coefficients satisfy, see [6].) When a, b s, l and m are integers such that $s \geq a \geq 0$ and $s \geq l$, we write

$$\beta_m(s,l,a,b) = \sum_{n=0}^{|b|+m} \binom{s-a}{b_+ + m - n} \binom{a}{b_- + m - n} \binom{s-l+n}{n}, \tag{1.2}$$

where b_+ and b_- are defined by

$$b_+ + b_- = |b|, \qquad b_+ - b_- = b. \tag{1.3}$$

In other word $b_\pm$ is defined to be $\frac{|b| \pm b}{2}$. The aim of this paper is to give an elementary proof of the following theorem.

Theorem 1.1. *Let s and l be non-negative integers such that $s \geq l \geq 0$. Let j be an integer. Then we have*

$$\sum_{m=0}^{\lfloor (l-1)/2 \rfloor} \sum_{i=2m}^{l-1} (-2)^{i-2m} \left\{ \binom{s-l}{i-2m, j-l+m, s-i-j+m} \times \beta_m(s,l,i+j-l,l-i) + \binom{s-l}{i-2m, j-i+m, s-l-j+m} \beta_m(s,l,l+j-i,i-l) \right\}$$

(1.4)

$$+\sum_{m=0}^{\lfloor l/2\rfloor}(-2)^{l-2m}\binom{s-l}{l-2m,j-l+m,s-l-j+m}\beta_m(s,l,j,0)=\binom{s}{j},$$

where $\lfloor x\rfloor$ stands for the largest integer less than or equal to x for any real number x. It is amusing that the left-hand side includes the parameter l, which eventually equals the right-hand side that is independent of l. This fact is highly nontrivial from the appearance of the left-hand side.

2. Proof of the identity

First we summarize certain recurrence properties of β_m as follows.

Lemma 2.1. *Let s, l, a and b be integers such that $s \geq l$ and $s \geq a \geq 0$. Then the following identities hold.*

(i) *If $b > 0$, then*

$$\beta_m(s,l,a,b)=\beta_m(s+1,l,a,b)-\beta_m(s+1,l,a+1,b-1). \tag{2.1}$$

(ii) *If $a > 0$ and $b \geq 0$, then*

$$\beta_m(s,l,a-1,b)=\beta_m(s+1,l,a,b)-\beta_{m-1}(s+1,l,a-1,b+1). \tag{2.2}$$

(iii) *If $b \leq 0$, then*

$$\beta_m(s,l,a,b)=\beta_m(s+1,l,a,b)-\beta_{m-1}(s+1,l,a+1,b-1). \tag{2.3}$$

(iv) *If $a > 0$ and $b < 0$, then*

$$\beta_m(s,l,a-1,b)=\beta_m(s+1,l,a,b)-\beta_m(s+1,l,a-1,b+1). \tag{2.4}$$

Proof. We only use the well-known recurrence equation of binomial coefficients which reads

$$\binom{n+1}{r}=\binom{n}{r}+\binom{n}{r-1}, \tag{2.5}$$

and perform direct computations to prove these identities. First we prove (2.1). Applying the recurrence (2.5) to the first and third binomial coefficients of (1.2), we obtain that $\beta_m(s,l,a,b)$ equals

$$\sum_{n\geq 0}\binom{s-a+1}{b+m-n}\binom{a}{m-n}\binom{s-l+n+1}{n}$$
$$-\sum_{n\geq 0}\binom{s-a}{b+m-n-1}\binom{a}{m-n}\binom{s-l+n+1}{n}$$
$$-\sum_{n\geq 0}\binom{s-a}{b+m-n}\binom{a}{m-n}\binom{s-l+n}{n-1}.$$

If we apply the recurrence $\binom{a}{m-n} = \binom{a+1}{m-n} - \binom{a}{m-n-1}$ to the second term, then we obtain this equals

$$\beta_m(s+1,l,a,b) - \sum_{n\ge 0}\binom{s-a}{b+m-n-1}\binom{a+1}{m-n}\binom{s-l+n+1}{n}$$
$$-\sum_{n\ge 0}\binom{s-a}{b+m-n}\binom{a}{m-n}\binom{s-l+n}{n-1}$$
$$+\sum_{n\ge 0}\binom{s-a}{b+m-n-1}\binom{a}{m-n-1}\binom{s-l+n+1}{n}.$$

The last two terms kill each other and consequently we obtain

$$\beta_m(s,l,a,b) = \beta_m(s+1,l,a,b) - \beta_m(s+1,l,a+1,b-1).$$

This proves the first identity. The other identities can be proven similarly. The details are left to the reader. □

Let s, l, m, j be integers such that $s \ge l$ and $m \ge 0$. We define $\Lambda_m(s,l,j)$ by

$$\Lambda_m(s,l,j) = \sum_{i=2m}^{l-1}(-2)^{i-2m}\binom{s-l}{i-2m,\, j-l+m,\, s-i-j+m}\beta_m(s,l,i+j-l,l-i)$$
$$+\sum_{i=2m}^{l}(-2)^{i-2m}\binom{s-l}{i-2m,\, j-i+m,\, s-l-j+m}\beta_m(s,l,l+j-i,i-l). \tag{2.6}$$

Then $\Lambda_m(s,l,j)$ satisfies the following recurrence equation.

Lemma 2.2. *Let s, l, m, j be integers such that $s \ge l$ and $j \ge 0$. Then*

$$\Lambda_m(s,l,j) + \Lambda_m(s,l,j-1) = \Lambda_m(s+1,l,j) + \Phi_m(s,l,j) - \Phi_{m-1}(s,l,j) \tag{2.7}$$

where

$$\Phi_m(s,l,j) = \sum_{i=2m+1}^{l-1}(-2)^{i-2m-1}\Bigg\{\binom{s-l}{i-2m-1,\, j-l+m,\, s-i-j+m+1} \times \beta_m(s+1,l,i+j-l,l-i)$$
$$+\binom{s-l}{i-2m-1,\, j-i+m,\, s-l-j+m+1}\beta_m(s+1,l,l+j-i,i-l)\Bigg\}. \tag{2.8}$$

Proof. By (2.1) and (2.3), we obtain $\Lambda_m(s,l,j)$ equals

$$\sum_{i=2m}^{l-1}(-2)^{i-2m}\binom{s-l}{i-2m,j-l+m,s-i-j+m}$$
$$\times\left\{\beta_m(s+1,l,i+j-l,l-i)-\beta_m(s+1,l,i+j-l+1,l-i-1)\right\}$$
$$+\sum_{i=2m}^{l}(-2)^{i-2m}\binom{s-l}{i-2m,j-i+m,s-l-j+m}$$
$$\times\left\{\beta_m(s+1,l,l+j-i,i-l)-\beta_{m-1}(s+1,l,l+j-i+1,i-l-1)\right\}.$$

Similarly, using (2.2) and (2.4), we can rewrite $\Lambda_m(s,l,j-1)$ as

$$\sum_{i=2m}^{l}(-2)^{i-2m}\binom{s-l}{i-2m,j-l+m-1,s-i-j+m+1}$$
$$\times\left\{\beta_m(s+1,l,i+j-l,l-i)-\beta_{m-1}(s+1,l,i+j-l-1,l-i+1)\right\}$$
$$+\sum_{i=2m}^{l-1}(-2)^{i-2m}\binom{s-l}{i-2m,j-i+m-1,s-l-j+m+1}$$
$$\times\left\{\beta_m(s+1,l,l+j-i,i-l)-\beta_m(s+1,l,l+j-i-1,i-l+1)\right\}.$$

Adding these two identities, we obtain that $\Lambda_m(s,l,j)+\Lambda_m(s,l,j-1)$ is equal to

$$\sum_{i=2m}^{l}(-2)^{i-2m}A\,\beta_m(s+1,l,i+j-l,l-i)$$
$$+\sum_{i=2m}^{l-1}(-2)^{i-2m}B\,\beta_m(s+1,l,l+j-i,i-l)$$
$$-\sum_{i=2m}^{l-1}(-2)^{i-2m}\binom{s-l}{i-2m,j-l+m,s-i-j+m}$$
$$\times\beta_m(s+1,l,i+j-l+1,l-i-1)$$
$$-\sum_{i=2m}^{l}(-2)^{i-2m}\binom{s-l}{i-2m,j-i+m,s-l-j+m}$$
$$\times\beta_{m-1}(s+1,l,l+j-i+1,i-l-1)$$

$$
\begin{aligned}
&- \sum_{i=2m}^{l} (-2)^{i-2m} \binom{s-l}{i-2m,\, j-l+m-1,\, s-i-j+m+1} \\
&\qquad \times \beta_{m-1}(s+1, l, i+j-l-1, l-i+1) \\
&- \sum_{i=2m}^{l-1} (-2)^{i-2m} \binom{s-l}{i-2m,\, j-i+m-1,\, s-l-j+m+1} \\
&\qquad \times \beta_{m}(s+1, l, l+j-i-1, i-l+1),
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \binom{s-l}{i-2m,\, j-l+m,\, s-i-j+m} \\
&\quad + \binom{s-l}{i-2m,\, j-l+m-1,\, s-i-j+m+1}, \\
B &= \binom{s-l}{i-2m,\, j-i+m,\, s-l-j+m} \\
&\quad + \binom{s-l}{i-2m,\, j-i+m-1,\, s-l-j+m+1}.
\end{aligned}
$$

If we replace i by $i+1$ or $i-1$ in the last four terms, then this sum becomes

$$
\begin{aligned}
&\sum_{i=2m}^{l} (-2)^{i-2m} A\, \beta_m(s+1, l, i+j-l, l-i) \\
&+ \sum_{i=2m}^{l-1} (-2)^{i-2m} B\, \beta_m(s+1, l, l+j-i, i-l) \\
&- \sum_{i=2m+1}^{l} (-2)^{i-2m-1} \binom{s-l}{i-2m-1,\, j-l+m,\, s-i-j+m+1} \\
&\qquad \times \beta_m(s+1, l, i+j-l, l-i) \\
&- \sum_{i=2m-1}^{l-1} (-2)^{i-2m+1} \binom{s-l}{i-2m+1,\, j-i+m-1,\, s-l-j+m} \\
&\qquad \times \beta_{m-1}(s+1, l, l+j-i, i-l) \\
&- \sum_{i=2m-1}^{l-1} (-2)^{i-2m+1} \binom{s-l}{i-2m+1,\, j-l+m-1,\, s-i-j+m} \\
&\qquad \times \beta_{m-1}(s+1, l, i+j-l, l-i)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{i=2m+1}^{l} (-2)^{i-2m-1} \binom{s-l}{i-2m-1, j-i+m, s-l-j+m+1} \\
&\qquad \times \beta_m(s+1, l, l+j-i, i-l).
\end{aligned}
$$

Using

$$
\begin{aligned}
&A + \binom{s-l}{i-2m-1, j-l+m, s-i-j+m+1} \\
&= \binom{s-l+1}{i-2m, j-l+m, s-i-j+m+1}, \\
&B + \binom{s-l}{i-2m-1, j-i+m, s-l-j+m+1} \\
&= \binom{s-l+1}{i-2m, j-i+m, s-l-j+m+1},
\end{aligned}
$$

we see that $\Lambda_m(s,l,j) + \Lambda_m(s,l,j-1)$ is equal to

$$
\begin{aligned}
&\sum_{i=2m}^{l-1} (-2)^{i-2m} \binom{s-l+1}{i-2m, j-l+m, s-i-j+m+1} \\
&\qquad \times \beta_m(s+1, l, i+j-l, l-i) \\
&+ \sum_{i=2m}^{l} (-2)^{i-2m} \binom{s-l+1}{i-2m, j-i+m, s-l-j+m+1} \\
&\qquad \times \beta_m(s+1, l, l+j-i, i-l) \\
&+ \sum_{i=2m+1}^{l-1} (-2)^{i-2m-1} \binom{s-l}{i-2m-1, j-l+m, s-i-j+m+1} \\
&\qquad \times \beta_m(s+1, l, i+j-l, l-i) \\
&- \sum_{i=2m-1}^{l-1} (-2)^{i-2m+1} \binom{s-l}{i-2m+1, j-i+m-1, s-l-j+m} \\
&\qquad \times \beta_{m-1}(s+1, l, l+j-i, i-l) \\
&- \sum_{i=2m-1}^{l-1} (-2)^{i-2m+1} \binom{s-l}{i-2m+1, j-l+m-1, s-i-j+m} \\
&\qquad \times \beta_{m-1}(s+1, l, i+j-l, l-i) \\
&+ \sum_{i=2m+1}^{l-1} (-2)^{i-2m-1} \binom{s-l}{i-2m-1, j-i+m, s-l-j+m+1}
\end{aligned}
$$

$$\times\, \beta_m(s+1, l, l+j-i, i-l),$$

which is equal to the right-hand side of (2.7). This completes the proof of the lemma. □

Proof of Theorem 1.1. Assume $s \geq l \geq 0$. If we put

$$\Gamma(s, l, j) = \sum_{m=0}^{\infty} \Lambda_m(s, l, j),$$

then, by (2.7), it is easy to see that

$$\Gamma(s+1, l, j) = \Gamma(s, l, j) + \Gamma(s, l, j-1) \tag{2.9}$$

holds. In addition, if $j < 0$ or $j > s$, then we have $\Lambda_m(s, l, j) = 0$ for all $m \geq 0$ since the multinomial coefficients vanish in the definition (2.6). If $j = 0$, then we also have

$$\Lambda_m(s, l, j) = \begin{cases} \beta_0(s, l, l, -l) = 1 & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

Hence, we have

$$\Gamma(s, l, j) = \begin{cases} 0 & \text{if } j < 0 \text{ or } j > s, \\ 1 & \text{if } j = 0. \end{cases} \tag{2.10}$$

From (2.9) and (2.10), we conclude that $\Gamma(s, l, j) = \binom{s}{j}$. This completes the proof. □

3. Concluding remarks

An interesting question we can ask is "Can we make a q-analogue of the identity (1.4)?". (For q-series, the reader can refer to [1].) We had a trial in this direction which is not yet complete. For example, define $\beta_m[s, l, a, b; q]$ by

$$\begin{aligned} &\beta_m[s, l, a, b; q] \\ &= \sum_{n=0}^{|b|+m} q^{n(n-|b|+l-2m)} \begin{bmatrix} s-a \\ b_+ + m - n \end{bmatrix}_q \begin{bmatrix} a \\ b_- + m - n \end{bmatrix}_q \begin{bmatrix} s-l+n \\ n \end{bmatrix}_q, \end{aligned} \tag{3.1}$$

as a q-analogue of $\beta_m(s, l, a, b)$, where

$$\begin{bmatrix} r_1 + \cdots + r_k \\ r_1, \ldots, r_k \end{bmatrix}_q = \begin{cases} \frac{[r_1+\cdots+r_k]_q!}{[r_1]_q!\cdots[r_k]_q!} & \text{if all } r_i \geq 0, \\ 0 & \text{otherwise}, \end{cases}, \qquad \begin{bmatrix} n \\ r \end{bmatrix}_q = \begin{bmatrix} n \\ r, n-r \end{bmatrix}_q,$$

with $[n]_q! = (1+q)\cdots(1+q+\cdots+q^{n-1})$. Then one can prove that $\beta_m[s,l,a,b;q]$ satisfies the following simple recurrence equations, which can be considered as a q-analogue of the recurrence equations in Lemma 2.1.

Proposition 3.1. *Let s, l, a and b be integers such that $s \geq l$ and $s \geq a \geq 0$. Then the following identities hold.*

(i) *If $b > 0$, then*

$$\beta_m[s,l,a,b;q] = \beta_m[s+1,l,a,b;q] - q^{s-a-b-m+1}\beta_m[s+1,l,a+1,b-1;q]. \tag{3.2}$$

(ii) *If $a > 0$ and $b \geq 0$, then*

$$\beta_m[s,l,a-1,b;q] = \beta_m[s+1,l,a,b;q] - q^{a-m}\beta_{m-1}[s+1,l,a-1,b+1;q]. \tag{3.3}$$

(iii) *If $b \leq 0$, then*

$$\beta_m[s,l,a,b;q] = \beta_m[s+1,l,a,b;q] - q^{s-a-m+1}\beta_{m-1}[s+1,l,a+1,b-1;q]. \tag{3.4}$$

(iv) *If $a > 0$ and $b < 0$, then*

$$\beta_m[s,l,a-1,b;q] = \beta_m[s+1,l,a,b;q] - q^{a+b-m}\beta_m[s+1,l,a-1,b+1;q]. \tag{3.5}$$

Nevertheless, at this point, we do not know how to define a q-analogue of $\Lambda_m(s,l,j)$ which has a simple recurrence equation as in Lemma 2.2.

Another interesting question we can ask is the following. One can see that the left-hand side of (1.4) is a sum (a double sum or triple sum), but the right-hand side is so simple, i.e., just a binomial coefficient $\binom{s}{j}$. Does Gasper's algorithm in [7] work to prove this identity? The left-hand side of (1.4) is a finite sum of a product of binomial coefficients which is a hypergeometric summation. But it is not a single sum and there are several cases in the definition of $\beta_m(s,l,a,b)$. So we have no idea how to prove the main result by the creative telescoping using a computer.

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