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INDECOMPOSABILITY OF VARIOUS PROFINITE GROUPS ARISING FROM HYPERBOLIC CURVES

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Abstract. In this paper, we prove that the étale fundamental group of a hyperbolic curve over an arithmetic field [e.g., a finite extension field of $\mathbb{Q}$ or $\mathbb{Q}_p$] or an algebraically closed field is indecomposable [i.e., cannot be decomposed into the direct product of nontrivial profinite groups]. Moreover, in the case of characteristic zero, we also prove that the étale fundamental group of the configuration space of a curve of the above type is indecomposable. Finally, we consider the topic of indecomposability in the context of the comparison of the absolute Galois group of $\mathbb{Q}$ with the Grothendieck-Teichmüller group GT and pose the question: Is GT indecomposable? We give an affirmative answer to a pro-l version of this question.

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Introduction

In Introduction, we shall write Π_S for the étale fundamental group of a connected noetherian scheme S. Moreover, for a field K, we shall write $\overline{K}$ (respectively, G_K) for an algebraic closure of K (respectively, the absolute Galois group of K). In [9], [10], Grothendieck introduced the notion of an "anabelian variety". He refers to a variety V over a finitely generated

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extension field F of $\mathbb{Q}$ which may be "reconstructed" from the natural [outer] surjection

$$\Pi_V \twoheadrightarrow G_F$$

as an "anabelian variety". But we do not know any general [rigorous] conditions which characterize the "anabelianity". On the other hand, it is considered that the notion of the slimness of profinite groups is deeply related to the "anabelianity" [cf. [16]]. [We shall say that a profinite group G is slim if every open subgroup of G is center-free.] In fact, in the one dimensional case, the following equivalences are known: For a geometrically connected, smooth curve C over F,

$$C \text{ is anabelian} \ \Leftrightarrow\ C \text{ is hyperbolic} \ \Leftrightarrow\ \Pi_{C\times_F\overline{F}} \text{ is nontrivial and slim}$$

[cf., e.g., [27]].

In this paper, as a notion which is deeply related to the "anabelianity", we adopt the strong indecomposability of profinite groups. The term strong indecomposability is defined as follows [cf. Definition 1.1]:

> We shall say that a profinite group G is *indecomposable* if, for any isomorphism of profinite groups $G \cong G_1 \times G_2$, where G_1, G_2 are profinite groups, it follows that either G_1 or G_2 is the trivial group. We shall say that G is *strongly indecomposable* if every open subgroup of G is indecomposable.

Indeed, there is a similarity between the "anabelianity" [of a variety V] and the strong indecomposability [of a profinite group G], as follows:

(a) Let $\rho : G_F \to \mathrm{Out}(\Pi_{V\times_F\overline{F}})$ be the natural outer Galois representation associated to V. To test the "anabelianity" of V, we need to consider the Galois centralizer

$$Z_{\mathrm{Out}(\Pi_{V\times_F\overline{F}})}(\mathrm{Im}(\rho))$$

[cf. [16]]. In other words, we need to consider a *subgroup* $\mathrm{Im}(\rho) \subseteq \mathrm{Out}(\Pi_{V\times_F\overline{F}})$ and a *subgroup* $Z_{\mathrm{Out}(\Pi_{V\times_F\overline{F}})}(\mathrm{Im}(\rho)) \subseteq \mathrm{Out}(\Pi_{V\times_F\overline{F}})$, which *commutes* with $\mathrm{Im}(\rho)$.

(b) To test the strong indecomposability of G, we need to consider, for every open subgroup H of G, whether or not H has a decomposition

$$H = H_1 \times H_2$$

— where H_1, H_2 are nontrivial. In other words, we need to consider a *subgroup* $H_1 \subseteq H$ and a *subgroup* $H_2 \subseteq H$, which *commutes* with H_1.

In this paper, we prove that various profinite groups, which appear in anabelian geometry, are, in fact, strongly indecomposable. In the following, for a prime number l, we shall write $G^{(l)}$ for the maximal pro-l quotient of a profinite group G. First, in the one dimensional case, we prove the following [cf. Theorems 3.1, 3.6; Proposition 3.2]:

Theorem A. *Let k be an algebraically closed field of characteristic $p \geq 0$; $l \neq p$ a prime number; X a smooth curve of type (g, r) over k. Then the following hold:*

(i) *Suppose that $p = 0$. If $2g - 2 + r > 0$, then Π_X, $\Pi_X^{(l)}$ are slim and strongly indecomposable.*

(ii) *Suppose that $p > 0$. If $(g, r) \neq (0, 0)$, $(1, 0)$ (respectively, $2g-2+r > 0$), then Π_X (respectively, $\Pi_X^{(l)}$) is slim and strongly indecomposable.*

We note that the characteristic zero case and the pro-l case of Theorem A are well-known [cf. [25], Proposition 3.2].

Next, we consider the case that the base field is non-algebraically closed. Let k be a field; V a geometrically connected scheme of finite type over k. In particular, the composite $V \times_k \overline{k} \to V \to \mathrm{Spec}(k)$ induces the following exact sequence of profinite groups

$$1 \longrightarrow \Pi_{V \times_k \overline{k}} \longrightarrow \Pi_V \longrightarrow G_k \longrightarrow 1$$

[cf. [8], EXPOSÉ IX, Théorème 6.1]. In the following, for any prime number l, we shall write

$$\Pi_V^l \stackrel{\mathrm{def}}{=} \Pi_V / \mathrm{Ker}(\Pi_{V \times_k \overline{k}} \twoheadrightarrow \Pi_{V \times_k \overline{k}}^{(l)})$$

— where $\Pi_{V \times_k \overline{k}} \twoheadrightarrow \Pi_{V \times_k \overline{k}}^{(l)}$ is the natural surjection. Then we can prove the following theorem [cf. Theorem 4.3]:

Theorem B. *Let k be a field of characteristic $p \geq 0$ such that G_k is slim and strongly indecomposable; X a smooth curve of type (g, r) over k. Suppose that there exists a prime number $l \neq p$ satisfying the following condition:*

> $(*_k^l)$ *For any finite extension field k' of k, the l-adic cyclotomic character $\chi_{k'} : G_{k'} \to \mathbb{Z}_l^\times$ of k' is nontrivial.*

Then the following hold:

(i) *Suppose that $p = 0$. If $2g - 2 + r > 0$, then Π_X, Π_X^l are slim and strongly indecomposable.*

(ii) *Suppose that* $p > 0$. *If* $(g, r) \neq (0, 0)$, $(1, 0)$ *(respectively,* $2g-2+r > 0$*), then* Π_X *(respectively,* Π^l_X*) is slim and strongly indecomposable.*

Next, we consider the higher dimensional case. We note that, in the higher dimensional case, configuration spaces of hyperbolic curves are typical examples of anabelian varieties [cf. [16]]. In this case, we prove the following [cf. Theorem 4.4]:

Theorem C. *Let* n *be a positive integer;* k *a field of characteristic* $p \geq 0$ *such that* G_k *is slim and strongly indecomposable;* $l \neq p$ *a prime number;* X *a hyperbolic curve over* k*;* X_n *the* n*-th configuration space associated to* X*. Then the following hold:*

(i) *Suppose that* k *is algebraically closed. If* $p = 0$ *(respectively,* $p > 0$*), then* Π_{X_n}, $\Pi^{(l)}_{X_n}$ *are (respectively,* $\Pi^{(l)}_{X_n}$ *is) slim and strongly indecomposable.*

(ii) *Suppose that* k *satisfies condition* $(*^l_k)$ *appearing in the statement of Theorem B. If* $p = 0$ *(respectively,* $p > 0$*), then* Π_{X_n}, $\Pi^l_{X_n}$ *are (respectively,* $\Pi^l_{X_n}$ *is) slim and strongly indecomposable.*

For instance, Theorems B and C imply the following [cf. Corollary 4.6]:

Corollary D. *Let* n *be a positive integer;* k *a field of characteristic* $p \geq 0$*;* $l \neq p$ *a prime number;* X *a smooth curve of type* (g, r) *over* k*;* X_n *the* n*-th configuration space associated to* X*. Then the following hold:*

(i) *Suppose that* k *is a finitely generated extension field of either a number field or a mixed characteristic local field. If* $2g - 2 + r > 0$*, then* Π_{X_n}, $\Pi^l_{X_n}$ *are slim and strongly indecomposable.*

(ii) *Suppose that* k *is a finitely generated transcendental extension field of a finite field. If* $(g, r) \neq (0, 0)$, $(1, 0)$ *(respectively,* $2g-2+r > 0$*), then* Π_X *(respectively,* $\Pi^l_{X_n}$*) is slim and strongly indecomposable.*

Theorem C also implies the following geometric result [cf. Corollary 5.7]:

Theorem E. *Let* n *be a positive integer;* k *a field;* X *a hyperbolic curve over* k*;* X_n *the* n*-th configuration space associated to* X*. Suppose that there exists an isomorphism of* k*-schemes*

$$X_n \xrightarrow{\sim} Y \times_k Z$$

— where Y, Z *are* k*-schemes. Then it follows that either*

$$Y \cong \mathrm{Spec}(k) \quad \textit{or} \quad Z \cong \mathrm{Spec}(k).$$

Finally, we consider the Grothendieck-Teichmüller group GT. One fundamental problem in the theory of GT is the issue of whether or not the well-known injection

$$G_{\mathbb{Q}} \hookrightarrow \mathrm{GT}$$

is bijective. In this paper, in connection with this problem, we consider the following problem [cf. [33], §1.4]:

> Suppose that $G_{\mathbb{Q}}$ satisfies a [profinite] group-theoretic property (P). Then does GT also satisfy the property (P)?

In particular, we pose the question:

Is GT strongly indecomposable?

Here, we note that $G_{\mathbb{Q}}$ is strongly indecomposable [cf. Corollary 2.3]. In this paper, we give an affirmative answer to a pro-l version of this question. More precisely, we prove the following result [cf. Corollary 6.3]:

Theorem F. *For any prime number l, the pro-l Grothendieck-Teichmüller group GT_l is strongly indecomposable.*

We note that, in the proof of Theorem F, the above similarity between the "anabelianity" and the strong indecomposability is effectively used.

0. Notations and Conventions

Fields: A finite extension field of $\mathbb{Q}$ will be referred to as a *number field*. A finite extension field of $\mathbb{Q}_p$ for some prime number p will be referred to as a *mixed characteristic local field*.

Topological groups: Let G be a Hausdorff topological group, and $H \subseteq G$ a closed subgroup. Let us write $Z_G(H)$ for the *centralizer* of H in G. We shall write $Z(G) \stackrel{\mathrm{def}}{=} Z_G(G)$ for the *center* of G.

We shall say that a profinite group G is *elastic* if it holds that every topologically finitely generated closed normal subgroup $N \subseteq H$ of an open subgroup $H \subseteq G$ of G is either trivial or of finite index in G. If G is elastic, but not topologically finitely generated, then we shall say that G is *very elastic*.

We shall say that a profinite group G is *slim* if for every open subgroup $H \subseteq G$, the centralizer $Z_G(H)$ is trivial. A profinite group G is slim if and only if every open subgroup of G has trivial center [cf. [21], Remark 0.1.3]. Note that every open subgroup of a slim profinite group is also slim. It is easily verified that every finite closed normal subgroup $N \subseteq G$ of a slim profinite group G is trivial.

Let p be a prime number. Then we shall write $G^{(p)}$ for the *maximal pro-p quotient* of a profinite group G. If G admits an open subgroup which is pro-p, then we shall say that G is *almost pro-p*.

We shall write G^{ab} for the *abelianization* of a profinite group G, i.e., the quotient of G by the closure of the commutator subgroup of G.

If G is a topologically finitely generated profinite group, then one verifies easily that the topology of G admits a basis of characteristic open subgroups, which thus induces a profinite topology on $\mathrm{Aut}(G)$ [cf. [31], Proposition 4.4.3], hence also on $\mathrm{Out}(G)$.

Let X be a connected noetherian scheme. Then we shall write Π_X for the *étale fundamental group* of X [for some choice of basepoint]. For any field k, we shall write $G_k \stackrel{\mathrm{def}}{=} \Pi_{\mathrm{Spec}(k)}$ for the *absolute Galois group* of k.

Curves: Let S be a scheme and X a scheme over S. If (g, r) is a pair of nonnegative integers, then we shall say that $X \to S$ is a *smooth curve of type* (g, r) over S if there exist an S-scheme $\overline{X}$ which is smooth, proper, of relative dimension 1 with geometrically connected fibers of genus g, and a closed subscheme $D \subseteq \overline{X}$ which is finite étale of degree r over S such that the complement of D in $\overline{X}$ is isomorphic to X over S. By abuse of terminology, we shall refer to g as the *genus* of X.

We shall say that X is a *hyperbolic curve* over S if there exists a pair (g, r) of nonnegative integers with $2g - 2 + r > 0$ such that X is a smooth curve of type (g, r) over S.

Let $X \to S$ be a smooth curve of type (g, r). For positive integers n, i, j such that $i < j \leq n$, write

$$p_{i,j} : P_n \stackrel{\mathrm{def}}{=} X \times_S \ldots \times_S X \to X \times_S X$$

for the projection of the product P_n of n copies of $X \to S$ to the i-th and j-th factors. Then we shall refer to as the *n-th configuration space* associated to $X \to S$ the S-scheme

$$X_n \to S$$

which is the open subscheme determined by the complement in P_n of the union of the various inverse images via the $p_{i,j}$ [as (i, j) ranges over the pairs of positive integers $\leq n$ such that $i < j$] of the image of the diagonal embedding $X \hookrightarrow X \times_S X$.

1. Indecomposability of Profinite Groups

In this section, we introduce the notion of the indecomposability of profinite groups, and prove [profinite] group-theoretic results which are needed in §4, §5.

Definition 1.1. (cf. [25], Definition 3.1) We shall say that a profinite group G is *indecomposable* if, for any isomorphism of profinite groups $G \cong G_1 \times G_2$, where G_1, G_2 are profinite groups, it follows that either G_1 or G_2 is the trivial group. We shall say that G is *strongly indecomposable* if every open subgroup of G is indecomposable.

Lemma 1.2. *Let G be a profinite group. If G is elastic, slim, and topologically finitely generated, then G is strongly indecomposable.*

Proof. First, we note that any open subgroup of G is also elastic, slim, and topologically finitely generated. Thus, to verify the assertion, it suffices to show that G is indecomposable. Suppose that we have an isomorphism of profinite groups $G \cong G_1 \times G_2$ such that $G_1 \neq \{1\}$. Then since G_1 is a nontrivial topologically finitely generated closed normal subgroup of G, [by the elasticity of G] G_1 is of finite index in G. In particular, G_1 is an open subgroup of G. Thus, by the slimness of G, we have $G_2 \subseteq Z_G(G_1) = \{1\}$. □

Remark 1.3. Lemma 1.2 is implicitly used in [25], Remark 3.2.1.

Proposition 1.4. *Let*

$$1 \longrightarrow \Delta \longrightarrow \Pi \xrightarrow{p} G \longrightarrow 1$$

be an exact sequence of profinite groups. Then the following hold:

(i) *Suppose that Δ is indecomposable, and G is center-free and indecomposable. Then if the natural outer Galois representation*

$$G \to \mathrm{Out}(\Delta)$$

associated to the above exact sequence is nontrivial, then Π is also indecomposable.

(ii) *Suppose that* Δ *is nontrivial and center-free, and that* G *is nontrivial. Then if* Π *is indecomposable, then the natural outer Galois representation*

$$G \to \mathrm{Out}(\Delta)$$

associated to the above exact sequence is nontrivial.

Proof. (i) Suppose that $\Pi = \Pi_1 \times \Pi_2$, where Π_1, Π_2 are nontrivial closed normal subgroups of Π. Then since G is center-free, it follows from [25], Proposition 3.3 that there exist normal closed subgroups $H_i \subseteq \Pi_i$ [for $i = 1$, 2] such that $\Pi_1/H_1 \times \Pi_2/H_2 \xrightarrow{\sim} G$. In particular, since G is indecomposable, we obtain that either $\Pi_1/H_1 = \{1\}$ or $\Pi_2/H_2 = \{1\}$. Without loss of generality, we may assume that $\Pi_1/H_1 = \{1\}$, so $\Pi_1 = H_1$, $\Pi_2/H_2 \xrightarrow{\sim} G$. Thus, we have $\Pi_1 \times H_2 \xrightarrow{\sim} \Delta$.

Now I claim that $H_2 \neq \{1\}$. Indeed, suppose that $H_2 = \{1\}$, so $\Pi_1 \xrightarrow{\sim} \Delta$, $\Pi_2 \xrightarrow{\sim} G$. Then the extension determined by the exact sequence that appears in the statement of Proposition 1.4 is isomorphic to the trivial extension of G by Δ

$$1 \longrightarrow \Delta \longrightarrow \Delta \times G \longrightarrow G \longrightarrow 1.$$

Thus, the natural outer Galois representation $G \to \mathrm{Out}(\Delta)$ induced by the conjugation action of G on Δ is trivial. But this contradicts the assumption that the outer representation $G \to \mathrm{Out}(\Delta)$ is nontrivial. This completes the proof of the claim.

Thus, $\Delta \cong \Pi_1 \times H_2$ gives a nontrivial decomposition, which contradicts the indecomposability of Δ. This completes the proof that Π is indecomposable.

(ii) Suppose that the representation $G \to \mathrm{Out}(\Delta)$ is trivial. Here, note that both Δ and $Z_\Pi(\Delta)$ are normal closed subgroups of Π. Now I claim that Π is generated by Δ and $Z_\Pi(\Delta)$. Indeed, let $\pi \in \Pi$. Then, by the triviality of $G \to \mathrm{Out}(\Delta)$, there exists an element $\delta \in \Delta$ such that

$$\pi \cdot x \cdot \pi^{-1} = \delta \cdot x \cdot \delta^{-1}$$

for any $x \in \Delta$. In particular, we have $\delta^{-1} \cdot \pi \in Z_\Pi(\Delta)$, so $\pi = \delta \cdot (\delta^{-1} \cdot \pi) \in \Delta \cdot Z_\Pi(\Delta)$. This completes the proof of the claim. Thus, since Δ is center-free, i.e., $\Delta \cap Z_\Pi(\Delta) = Z(\Delta) = \{1\}$, we obtain that $\Pi \cong \Delta \times Z_\Pi(\Delta)$. Here, we note that since $p(Z_\Pi(\Delta)) = G$ is nontrivial, we have $Z_\Pi(\Delta) \neq \{1\}$. Therefore, since Δ is nontrivial, we conclude that Π is not indecomposable, a contradiction. $\square$

Remark 1.5. (i) One cannot drop the center-freeness assumption in the statement of Proposition 1.4, (i). Indeed, let $\mathfrak{S}_3$ be the symmetric group on 3 letters; $\mathrm{sgn} : \mathfrak{S}_3 \to \{\pm 1\}$ the homomorphism obtained by taking signatures; $\Phi : \mathfrak{S}_3 \times \mathfrak{S}_3 \twoheadrightarrow \{\pm 1\}$ the homomorphism given by assigning $(\sigma_1, \sigma_2) \mapsto \mathrm{sgn}(\sigma_1) \cdot \mathrm{sgn}(\sigma_2)$. In particular, we have the following exact sequence

$$1 \longrightarrow \mathrm{Ker}(\Phi) \longrightarrow \mathfrak{S}_3 \times \mathfrak{S}_3 \xrightarrow{\Phi} \{\pm 1\} \longrightarrow 1.$$

Here, note that $\{\pm 1\}$ is not center-free. Then although $\mathrm{Ker}(\Phi)$ and $\{\pm 1\}$ are indecomposable, and, moreover, the natural outer representation associated to this exact sequence is nontrivial, $\mathfrak{S}_3 \times \mathfrak{S}_3$ is not indecomposable.

(ii) One cannot drop the center-freeness assumption in the statement of Proposition 1.4, (ii). Indeed, let us consider the exact sequence

$$1 \longrightarrow \mathbb{Z}_l \xrightarrow{\times l} \mathbb{Z}_l \longrightarrow \mathbb{Z}_l / l\mathbb{Z}_l \longrightarrow 1.$$

Here, note that $\mathbb{Z}_l$ is not center-free. Then although $\mathbb{Z}_l$ is indecomposable, the natural outer representation associated to this exact sequence is trivial.

The following Lemma 1.6 (respectively, Lemma 1.7) is a variant of [13], Lemma 5 (respectively, [13], Lemma 23).

Lemma 1.6. *Let G be a slim profinite group; H an open subgroup of G; α an automorphism of G. Suppose that it holds that $\alpha(h) = h$ for any $h \in H$. Then $\alpha = \mathrm{id}_G$.*

Proof. Write $N \stackrel{\text{def}}{=} \bigcap_{g \in G} (g \cdot H \cdot g^{-1})$. Here, we observe the following facts:

(1) N is a normal open subgroup of G;
(2) It holds that $\alpha(n) = n$ for any $n \in N$.

Now I claim that, for any $g \in G$, it holds that $\alpha(g) \cdot g^{-1} \in Z_G(N)$. Indeed, for any $n \in N$, we have

$$\begin{aligned} \alpha(g) \cdot g^{-1} \cdot n \cdot (\alpha(g) \cdot g^{-1})^{-1} &= \alpha(g) \cdot g^{-1} \cdot n \cdot g \cdot \alpha(g)^{-1} \\ &= \alpha(g) \cdot \alpha(g^{-1} \cdot n \cdot g) \cdot \alpha(g)^{-1} \\ &= \alpha(g) \cdot \alpha(g)^{-1} \cdot \alpha(n) \cdot \alpha(g) \cdot \alpha(g)^{-1} \\ &= \alpha(n) \\ &= n \end{aligned}$$

— where the second (respectively, fifth) equality follows from the facts (1), (2) (respectively, fact (2)). In light of the claim, it follows from the slimness of G that $\alpha(g) \cdot g^{-1} = 1$, hence that $\alpha(g) = g$. Therefore, we conclude that $\alpha = \mathrm{id}_G$. □

Lemma 1.7. *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta' & \longrightarrow & \Pi' & \longrightarrow & G' & \longrightarrow & 1 \\ & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & \Pi & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

be a commutative diagram of profinite groups, where the horizontal sequences are exact. Write

$$\rho : G \to \mathrm{Out}(\Delta) \quad (\textit{respectively, } \rho' : G' \to \mathrm{Out}(\Delta'))$$

for the natural outer representation associated to the lower (respectively, upper) horizontal sequence. Then the following hold:

(i) *Suppose that Δ is slim, and that the three vertical arrows are open injections. Then $\mathrm{Ker}(\rho')$ is an open subgroup of $\mathrm{Ker}(\rho)$. In particular, it holds that $\mathrm{Im}(\rho)$ is infinite if and only if $\mathrm{Im}(\rho')$ is infinite.*

(ii) *Suppose that α is a surjection. Then it holds that*

$$\gamma(\mathrm{Ker}(\rho')) \subseteq \mathrm{Ker}(\rho).$$

In particular, if γ is an open injection, and $\mathrm{Im}(\rho)$ is infinite, then $\mathrm{Im}(\rho')$ is infinite.

Proof. First, we consider assertion (i). We begin by observing that it follows from Lemma 1.6, that

$$Z_{\Pi'}(\Delta') \subseteq Z_{\Pi}(\Delta).$$

In particular, we have

$$\mathrm{Ker}(\rho') = p(Z_{\Pi'}(\Delta')) \subseteq p(Z_{\Pi}(\Delta)) = \mathrm{Ker}(\rho).$$

Thus, it suffices to verify that $\mathrm{Ker}(\rho')$ is open in $\mathrm{Ker}(\rho)$. Let

$$\phi : N_{\Delta}(\Delta')/\Delta' \to \mathrm{Out}(\Delta')$$

be the outer representation induced by conjugation. Here, note that since $N_{\Delta}(\Delta')/\Delta'$ is finite [cf. the fact that Δ' is open in Δ], $\mathrm{Im}(\phi)$ is also finite. Now we observe that it holds that $\rho'(G' \cap \mathrm{Ker}(\rho)) \subseteq \mathrm{Im}(\phi)$. In particular, we obtain the following exact sequence of profinite groups

$$1 \longrightarrow \mathrm{Ker}(\rho') \longrightarrow G' \cap \mathrm{Ker}(\rho) \xrightarrow{\rho'} \mathrm{Im}(\phi).$$

Thus, since $\mathrm{Im}(\phi)$ is finite, it follows that $\mathrm{Ker}(\rho')$ is open in $G' \cap \mathrm{Ker}(\rho)$. On the other hand, since $G' \cap \mathrm{Ker}(\rho)$ is open in $\mathrm{Ker}(\rho)$ [cf. the fact that G' is open in G], we conclude that $\mathrm{Ker}(\rho')$ is open in $\mathrm{Ker}(\rho)$. This completes the proof of assertion (i). Next, we consider assertion (ii). Let us verify the inclusion $\gamma(\mathrm{Ker}(\rho')) \subseteq \mathrm{Ker}(\rho)$. Let $g' \in \mathrm{Ker}(\rho')$; $\pi' \in \Pi'$ an element which mapped to g' via the surjection $\Pi' \twoheadrightarrow G'$. In particular, there exists an element $\delta' \in \Delta'$ such that it holds that

$$\pi' \cdot x \cdot (\pi')^{-1} = \delta' \cdot x \cdot (\delta')^{-1}.$$

for any $x \in \Delta'$. Thus, since α is surjective, it holds that

$$\beta(\pi') \cdot y \cdot (\beta(\pi'))^{-1} = \alpha(\delta') \cdot y \cdot (\alpha(\delta'))^{-1}.$$

for any $y \in \Delta$. Therefore, since $\beta(\pi')$ is mapped to $\gamma(g')$ via the surjection $\Pi \twoheadrightarrow G$, we conclude that $\gamma(g') \in \mathrm{Ker}(\rho)$, hence that $\gamma(\mathrm{Ker}(\rho')) \subseteq \mathrm{Ker}(\rho)$. This completes the proof of assertion (ii). $\square$

Proposition 1.8. *Let*

$$1 \longrightarrow \Delta \longrightarrow \Pi \xrightarrow{p} G \longrightarrow 1$$

be an exact sequence of profinite groups. Then the following hold:

(i) *If Δ, G are center-free (respectively, slim), then Π is also center-free (respectively, slim).*

(ii) *Suppose that Δ is slim and strongly indecomposable, and that G is slim and strongly indecomposable. Then if the image of the natural outer representation*

$$G \to \mathrm{Out}(\Delta)$$

associated to the above exact sequence is infinite, then Π is also slim and strongly indecomposable.

Proof. First, let us consider assertion (i). To verify assertion (i), it suffices to verify the center-freeness portion of assertion (i). Suppose that Δ, G are center-free. Let $\pi \in Z(\Pi)$. Then it follows that $p(\pi) \in Z(G) = \{1\}$, hence that $\pi \in \Delta \cap Z(\Pi) \subseteq Z(\Delta) = \{1\}$. Thus, we conclude that Π is also center-free. This completes the proof of assertion (i). Next, we consider assertion (ii). The slimness portion of assertion (ii) follows from assertion (i). The strong indecomposability portion of assertion (ii) follows from Proposition 1.4, (i); Lemma 1.7, (i). $\square$

2. Indecomposability of Various Absolute Galois Groups

In this section, we review [profinite] group-theoretic properties of various absolute Galois groups.

Theorem 2.1. *Let k be a Hilbertian field [cf.* [5]*, Chapter 12]. Then G_k is very elastic, slim, and strongly indecomposable.*

Proof. The very elasticity portion of Theorem 2.1 follows from [5], Lemma 16.11.5; [5], Proposition 16.11.6. Note that for any open subgroup H of G_k, there exists a finite separable extension k_H of k such that $G_{k_H} \xrightarrow{\sim} H$. Here, by [5], Corollary 12.2.3, k_H is also a Hilbertian field. Thus, to verify the slimness and the strong indecomposability portions of Theorem 2.1, it suffices to show that G_k is center-free and indecomposable. But this center-freeness (respectively, indecomposability) follows from [5], Proposition 16.11.6 (respectively, a theorem of Haran-Jarden [cf. [11], Corollary 2.5]). □

Remark 2.2. Let k be either a finite field or a mixed characteristic local field. Then k is always non-Hilbertian. Indeed, G_k is topologically finitely generated [cf. Proposition 2.4, below; [5], Lemma 16.11.5].

Corollary 2.3. *The following types of fields are Hilbertian:*

(i) *finitely generated extension fields of $\mathbb{Q}$,*

(ii) *finitely generated transcendental extension fields of an arbitrary field.*

In particular, their absolute Galois groups are very elastic, slim, and strongly indecomposable.

Proof. The first statement follows from [5], Theorem 13.4.2. The second statement follows from the first, together with Theorem 2.1. □

Proposition 2.4. *Let k be a mixed characteristic local field. Then G_k is elastic, slim, topologically finitely generated, and strongly indecomposable.*

Proof. The assertions follow from Lemma 1.2; [24], Theorem 1.7, (ii); [28], Theorem 7.4.1. □

Remark 2.5. Let k be a positive characteristic local field, i.e., a field isomorphic to the field of formal Laurent series $\mathbb{F}((t))$ over a finite field $\mathbb{F}$. Then, by applying the same argument as the argument applied in the proof of [21], Theorem 1.1.1, (ii), one may verify easily that G_k is slim. On the other hand, it is not clear to the author at the time of writing whether or not the elasticity, as well as the [strong] indecomposability of G_k holds. [Note that, in this case, G_k is not topologically finitely generated [cf., e.g., [28], Theorem 7.5.10].]

3. Indecomposability of Geometric Fundamental Groups of Curves

In this section, we discuss the indecomposability of the geometric fundamental group of a smooth [hyperbolic] curve.

First, let us recall the following well-known fact.

Theorem 3.1. *Let k be an algebraically closed field of characteristic zero; l a prime number; X a hyperbolic curve over k. Then Π_X, $\Pi_X^{(l)}$ are elastic, slim, and topologically finitely generated. In particular, Π_X, $\Pi_X^{(l)}$ are strongly indecomposable.*

Proof. The fact that Π_X, $\Pi_X^{(l)}$ are elastic (respectively, slim; topologically finitely generated) follows from [25], Theorem 1.5 (respectively, [25], Proposition 1.4; [8], EXPOSÉ XIII, Corollaire 2.12). In particular, the strong indecomposability portion of Theorem 3.1 follows from Lemma 1.2 [cf. also [25], Proposition 3.2; [25], Remark 3.2.1]. □

Thus, for the rest of this section, we consider the case of positive characteristic. The following Proposition is an immediate consequence of Theorem 3.1.

Proposition 3.2. *Let k be an algebraically closed field of characteristic $p > 0$; $l \neq p$ a prime number; X a hyperbolic curve over k. Then $\Pi_X^{(l)}$ is elastic, slim, topologically finitely generated, and strongly indecomposable.*

Proof. This follows from Theorem 3.1; [8], EXPOSÉ XIII, Corollaire 2.12. □

The following Lemmas 3.3, 3.4 are well-known, but we review them briefly for the sake of completeness.

Lemma 3.3. *Let k be an algebraically closed field of characteristic $p > 0$; X a smooth curve of type (g, r) over k such that the pair (g, r) satisfies $(g, r) \neq (0, 0)$, $(1, 0)$. Then there exists a normal open subgroup N of Π_X such that the Galois finite étale covering $X_N \to X$ corresponding to N has genus ≥ 2.*

Proof. If $g \geq 2$, then there is nothing to prove. Thus, we may assume that $g \leq 1$. First, we consider the case where $g = 0$, i.e., the unique smooth compactification of X is isomorphic to $\mathbb{P}^1_k$. Here, note that if we identify the function field of $\mathbb{P}^1_k$ with $k(t)$, where t is an indeterminate, then for any Artin-Schreier equation

$$x^p - x = t^m \quad (m \in \mathbb{Z}_{>0},\ p \nmid m),$$

one computes easily that the normalization of $\mathbb{P}^1_k$ in the extension field $k(t)[x]/(x^p - x - t^m)$ of $k(t)$ determines a Galois finite ramified covering $\phi_m : C_m \to \mathbb{P}^1_k$ of $\mathbb{P}^1_k$ branched only at ∞, where C_m is a smooth, proper curve of genus $\frac{(m-1)(p-1)}{2}$ [cf., e.g., [35], Example 8.16]. Thus, for any curve X of type $(0, r)$, where $r > 0$, by taking m to be sufficiently large and pulling back ϕ_m via an embedding $X \hookrightarrow \mathbb{A}^1_k \subset \mathbb{P}^1_k$, we obtain a Galois finite étale covering $X' \to X$ of X such that the genus of X' is ≥ 2. Next, we consider the case where $g = 1$, i.e., the unique smooth compactification of X is an elliptic curve E. Note that by applying the Riemann-Roch Theorem to E, we obtain a finite morphism $E_1 \stackrel{\text{def}}{=} E \setminus \{p\} \to \mathbb{A}^1_k$ over k, where $p \in E \setminus X$ is a closed point of E. Next, let us observe that it follows from the genus 0 case, which has already been verified, that there exists a Galois finite étale covering $C \to \mathbb{A}^1_k$ of $\mathbb{A}^1_k$ such that the genus of C is ≥ 2. Then any connected component of $E_1 \times_{\mathbb{A}^1_k} C$ determines a Galois finite étale covering $C' \to E_1$ of E_1. Moreover, by applying the Hurwitz formula to the compactification of the finite morphism $C' \hookrightarrow E_1 \times_{\mathbb{A}^1_k} C \to C$, it follows that the genus of C' is also ≥ 2. Thus, for any curve X of type $(1, r)$, where $r > 0$, by pulling back $C' \to E_1$ via the natural open immersion $X \hookrightarrow E_1$, we obtain a Galois finite étale covering $X' \to X$ of X such that the genus of X' is ≥ 2. $\square$

Lemma 3.4. *In the notation of Lemma 3.3, let $l \neq p$ be a prime number. Then for any normal open subgroup N of Π_X such that the connected finite étale covering $X_N \to X$ corresponding to N has genus ≥ 2, the conjugation action of Π_X/N on $N^{\text{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_l$ is faithful.*

Proof. The assertion follows immediately from the argument given in [3], Lemma 1.14. □

Remark 3.5. It is well-known that, if we replace "$(g, r) \neq (0, 0)$, $(1, 0)$" by "$2g - 2 + r > 0$", then the "characteristic zero versions" of Lemmas 3.3, 3.4 also hold. In fact, such "characteristic zero versions" are used in the proof of [25], Proposition 3.2.

The following Theorem 3.6 is the main result of this section. We note that the slimness portion of Theorem 3.6 is well-known [cf. [34], Proposition 1.11]. But for the convenience of the reader, we also give a proof of the slimness portion of Theorem 3.6, as an application of Lemmas 3.3, 3.4.

Theorem 3.6. *In the notation of Lemma 3.3, $G \stackrel{\mathrm{def}}{=} \Pi_X$ is slim and strongly indecomposable.*

Proof. First, I claim that

> $(*_1)$ Let H be an open subgroup of G; $X_H \to X$ the connected finite étale covering corresponding to H. Then X_H is a smooth curve of type $\neq (0, 0)$, $(1, 0)$ over k.

Indeed, this follows immediately from the Hurwitz formula. In light of the claim $(*_1)$, to verify the slimness (respectively, strong indecomposability) of G, it suffices to show that G is center-free (respectively, indecomposable).

Thus, let us first verify the center-freeness of G. Here, observe that, for any open subgroups $J \subseteq G$, there exists a normal open subgroup J' of G such that $J' \subseteq J$, and, moreover, the Galois finite étale covering $X_{J'} \to X$ corresponding to J' has genus ≥ 2. Indeed, it follows from the claim $(*_1)$ and Lemma 3.3 that there exists an open subgroup $J'' \subseteq J$ such that the connected finite étale covering $X_{J''} \to X$ corresponding to J'' [$\subseteq G$] has genus ≥ 2. Then $J' \stackrel{\mathrm{def}}{=} \bigcap_{g \in G}(g \cdot J'' \cdot g^{-1})$ is a normal open subgroup of G such that $J' \subseteq J$, and, moreover, the Galois finite étale covering $X_{J'} \to X$ corresponding to J' has genus ≥ 2 [cf. the Hurwitz formula]. In particular, the center-freeness of G follows from this observation, together with Lemma 3.4. This completes the proof the center-freeness of G, hence also the slimness of G.

Thus, it remains to show that G is indecomposable. Suppose that we have an isomorphism of profinite groups $G \cong G_1 \times G_2$ such that $G_1 \neq \{1\}$, $G_2 \neq \{1\}$. In particular, by the slimness of G, it follows that G_1, G_2 are infinite [cf. §0]. Then, by applying Lemma 3.3, we obtain a normal open subgroup

$K \subseteq G$ such the Galois finite étale covering $X_K \to X$ corresponding to K has genus ≥ 2. Take an open subgroup $K' \subseteq K$, which may be identified with

$$G_1' \times G_2'$$

— where, for $i = 1, 2$, G_i' is an open subgroup of G_i. In particular, G_i' is nontrivial [cf. the infiniteness of G_i]. Moreover, it follows from the Hurwitz formula that the connected finite étale covering $X_{K'} \to X_K$ corresponding to K' [$\subseteq K$] has genus ≥ 2 [cf. the fact that the genus of X_K is ≥ 2]. Thus, by replacing G (respectively, G_i) by K' (respectively, G_i'), we may assume, without loss of generality, that the genus of X is ≥ 2. Now I claim that

> $(*_2)$ for every prime number $l \neq p$, there exist finite quotients $G_1 \twoheadrightarrow Q_1$, $G_2 \twoheadrightarrow Q_2$ such that l divides the orders of Q_1, Q_2.

Indeed, suppose that l does not divide the order of any finite quotient of G_1. Now let $N_1 \subsetneq G_1$ be a proper normal open subgroup of G_1. Note that by assumption, we have $N_1^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_l = \{1\}$. Write $N \overset{\mathrm{def}}{=} N_1 \times G_2$. Then since the conjugation action of $G/N \cong G_1/N_1 \times \{1\}$ on

$$N^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_l \cong (N_1^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_l) \times (G_2^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_l) \cong \{1\} \times (G_2^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}_l)$$

is trivial, we conclude from Lemma 3.4 that $G/N = \{1\}$, a contradiction. This completes the proof of the claim $(*_2)$. In light of the claim $(*_2)$, we obtain an open subgroup $U_i \subseteq G_i$ such that there exists a surjection

$$U_i \twoheadrightarrow \mathbb{Z}/l\mathbb{Z}.$$

for $i = 1, 2$. In particular, it follows that $U_i^{(l)}$ is nontrivial. Let U be the open subgroup of G that may be identified with $U_1 \times U_2$ [$\subseteq G_1 \times G_2$]. Thus, we have

$$U^{(l)} \cong U_1^{(l)} \times U_2^{(l)}.$$

On the other hand, since the connected finite étale covering $X_U \to X$ corresponding to U has genus ≥ 2 [cf. the Hurwitz formula], it follows from Proposition 3.2, that $U^{(l)}$ is indecomposable — a contradiction. This completes the proof of the indecomposability of G, hence also the strong indecomposability of G. $\square$

4. Indecomposability of Various Fundamental Groups

In this section, by applying the results of §1, §2 and §3, we prove the indecomposability of various fundamental groups.

Let k be a field; $\overline{k}$ an algebraic closure of k; V a geometrically connected scheme of finite type over k. In particular, the composite $V \times_k \overline{k} \to V \to \mathrm{Spec}(k)$ induces the following exact sequence of profinite groups

$$1 \longrightarrow \Pi_{V \times_k \overline{k}} \longrightarrow \Pi_V \longrightarrow G_k \longrightarrow 1$$

[cf. [8], EXPOSÉ IX, Théorème 6.1]. In the following, for any prime number l, we shall write

$$\Pi_V^l \stackrel{\mathrm{def}}{=} \Pi_V/\mathrm{Ker}(\Pi_{V \times_k \overline{k}} \twoheadrightarrow \Pi_{V \times_k \overline{k}}^{(l)})$$

— where $\Pi_{V \times_k \overline{k}} \twoheadrightarrow \Pi_{V \times_k \overline{k}}^{(l)}$ is the natural surjection.

Definition 4.1. Let k be a field of characteristic $p \geq 0$; $l \neq p$ a prime number. Then for the pair (k, l), we consider the following condition:

$(*_k^l)$ For any finite extension field k' of k, the l-adic cyclotomic character $\chi_{k'} : G_{k'} \to \mathbb{Z}_l^\times$ of k' is nontrivial.

We shall say that k is *l-cyclotomically full* if the pair (k, l) satisfies condition $(*_k^l)$.

Lemma 4.2. *In the notation of Definition 4.1, the following hold:*

(i) *k is l-cyclotomically full if and only if for any finite extension field k' of k, there exists a positive integer n such that k' does not contain a primitive l^n-th root of unity.*

(ii) *Let K be an extension field of k. Then if K is l-cyclotomically full, then the same is true of k. Suppose further that K is a finitely generated extension field of k. Then if k is l-cyclotomically full, then the same is true of K.*

(iii) *k is l-cyclotomically full if and only if the image of the l-adic cyclotomic character $\chi_k : G_k \to \mathbb{Z}_l^\times$ of k is infinite.*

(iv) *Let X be a smooth curve of type (g, r) over k such that the pair (g, r) satisfies $(g, r) \neq (0, 0)$, $(0, 1)$ (respectively, $(g, r) \neq (0, 0)$) if $p = 0$ (respectively, $p > 0$); $\overline{k}$ an algebraic closure of k. Write $X_{\overline{k}} \stackrel{\mathrm{def}}{=} X \times_k \overline{k}$. Suppose, moreover, that k is l-cyclotomically full. Then the image of the natural outer Galois representation*

$$\rho_k : G_k \to \mathrm{Out}(\Pi_{X_{\overline{k}}})$$

associated to the exact sequence

$$1 \longrightarrow \Pi_{X_{\overline{k}}} \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1$$

is infinite. If, moreover, $(g, r) \neq (0, 1)$*, then the image of the naturally induced pro-*l *outer Galois representation*

$$\rho_k^{(l)} : G_k \to \mathrm{Out}(\Pi_{X_{\overline{k}}}^{(l)})$$

is infinite.

(v) *Let* k *be either a number field, or a mixed characteristic local field, or a finite field. Suppose that* K *is a finitely generated extension field of* k*. Then* K *is* l*-cyclotomically full.*

Proof. Assertion (i) follows immediately from the definitions.

The first statement of assertion (ii) follows from assertion (i). The second statement of assertion (ii) follows from assertion (i) and the following well-known fact:

If K is a finitely generated extension field of k, then the algebraic closure of k in K is a finite extension of k.

[In fact, let $E \subseteq K$ be the algebraic closure of k in K; $\{x_1, \ldots, x_n\} \subseteq K$ a transcendence basis of K/k. Then we obtain that $[E : k] = [E(x_1, \ldots, x_n) : k(x_1, \ldots, x_n)] \leq [K : k(x_1, \ldots, x_n)] < +\infty$.]

We consider assertion (iii). First, let us prove necessity. Suppose that the image of χ_k is finite. Then the kernel H of χ_k is an open subgroup of G_k. Thus, there exists a finite extension k' of k such that $G_{k'} \xrightarrow{\sim} H$. In particular, the l-adic cyclotomic character $\chi_{k'} : G_{k'} \xrightarrow{\sim} H \hookrightarrow G_k \to \mathbb{Z}_l^\times$ of k' is trivial — a contradiction. Next, we prove sufficiency. To this end, let k' be a finite extension field of k. Write $\chi_{k'} : G_{k'} \to \mathbb{Z}_l^\times$ for the l-adic cyclotomic character of k', H for the kernel of χ_k. Then if we identify $G_{k'}$ with an open subgroup of G_k, then $G_{k'}/G_{k'} \cap H$ [$\xrightarrow{\sim} \mathrm{Im}(\chi_{k'})$] corresponds to an open subgroup of G_k/H [$\xrightarrow{\sim} \mathrm{Im}(\chi_k)$]. On the other hand, since $\mathrm{Im}(\chi_k)$ is infinite, we thus conclude that $\mathrm{Im}(\chi_{k'})$ is also infinite, hence, in particular, nontrivial. This completes the proof of assertion (iii).

Next, we consider assertion (iv). First, suppose that $(g, r) = (0, 1)$ [so $p > 0$]. Then note that there exists an open subgroup N of $\Pi_{X_{\overline{k}}}$ such that the connected finite étale covering $Z \to X_{\overline{k}}$ corresponding to N has genus ≥ 2 [cf. Lemma 3.3]. Now, by applying [26], Proposition (1.4.1), (i), we have an open subgroup H of Π_X such that $H \cap \Pi_{X_{\overline{k}}} = N$. Let $Y \to X$ be the connected finite étale covering corresponding to H [$\subseteq \Pi_X$]; k' the finite extension field of k corresponding to the image of H via the surjection $\Pi_X \twoheadrightarrow G_k$. Here, observe that there exists a finite extension field k'' of k' such that $Y' \stackrel{\mathrm{def}}{=} Y \times_{k'} k''$ is a hyperbolic curve over k''. Thus, we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{Y'\times_{k''}\overline{k}} & \longrightarrow & \Pi_{Y'} & \longrightarrow & G_{k''} & \longrightarrow & 1 \\
 & & \downarrow\wr & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Pi_{Y\times_{k'}\overline{k}} & \longrightarrow & \Pi_{Y} & \longrightarrow & G_{k'} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Pi_{X\times_{k}\overline{k}} & \longrightarrow & \Pi_{X} & \longrightarrow & G_{k} & \longrightarrow & 1
\end{array}$$

— where the vertical arrows are open injections; the horizontal sequences are exact. Here, note that, by applying the Hurwitz formura to

$$[Y' \times_{k''} \overline{k} \ \overset{\sim}{\to}] \ Y \times_{k'} \overline{k} \ \to \ Z,$$

it follows that the genus of Y' is ≥ 2. Thus, since $\Pi_{X\times_k\overline{k}}$ is slim [cf. Lemma 3.6], it follows from Lemma 1.7, (i), that the infiniteness of the image of the natural outer representation

$$G_{k''} \to \mathrm{Out}(\Pi_{Y'\times_{k''}\overline{k}})$$

implies the infiniteness of $\mathrm{Im}(\rho_k)$. Therefore, the case where $(g,r) = (0,1)$ follows from the case where $g \geq 2$. Thus, in the remainder of the proof of assertion (iv), we may assume without loss of generality that $(g,r) \neq (0,1)$. Next, observe that to verify the infiniteness of $\mathrm{Im}(\rho_k)$, it suffices to verify the infiniteness of $\mathrm{Im}(\rho_k^{(l)})$. Moreover, by replacing k by a suitable finite extension of k, it suffices to verify that $\rho_k^{(l)}$ is nontrivial. Suppose that $\rho_k^{(l)}$ is trivial. First, we assume that $g \geq 1$. We write $\overline{X}$ for the smooth compactification of X; $J(\overline{X})$ for the Jacobian variety of $\overline{X}$;

$$\Delta \overset{\mathrm{def}}{=} \Pi^{(l)}_{X\times_k\overline{k}}; \quad \overline{\Delta} \overset{\mathrm{def}}{=} \Pi^{(l)}_{\overline{X}\times_k\overline{k}}.$$

In particular, we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta & \longrightarrow & \Pi^l_X & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \overline{\Delta} & \longrightarrow & \Pi^l_{\overline{X}} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are exact; the left-hand and central vertical arrows are the surjections induced by the open immersion $X \hookrightarrow \overline{X}$ [cf. the definition of $\Pi^l_{(-)}$]. Thus, it follows from Lemma 1.7, (ii), that the natural outer Galois representation

$$\overline{\rho} : G_k \to \mathrm{Out}(\overline{\Delta})$$

associated to $\overline{X}$ is trivial, hence that the composite

$$G_k \overset{\overline{\rho}}{\to} \mathrm{Out}(\overline{\Delta}) \to \mathrm{Aut}(\overline{\Delta}^{\mathrm{ab}}) \overset{\sim}{\to} \mathrm{Aut}(T_l(J(\overline{X})))$$

— where we write $T_l(J(\overline{X}))$ for the l-adic Tate module of $J(\overline{X})$; the second arrow is the homomorphism induced by the abelianization; the third arrow is the isomorphism induced by the natural isomorphism $\overline{\Delta}^{\mathrm{ab}} \overset{\sim}{\to} T_l(J(\overline{X}))$ — i.e, the natural l-adic Galois representation associated to $J(\overline{X})$ is trivial. Then since, as is well-known [cf. the natural isomorphisms

$$\bigwedge^{2g} H^1_{\text{ét}}(J(\overline{X}) \times_k \overline{k}, \mathbb{Z}_l) \overset{\sim}{\to} H^{2g}_{\text{ét}}(J(\overline{X}) \times_k \overline{k}, \mathbb{Z}_l) \overset{\sim}{\to} \mathbb{Z}_l(-g)$$

of $\mathbb{Z}_l[G_k]$-modules discussed in [18], Remark 15.5; [17], Chapter VI, Theorem 11.1, (a)], the determinant of the l-adic Galois representation associated to $J(\overline{X})$ is a positive power of the l-adic cyclotomic character of k, we conclude that some positive power of the l-adic cyclotomic character of k is trivial. But this contradicts (iii). Next, we assume that $g = 0$ and $r \geq 2$. Let us first observe that, by replacing k by a suitable finite extension field of k, we may assume without loss of generality that X is obtained by removing $r-2$ k-rational point(s) from $\mathbb{A}^1_k \setminus \{0\}$. Then we note that, in the above notation, we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta & \longrightarrow & \Pi^l_X & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \Vert & & \\
1 & \longrightarrow & \Pi^{(l)}_{\mathbb{A}^1_{\overline{k}}\setminus\{0\}} & \longrightarrow & \Pi^l_{\mathbb{A}^1_k\setminus\{0\}} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are exact; the left-hand and central vertical arrows are the surjections induced by the open immersion $X \hookrightarrow \mathbb{A}^1_k \setminus \{0\}$ [cf. the definition of $\Pi^l_{(-)}$]. Thus, it follows from Lemma 1.7, (ii), that the natural outer Galois representation

$$G_k \to \mathrm{Out}(\Pi^{(l)}_{\mathbb{A}_{\overline{k}}\setminus\{0\}})$$

associated to $\mathbb{A}^1_k \setminus \{0\}$ is trivial, hence that the l-adic cyclotomic character of k is trivial, a contradiction. [Here, we recall that $H^1_{\text{ét}}(\mathbb{A}^1_{\overline{k}} \setminus \{0\}, \mathbb{Z}_l) \cong \mathbb{Z}_l(-1)$.]

Finally, we consider assertion (v). To verify the assertion, it suffices to show that k is l-cyclotomically full [cf. (ii)]. Thus, to verify the assertion, it suffices to show that, for any finite extension field k' of k, there exists a positive integer n such that k' does not contain a primitive l^n-th root of unity [cf. (i)]. But this follows from the well-known fact that for any finite extension field k' of k, the group of roots of unity in k' is finite [cf. [19], Chapter 5; [30], Chapter 2, §4.3, §4.4]. $\square$

Theorem 4.3. *Let k be a field of characteristic $p \geq 0$ such that G_k is slim and strongly indecomposable; X a smooth curve of type (g, r) over k. Then if k is l-cyclotomically full for a prime number $l \neq p$, then the following hold:*

(i) *Suppose that $p = 0$. If $2g - 2 + r > 0$, then Π_X, Π^l_X are slim and strongly indecomposable.*

(ii) *Suppose that $p > 0$. If $(g, r) \neq (0, 0)$, $(1, 0)$ (respectively, $2g-2+r > 0$), then Π_X (respectively, Π^l_X) is slim and strongly indecomposable.*

Proof. Let $\overline{k}$ be an algebraic closure of k. Then we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{X \times_k \overline{k}} & \longrightarrow & \Pi_X & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Pi^{(l)}_{X \times_k \overline{k}} & \longrightarrow & \Pi^l_X & \longrightarrow & G_k & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are exact [cf. the definition of Π^l_X]. Therefore, Theorem 4.3 follows from Proposition 1.8, (ii); Theorem 3.1; Proposition 3.2; Theorem 3.6; Lemma 4.2, (iv). ☐

Theorem 4.4. *Let n be a positive integer; k a field of characteristic $p \geq 0$ such that G_k is slim and strongly indecomposable; $l \neq p$ a prime number; X a hyperbolic curve over k; X_n the n-th configuration space associated to X. Then the following hold:*

(i) *Suppose that k is algebraically closed. If $p = 0$ (respectively, $p > 0$), then Π_{X_n}, $\Pi^{(l)}_{X_n}$ are (respectively, $\Pi^{(l)}_{X_n}$ is) slim and strongly indecomposable.*

(ii) *Suppose that k is l-cyclotomically full. If $p = 0$ (respectively, $p > 0$), then Π_{X_n}, $\Pi^l_{X_n}$ are (respectively, $\Pi^l_{X_n}$ is) slim and strongly indecomposable.*

Proof. First, let us consider assertion (i). For $n \geq 2$, let $X_n \to X_{n-1}$ be the projection morphism obtained by forgetting the factor labeled n; $\overline{x}$ a geometric point of X_{n-1}; $(X_n)_{\overline{x}}$ the fiber of $X_n \to X_{n-1}$ over $\overline{x}$. Then if $p = 0$ (respectively, $p \geq 0$), then we have the following exact sequence of profinite groups

$$1 \longrightarrow \Pi_{(X_n)_{\overline{x}}} \longrightarrow \Pi_{X_n} \longrightarrow \Pi_{X_{n-1}} \longrightarrow 1$$

(respectively,

$$1 \longrightarrow \Pi^{(l)}_{(X_n)_{\overline{x}}} \longrightarrow \Pi^{(l)}_{X_n} \longrightarrow \Pi^{(l)}_{X_{n-1}} \longrightarrow 1\ )$$

[cf. [25], Proposition 2.2, (i)]. We note that $\Pi_{(X_n)_{\overline{x}}}$, Π_{X_1}, $\Pi^{(l)}_{(X_n)_{\overline{x}}}$, and $\Pi^{(l)}_{X_1}$ are slim and strongly indecomposable [cf. Theorem 3.1; Proposition 3.2]. Thus, since the natural outer representation

$$\Pi_{X_{n-1}} \to \mathrm{Out}(\Pi_{(X_n)_{\overline{x}}}) \quad (\text{respectively, } \Pi^{(l)}_{X_{n-1}} \to \mathrm{Out}(\Pi^{(l)}_{(X_n)_{\overline{x}}}))$$

associated to the above exact sequence is injective [cf. [2], Theorem 1; [2], the Remark following the proof of Theorem 1], by applying induction on n, it follows from Proposition 1.8, (ii), that Π_{X_n}, $\Pi^{(l)}_{X_n}$ are slim and strongly indecomposable. This completes the proof of assertion (i). Next, we consider assertion (ii). Let $\overline{k}$ be an algebraic closure of k. We note that we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{X_n \times_k \overline{k}} & \longrightarrow & \Pi_{X_n} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Pi^{(l)}_{X_n \times_k \overline{k}} & \longrightarrow & \Pi^{l}_{X_n} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are exact [cf. the definition of $\Pi^l_{X_n}$]. Here, let us observe that $X_n \times_k \overline{k}$ may be naturally identified with the n-th configuration space of $X \times_k \overline{k}$. Write $\Delta_n \stackrel{\mathrm{def}}{=} \Pi_{(X \times_k \overline{k})_n}$. Now I claim that, if $p = 0$ (respectively, $p \geq 0$), then the image of the outer representation

$$\rho_n : G_k \to \mathrm{Out}(\Delta_n) \quad (\text{respectively, } \rho^{(l)}_n : G_k \to \mathrm{Out}(\Delta^{(l)}_n))$$

is infinite. Indeed, since k is l-cyclotomically full, it follows from Lemma 4.2, (iv), that the image of the outer representaion

$$\rho_1 : G_k \to \mathrm{Out}(\Delta_1) \quad (\text{respectively, } \rho^{(l)}_1 : G_k \to \mathrm{Out}(\Delta^{(l)}_1))$$

is infinite. Here, note that we have the following commutative diagram of profinite groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_n & \longrightarrow & \Pi_{X_n} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta_1 & \longrightarrow & \Pi_{X} & \longrightarrow & G_k & \longrightarrow & 1
\end{array}$$

(respectively,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_n^{(l)} & \longrightarrow & \Pi_{X_n}^l & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta_1^{(l)} & \longrightarrow & \Pi_X^l & \longrightarrow & G_k & \longrightarrow & 1\ ) \end{array}$$

— where the horizontal sequences are exact; the left-hand and central vertical arrows are the surjections induced by the projection $X_n \to X$ obtained by forgetting the factors with labels $\neq n$. Thus, we conclude from Lemma 1.7, (ii), that $\mathrm{Im}(\rho_n)$ (respectively, $\mathrm{Im}(\rho_n^{(l)})$) is infinite. This completes the proof of the claim. In light of the claim, assertion (ii) follows from assertion (i) and Proposition 1.8, (ii). ☐

Corollary 4.5. *Let n be a positive integer; k a Hilbertian field of characteristic $p \geq 0$; X a smooth curve of type (g, r) over k; X_n the n-th configuration space associated to X. Suppose that there exists a prime number $l \neq p$ such that k is l-cyclotomically full. Then the following hold:*

(i) *Suppose that $p = 0$. If $2g - 2 + r > 0$, then Π_{X_n}, $\Pi_{X_n}^l$ are slim and strongly indecomposable.*

(ii) *Suppose that $p > 0$. If $(g, r) \neq (0, 0)$, $(1, 0)$ (respectively, $2g-2+r > 0$), then Π_X (respectively, $\Pi_{X_n}^l$) is slim and strongly indecomposable.*

Proof. These assertions follow immediately from Theorems 2.1, 4.3, 4.4. ☐

Corollary 4.6. *Let n be a positive integer; k a field of characteristic $p \geq 0$; $l \neq p$ a prime number; X a smooth curve of type (g, r) over k; X_n the n-th configuration space associated to X. Then the following hold:*

(i) *Suppose that k is a finitely generated extension field of either a number field or a mixed characteristic local field. If $2g - 2 + r > 0$, then Π_{X_n}, $\Pi_{X_n}^l$ are slim and strongly indecomposable.*

(ii) *Suppose that k is a finitely generated transcendental extension field of a finite field. If $(g, r) \neq (0, 0)$, $(1, 0)$ (respectively, $2g-2+r > 0$), then Π_X (respectively, $\Pi_{X_n}^l$) is slim and strongly indecomposable.*

Proof. First, we note that every field k which appears in Corollary 4.6 is l-cyclotomically full [cf. Lemma 4.2, (v)]. Thus, in the case that k is Hilbertian [cf. Corollary 2.3] (respectively, non-Hilbertian, i.e., mixed characteristic local), the assertions follow from Corollary 4.5 (respectively, Proposition 2.4 and Theorem 4.4). ☐

Remark 4.7. Let k_0 be a positive characteristic local field, i.e., a field isomorphic to the field of formal Laurent series $\mathbb{F}((t))$ over a finite field $\mathbb{F}$. In the notation of Corollary 4.6, suppose that k is a finitely generated extension field of k_0. Here, we note that k is l-cyclotomically full. [Indeed, to verify that k is l-cyclotomically full, we may assume, without loss of generality, that $k = \mathbb{F}((t))$ [cf. Lemma 4.2, (ii)]. Then since the l-adic cyclotomic character $\chi_k : G_k \to \mathbb{Z}_l^\times$ of k factors as the composite of the natural [outer] surjection $G_k \twoheadrightarrow G_{\mathbb{F}}$ — which is induced by the natural injection $\mathbb{F} \hookrightarrow k$ — and the l-adic cyclotomic character $G_{\mathbb{F}} \to \mathbb{Z}_l^\times$ of $\mathbb{F}$, we conclude that the image of χ_k is infinite [cf. Lemma 4.2, (iii), (v)], hence that k is l-cyclotomically full [cf. Lemma 4.2, (iii)].] In particular:

- Suppose that $[k : k_0] = +\infty$. In this case, since k is Hilbertian [cf. Corollary 2.3], we can prove the following [cf. Corollary 4.5]:

  If $(g, r) \neq (0, 0)$, $(1, 0)$ (respectively, $2g - 2 + r > 0$), then Π_X (respectively, $\Pi^l_{X_n}$) is slim and strongly indecomposable.

- Suppose that $[k : k_0] < +\infty$. In this case, note that G_k is slim [cf. Remark 2.5]. Thus, by applying the same arguments as the arguments applied in the proofs of Theorems 4.3, 4.4 [cf. also Proposition 1.8, (i)], we can prove the following:

  If $(g, r) \neq (0, 0)$, $(1, 0)$ (respectively, $2g - 2 + r > 0$), then Π_X (respectively, $\Pi^l_{X_n}$) is slim.

  On the other hand, it is not clear to the author at the time of writing whether or not Π_X (respectively, $\Pi^l_{X_n}$) is [strongly] indecomposable.

5. Indecomposability of k-schemes

In this section, we introduce the notion of the indecomposability of k-schemes, and give a criterion so that a k-scheme is indecomposable. As an application, we prove that the configuration space of a hyperbolic curve over a field of characteristic zero is indecomposable.

Definition 5.1. Let k be a field. We shall say that a k-scheme V is *indecomposable* if, for any k-isomorphism of k-schemes $V \cong Y \times_k Z$, where Y, Z are k-schemes, it follows that either

$$Y \cong \mathrm{Spec}(k) \quad \text{or} \quad Z \cong \mathrm{Spec}(k).$$

Definition 5.2. (cf. [14], Definition 2.5; [32], Definition 2.25) Let k be a field; $\overline{k}$ an algebraic closure of k; l a prime number; X a geometrically connected, separated scheme of finite type over k. Then we shall say that X is of *LFG-type* (respectively, *l-LFG-type*) if, for any connected, normal, separated scheme Y of finite type over $\overline{k}$ and any nonconstant morphism $Y \to X \times_k \overline{k}$ over $\overline{k}$, the image of the outer homomorphism

$$\Pi_Y \to \Pi_{X \times_k \overline{k}} \quad (\text{respectively, } \Pi_Y^{(l)} \to \Pi_{X \times_k \overline{k}}^{(l)})$$

is infinite.

Remark 5.3. In the notation of Definition 5.2, the implication

$$X \text{ is of } l\text{-LFG-type} \;\Rightarrow\; X \text{ is of LFG-type}$$

holds.

The following Proposition is essentially proved in [32], Proposition 2.28 [cf. also [14], Proposition 2.7]. [Note that, in [32], the characteristic of the base field is assumed to be zero.]

Proposition 5.4. *Let n be a positive integer; k a field of characteristic $p \geq 0$; X a hyperbolic curve over k; X_n the n-th configuration space of X. Then, for any prime number $l \neq p$, X_n is of l-LFG-type. In particular, X_n is of LFG-type.*

Proof. We may assume that k is algebraically closed. First, suppose that $n = 1$. Let Y be a connected, normal, separated scheme of finite type over k. Then since any nonconstant k-morphism $Y \to X$ is dominant, it follows from [14], Lemma 1.3, that $\Pi_Y \to \Pi_X$ is open. In particular, $\Pi_Y^{(l)} \to \Pi_X^{(l)}$ is also open. Thus, we conclude from the well-known fact that $\Pi_X^{(l)}$ is infinite that the image of $\Pi_Y^{(l)} \to \Pi_X^{(l)}$ is infinite.

Next, suppose that $n > 1$, and that the induction hypothesis is in force. Let $f : Y \to X_n$ be a nonconstant k-morphism. We consider the composite

$$g \;:\; Y \;\xrightarrow{f}\; X_n \;\to\; X_{n-1},$$

where the second arrow is the projection morphism obtained by forgetting the factor labeled n. If g is nonconstant, then it follows from the induction hypothesis that the image of $\Pi_Y^{(l)} \to \Pi_{X_{n-1}}^{(l)}$ is infinite, hence that the image of $\Pi_Y^{(l)} \to \Pi_{X_n}^{(l)}$ is infinite. If g is constant, then we write $\overline{x} \to X_{n-1}$ for the geometric point of X_{n-1} through which g factors. We write $(X_n)_{\overline{x}}$ for the fiber of $X_n \to X_{n-1}$ over $\overline{x}$. In particular, f factors as the composite

$$Y \ \to \ (X_n)_{\overline{x}} \ \to \ X_n,$$

where the first arrow is a nonconstant k-morphism; the second arrow is the natural projection. We note that it follows from the case where $n = 1$ that the image of $\Pi_Y^{(l)} \to \Pi_{(X_n)_{\overline{x}}}^{(l)}$ is infinite. Then since $\Pi_{(X_n)_{\overline{x}}}^{(l)} \to \Pi_{X_n}^{(l)}$ is injective [cf. [25], Proposition 2.2, (i)], we conclude that the image of $\Pi_Y^{(l)} \to \Pi_{X_n}^{(l)}$ is also infinite. □

Lemma 5.5. *Let k be a field; V a geometrically integral, separated scheme of finite type over k. Suppose that there exists an isomorphism of k-schemes*

$$V \xrightarrow{\sim} Y \times_k Z$$

— where Y, Z are k-schemes. Then Y, Z are geometrically integral, separated schemes of finite type over k.

Proof. In the following, we shall identify V and $Y \times_k Z$ via the above isomorphism. First, note that since V is separated and of finite type over k, it follows from [6], Proposition (5.5.1), (v); [6], Corollaire (6.3.9), that the natural projections $V \to Y$, $V \to Z$ are separated and of finite type. Now I claim that Y, Z are quasi-compact. Indeed, let us first observe that [since V is integral] V is nonempty, hence that Y, Z are nonempty. In particular, it follows that the structure morphisms $Y \to \mathrm{Spec}(k)$, $Z \to \mathrm{Spec}(k)$ are surjective, hence that the natural projections $V \to Y$, $V \to Z$ are surjective. Thus, since V is quasi-compact, we conclude that Y, Z are quasi-compact. This completes the proof of the claim. In light of the claim, it follows immediately that the structure morphisms $Y \to \mathrm{Spec}(k)$, $Z \to \mathrm{Spec}(k)$ are faithfully flat and quasi-compact. Thus, we conclude from [7], Proposition (2.7.1), that $Y \to \mathrm{Spec}(k)$, $Z \to \mathrm{Spec}(k)$ are separated and of finite type. To verify that Y, Z are geometrically integral over k, it suffices to show that, for an algebraic closure $\overline{k}$ of k,

$$Y \times_k \overline{k}, \quad Z \times_k \overline{k}$$

are integral. But since the natural projections $V \times_k \overline{k} \to Y \times_k \overline{k}$, $V \times_k \overline{k} \to Z \times_k \overline{k}$ are faithfully flat, the integrality of $Y \times_k \overline{k}$, $Z \times_k \overline{k}$ follows from the integrality of $V \times_k \overline{k}$. □

Theorem 5.6. *Let k be a field of characteristic $p \geq 0$; $l \neq p$ a prime number; $\overline{k}$ an algebraic closure of k; V a geometrically integral, separated scheme of finite type over k. Suppose that one of the following conditions is satisfied:*

(1) *The following conditions are satisfied:*
 (1-i) *V is proper over k.*
 (1-ii) *V is of LFG-type.*
 (1-iii) *$\Pi_{V \times_k \overline{k}}$ is indecomposable.*

(2) *The following conditions are satisfied:*
 (2-i) *$p = 0$.*
 (2-ii) *V is of LFG-type.*
 (2-iii) *$\Pi_{V \times_k \overline{k}}$ is indecomposable.*

(3) *The following conditions are satisfied:*
 (3-i) *V is of l-LFG-type.*
 (3-ii) *$\Pi^{(l)}_{V \times_k \overline{k}}$ is indecomposable.*

Then V is indecomposable.

Proof. To verify Theorem 5.6, it suffices to show the following claim

($*$) Suppose that there exists an isomorphism of k-schemes

$$V \xrightarrow{\sim} Y \times_k Z$$

— where Y, Z are k-schemes. Then it follows that either

$$Y \cong \mathrm{Spec}(k) \quad \text{or} \quad Z \cong \mathrm{Spec}(k).$$

First, note that Y, Z are geometrically integral, separated schemes of finite type over k [cf. Lemma 5.5]. Thus, to verify the claim ($*$), we may assume that k is algebraically closed. Then to verify the claim ($*$), it suffices to show that either

$$\dim(Y) = 0 \quad \text{or} \quad \dim(Z) = 0.$$

Now we observe that, if either condition (1) or condition (2) (respectively, condition (3)) is satisfied, then by the Künneth formula [cf. [8], EXPOSÉ X, Corollaire 1.7; [8], EXPOSÉ XIII, Proposition 4.6; [29], Proposition 4.7], there exists an isomorphism of profinite groups

$$\Pi_V \xrightarrow{\sim} \Pi_Y \times \Pi_Z \quad \text{(respectively, } \Pi^{(l)}_V \xrightarrow{\sim} \Pi^{(l)}_Y \times \Pi^{(l)}_Z\text{)}.$$

Then since Π_V (respectively, $\Pi^{(l)}_V$) is indecomposable, we may assume without loss of generality that $\Pi_Y = \{1\}$ (respectively, $\Pi^{(l)}_Y = \{1\}$). Now we fix a k-rational point $z \in Z(k)$ of Z. Then we obtain a closed immersion

$Y \xrightarrow{\sim} Y \times_k \{z\} \hookrightarrow Y \times_k Z \xrightarrow{\sim} V$. Write $Y' \to Y$ for the [surjective] morphism obtained by normalizing Y. Here, if we assume that $\dim(Y) \geq 1$, then the composite $Y' \to Y \hookrightarrow V$ is nonconstant. Thus, since V is of LFG-type (respectively, l-LFG-type), the image of the outer homomorphism

$$\Pi_{Y'} \to \Pi_V \quad (\text{respectively, } \Pi_{Y'}^{(l)} \to \Pi_V^{(l)})$$

is infinite — a contradiction. Therefore, we conclude that $\dim(Y) = 0$. This completes the proof of the claim $(*)$, hence also of Theorem 5.6. ☐

Corollary 5.7. *In the notation of Proposition 5.4, X_n is indecomposable.*

Proof. This follows from Theorem 4.4, (i); Proposition 5.4; Theorem 5.6. ☐

Remark 5.8. It is not clear to the author at the time of writing whether or not there exists a geometric proof of Corollary 5.7.

6. Indecomposability of the Pro-l Grothendieck-Teichmüller Group

In this section, we verify the indecomposability of the pro-l Grothendieck-Teichmüller group GT_l [cf. Corollary 6.3] as a consequence of a certain anabelian result over finite fields [cf. [12], Remark 6, (iv)].

Let l be a prime number; k an algebraically closed field of characteristic zero; $\mathbb{F}$ a field of characteristic $\mathrm{char}(\mathbb{F}) \neq l$; $\overline{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$; $\Pi \stackrel{\mathrm{def}}{=} \Pi_{\mathbb{P}^1_k \setminus \{0,1,\infty\}}$; $\mathfrak{S}_3$ the symmetric group on 3 letters. Here, we observe that the natural action of $\mathfrak{S}_3$ on $\mathbb{P}^1_k \setminus \{0,1,\infty\}$ induces injections

$$\mathfrak{S}_3 \hookrightarrow \mathrm{Out}(\Pi); \quad \mathfrak{S}_3 \hookrightarrow \mathrm{Out}(\Pi^{(l)}).$$

Thus, in the following, via these injections, we regard $\mathfrak{S}_3$ as a subgroup of $\mathrm{Out}(\Pi)$ and $\mathrm{Out}(\Pi^{(l)})$. Moreover, if $\mathrm{char}(\mathbb{F}) = 0$ (respectively, $\mathrm{char}(\mathbb{F}) \geq 0$), then let

$$\rho_{\mathbb{F}} : G_{\mathbb{F}} \to \mathrm{Out}(\Pi_{\mathbb{P}^1_{\overline{\mathbb{F}}} \setminus \{0,1,\infty\}}) \xrightarrow{\sim} \mathrm{Out}(\Pi)$$

$$(\text{respectively, } \rho_{\mathbb{F}}^{(l)} : G_{\mathbb{F}} \to \mathrm{Out}(\Pi^{(l)}_{\mathbb{P}^1_{\overline{\mathbb{F}}} \setminus \{0,1,\infty\}}) \xrightarrow{\sim} \mathrm{Out}(\Pi^{(l)}))$$

be the composite of the natural [pro-l] outer Galois representation with the isomorphism induced by the natural [outer] isomorphism

$$\Pi_{\mathbb{P}^1_{\overline{\mathbb{F}}} \setminus \{0,1,\infty\}} \xrightarrow{\sim} \Pi \quad (\text{respectively, } \Pi^{(l)}_{\mathbb{P}^1_{\overline{\mathbb{F}}} \setminus \{0,1,\infty\}} \xrightarrow{\sim} \Pi^{(l)}).$$

Theorem 6.1. *Let N be a closed subgroup of* $\mathrm{Out}(\Pi)$ *(respectively,* $\mathrm{Out}(\Pi^{(l)})$*) satisfying the following conditions:*

(a) *N contains an open subgroup of* $\mathrm{Im}(\rho_{\mathbb{Q}})$ *(respectively,* $\mathrm{Im}(\rho_{\mathbb{Q}}^{(l)})$*).*

(b) *It holds that $N \subseteq Z_{\mathrm{Out}(\Pi)}(\mathfrak{S}_3)$ (respectively, $N \subseteq Z_{\mathrm{Out}(\Pi^{(l)})}(\mathfrak{S}_3)$).*

Then N is slim (respectively, slim and strongly indecomposable).

Proof. First, let us consider the slimness portion of Theorem 6.1. To verify the slimness of N, it suffices to show that for any open subgroup U of N, U is center-free. Let $\sigma \in Z(U)$. We note that, by condition (a), there exists a finite extension field E of $\mathbb{Q}$ such that $\mathrm{Im}(\rho_E) \subseteq U$ (respectively, $\mathrm{Im}(\rho_E^{(l)}) \subseteq U$). Thus, it follows from [20], Theorem A, that

$$\sigma \in Z_{\mathrm{Out}(\Pi)}(\mathrm{Im}(\rho_E)) \xleftarrow{\sim} \mathrm{Aut}_E(\mathbb{P}^1_E \setminus \{0,1,\infty\}) \xleftarrow{\sim} \mathfrak{S}_3$$

$$(\text{respectively, } \sigma \in Z_{\mathrm{Out}(\Pi^{(l)})}(\mathrm{Im}(\rho_E^{(l)})) \xleftarrow{\sim} \mathrm{Aut}_E(\mathbb{P}^1_E \setminus \{0,1,\infty\}) \xleftarrow{\sim} \mathfrak{S}_3),$$

hence that, $\sigma \in Z(\mathfrak{S}_3) = \{1\}$ [cf. condition (b)]. Therefore, we conclude that U is center-free, hence also that N is slim.

Next, let us verify the strong indecomposability of N [$\subseteq \mathrm{Out}(\Pi^{(l)})$]. To verify the strong indecomposability of N, it suffices to show that for any open subgroup U of N, U is indecomposable. Let $p \neq l$ be a prime number. Here, we note that since the restriction

$$\rho_{\mathbb{Q}}^{(l)}|_{G_{\mathbb{Q}_p}} : G_{\mathbb{Q}_p} \ \to \ \mathrm{Out}(\Pi^{(l)})$$

factors as the composite

$$G_{\mathbb{Q}_p} \ \twoheadrightarrow \ G_{\mathbb{F}_p} \ \overset{\rho_{\mathbb{F}_p}^{(l)}}{\to} \ \mathrm{Out}(\Pi^{(l)})$$

— where the first arrow is the natural [outer] surjection — it follows from condition (a) that N contains an open subgroup of $\mathrm{Im}(\rho_{\mathbb{F}_p}^{(l)})$. Thus, there exists a finite extension field F of $\mathbb{F}_p$ such that

$$G \ \overset{\mathrm{def}}{=} \ \mathrm{Im}(\rho_F^{(l)}) \ \subseteq \ U.$$

Moreover, since $\mathrm{Out}(\Pi^{(l)})$ is almost pro-l [cf. [1], Corollary 7], by replacing F by a suitable finite extension field of F, we may assume without loss of generality that $\rho_F^{(l)}$ factors through the maximal pro-l quotient $G_F \twoheadrightarrow G_F^{(l)}$ of G_F. Here, we note that since G is infinite [cf. Lemma 4.2, (iv), (v)], we have $G \cong \mathbb{Z}_l$.

Now suppose that we have an isomorphism of profinite groups $U \cong H_1 \times H_2$. In the following, we shall identify U and $H_1 \times H_2$ via this isomorphism. Then I claim that it holds that

$$\text{either} \quad G \cap H_1 \neq \{1\} \quad \text{or} \quad G \cap H_2 \neq \{1\}.$$

Indeed, suppose that $G \cap H_1 = \{1\}$ and $G \cap H_2 = \{1\}$. In particular, it follows that, for $i = 1, 2$, the composite

$$G \hookrightarrow U = H_1 \times H_2 \overset{\mathrm{pr}_i}{\twoheadrightarrow} H_i$$

— where pr_i is i-th projection — is injective. Thus, if we write $K_i \subseteq H_i$ for the image of the above composite, we obtain that $G \overset{\sim}{\to} K_i$ $[\cong \mathbb{Z}_l]$. Here, note that we have inclusions

$$G \subseteq K \overset{\mathrm{def}}{=} K_1 \times K_2 \subseteq H_1 \times H_2.$$

Now let us observe that it follows from Lemma 4.2, (iv), (v); [22], Corollary 2.7, (i); condition (b); [15], Lemma 3.5, that

$$Z_U(G) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{(l)})$$

— where we write $\mathrm{Out}^{|\mathrm{C}|}(\Pi^{(l)})$ for the group of outer automorphisms of $\Pi^{(l)}$ which fixes the conjugacy class of each cuspidal inertia subgroup of $\Pi^{(l)}$. Thus, since K $[\cong \mathbb{Z}_l \times \mathbb{Z}_l]$ is abelian, we obtain that

$$K \subseteq Z_U(G) \hookrightarrow \mathbb{Z}_l^{\times}$$

— where "$\hookrightarrow$" is induced by the morphism "$\deg_{\mathbb{P}}$" of [12], Definition 3.1, which is injective by [12], Remark 6, (iv). Take an open subgroup J of $\mathbb{Z}_l^{\times}$ such that $J \cong \mathbb{Z}_l$. In particular, if we identify K with $\mathbb{Z}_l \times \mathbb{Z}_l$, then there exists a positive integer m such that $l^m\mathbb{Z}_l \times l^m\mathbb{Z}_l \subseteq J \cap K \subseteq K$. On the other hand, since $l^m\mathbb{Z}_l \times l^m\mathbb{Z}_l \neq \{0\}$ is a closed subgroup of J $[\cong \mathbb{Z}_l]$, we obtain that $l^m\mathbb{Z}_l \times l^m\mathbb{Z}_l \cong \mathbb{Z}_l$, a contradiction. This completes the proof of the claim.

In light of the claim, we may assume without loss of generality that

$$G \cap H_1 \neq \{1\}.$$

Then since $G \cap H_1 \subseteq G$ is a nontrivial closed subgroup of $G \cong \mathbb{Z}_l$, it follows that $G \cap H_1$ is open in G. Thus, by replacing F by a suitable finite extension, we may assume without loss of generality that $G \subseteq H_1$. In particular, we obtain that

$$H_2 \subseteq Z_U(G) \hookrightarrow \mathbb{Z}_l^{\times}$$

— where "$\hookrightarrow$" denotes the arrow "$\hookrightarrow$" in the final display of the proof of the above claim. Thus, it follows that H_2 is abelian. On the other hand, since H_2 is center-free [cf. the fact that N, hence also U is slim], we obtain that $H_2 = \{1\}$. Therefore, we conclude that U is indecomposable, as desired. $\square$

In the following, we shall denote by

$$\mathrm{pr}_i : (\mathbb{P}^1_k \setminus \{0,1,\infty\})_2 \ \to \ \mathbb{P}^1_k \setminus \{0,1,\infty\}$$

— where we denote by $(\mathbb{P}^1_k \setminus \{0,1,\infty\})_2$ the second configuration space of $\mathbb{P}^1_k \setminus \{0,1,\infty\}$ — the i-th projection for $i \in \{1,2\}$. We shall write

$$\Pi_2 \ \stackrel{\mathrm{def}}{=} \ \Pi_{(\mathbb{P}^1_k \setminus \{0,1,\infty\})_2}.$$

In particular, pr_i induces a(n) [outer] surjection

$$\Pi_2 \ \twoheadrightarrow \ \Pi \quad (\text{respectively, } \ \Pi_2^{(l)} \ \twoheadrightarrow \ \Pi^{(l)})$$

which we denote by p_i (respectively, $p_i^{(l)}$). Then we shall write

$$\mathrm{Out}^{\mathrm{F}}(\Pi_2) \quad (\text{respectively, } \ \mathrm{Out}^{\mathrm{F}}(\Pi_2^{(l)}))$$

for the group of outer automorphisms σ of Π_2 (respectively, $\Pi_2^{(l)}$) such that $\sigma(\mathrm{Ker}(p_i)) = \mathrm{Ker}(p_i)$ (respectively, $\sigma(\mathrm{Ker}(p_i^{(l)})) = \mathrm{Ker}(p_i^{(l)})$) for $i \in \{1,2\}$ [cf. [23], Definition 1.1, (ii)].

Now let us recall the Grothendieck-Teichmüller group

$$\mathrm{GT}$$

which was introduced by Drinfeld [cf. [4]]. This group is a subgroup of the automorphism group of the free profinite group of rank 2 and important, e.g., from the point of view of the comparison with $G_{\mathbb{Q}}$ [cf. [33], §1.4]. Let us, moreover, recall that GT may be identified with

$$\mathrm{Im}(\mathrm{Out}^{\mathrm{F}}(\Pi_2) \to \mathrm{Out}(\Pi)) \ \cap \ Z_{\mathrm{Out}(\Pi)}(\mathfrak{S}_3) \quad [\subseteq \mathrm{Out}(\Pi)]$$

— where $\mathrm{Out}^{\mathrm{F}}(\Pi_2) \to \mathrm{Out}(\Pi)$ is the injection induced by p_1 [cf. [23], Remark 1.11.1; [23], Corollary 1.12, (ii); [15], Theorem 2.3, (ii)]. Now we define a pro-l analogue of GT as follows:

Definition 6.2. We shall write

$$\mathrm{GT}_l \ \stackrel{\mathrm{def}}{=} \ \mathrm{Im}(\mathrm{Out}^{\mathrm{F}}(\Pi_2^{(l)}) \to \mathrm{Out}(\Pi^{(l)})) \ \cap \ Z_{\mathrm{Out}(\Pi^{(l)})}(\mathfrak{S}_3) \quad [\subseteq \mathrm{Out}(\Pi^{(l)})]$$

and refer to GT_l as the *pro-l Grothendieck-Teichmüller group* [cf. [15], Definition 3.4, (i)].

Corollary 6.3. GT *(respectively,* GT_l*) is slim (respectively, slim and strongly indecomposable).*

Proof. To verify Corollary 6.3, it suffices to show that GT and GT_l satisfy conditions (a), (b) of Theorem 6.1. It is immediate that GT and GT_l satisfy condition (b). Now I claim that the following inclusions hold:

$$\mathrm{Im}(\rho_{\mathbb{Q}}) \subseteq \mathrm{GT}, \quad \mathrm{Im}(\rho_{\mathbb{Q}}^{(l)}) \subseteq \mathrm{GT}_l.$$

Let us verify the first inclusion. [In a similar vein, one can verify the second inclusion.] To verify the inclusion, we may assume that $k = \overline{\mathbb{Q}}$, and that $\rho_{\mathbb{Q}}$ is the natural outer Galois representaion associated to $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$. First, let us observe that the image of the natural outer Galois representation

$$G_{\mathbb{Q}} \to \mathrm{Out}(\Pi_2)$$

associated to $(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\})_2$ is contained in $\mathrm{Out}^{\mathrm{F}}(\Pi_2) \subseteq \mathrm{Out}(\Pi_2)$, hence that the homomorphism $G_{\mathbb{Q}} \to \mathrm{Out}(\Pi_2)$ factors as the composite of a homomorphism $\alpha_{\mathbb{Q}} : G_{\mathbb{Q}} \to \mathrm{Out}^{\mathrm{F}}(\Pi_2)$ and the natural injection $\mathrm{Out}^{\mathrm{F}}(\Pi_2) \hookrightarrow \mathrm{Out}(\Pi_2)$. [Indeed, this observation follows from the fact that the natural action of $G_{\mathbb{Q}}$ on $(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\})_2$ is compatible with pr_i for $i \in \{1, 2\}$.] In particular, we obtain a commutative diagram

$$\begin{array}{ccc}
G_{\mathbb{Q}} & \xrightarrow{\alpha_{\mathbb{Q}}} & \mathrm{Out}^{\mathrm{F}}(\Pi_2) \\
\| & & \downarrow \\
G_{\mathbb{Q}} & \xrightarrow{\rho_{\mathbb{Q}}} & \mathrm{Out}(\Pi)
\end{array}$$

— where the right-hand vertical arrow is the injection induced by pr_1. In light of the commutativity of this diagram, we conclude that

$$\mathrm{Im}(\rho_{\mathbb{Q}}) \ \subseteq \ \mathrm{Im}(\mathrm{Out}^{\mathrm{F}}(\Pi_2) \to \mathrm{Out}(\Pi)).$$

Thus, to verify the inclusion $\mathrm{Im}(\rho_{\mathbb{Q}}) \subseteq \mathrm{GT}$, it suffices to verify the inclusion $\mathrm{Im}(\rho_{\mathbb{Q}}) \subseteq Z_{\mathrm{Out}(\Pi)}(\mathfrak{S}_3)$. On the other hand, this follows immediately from the fact that, for any $\sigma \in \mathfrak{S}_3$, $g \in G_{\mathbb{Q}}$, the automorphism of $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ induced by σ commutes with the automorphism of $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ induced by g. [Here, we note that $\mathfrak{S}_3$ and $G_{\mathbb{Q}}$ act naturally on $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$.] This completes the proof of the claim. Therefore, we conclude that GT and GT_l satisfy condition (a). □

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