Math. J. Okayama Univ. **60** (2018), 59–72

ABSOLUTE CONTINUITY OF THE REPRESENTING MEASURES OF THE TRANSMUTATION OPERATORS ATTACHED TO THE ROOT SYSTEM OF TYPE BC_2

Khalifa Trimèche

Abstract. We prove in this paper the absolute continuity of the representing measures of the transmutation operators $V_k, {}^tV_k$ and $V_k^W, {}^tV_k^W$ associated respectively to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type BC_2.

1. Introduction

We consider the differential-difference operators $T_j, j = 1, 2, ..., d$, associated with a root system $\mathcal{R}$, a Weyl group W and a multiplicity function k, introduced by I. Cherednik in [2], and called the Cherednik operators in the literature. These operators are helpful for the extension and simplification of the theory of Heckman-Opdam, which is a generalization of the harmonic analysis on the symmetric spaces $G|K$ (see [3, 4, 5, 7]).

The notion of transmutation operators called also the trigonometric Dunkl intertwining operators and their dual introduced in [8] are fundamental in the harmonic analysis associated to the Cherednik operators and the Heckman-Opdam theory. We have considered in [8, 9] the transmutation operators $V_k, {}^tV_k$ and $V_k^W, {}^tV_k^W$ associated respectively to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type BC_2, and we have proved that these operators are integral transforms, more precisely, for all function g in $\mathcal{E}(\mathbb{R}^2)$ (the space of C^∞-functions on $\mathbb{R}^2\}$ we have

$$\forall\, x \in \mathbb{R}^2, V_k(g)(x) = \int_{\mathbb{R}^2} g(y)d\mu_x(y), \tag{1.1}$$

where μ_x is a positive measure with compact support contained in the closed ball $\bar{B}(0, \|x\|)$ of center 0 and radius $\|x\|$, and of norm less than or equal to 1.
And for all function f in $\mathcal{D}(\mathbb{R}^2)$ (the space of C^∞-functions on $\mathbb{R}^2$, with compact support) we have

$$\forall\, y \in \mathbb{R}^2,\ {}^tV_k(f)(y) = \int_{\mathbb{R}^2} f(x)d\nu_y(x), \tag{1.2}$$

Mathematics Subject Classification. Primary 51F15; Secondary 33E30; 47B34.

Key words and phrases. Transmutation operators ; Absolute continuity of the representing measures ; Cherednik operators ; Heckman-Opdam theory.

where ν_y is a positive measure with support in the set $\{x \in \mathbb{R}^2 \,;\, \|x\| \geq \|y\|\}$. From the previous results we have deduced that for all functions g in $\mathcal{E}(\mathbb{R}^2)^W$ (the subspace of $\mathcal{E}(\mathbb{R}^2)$ of W-invariant functions) and f in $\mathcal{D}(\mathbb{R}^2)^W$ (the subspace of $\mathcal{D}(\mathbb{R}^2)$ of W-invariant functions) we have

$$\forall\, x \in \mathbb{R}^2,\ V_k^W(g)(x) = \int_{\mathbb{R}^2} g(y)d\mu_x^W(y), \tag{1.3}$$

and

$$\forall\, y \in \mathbb{R}^2,\ {}^tV_k(f)(y) = \int_{\mathbb{R}^2} f(x)d\nu_y^W(x), \tag{1.4}$$

where

$$\mu_x^W = \frac{1}{|W|}\sum_{w\in W}\mu_{wx} \tag{1.5}$$

and

$$\nu_y^W = \frac{1}{|W|}\sum_{w\in W}\nu_{wy} \tag{1.6}$$

In this paper we prove that for all $x \in \mathbb{R}^2_{reg}$ (the regular part of $\mathbb{R}^2$) and $y \in \mathbb{R}^2$, the measures μ_x, μ_x^W and ν_y, ν_y^W are absolute continuous with respect to Lebesgue measure on $\mathbb{R}^2$. More precisely there exist positive functions $\mathcal{K}(x,y)$ and $\mathcal{K}^W(x,y)$ such that

$$d\mu_x(y) = \mathcal{K}(x,y)dy, \tag{1.7}$$

$$d\mu_x^W(y) = \mathcal{K}^W(x,y)dy, \tag{1.8}$$

$$d\nu_y(x) = \mathcal{K}(x,y)\mathcal{A}_k(x)dx, \tag{1.9}$$

$$d\nu_y^W(x) = \mathcal{K}^W(x,y)\mathcal{A}_k(x)dx, \tag{1.10}$$

where $\mathcal{A}_k$ is a weight function on $\mathbb{R}^2$ which will be given in the following section (see (2.7)).

The function $y \to \mathcal{K}(x,y)$ and $y \to \mathcal{K}^W(x,y)$ have their support contained in the closed ball $\bar{B}(0,\|x\|)$ and satisfy

$$\int_{\mathbb{R}^2}\mathcal{K}(x,y)dy \leq 1, \tag{1.11}$$

and

$$\int_{\mathbb{R}^2}\mathcal{K}^W(x,y)dy \leq 1. \tag{1.12}$$

As applications of the previous results, we prove that for all $\lambda \in \mathbb{C}^2$ the Opdam-Cherednik kernel G_λ and the Heckman-Opdam hypergeometric function F_λ possess the following integral representations

$$\forall\, x \in \mathbb{R}^2_{reg},\ G_\lambda(x) = \int_{\mathbb{R}^2}\mathcal{K}(x,y)e^{-i\langle\lambda,y\rangle}dy, \tag{1.13}$$

and

$$\forall\, x \in \mathbb{R}^2_{reg},\ F_\lambda(x) = \int_{\mathbb{R}^2} \mathcal{K}(x,y)^W e^{-i\langle\lambda,y\rangle} dy. \tag{1.14}$$

2. The Cherednik operators attached to the root system of type BC_2

We consider $\mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$, and inner product $\langle .,.\rangle$ for which this basis is orthonormal. We extend this inner product to a complex bilinear form on $\mathbb{C}^2$.

2.1. The root system of type BC_2 and the Cherednik operators on $\mathbb{R}^2$. The root system of type BC_2 can be identified with the set $\mathcal{R}$ given by

$$\mathcal{R} = \{\pm e_1, \pm e_2, \pm 2e_1, \pm 2e_2\} \cup \{\pm e_1 \pm e_2\}, \tag{2.1}$$

which can also be written in the form

$$\mathcal{R} = \{\pm\alpha_i, i = 1, 2, ..., 6\},$$

with

$$\alpha_1 = e_1, \alpha_2 = e_2, \alpha_3 = 2e_1, \alpha_4 = 2e_2\ , \alpha_5 = (e_1 - e_2), \alpha_6 = (e_1 + e_2). \tag{2.2}$$

We denote by $\mathcal{R}_+$ the set of positive roots.

$$\mathcal{R}_+ = \{\alpha_i, i = 1, 2, ..., 6\}. \tag{2.3}$$

For $\alpha \in \mathcal{R}$, we consider

$$r_\alpha(x) = x - \langle\breve{\alpha}, x\rangle\alpha, \text{ with } \breve{\alpha} = \frac{2\alpha}{\|\alpha\|^2}, \tag{2.4}$$

the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^2$ orthogonal to α.

The reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group W called the Weyl group associated with $\mathcal{R}$. In this case W is isomorphic to the hyperoctahedral group which is generated by permutations and sign changes of the $e_i, i = 1, 2$.

The multiplicity function $k : \mathcal{R} \to]0, +\infty[$ can be written in the form $k = (k_1, k_2, k_3)$ where k_1 and k_2 are the values on the roots α_1, α_2, and α_3, α_4 respectively, and k_3 is the value on the roots α_5, α_6.

The positive Weyl chamber denoted by $\mathfrak{a}^+$ is given by

$$\mathfrak{a}^+ = \{x \in \mathbb{R}^2\ ; \quad \forall\, \alpha \in \mathcal{R}_+, \langle\alpha, x\rangle > 0\}. \tag{2.5}$$

it can also be written in the form

$$\mathfrak{a}^+ = \{(x_1, x_2) \in \mathbb{R}^2\ ; x_1 > x_2 > 0\}. \tag{2.6}$$

Moreover, let $\mathcal{A}_k$ be the weight function

$$\forall\, x \in \mathbb{R}^2,\ \mathcal{A}_k(x) = \prod_{\alpha\in\mathcal{R}_+} |\sinh\langle\frac{\alpha}{2}, x\rangle|^{2k(\alpha)}. \tag{2.7}$$

The Cherednik operators $T_j, j = 1,2$, are defined for functions f of class C^1 on $\mathbb{R}^2$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha\in\mathcal{R}_+} \frac{k(\alpha)\langle\alpha, e_j\rangle}{1-e^{-\langle\alpha,x\rangle}}\{f(x)-f(r_\alpha x)\} - \rho_j f(x), \quad (2.8)$$

with

$$\rho_j = \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} k(\alpha)\langle\alpha, e_j\rangle, \quad j=1,2. \quad (2.9)$$

These operators can also be written in the following form

$$T_1 f(x) = \frac{\partial}{\partial x_1} f(x) + k_1\, \frac{\{f(x)-f(r_{\alpha_1}x)\}}{1-e^{-\langle\alpha_1,x\rangle}} + 2k_2 \frac{\{f(x)-f(r_{\alpha_3}x)\}}{1-e^{-\langle\alpha_3,x\rangle}}$$

$$+k_3\Big[\frac{f(x)-f(r_{\alpha_5}x)}{1-e^{-\langle\alpha_5,x\rangle}} + \frac{f(x)-f(r_{\alpha_6}x)}{1-e^{-\langle\alpha_6,x\rangle}}\Big] - (\frac{1}{2}k_1+k_2+k_3)f(x), \quad (2.10)$$

$$\begin{aligned} T_2 f(x) \quad &= \frac{\partial}{\partial x_2} f(x) + \frac{\{f(x)-f(r_{\alpha_2}x)\}}{1-e^{-\langle\alpha_2,x\rangle}} + 2k_2 \frac{\{f(x)-f(r_{\alpha_4}x)\}}{1-e^{-\langle\alpha_4 x)\rangle}} \\ &+k_3\Big[-(\frac{f(x)-f(r_{\alpha_5}x)}{1-e^{-\langle\alpha_5,x\rangle}}) + (\frac{f(x)-f(r_{\alpha_6}x)}{1-e^{-\langle\alpha_6,x\rangle}})\Big] - (\frac{1}{2}k_1+k_2)f(x). \end{aligned} \quad (2.11)$$

2.2. **The Opdam-Cherednik kernel and the Heckman-Opdam hypergeometric function (see** [3, 4, 5, 7]**).** We denote by $G_\lambda, \lambda\in\mathbb{C}^2$, the eigenfunction of the operators $T_j, j=1,2$. It is the unique analytic function on $\mathbb{R}^2$ which satisfies the differential-difference system

$$\begin{cases} T_j G_\lambda(x) &= -i\lambda_j G_\lambda(x), j=1,2, x\in\mathbb{R}^2, \\ G_\lambda(0) &= 1. \end{cases} \quad (2.12)$$

It is called the Opdam-Cherednik kernel.

We consider the function $F_\lambda, \lambda\in\mathbb{C}^2$, defined by

$$\forall\, x\in\mathbb{R}^2, \;\; F_\lambda(x) = \frac{1}{|W|}\sum_{w\in W} G_\lambda(wx). \quad (2.13)$$

It is called the Heckman-Opdam hypergeometric function.

The functions G_λ and F_λ possess the following properties .

i) For all $x\in\mathbb{R}^2$ the function $\lambda\to G_\lambda(x)$ is entire on $\mathbb{C}^2$.

ii) We have

$$\forall\, x\in\mathbb{R}^2, \;\; \forall\,\lambda\in\mathbb{C}^2, |G_\lambda(x)| \le G_{Im(\lambda)}(x). \quad (2.14)$$

iii) We have

$$\forall\, x\in\mathbb{R}^2, \; \forall\,\lambda\in\mathbb{R}^2, \; |G_\lambda(x)| \le 1. \quad (2.15)$$

(See [9]).

vi) The function $G_\lambda, \lambda \in \mathbb{C}^2$, admits the following Laplace type representation

$$\forall\, x \in \mathbb{R}^2, G_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle\lambda,y\rangle} d\mu_x(y), \tag{2.16}$$

where μ_x is the positive measure given by (1.1).

v) From (2.13), (2.16) we deduce that the function $F_\lambda, \lambda \in \mathbb{C}^2$, possesses the Laplace type representation

$$\forall\, x \in \mathbb{R}^2, F_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle\lambda,y\rangle} d\mu_x^W(y), \tag{2.17}$$

where μ_x^W is the measure given by (1.3).

3. The transmutation operator and its dual associated with the Cherednik operators attached to the root system of type BC_2

Notations. We denote by

- $\mathcal{E}(\mathbb{R}^2)$ the space of C^∞-functions on $\mathbb{R}^2$. Its topology is defined by the semi-norms

$$q_{n,K}(\varphi) = \sup_{\substack{|\mu|\leq n \\ x\in K}} |D^\mu \varphi(x)|.$$

where K is a compact of $\mathbb{R}^2$, $n \in \mathbb{N}$ and

$$D^\mu = \frac{\partial^{|\mu|}}{\partial^{\mu_1} x_1 \partial^{\mu_2} x_2 \partial^{\mu_3} x_3}, \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{N}^3, |\mu| = \mu_1 + \mu_2 + \mu_3.$$

- $\mathcal{D}(\mathbb{R}^2)$ the space of C^∞-functions on $\mathbb{R}^2$ with compact support. We have

$$\mathcal{D}(\mathbb{R}^2) = \cup_{a>0} \mathcal{D}_a(\mathbb{R}^2),$$

where $\mathcal{D}_a(\mathbb{R}^2)$ is the space of C^∞-functions on $\mathbb{R}^2$ with support in the closed ball $B(0,a)$ of center 0 and radius a. The topology of $\mathcal{D}_a(\mathbb{R}^2)$ is defined by the semi-norms

$$P_n(\psi) = \sup_{\substack{|\mu|\leq n \\ x\in B(0,a)}} |D^\mu \psi(x)|, n \in \mathbb{N}.$$

The space $\mathcal{D}(\mathbb{R}^2)$ is equipped with the inductive limit topology.

By using the measure μ_x given by (1.1) we define by applying the same method as in [8],the transmutation operator V_k called also the trigonometric Dunkl intertwining operator relating to the root system of type BC_2 by

$$\forall\, x \in \mathbb{R}^2, \quad V_k(g)(x) = \int_{\mathbb{R}^2} g(y) d\mu_x(y), \quad g \in \mathcal{E}(\mathbb{R}^2). \tag{3.1}$$

The operator V_k is the unique linear topological isomorphism from $\mathcal{E}(\mathbb{R}^2)$ onto itself satisfying the transmutation relations

$$\forall\, x \in \mathbb{R}^2,\ \ T_j V_k(g)(x) = V_k(\frac{\partial}{\partial y_j} g)(x),\ \ j = 1, 2. \tag{3.2}$$

and the condition

$$V_k(g)(0) = g(0). \tag{3.3}$$

The dual ${}^tV_k$ of the operator V_k is defined by the following duality relation

$$\int_{\mathbb{R}^2} {}^tV_k(f)(y)g(y)dy = \int_{\mathbb{R}^2} V_k(g)(x)f(x)\mathcal{A}_k(x)dx, \tag{3.4}$$

with f in $\mathcal{D}(\mathbb{R}^2)$ and g in $\mathcal{E}(\mathbb{R}^2)$.

The operator ${}^tV_k$ is a linear topological isomorphism from $\mathcal{D}(\mathbb{R}^2)$ onto itself satisfying the transmutation relations

$$\forall\, y \in \mathbb{R}^2, {}^tV_k((T_j + S_j)f)(y) = \frac{\partial}{\partial y_j} {}^tV_k(f)(y), j = 1, 2, \tag{3.5}$$

where S_j is the operator on $\mathcal{D}(\mathbb{R}^2)$ given by

$$\forall\, x \in \mathbb{R}^2, S_j(h)(x) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\ \langle \alpha, e_j \rangle h(r_\alpha x).$$

The operator ${}^tV_k$ is an integral transform, more precisely we have

$$\forall\, y \in \mathbb{R}^2,\ {}^tV_k(f)(y) = \int_{\mathbb{R}^2} f(x)d\nu_y(x), \quad f \in \mathcal{D}(\mathbb{R}^2), \tag{3.6}$$

where ν_y is the measure given by (1.2).

Remark 3.1. *By using the measure μ_x given by (1.1) we have defined and studied in* [8] *the transmutation operator V_k^W on $\mathcal{E}(\mathbb{R}^2)^W$ relating to the root system of type BC_2, and we have studied also its dual ${}^tV_k^W$ on $\mathcal{D}(\mathbb{R}^2)^W$. We have given some properties of these operators and we have proved that they are positive integral transforms.*

Notation. We denote by $B(c, a)$ the open ball of $\mathbb{R}^2$ of center $c \in \mathbb{R}^2$ and radius $a > 0$, and by $\bar{B}(c, a)$ its closure.

Proposition 3.2. *Let $y_0 \in \mathbb{R}^2$ and $a > 0$. We consider the sequence $\{g_n\}_{n \in \mathbb{N}\backslash\{0\}}$ of functions in $\mathcal{D}(\mathbb{R}^2)$, positive, increasing such that :*

$$\forall\, n \in \mathbb{N}\backslash\{0\}, \operatorname{supp} g_n \subset \bar{B}(y_0, a), \forall\, y \in B(y_0, a - \frac{1}{n}), g_n(y) = 1,$$

and

$$\forall\, y \in \mathbb{R}^2,\ \lim_{n \to +\infty} g_n(y) = 1_{B(y_0,a)}(y),$$

where $1_{B(y_0,a)}$ is the characteristic function of the ball $B(y_0,a)$. We have

$$\begin{aligned}\forall\, x \in \mathbb{R}^2, \lim_{n\to+\infty} V_k(g_n)(x) &= \lim_{n\to+\infty}\int_{\mathbb{R}^2} g_n(y)d\mu_x(y)\\ &= \int_{\mathbb{R}^2} 1_{B(y_0,a)}(y)d\mu_x(y).\end{aligned}$$

The function $x \to \mu_x(B(y_0,a)) = \int_{\mathbb{R}^2} 1_{B(y_0,a)}(y)d\mu_x(y)$, which can also be denoted by $V_k(1_{B(y_0,a)})(x)$ is defined almost every where on $\mathbb{R}^2$ (see [1, p. 17]*), measurable and for all f in $\mathcal{D}(\mathbb{R}^2)$, we have*

$$\int_{\mathbb{R}^2} \mu_x(B(y_0,a))f(x)\mathcal{A}_k(x)dx = \int_{B(y_0,a)} {}^tV_k(f)(y)dy. \tag{3.7}$$

Proof. For all $n \in \mathbb{N}\backslash\{0\}$, the function $V_k(g_n)$ belongs to $\mathcal{E}(\mathbb{R}^2)$. Then we obtain the results of this proposition from the continuity of the operator V_k from $\mathcal{E}(\mathbb{R}^2)$ into itself, the monotonic convergence theorem and the relation (3.4). □

Remark 3.3. *There exists a σ-algebra $\mathfrak{m}$ in $\mathbb{R}^2$ which contains all Borel sets in $\mathbb{R}^2$. Then for all $E \in \mathfrak{m}$, the function $x \to \mu_x(E)$ is defined almost every where on $\mathbb{R}^2$, measurable and we have the following relation*

$$\int_{\mathbb{R}^2} \mu_x(E)f(x)\mathcal{A}_k(x)dx = \int_E {}^tV_k(f)(y)dy, \quad f \in \mathcal{D}(\mathbb{R}^2). \tag{3.8}$$

Proposition 3.4. *Let $x_0 \in \mathbb{R}^2$ and $a > 0$. We consider the sequence $\{f_n\}_{n\in\mathbb{N}\backslash\{0\}}$ of functions in $\mathcal{D}(\mathbb{R}^2)$, positive, increasing such that :*

$$\forall\, n \in \mathbb{N}\backslash\{0\},\ \operatorname{supp} f_n \subset \bar{B}(x_0,a),\ \forall\, x \in B(x_0, a-\frac{1}{n}),\ f_n(x) = 1,$$

and

$$\forall\, x \in \mathbb{R}^2,\ \lim_{n\to+\infty} f_n(x) = 1_{B(x_0,a)}(x),$$

where $1_{B(x_0,a)}$ is the characteristic function of the ball $B(x_0,a)$. We have

$$\begin{aligned}\forall\, y \in \mathbb{R}^2, \lim_{n\to+\infty} {}^tV_k(f_n)(y) &= \lim_{n\to+\infty}\int_{\mathbb{R}^2} f_n(x)d\nu_y(x)\\ &= \int_{\mathbb{R}^2} 1_{B(x_0,a)}(x)d\nu_y(x).\end{aligned}$$

The function $y \to \nu_y(B(x_0,a)) = \int_{\mathbb{R}^2} 1_{B(x_0,a)}(x)d\nu_y(x)$, which can also be denoted by ${}^tV_k(1_{B(x_0,a)})(y)$ is defined almost every where on $\mathbb{R}^2$ (see [1, p.

17]*), measurable and for all* g *in* $\mathcal{E}(\mathbb{R}^2)$*, we have*

$$\int_{\mathbb{R}^2} \nu_y(B(x_0,a))g(y)dy = \int_{B(x_0,a)} V_k(g)(x)\mathcal{A}_k(x)dx. \tag{3.9}$$

Proof. For all $n \in \mathbb{N}\backslash\{0\}$, the function ${}^tV_k(f_n)$ belongs to $\mathcal{D}(\mathbb{R}^2)$. Then the continuity of the operator ${}^tV_k$ from $\mathcal{D}(\mathbb{R}^2)$ into itself, the monotonic convergence theorem and the relation (3.4) imply the results of this proposition. □

3.1. **Absolute continuity of the measure ν_y.** The purpose of this subsection is to prove that for all $y \in \mathbb{R}^2$, the measure ν_y is absolute continuous will respect to the Lebesgue measure on $\mathbb{R}^2$. **Notation**. We denote by λ the Lebesgue measure on $\mathbb{R}^2$.

Proposition 3.5. *For* $x \in \mathbb{R}^2_{reg}$*, there exists a unique positive function* $\mathcal{K}(x,.)$ *integrable with respect to the Lebesgue measure* λ*, and a positive measure* μ_x^s *on* $\mathbb{R}^2$ *such that for every Borel set* E*, we have*

$$\mu_x(E) = \int_E \mathcal{K}(x,y)dy + \mu_x^s(E). \tag{3.10}$$

Proof. We deduce (3.10) from (1.1) and Theorem 6.9 of [6, p.129-130] and Theorem 8.6 and its Corollary of [6, p. 166]. □

Remark 3.6. i) *The supports of the function* $y \to \mathcal{K}(x,y)$ *and the measure* μ_x^s *are contained in the ball* $\bar{B}(0,\|x\|)$.

ii) *The measures* μ_x^s *and the Lebesgue measure* λ *are mutually singular.*

iii) *From Theorem 8.6, p. 166 and Definition 8.3, p.164, of* [6]*, we have*

$$\mathcal{K}(x,y) = \lim_{a\to 0} \frac{\mu_x(B(y,\mathfrak{a}))}{\lambda(B(y,\mathfrak{a}))}\ . \tag{3.11}$$

Proposition 3.7. *We consider* $x \in \mathbb{R}^2_{reg}$ *and a positive function* f *in* $\mathcal{D}(\mathbb{R}^2)$ *with support contained in the ball* $\bar{B}(0,R), R > 0$.

i) *For all Borel set* E*, we have*

$$\int_E \mathcal{N}^f(y)dy = \int_{\bar{B}(0,R)} \mu_x^s(E)f(x)\mathcal{A}_k(x)dx, \tag{3.12}$$

where

$$\mathcal{N}^f(y) = {}^tV_k(f)(y) - \int_{\bar{B}(0,R)} \mathcal{K}(x,y)f(x)\mathcal{A}_k(x)dx. \tag{3.13}$$

ii) *We have*

$$\forall\, y \in \mathbb{R}^2,\ \mathcal{N}^f(y) \geq 0. \tag{3.14}$$

Proof. i) By using the relations (3.8), (3.10), we obtain

$$\begin{aligned}\int_E {}^tV_k(f)(y)dy &= \int_{\bar{B}(0,R)} \mu_x(E)f(x)\mathcal{A}_k(x)dx \\ &= \int_{\bar{B}(0,R)} \Big[\int_E \mathcal{K}(x,y)dy + \mu_x^s(E)\Big] f(x)\mathcal{A}_k(x)dx.\end{aligned}$$

We deduce (3.12) by applying Fubini-Tonelli's theorem to the second member.

ii) From the relation (3.12), the positivity of the measure μ_x^s implies that for all Borel set E, we have

$$\int_E \mathcal{N}^f(y)dy \geq 0.$$

Thus

$$\forall\, y \in \mathbb{R}^2, \mathcal{N}^f(y) \geq 0.$$

□

Proposition 3.8. *The measure Λ^f on $\mathbb{R}^2$, given for all Borel set E by*

$$\Lambda^f(E) = \int_E \mathcal{N}^f(y)dy, \tag{3.15}$$

is positive and bounded.

Proof. - The relation (3.14) gives the positivity of the measure Λ^f.

- From the relations (3.15), (3.12), for all Borel set E we have

$$\Lambda^f(E) \leq \int_{\bar{B}(0,R)} \|\mu_x^s\| f(x)\mathcal{A}_k(x)dx. \tag{3.16}$$

On the other hand by using (3.10), we obtain for $x \in \mathbb{R}^2_{reg}$,

$$\mu_x^s(E) \leq \mu_x(E),$$

thus

$$\|\mu_x^s\| \leq \|\mu_x\| \leq 1.$$

By using this result, the relation (3.16) implies that for all Borel set E, we have

$$\Lambda^f(E) \leq M_f,$$

where

$$M_f = \int_{\bar{B}(0,R)} f(x)\mathcal{A}_k(x)dx.$$

Then the measure Λ^f is bounded. □

Proposition 3.9. *Let $x \in \mathbb{R}^2_{reg}$ and f the function given in Proposition 3.7.*

i) *For all Borel set E we have*

$$\Lambda^f(E) = 0. \tag{3.17}$$

ii) *For $y \in \mathbb{R}^2$, we have*

$${}^tV_k(f)(y) = \int_{\bar{B}(0,R)} \mathcal{K}(x,y)f(x)\mathcal{A}_k(x)dx. \tag{3.18}$$

Proof. i) From the relations (3.15), (3.12), for all Borel set E the measure Λ^f possesses also the following form

$$\Lambda^f(E) = \int_{\bar{B}(0,R)} \mu_x^s(E)f(x)\mathcal{A}_k(x)dx. \tag{3.19}$$

On the other hand from Proposition 3.8 the measure Λ^f is absolute continuous with respect to the Lebesgue measure λ and from Remark 3.6 ii) the measure $\mu_x^s, x \in \bar{B}(0,R)$, and the Lebesgue measure λ are mutually singular. Then from Proposition 6.8 (f), p.129, of [6], the measure Λ^f and $\mu_x^s, x \in \bar{B}(0,R)$, are mutually singular. By using the definition of measures mutually singular (see p. 128 of [6]), we deduce (3.17) from (3.19).

ii) By using the i) and (3.15), (3.13), we obtain (3.18).

□

Theorem 3.10. *For all f in $\mathcal{D}(\mathbb{R}^2)$ we have*

$$\forall\, y \in \mathbb{R}^2,\ {}^tV_k(f)(y) = \int_{\mathbb{R}^2} \mathcal{K}(x,y)f(x)\mathcal{A}_k(x)dx. \tag{3.20}$$

Proof. We obtain (3.20) by writing $f = f^+ - f^-$ and by using Proposition 3.9 ii). □

Remark 3.11. *Theorem 3.10 shows that for all $y \in \mathbb{R}^2$ the measure ν_y given by the relation (1.2), is absolute continuous with respect to the measure $\mathcal{A}_k(x)dx$. More precisely we have*

$$d\nu_y(x) = \mathcal{K}(x,y)\mathcal{A}_k(x)dx. \tag{3.21}$$

3.2. **Absolute continuity of the measure μ_x.** The purpose of this subsection is to prove that for all $x \in \mathbb{R}^2_{reg}$ the measure μ_x is absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^2$.

Theorem 3.12. *For all g in $\mathcal{E}(\mathbb{R}^2)$ and $x_0 \in \mathbb{R}^2_{reg}$, we have*

$$V_k(g)(x_0) = \int_{\mathbb{R}^2} \mathcal{K}(x_0,y)g(y)dy. \tag{3.22}$$

Proof. By writing $g = g^+ - g^-$ it suffices to prove the theorem for g positive. From the relation (3.9) we have

$$\frac{1}{\lambda(B(x_0,a))}\int_{B(x_0,a)} V_k(g)(x)\mathcal{A}_k(x)dx = \int_{\mathbb{R}^2} g(y)\,\frac{\nu_y(B(x_0,a))}{\lambda(B(x_0,a))}dy\ . \quad (3.23)$$

By using the relation (3.20), and by applying Fubini-Tonelli's theorem to the second member of (3.23), we obtain

$$\frac{1}{\lambda(B(x_0,a))}\int_{B(x_0,a)} V_k(g)(x)\mathcal{A}_k(x)dx =$$
$$\frac{1}{\lambda(B(x_0,a))}\int_{B(x_0,a)}\left[\int_{\mathbb{R}^2}\mathcal{K}(x,y)g(y)dy\right]\mathcal{A}_k(x)dx.$$

By applying the relation (2) of [6, p.168], to the two members of this relation we get

$$\mathcal{A}_k(x_0)V_k(g)(x_0) = \mathcal{A}_k(x_0)\int_{\mathbb{R}^2}\mathcal{K}(x_0,y)g(y)dy,$$

as

$$\mathcal{A}_k(x_0) \neq 0 \Leftrightarrow x_0 \in \mathbb{R}^2_{reg},$$

thus for $x_0 \in \mathbb{R}^2_{reg}$, we have

$$V_k(g)(x_0) = \int_{\mathbb{R}^2}\mathcal{K}(x_0,y)g(y)dy.$$

□

Remark 3.13. *From Theorem 3.12 and the relation (1.1) we deduce that for all $x \in \mathbb{R}^2_{reg}$ the measure μ_x is absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. More precisely we have*

$$d\mu_x(y) = \mathcal{K}(x,y)dy. \quad (3.24)$$

Corollary 3.14. i) *For all $\lambda \in \mathbb{C}^2$ and $x \in \mathbb{R}^2_{reg}$, we have*

$$G_\lambda(x) = \int_{\mathbb{R}^2}\mathcal{K}(x,y)e^{-i\langle\lambda,y\rangle}dy. \quad (3.25)$$

ii) *For all $x \in \mathbb{R}^2_{reg}$, we have*

$$\int_{\mathbb{R}^2}\mathcal{K}(x,y)dy \leq 1. \quad (3.26)$$

iii) *For all $x \in \mathbb{R}^2_{reg}$, we have*

$$supp\mathcal{K}(x,.) \subset \bar{B}(0,\|x\|). \quad (3.27)$$

Proof. We deduce the results of this Corollary from (1.1), (2.17), and Theorem 3.12. □

Theorem 3.15. *We have*

$$\forall\, x \in \mathbb{R}^2_{reg},\ \lim_{\|\lambda\|\to+\infty} G_\lambda(x) = 0. \tag{3.28}$$

Proof. From the relation (3.26) the function $\mathcal{K}(x,.)$ is integrable on $\mathbb{R}^2$ with respect to the Lebesgue measure on $\mathbb{R}^2$. Then we deduce (3.28) from the relation (3.25) and Riemann-Lebesgue Lemma for the usual Fourier transform on $\mathbb{R}^2$. ☐

3.3. Absolute continuity of the measures ν_y^W and μ_x^W.

Theorem 3.16. *For all f in $\mathcal{D}(\mathbb{R}^2)^W$, we have*

$$\forall\, y \in \mathbb{R}^2,\ {}^tV_k^W(f)(y) = \int_{\mathbb{R}^2} \mathcal{K}^W(x,y) f(x)\mathcal{A}_k(x)dx, \tag{3.29}$$

where $\mathcal{K}^W(x,y)$ is the function given by

$$\mathcal{K}^W(x,y) = \frac{1}{|W|^2}\sum_{w,w'\in W}\mathcal{K}(wx, w'y). \tag{3.30}$$

Proof. The relations (1.4), (1.6) and Theorem 3.10 imply the relations (3.29), (3.30). ☐

Theorem 3.17. *For all g in $\mathcal{E}(\mathbb{R}^2)^W$, we have*

$$\forall\, x \in \mathbb{R}^2_{reg},\ V_k^W(g)(x) = \int_{\mathbb{R}^2} \mathcal{K}^W(x,y) g(y)dy, \tag{3.31}$$

where $\mathcal{K}^W(x,y)$ is the function given by the relation (3.30).

Proof. We deduce (3.31) from the relations (1.3), (1.5) and Theorem 3.12. ☐

Remark 3.18. *Theorems 3.16, 3.17 show that the measures $\nu_y^W, y \in \mathbb{R}^2$ and $\mu_x^W, x \in \mathbb{R}^2_{reg}$, given respectively by the relations (1.4), (1.3), are absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. More precisely we have*

$$d\nu_y^W(x) = \mathcal{K}^W(x,y)\mathcal{A}_k(x)dx, \tag{3.32}$$

and

$$d\mu_x^W(y) = \mathcal{K}^W(x,y)dy. \tag{3.33}$$

Corollary 3.19. i) *For all $\lambda \in \mathbb{C}^2$ and $x \in \mathbb{R}^2_{reg}$, we have*

$$F_\lambda(x) = \int_{\mathbb{R}^2} \mathcal{K}^W(x,y) e^{-i\langle\lambda,y\rangle}dy. \tag{3.34}$$

ii) *For all* $x \in \mathbb{R}^2_{reg}$, *we have*

$$\int_{\mathbb{R}^2} \mathcal{K}^W(x,y)dy \leq 1. \tag{3.35}$$

iii) *For all* $x \in \mathbb{R}^2_{reg}$, *we have*

$$supp\mathcal{K}^W(x,.) \subset \bar{B}(0, \|x\|). \tag{3.36}$$

Proof. The relations (1.3), (2.18) and Theorem 3.17, imply the results of this Corollary. □

Theorem 3.20. *We have*

$$\forall\, x \in \mathbb{R}^2_{reg}, \lim_{\|\lambda\| \to +\infty} F_\lambda(x) = 0. \tag{3.37}$$

Proof. We deduce (3.37) from the relations (3.34), (3.35) and Riemann-Lebesgue Lemma for the usual Fourier transform on $\mathbb{R}^2$. □

Acknowledgments

The author is grateful to the referee for careful reading and useful remarks.

References

[1] N. Bourbaki. *Elements de Mathematiques Fascule XXI.* Intégration des mesures. Chapitre 5, Hermann, Paris, 1967.

[2] I. Cherednik. *A unification of Knizhnik-Zamolodchnikov equations and Dunkl operators via affine Hecke algebras.* Invent.Math. **106**, (1991), pp. 411-432.

[3] G.H.Heckman and E.M.Opdam. *Root systems and hypergeometric functions. I.* Compositio. Math. **64**, (1987), pp. 329-352.

[4] G.J.Heckman and H. Schlichtkrull. *Harmonic analysis and special functions on symmetric spaces.* Academic Press 1994.

[5] E.M.Opdam. *Harmonic analysis for certain representations of graded Hecke algebras.* Acta Math. **175**, (1995), pp. 75-121.

[6] W. Rudin. *Real and complex analysis.* Mc Graw Hill Inc. second edition. 1974.

[7] B.Schapira. *Contributions to the hypergeometric function theory of Heckman and Opdam : sharp estimates, Schwartz space, heat kernel.* Geom. Funct. Anal. **18**, (2008), pp. 222-250.

[8] K. Trimèche. *The positivity of the transmutation operators associated with the Cherednik operators for the root system of type BC_2.* Math. J. Okayama Univ. **58** (2016), 183–198.

[9] K. Trimèche. *The positivity of the hypergeometric translation operators associated to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type BC_2.* Int. J. Appl. Math. **29** (2016), 687–715.

Khalifa Trimèche
Department of Mathematics
Faculty of sciences of Tunis
University Tunis El-Manar- Tunisia
CAMPUS, 2092 Tunis, Tunisia
e-mail address: khlifa.trimeche@fst.rnu.tn
e-mail address: khalifa.trimeche@gmail.com

(*Received July 22, 2014*)
(*Accepted February 17, 2015*)