

THE DEGREE OF SET-VALUED MAPPINGS FROM ANR SPACES TO HOMOLOGY SPHERES

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ABSTRACT. An admissible mapping is a set-valued mapping which has a selected pair of continuous mappings. In this paper, we study the degree of admissible mappings from ANR spaces to homology spheres and prove the uniqueness of the degree under some conditions.

1. INTRODUCTION

For every point x in a topological space X , a non-empty closed set $\varphi(x)$ in a topological space Y is assigned, the correspondence is called a set-valued mapping which is written by the Greek alphabet $\varphi : X \rightarrow Y$. For a single-valued mapping, we write $f : X \rightarrow Y$ etc. by the Roman alphabet. In this paper, we assume that set-valued mappings are upper semi-continuous (cf. [6]).

Fixed point theorems and equivariant point theorems are studied by many mathematicians (cf. [3], [11], [12], [13]). Fixed point theorems and equivariant point theorems of single-valued mappings are generalized for set-valued mappings by many topologists (cf. [5], [6], [15], [16]).

An admissible mapping $\varphi : X \rightarrow Y$ is a set-valued mapping which has a selected pair $p : \Gamma \rightarrow X$ and $q : \Gamma \rightarrow Y$ (cf. Definition 2). When an ANR spaces X satisfies $H^n(X; Z) \cong Z$, we define the degree of φ by the set

$$\deg(\varphi) = \{ \deg((p^*)^{-1}q^*) : (p^*)^{-1}q^* : H^n(N; Z) \rightarrow H^n(X; Z) \}$$

where N is an n -dimensional homology sphere. In this paper, we shall study the degree of $\varphi : X \rightarrow N$ under the conditions $\varphi(x) \cap \varphi(T(x)) = \emptyset$ and $T'\varphi(x) \cap \varphi(T(x)) = \emptyset$ where T and T' are involutions.

Y.Hara and Y.Moriwaki [8] determined the degree of admissible mappings $\varphi : M \rightarrow S^n$ where M is an n -dimensional smooth manifold and S^n is the n -dimensional sphere. Their method uses essentially the character of the sphere. In this paper, we shall give a generalization of their results. Our technic is homological method and more elementary than [14], [15]. Though our main theorems are proved partly by Theorem 5.5 and Theorem 6.3 of [15], the uniqueness of the degree is not proved there. Our main results are stated as follows (cf. Theorem 3.4, Theorem 4.1).

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Main Theorem 1. *Let X be an ANR space with a free involution T and N be an n -dimensional homology sphere. Suppose that $\dim X = n$ and $H^n(X; Z) \cong Z$ and $c(X, T)^n \neq 0$. If an admissible mappings $\varphi : X \rightarrow N$ satisfies $\varphi(x) \cap \varphi(Tx) = \emptyset$ for any $x \in X$, then there exists a unique odd number m such that $\deg \varphi = \{m\}$.*

For the case $c(X, T)^n = 0$, then there exists a unique even number m such that $\deg(\varphi) = \{m\}$ (cf. Theorem 3.5).

Main Theorem 2. *Let X be an ANR space with a free involution T and N be an n -dimensional homology sphere with a non trivial involution T' . Suppose that $\dim X = n$ and $H^n(X; Z) \cong Z$ and an admissible mappings $\varphi : X \rightarrow N$ satisfies $T'\varphi(x) \cap \varphi(Tx) = \emptyset$ for any $x \in X$. Then there exists a unique even number m such that $\deg(\varphi) = \{m\}$. In particular, if T' is an orientation reversing involution, then $\deg \varphi = \{0\}$.*

Assume that an involution T' on N is free in the above main theorem. For the case that n is an odd number, there exists a unique even number m such that $\deg \varphi = \{m\}$. For the case that n is an even number, then $\deg \varphi = \{0\}$ (cf. Corollary 4.2).

2. PRELIMINARIES

In this paper, we shall use the Alexander-Spanier cohomology theory $\bar{H}^*(-; G)$ and the singular cohomology theory $H^*(-; G)$. If the singular cohomology theory satisfies the continuity condition (cf. Theorem 6.9.1 in [17]), the Alexander-Spanier cohomology theory is isomorphic to the singular cohomology theory, that is,

$$(2.1) \quad \mu; \bar{H}^*(X; G) \cong H^*(X; G).$$

In particular, $\bar{H}^*(-; G)$ and $H^*(-; G)$ are isomorphic for ANR spaces by Corollary 6.9.5 of [17]. In this paper, we use $\bar{H}^*(-)$ and $H^*(-)$ for $\bar{H}^*(-; \mathbf{F}_2)$ and $H^*(-; \mathbf{F}_2)$ respectively where \mathbf{F}_2 is the prime field of the characteristic 2.

Let $f : X \rightarrow Y$ be a continuous mapping. When $f^{-1}(K)$ is a compact set for any compact subset $K \subset Y$, f is called a proper mapping. f is called a perfect mapping, if f is a closed mapping and any preimage $f^{-1}(y)$ is a compact set for each $y \in Y$. A perfect mapping is a proper mapping by Theorem 3.7.2 in R. Engelking [4]. For the case that Y is a metric space, a proper mapping f is a closed mapping (cf. Proposition 1.8.1 in [6]).

A mapping $f : X \rightarrow Y$ is called a compact mapping, if $f(X)$ is contained in a compact set of Y , or equivalently its closure $f(X)$ is compact.

Definition 1. Let X and Y be paracompact Hausdorff spaces. A mapping $f : X \rightarrow Y$ is called a Vietoris mapping, if it satisfies the following conditions;

- (1) f is a perfect and onto continuous mapping.
- (2) $f^{-1}(y)$ is an acyclic space for any $y \in Y$, that is, it is a connected space and $\bar{H}^*(f^{-1}(y); G) = 0$ for positive dimensions.

When f is closed and onto continuous mapping and satisfies the condition (2), we call it weak Vietoris mapping.

The following theorem is called the Vietoris-Begle mapping theorem and is important for our purpose (cf. Theorem 6.9.15 in [17]).

Theorem 2.1. *Let $f : X \rightarrow Y$ be a weak Vietoris mapping between paracompact Hausdorff spaces X and Y . Then,*

$$(2.2) \quad f^* : \bar{H}^m(Y; G) \rightarrow \bar{H}^m(X; G)$$

is an isomorphism for all $m \geq 0$.

The graph of a set-valued mapping $\varphi : X \rightarrow Y$ is defined by $\Gamma_\varphi = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$. If φ is upper semi-continuous, Γ_φ is closed, but the converse is not true. If the image $\varphi(X)$ is contained in a compact set, the converse is true (cf. §14 in [6]).

Definition 2. An upper semi-continuous mapping $\varphi : X \rightarrow Y$ is called an admissible mapping, if there exist a paracompact Hausdorff space Γ and continuous mappings $p : \Gamma \rightarrow X$ and $q : \Gamma \rightarrow Y$ satisfying the following conditions:

- (1) $p : \Gamma \rightarrow X$ is a Vietoris mapping
- (2) $\varphi(x) \supset q(p^{-1}(x))$ for each $x \in X$.

A pair (p, q) of mappings p and q is called a selected pair of φ .

For an admissible mapping $\varphi : X \rightarrow Y$, we define $\varphi^* : \bar{H}^*(Y; G) \rightarrow \bar{H}^*(X; G)$ by the set

$$\varphi^* = \{(p^*)^{-1}q^*\} \mid (p, q) \text{ is a selected pair of } \varphi\}.$$

φ_* is similarly defined by the set $\{q_*(p_*)^{-1}\}$.

When an ANR spaces X satisfies $H^n(X; Z) \cong Z$ and N is an n -dimensional homology sphere, the degree $\deg(\varphi)$ of φ is also defined (see Introduction).

Note that $\deg(\varphi)$ is a subset of the set Z of integers.

3. THE DEGREE OF ADMISSIBLE MAPPINGS 1

In this paper, manifolds are paracompact Hausdorff topological manifolds. An n -dimensional homology sphere N means that it is an n -dimensional topological manifold and has the homology groups of the n -dimensional sphere i.e. $H^*(N; Z) \cong H^*(S^n; Z)$. It is easily seen $H^*(N; G) \cong H^*(S^n; G)$ for any abelian group G .

Let τ be the group of the order 2. When a paracompact Hausdorff space X has a free involution T , the group τ generated by T acts on X . Similarly when Y has a free involution T' , the group τ generated by T' acts on Y . Their orbit spaces are denoted by X_τ, Y_τ . An equivariant mapping $f : X \rightarrow Y$ creates the induced mapping $f_\tau : X_\tau \rightarrow Y_\tau$ between their orbits spaces.

When an equivariant mapping $f : X \rightarrow Y$ is a perfect mapping, the induced mapping $f_\tau : X_\tau \rightarrow Y_\tau$ is also a perfect mapping. If $f : X \rightarrow Y$ is a Vietoris mapping, $f_\tau : X_\tau \rightarrow Y_\tau$ is also a Vietoris mapping.

There exist the standard covering projections $\pi_n : S^n \rightarrow RP^n$ and $\pi_\infty : S^\infty \rightarrow RP^\infty$. For a covering projection $\pi_X : X \rightarrow X_\tau$, there exist a classifying mapping $f_\tau : X_\tau \rightarrow RP^\infty$ and $f : X \rightarrow S^\infty$ such that $\pi_\infty f = f_\tau \pi_X$ by Theorem 4.12.2 of [9]. The first Stiefel-Whitney class $c(X, T) \in \bar{H}^1(X_\tau)$ (or $c(X, T) \in H^1(X_\tau)$) is defined by $f_\tau^*(\omega)$ where ω is the generator of $H^1(RP^\infty)$ i.e. $c(X, T) = f_\tau^*(\omega)$.

X is called an ANR space, if X is a metric space and for any metric space Y and any embedding $h : X \rightarrow Y$ $h(X)$ is a neighborhood retract of Y . (cf. [10], or Definition 1.7 of [6]). If X is an ANR space, X_τ is also an ANR space by Theorem 4.4.17 of [4] and by Theorem 3.3 of [7].

When X is an ANR space, there exist a simplicial complex K and mappings $h : X \rightarrow K, k : K \rightarrow X$ such that kh is homotopic to Id_X by Theorem 6.1 of [7] (cf. [10], or [1]). Moreover when X is n -dimensional, K is an n -dimensional complex.

$\dim X$ means the covering dimension of X . (cf. Chapter 7 of [4]). When X is a normal space, then $\dim X = n$ implies $\dim X_\tau = n$ by Theorem 7.1.7 and Theorem 7.2.4 of [4]. When an ANR space X satisfies $H^n(X; Z) \cong Z$ and $\dim X = n$, then $H^k(X; Z) = 0$ for $k > n$ and $H^n(X) \cong \mathbf{F}_2$ by Theorem 5.5.10 of [17]. In particular, if X is a compact ANR space, then $H_n(X; Z) \cong Z$ and $H_{n-1}(X; Z)$ is finitely generated free group by the universal coefficient theorem.

Proposition 3.1. *Let X be an ANR space with a free involution T which satisfies $\dim X = n$ and $H^n(X; Z) \cong Z$. Suppose that $c(X, T)^n \neq 0$. Then $T^* = \text{Id}_{H^n(X; Z)}$ for an odd number n and $T^* = -\text{Id}_{H^n(X; Z)}$ for an even number n .*

Proof. A covering projection $\pi_X : X \rightarrow X_\tau$ is induced by a classifying mapping $f_\tau : X_\tau \rightarrow RP^\infty$ and $f : X \rightarrow S^\infty$ such that $\pi_\infty f = f_\tau \pi_X$. By the above remark there exist a simplicial complex K and $h : X_\tau \rightarrow K$ and $k : K \rightarrow X_\tau$ such that kh is homotopic to Id_{X_τ} denoted by $kh \simeq Id_{X_\tau}$. We see $f_\tau \simeq f_\tau kh$. There exists a continuous mapping $g : K \rightarrow RP^n$ by the cellular approximation theorem such that g is homotopic to $f_\tau k$. Therefore we see that f_τ is homotopic to $gh : X_\tau \rightarrow RP^n$ by $f_\tau \simeq f_\tau kh \simeq gh$. We may assume $f_\tau : X_\tau \rightarrow RP^n$. That is, there exists the following diagram:

$$(3.1) \quad \begin{array}{ccc} X & \xrightarrow{f} & S^n \\ \downarrow \pi_X & & \downarrow \pi_n \\ X_\tau & \xrightarrow{f_\tau} & RP^n. \end{array}$$

Let $\sigma_n \in H^n(S^n)$ be the dual element of the generator of $H_n(S^n)$ and $\omega_n \in H^1(RP^n)$ be the generator. By the Gysin-Smith sequence of $\pi_n : S^n \rightarrow RP^n$, we see $(\pi_n)_!(\sigma_n) = \omega_n^n$ where $(\pi_n)_!$ is the transfer homomorphism of π_n .

Let μ be the generator of $H^n(X)$. Set $\omega' = c(X, T)$. By the Gysin-Smith sequence of $\pi_X : X \rightarrow X_\tau$:

$$(3.2) \quad \rightarrow H^n(X_\tau) \xrightarrow{\pi_X^*} H^n(X) \xrightarrow{(\pi_X)_!} H^n(X_\tau) \xrightarrow{\cup \omega'} H^{n+1}(X_\tau) \rightarrow$$

we have $(\pi_X)_! : H^n(X) \cong H^n(X_\tau) \cong \mathbf{F}_2$ and $(\pi_X)_!(\mu) = c(X, T)^n$. Moreover we see $f^*(\sigma_n) = \mu$ by $(\pi_X)_! f^*(\sigma_n) = f_\tau^*(\pi_n)_!(\sigma_n)$. Therefore $f^* : H^n(S^n; \mathbf{Z}) \rightarrow H^n(X; \mathbf{Z})$ is not trivial.

For the antipodal involution $T_n : S^n \rightarrow S^n$, we have $T_n^* = Id_{H^n(S^n; \mathbf{Z})}$ for an odd number n and $T_n^* = -Id_{H^n(S^n; \mathbf{Z})}$ for an even number n . Therefore we obtain our result by $fT = T_n f$. \square

Remark. An n -dimensional connected orientable closed manifold M satisfies $\dim M = n$ and $H^n(M; \mathbf{Z}) \cong \mathbf{Z}$. We can easily construct many ANR spaces satisfying the above conditions which are not manifolds.

Remark. The Gysin-Smith sequence is well-known for the singular cohomology theory (cf. [17]). The Gysin-Smith sequence for the Alexander-Spanier cohomology theory is proved by Corollary 6.5.2 of [17] (cf. §5 of [14] and Corollary 6.8.8 of [17]).

Let Δ_X be the diagonal set of $X^2 = X \times X$. There exists an involution $T_X : X^2 \rightarrow X^2$ defined by $T_X(x, y) = (y, x)$. T_X is a free involution on $X^2 - \Delta_X$.

Theorem 3.2. *Let N be an n -dimensional closed manifold. Suppose $H^*(N; Z) \cong H^*(S^n; Z)$. Then $H^*(N - \{x\}; G) \cong H^*(pt; G)$ for any $x \in N$ and $\pi_i^* : H^*(N^2 - \Delta_N; G) \cong H^*(N; G)$ for the projections $\pi_i : N^2 - \Delta_N \rightarrow N$ ($i = 1, 2$).*

Proof. Let U be a neighborhood of $x \in N$ homeomorphic to the open disk D^n . N is a union of $(N - U)$ and $\text{Cl}(U)$. $(N - U) \cap \text{Cl}(U)$ is homeomorphic to S^{n-1} . By using the Mayer-Vietoris sequence of $\{(N - U), \text{Cl}(U)\}$, we obtain $H^*(N - U; Z) \cong H^*(pt; Z)$ for any $x \in N$. Since $N - \{x\}$ is homotopy equivalent to $N - U$, we have $H^*(N - \{x\}; G) \cong H^*(pt; G)$ for any $x \in N$.

The projections $\pi_i : N^2 - \Delta_N \rightarrow N$ are a fiber bundle over N with a fiber $N - \{x\}$ (cf. Lemma 6.2.5 in E. Spanier [17]). That is, there exists a neighborhood U of x such that $\pi_i^{-1}(U)$ is homeomorphic to $U \times (N - \{x\})$. By comparing the Mayer-Vietoris sequences of $\{U\}$ and $\{\pi_i^{-1}(U)\}$ where $\{U\}$ is a covering of N , we obtain the theorem. \square

For a connected paracompact Hausdorff space X with a free involution T , $c(X, T) \in H^1(X_\tau)$ is not zero by the Gysin-Smith sequence.

If N is an n -dimensional homology sphere, $N^2 - \Delta_N$ satisfies $H^*(N^2 - \Delta_N) \cong H^*(N)$ by Theorem 3.2. Let $\nu \in H^n(N)$ be the dual element of the fundamental cycle $[N] \in H_n(N)$. $\nu' \in H^n(N^2 - \Delta_N)$ is defined by $\pi_1^*(\nu) = \nu'$. We obtain easily the following proposition.

Proposition 3.3. *Let N be an n -dimensional homology sphere and $T_N : N^2 - \Delta_N \rightarrow N^2 - \Delta_N$ be the free involution. Then $c(N^2 - \Delta_N, T_N)^n \neq 0$ and $\pi_!(\nu') = c(N^2 - \Delta_N, T_N)^n$ where $\pi : (N^2 - \Delta_N) \rightarrow (N^2 - \Delta_N)_\tau$.*

Let (p, q) be a selected pair of φ where $p : \Gamma \rightarrow X$ is a Vietoris mapping and $q : \Gamma \rightarrow N$ is a continuous mapping. When X has an involution T , define Γ_0 by

$$\Gamma_0 = \{(z, z') \in \Gamma \times \Gamma \mid p(z) = Tp(z')\}$$

and $p_0 : \Gamma_0 \rightarrow X$, $q_0 : \Gamma_0 \rightarrow N$ by $p_0(z, z') = p(z)$, $q_0(z, z') = q(z)$ respectively. $p_\Gamma : \Gamma_0 \rightarrow \Gamma$ is defined by $p_\Gamma(z, z') = z$. When X is a metric space, Γ_0 is a paracompact space by Theorem 3.7.9 and Theorem 5.1.35 of [4]. Easily seen, p_0 is a Vietoris mapping for the case. We remark $\deg((p_0^*)^{-1}q_0^*) = \deg((p^*)^{-1}q^*)$. When T is a free involution, we can define a free involution $T_0 : \Gamma_0 \rightarrow \Gamma_0$ by $T_0(z, z') = (z', z)$. Then p_0 is an equivariant mapping i.e. $Tp_0 = p_0T_0$.

In the following theorem, the degree is also determined by the composition of Theorem 6.3 of [15]. Here we use only an elementary method, essentially the Gysin-Smith sequence. And we shall prove the uniqueness of the degree.

Theorem 3.4. *Let X be an ANR space with a free involution T and N be an n -dimensional homology sphere. Suppose that $\dim X = n$, $H^n(X; Z) \cong Z$ and $c(X, T)^n \neq 0$. If an admissible mappings $\varphi : X \rightarrow N$ satisfies $\varphi(x) \cap \varphi(Tx) = \emptyset$ for any $x \in X$, then there exists a unique odd number m such that $\deg \varphi = \{m\}$.*

Proof. Let $Q : \Gamma_0 \rightarrow N^2 - \Delta_N$ be defined by $Q(z, z') = (q(z), q(z'))$. Q is well-defined by the assumption and Q is an equivariant mapping. Then there exists a classifying mapping $g_\tau : (N^2 - \Delta_N)_\tau \rightarrow RP^\infty$ for the covering projection $\pi : N^2 - \Delta_N \rightarrow (N^2 - \Delta_N)_\tau$.

Now consider the following diagram :

$$(3.3) \quad \begin{array}{ccccccc} X & \xleftarrow{p_0} & \Gamma_0 & \xrightarrow{Q} & N^2 - \Delta_N & \xrightarrow{g} & S^\infty \\ \downarrow \pi_X & & \downarrow \pi_{\Gamma_0} & & \downarrow \pi & & \downarrow \pi_\infty \\ X_\tau & \xleftarrow{(p_0)_\tau} & (\Gamma_0)_\tau & \xrightarrow{Q_\tau} & (N^2 - \Delta_N)_\tau & \xrightarrow{g_\tau} & RP^\infty. \end{array}$$

X_τ , $(\Gamma_0)_\tau$, $(N^2 - \Delta_N)_\tau$ are orbit spaces and $\pi_X, \pi_{\Gamma_0}, \pi$ and π_∞ are quotient mappings and $(p_0)_\tau$, Q_τ , g_τ are the induced mappings of p_0 , Q , g respectively.

We see

$$(3.4) \quad (p_0)_\tau^*(c(X, T)) = Q_\tau^* g_\tau^*(\omega)$$

by the naturality and Theorem 4.12.4 of [9]. Set $(p_0)_\tau^*(c(X, T)) = c(\Gamma_0, T_0)$. We obtain $c(\Gamma_0, T_0)^n \neq 0$ by $c(X, T)^n \neq 0$ and the naturality.

By the Gysin-Smith sequence of $\pi_X : X \rightarrow X_\tau$, we have $(\pi_X)_!(\mu) = c(X, T)^n$ where μ is the generator of $H^n(X)$. $c(N^2 - \Delta_N, T_N)^n \in H^n((N^2 - \Delta_N)_\tau)$ satisfies $c(N^2 - \Delta_N, T_N)^n = g_\tau^*(\omega^n)$. Then there exists the element $\nu' \in H^n(N^2 - \Delta_N)$ such that $c(N^2 - \Delta_N, T_N)^n = \pi_1(\nu')$ by Proposition 3.3.

By $(\pi_X)_!(\mu) = c(X, T)^n$ and $(p_0)_\tau^*(c(X, T)) = c(\Gamma_0, T_0)$, we obtain

$$(3.5) \quad Q^*(\nu') = p_0^*(\mu).$$

Therefore we have $(p_0^*)^{-1}Q^* : H^n(N^2 - \Delta_N) \cong H^n(X) \cong \mathbf{F}_2$.

Consider the following diagram:

$$(3.6) \quad \begin{array}{ccccc} X & \xleftarrow{p_0} & \Gamma_0 & \xrightarrow{Q} & N^2 - \Delta_N \\ \downarrow = & & \downarrow p_\Gamma & & \downarrow \pi_1 \\ X & \xleftarrow{p} & \Gamma & \xrightarrow{q} & N. \end{array}$$

Since p_Γ^* and π_1^* are isomorphisms, we see that $\deg((p^*)^{-1}q^*)$ is an odd number where $(p^*)^{-1}q^* : H^n(N; Z) \rightarrow H^n(X; Z)$.

We shall prove

$$(3.7) \quad \deg(T_N^*) = \begin{cases} 1 & \text{odd number } n \\ -1 & \text{even number } n . \end{cases}$$

By $QT_0 = T_N Q$ and $\deg(Q^*) \neq 0$, it holds $\deg(T_0^*) = \deg(T_N^*)$. By $p_0 T_0 = T p_0$ and $\deg(p_0^*) = \pm 1$, it holds $\deg(T_0^*) = \deg(T^*)$. Therefore we see

$$\deg(T_N^*) = \deg(T^*) = \pm 1.$$

By Proposition 3.1, we obtain our assertion (3.7).

Let $\hat{\nu}$ be the generator of $H^n(N; Z)$. $\hat{\nu}' \in H^n(N^2 - \Delta_N; Z)$ is defined by $\hat{\nu}' = \pi_1^*(\hat{\nu})$ where $\pi_1 : N^2 - \Delta_N \rightarrow N$. Note that ν and ν' are the reductions of $\hat{\nu}$ and $\hat{\nu}'$ respectively. From $\pi_i^* : H^n(N; Z) \cong H^n(N^2 - \Delta_N; Z) \cong Z$, we see

$$(3.8) \quad (\pi_1^*)^{-1} \pi_2^*(\hat{\nu}) = \begin{cases} +\hat{\nu} & \text{odd number } n \\ -\hat{\nu} & \text{even number } n \end{cases}$$

by $\pi_1 = \pi_2 T_N$ and the above remark (3.7).

We shall prove the uniqueness of $\deg((p^*)^{-1} q^*)$.

Let (p, q) and (p', q') be two selected pairs of φ , that is, $X \xleftarrow{p} \Gamma \xrightarrow{q} N$ and $X \xleftarrow{p'} \Gamma' \xrightarrow{q'} N$. Γ'_0 and $X \xleftarrow{p'_0} \Gamma'_0 \xrightarrow{q'_0} N$ are defined similarly to Γ_0 . Define Γ_{01} and Γ_{10} by

$$\Gamma_{01} = \{(z, z') \in \Gamma \times \Gamma' \mid p(z) = T p'(z')\}$$

and

$$\Gamma_{10} = \{(z', z) \in \Gamma' \times \Gamma \mid p'(z') = T p(z)\}.$$

$p_1 : \Gamma_{01} \rightarrow X$, $p'_1 : \Gamma_{10} \rightarrow X$ are defined by $p_1(z, z') = p(z)$, $p'_1(z', z) = p'(z')$ respectively. $\hat{\pi}_1 : \Gamma_{01} \rightarrow \Gamma$ and $\hat{\pi}'_1 : \Gamma_{10} \rightarrow \Gamma'$ are the projections to the first factors.

Now consider the following diagram:

$$\begin{array}{ccccc}
 X & \xleftarrow{p} & \Gamma & \xrightarrow{q} & N \\
 \uparrow = & & \uparrow \hat{\pi}_1 & & \uparrow \pi_1 \\
 X & \xleftarrow{p_1} & \Gamma_{01} & \xrightarrow{\hat{Q}} & N^2 - \Delta_N \\
 \downarrow T & & \downarrow \hat{T} & & \downarrow T_N \\
 X & \xleftarrow{p'_1} & \Gamma_{10} & \xrightarrow{\hat{Q}'} & N^2 - \Delta_N \\
 \downarrow = & & \downarrow \hat{\pi}'_1 & & \downarrow \pi_1 \\
 X & \xleftarrow{p'} & \Gamma' & \xrightarrow{q'} & N.
 \end{array}
 \tag{3.9}$$

where $\hat{Q}(z, z') = (q(z), q'(z'))$, $\hat{Q}'(z', z) = (q'(z'), q(z))$. $\hat{T} : \Gamma_{10} \rightarrow \Gamma_{10}$ is defined by $\hat{T}(z, z') = (z', z)$. By $\deg(T^*) = \deg(T_N^*)$ and the above diagram, we have $\deg((p^*)^{-1}q^*) = \deg((p'^*)^{-1}q'^*)$. \square

For the case of $c(X, T)^n = 0$ instead of $c(X, T)^n \neq 0$, we obtain the following theorem.

Theorem 3.5. *Let X be an ANR space with a free involution T and N be an n -dimensional homology sphere. Suppose that $\dim X = n$, $H^n(X; Z) \cong Z$ and $c(X, T)^n = 0$. If an admissible mappings $\varphi : X \rightarrow N$ satisfies $\varphi(x) \cap \varphi(Tx) = \emptyset$ for any $x \in X$, then there exists a unique even number m such that $\deg \varphi = \{m\}$.*

Proof. If $\deg((p^*)^{-1}q^*)$ is an odd degree, it holds $(p^*)^{-1}q^* : H^n(N) \cong H^n(X)$. By the diagram (3.6) of Theorem 3.4, we see $p_0^*(\mu) = Q^*(\nu')$. From $\pi_!(\nu') = c(N^2 - \Delta_N)^n$ and $Q_\tau^* \pi_!(\nu') = (\pi_{\Gamma_0})_! Q^*(\nu')$, we have $(\pi_X)_!(\mu) = c(X, T)^n$ by $p_0 T_0 = T p_0$ and the formula (3.4). By the assumption $c(X, T)^n = 0$, we see $(\pi_X)_!(\mu) = 0$.

On the other hand, by the Gysin-Smith sequence of $\pi_X : X \rightarrow X_\tau$, it hold $(\pi_X)_! : H^n(X) \cong H^n(X_\tau)$. We obtain $(\pi_X)_!(\mu) \neq 0$. This contradicts the above remark. Therefore $\deg((p^*)^{-1}q^*)$ is an even degree.

The uniqueness of $\deg(\varphi)$ is same as the proof of Theorem 3.4. \square

Remark. Let X be a simplicial complex with a simplicial free involution T . S^n has the antipodal involution. If X satisfies the condition of Theorem 3.4, any odd integer d is realized by an equivariant mapping $f : X \rightarrow S^n$ with $\deg(f) = d$.

Because there exists an equivariant mapping $f : X \rightarrow S^n$ by Chapter II Proposition 3.15 of T. tom Dieck [2]. When $c(X, T)^n \neq 0$, the degree of f

is odd. By Theorem 4. 11 of T. tom Dieck [2], there exists an equivariant mapping $g : X \rightarrow S^n$ with the degree $\deg(g) = d$.

Similarly if X satisfies the condition of Theorem 3.5 any even integer d is realized by an equivariant mapping $f : X \rightarrow S^n$ with $\deg(f) = d$. This is proved by Chapter II Proposition 3.15 and Theorem 4. 11 of T. tom Dieck [2]. The mappings defined above satisfy the condition $f(Tx) \neq f(x)$ for any $x \in X$.

4. THE DEGREE OF ADMISSIBLE MAPPINGS 2

Let X be a Hausdorff space with an involution T . We defined Δ_X and T_X in section 3. Set

$$\Delta'_X = \{(x, Tx) \in X^2 \mid x \in X\}.$$

If T is a free involution, then $\Delta_X \cap \Delta'_X = \emptyset$. Generally T_X is not a free involution on $X^2 - \Delta'_X$. When N is an n -dimensional homology sphere, it holds $\pi_i^* : H^*(N^2 - \Delta'_N) \cong H^*(N)$ ($i = 1, 2$) as Theorem 3.2.

In the following theorem, the degree is also determined by the contraposition of Theorem 5.5 of [15]. But the uniqueness is not proved there. Our method is more elementary than [14], [15].

Theorem 4.1. *Let X be an ANR space with a free involution T and N be an n -dimensional homology sphere with a non trivial involution T' . Suppose that $\dim X = n$ and $H^n(X; Z) \cong Z$ and an admissible mappings $\varphi : X \rightarrow N$ satisfies $T'\varphi(x) \cap \varphi(Tx) = \emptyset$ for any $x \in X$. Then there exists a unique even number m such that $\deg(\varphi) = \{m\}$. In particular, if T' is an orientation reversing involution, then $\deg \varphi = \{0\}$.*

Proof. Though $N^2 - \Delta'_N$ is a T_N -invariant set, T_N is not a free involution of $N^2 - \Delta'_N$. We need a device to work in the category of free involutions. As Proposition 3.3, we see $H^n(S^\infty \times (N^2 - \Delta'_N)) \cong \mathbf{F}_2$ whose generator is denoted by ν' .

Set $\omega = c(S^\infty \times_\tau (N^2 - \Delta'_N), T'_N)$ where $T'_N(x, z, z') = (Tx, z', z)$. At first we remark $\omega^k \neq 0$ for all k . Since T' is a non trivial involution, there exist a point $z_0 \in N$ such that $T'(z_0) \neq z_0$ and equivariant mappings

$$h : S^\infty \rightarrow S^\infty \times (N^2 - \Delta'_N), \quad k : S^\infty \times (N^2 - \Delta'_N) \rightarrow S^\infty$$

which are defined by $h(x) = (x, z_0, z_0)$ and $k(x, z, z') = x$. They satisfy $k_\tau h_\tau = Id_{RP^\infty}$. Therefore we obtain our assertion.

$\hat{q} : S^\infty \times \Gamma_0 \rightarrow S^\infty \times N^2$ is defined by $\hat{q}(x, z, z') = (x, q(z), q(z'))$. By our assumption \hat{q} is regarded as $\hat{q} : S^\infty \times \Gamma_0 \rightarrow S^\infty \times (N^2 - \Delta'_N)$. $\pi_{\Gamma_0} : S^\infty \times \Gamma_0 \rightarrow S^\infty \times_\tau \Gamma_0$, $\pi_{N^2} : S^\infty \times N^2 \rightarrow S^\infty \times_\tau N^2$ and $\pi : S^\infty \times (N^2 - \Delta'_N) \rightarrow S^\infty \times_\tau (N^2 - \Delta'_N)$ are covering projections. The generator of $H^n(N^2 - \Delta'_N) \cong \mathbf{F}_2$

is also denoted by ν' . Then we have $j^*(\nu \times 1) = \nu', j^*(1 \times \nu) = \nu'$ where $j : N^2 - \Delta'_N \rightarrow N^2$ is the inclusion.

Set $\tilde{j} : S^\infty \times (N^2 - \Delta'_N) \rightarrow S^\infty \times N^2$ defined by $\tilde{j}(x, z, z') = (x, j(z, z'))$.

Consider the following diagram:

$$(4.1) \quad \begin{array}{ccccc} \{0\} & \longrightarrow & H^*(S^\infty \times N^2) & \xrightarrow{\tilde{j}^*} & H^*(S^\infty \times (N^2 - \Delta'_N)) \\ \downarrow & & \downarrow (\pi_{N^2})! & & \downarrow \pi! \\ H^*(S^\infty \times N^2) & \xrightarrow{(\pi_{N^2})!} & H^*(S^\infty \times_\tau N^2) & \xrightarrow{\tilde{j}_\tau^*} & H^*(S^\infty \times_\tau (N^2 - \Delta'_N)) \\ \downarrow \hat{q}^* & & \downarrow \hat{q}_\tau^* & & \downarrow \hat{q}_\tau^* \\ \bar{H}^*(S^\infty \times \Gamma_0) & \xrightarrow{(\pi_{\Gamma_0})!} & \bar{H}^*(S^\infty \times_\tau \Gamma_0) & \xrightarrow{=} & \bar{H}^*(S^\infty \times_\tau \Gamma_0). \end{array}$$

We shall prove

$$(4.2) \quad \pi_!(\nu') = 0, \quad H^k(S^\infty \times_\tau (N^2 - \Delta'_N)) \cong \mathbf{F}_2 \oplus \mathbf{F}_2 \quad (k \geq n).$$

By the Gysin-Smith sequence

$$\rightarrow H^*(S^\infty \times_\tau (N^2 - \Delta'_N)) \xrightarrow{\pi^*} H^*(S^\infty \times (N^2 - \Delta'_N)) \xrightarrow{\pi_!} H^*(S^\infty \times_\tau (N^2 - \Delta'_N))$$

we obtain easily

$$H^k(S^\infty \times_\tau (N^2 - \Delta'_N)) \cong \mathbf{F}_2 \quad (0 \leq k \leq n-1).$$

If $\pi_!(\nu') \neq 0$, we have $\pi^* = 0 : H^n(S^\infty \times_\tau (N^2 - \Delta'_N)) \rightarrow H^n(S^\infty \times (N^2 - \Delta'_N))$ and $H^n(S^\infty \times_\tau (N^2 - \Delta'_N)) = \mathbf{F}_2$. Then we see $\pi_!(\nu') = \omega^n$ by the Gysin-Smith sequence. From $H^*(S^\infty \times (N^2 - \Delta'_N)) \cong H^*(N)$, we obtain $H^k(S^\infty \times_\tau (N^2 - \Delta'_N)) \cong 0$ for $k \geq n+1$. This contradicts the above result $\omega^k \neq 0$ for all k . Therefore we obtain $\pi_!(\nu') = 0$ and $H^k(S^\infty \times_\tau (N^2 - \Delta'_N)) \cong \mathbf{F}_2 \oplus \mathbf{F}_2$ for all $k \geq n$.

By $\tilde{j}^*(1 \times \nu) = \nu'$, we have $\tilde{j}_\tau^*(\pi_{N^2})!(1 \times \nu) = 0$. Therefore we see $\hat{q}_\tau^*((\pi_{N^2})!(1 \times \nu)) = 0$. We obtain

$$(4.3) \quad \hat{q}^*(1 \times \nu) = 0$$

by the isomorphism $(\pi_{\Gamma_0})! : \bar{H}^n(S^\infty \times \Gamma_0) \cong \bar{H}^n(S^\infty \times_\tau \Gamma_0)$. That $(\pi_{\Gamma_0})!$ is an isomorphism is proved by the Gysin-Smith sequence and $H^*(X) \cong \bar{H}^*(\Gamma_0)$. Similarly we have also $\hat{q}^*(\nu \times 1) = 0$.

Therefore we have $\hat{q}^* = 0 : H^n(S^\infty \times N^2) \rightarrow \bar{H}^n(S^\infty \times \Gamma_0)$ and $q^* = 0 : H^n(N) \rightarrow \bar{H}^n(\Gamma)$. Therefore we see that $\deg(q^*) = \{m\}$ where m is an even number.

Now we shall prove the uniqueness. Let (p, q) and (p', q') be two selected pairs of φ . Let $\hat{R} : \Gamma_{01} \rightarrow N^2 - \Delta_N$ and $\hat{R}' : \Gamma_{10} \rightarrow N^2 - \Delta_N$ be defined by

$\hat{R}(z, z') = (q(z), Tq'(z'))$ and $\hat{R}'(z', z) = (T'q'(z'), q(z))$ respectively. Note that \hat{R} and \hat{R}' are well defined by the assumption. When $q = q'$, \hat{R} is an equivariant mapping i.e. $\hat{R}T_0 = \hat{T}\hat{R}$ where $\hat{T}(z, z') = (T(z'), T(z))$. But \hat{T} is not a free involution on $N^2 - \Delta_N$.

Now consider the following diagram:

$$(4.4) \quad \begin{array}{ccccc} X & \xleftarrow{p} & \Gamma & \xrightarrow{q} & N \\ \uparrow = & & \uparrow \pi_1 & & \uparrow \pi_1 \\ X & \xleftarrow{p} & \Gamma_{01} & \xrightarrow{\hat{R}} & N^2 - \Delta_N \\ \downarrow T & & \downarrow T & & \downarrow T_N \\ X & \xleftarrow{p} & \Gamma_{10} & \xrightarrow{\hat{R}'} & N^2 - \Delta_N \\ \downarrow = & & \downarrow \pi_1 & & \downarrow \pi_1 \\ X & \xleftarrow{p'} & \Gamma' & \xrightarrow{T'q'} & N. \end{array}$$

Since π_i^* ($i = 1, 2$) are isomorphisms, we have

$$(4.5) \quad \deg((p^*)^{-1}q^*) = \deg((p'^*)^{-1}q'^*T'^*).$$

If T' is an orientation preserving mapping, we have $\deg((p^*)^{-1}q^*) = \deg((p'^*)^{-1}q'^*)$.

If T' is an orientation reversing mapping, we have $\deg((p^*)^{-1}q^*) = -\deg((p'^*)^{-1}q'^*)$ by taking $q'(z') = q(z')$. Therefore we have $\deg((p^*)^{-1}q^*) = 0$. In any cases, the degree of φ is uniquely determined. \square

Corollary 4.2. *Under the same conditions as Theorem 4.1, assume that an involution T' on N is free. If n is an even number, then $\deg \varphi = \{0\}$.*

Proof. By Proposition 3.1, T' is an orientation preserving involution for an odd number n and is an orientation reversing involution for an even number n . Therefore we obtain our result by Theorem 4.1. \square

Remark. There are many involutions on spheres, for example the antipodal involution, the reflections with respect to the hyper planes and their compositions etc.. Theorem 4.1 holds for these non trivial involutions.

In Corollary 4.2, it is necessary that T' is a free involution. For example, consider the involutions

$$T(x_0, x_1, x_2) = (-x_0, -x_1, -x_2), \quad T'(x_0, x_1, x_2) = (-x_0, -x_1, x_2)$$

on the unit sphere S^2 centered at the origin. Let $f : S^2 \rightarrow S^2$ defined by

$$f(x_0, x_1, x_2) = \frac{(x_0^2 - x_1^2, 2x_0x_1, x_2)}{\sqrt{(x_0^2 - x_1^2)^2 + (2x_0x_1)^2 + (x_2)^2}}$$

We see easily $f(T(x_0, x_1, x_2)) \neq T'f(x_0, x_1, x_2)$ and $\deg(f) = 2$. From this example, it is necessary that T' is a free involution in Corollary 4.2.

Remark. Let X be a simplicial complex with a simplicial free involution T . If X satisfies the condition of Theorem 4.1, any even integer $2d$ is realized by an equivariant mapping $f : X \rightarrow S^n$ such that $\deg(f) = 2d$. Here we consider the trivial involution T' on S^n .

Because there exists a mapping $f : X_\tau \rightarrow S^n$ such that $\deg f$ is any given integer by the Hopf classification theorem (cf. [17]). Let $f' : X \rightarrow S^n$ be defined by $f' = f\pi_X : X \rightarrow X_\tau \rightarrow S^n$. Since it holds $\pi_X^* = 0 : H^n(X_\tau) \rightarrow H^n(X)$ by the Gysin-Smith sequence, we see that the degree of f' is even and f' is an equivariant mapping with respect to T and T' .

By Theorem 4.11 of T. tom Dieck [2], for any even degree $2d$, there exists an equivariant mapping $g : X \rightarrow S^n$ with $\deg(g) = 2d$. g satisfies the condition $g(Tx) \neq T''g(x)$ for any $x \in X$ where T'' is the antipodal involution on S^n .

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