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RESTRICTION ON GALOIS GROUPS BY PRIME INERT CONDITION

Toru Komatsu

Abstract. In this paper, we study number fields K with the property that every prime factor of the degree of K remains prime in K. We determine all types of Galois groups of such K up to degree nine and find that Wang's non-existence in cyclic octic case is exceptionally undetermined by our group-theoretic criterion.

1. Introduction

Inverse Galois Problem (IGP) [8] asks whether a given finite group occurs as one of the Galois groups over a fixed field. Over a number field, the well known Hermite theorem tells that there are only finitely many field extensions with bounded degrees and ramifications. This indicates that arithmetic condition often restricts possible types of Galois groups to be realized in IGP. In view of IGP with arithmetic inert condition, the following result is of vital importance.

Proposition 1.1 (Wang [11], Swan [10, §5]). *There are no cyclic-Galois extensions of $\mathbb{Q}$ of degree 8 in which 2 remains prime.*

Jensen, Ledet and Yui [5, §2.6] applied Proposition 1.1 to show non-existence of a generic polynomial over $\mathbb{Q}$ with cyclic octic ($=\mathbb{Z}/8\mathbb{Z}$) Galois group. In this paper, we consider number fields K having the "degree-inert" property that every prime factor of the degree $[K:\mathbb{Q}]$ is inert in K. For the sake of convenience, we shall abbreviate this property as **deg.-inert property**. Writing K^{gc} for the Galois closure of K over $\mathbb{Q}$ and $\mathrm{Gal}(K^{\mathrm{gc}}/\mathbb{Q})$ for its Galois group, we let $G(K)$ denote the image of the associated transitive permutation representation $\rho : \mathrm{Gal}(K^{\mathrm{gc}}/\mathbb{Q}) \to S_n$, where S_n is the symmetric group on $n = [K:\mathbb{Q}]$ letters. (Note that the group $G(K)$ depends on K up to conjugacy in S_n.) Then, as a necessary condition, we have

Proposition 1.2. *The group $G(K)$ of a deg.-inert field K of degree n over $\mathbb{Q}$ contains a cycle of length n.*

Our main result of this paper is then:

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Theorem 1.3. *Every transitive subgroup G of S_n with an n-cycle occurs as the group $G(K)$ of a deg.-inert number field K, provided that $n < 10$ and G is not the cyclic octic group.*

We will deduce Proposition 1.2 from basic facts of algebraic number theory in §2, and demonstrate Theorem 1.3 in §3 using known lists of the permutation groups and tables of number fields. We provide explicit irreducible polynomials $f(x)$ over $\mathbb{Q}$ with $G(\mathbb{Q}[x]/(f))$ giving the transitive groups in Theorem 1.3. In §4 we observe higher degree cases and discuss a variant of our degree-inert property.

2. Necessary condition for deg.-inert field

In this section, we will deduce Proposition 1.2 from basic facts of algebraic number theory. We assume that K is a deg.-inert field of degree n and p is a prime factor of n. Let O be the ring of algebraic integers in K, and $\mathfrak{p}$ a unique prime ideal of K above p, that is, $\mathfrak{p} = pO$. The following Dedekind's lemma, a sufficient condition for a prime to be inert, may come to mind. Let $T(\theta, X) \in \mathbb{Z}[X]$ denote the minimal polynomial of an algebraic integer θ over $\mathbb{Q}$. Let l be a prime number, and $\mathbb{Z}/l$ its residue class field of $\mathbb{Z}$.

Lemma 2.1 (Dedekind, Cohen [2, Thm 4.8.13])**.** *If $T(\theta, X)$ is irreducible over $\mathbb{Z}/l$, then l is inert in $\mathbb{Q}(\theta)$.*

Unfortunately, the converse of Lemma 2.1 is false. The number $\theta = \sqrt{5}$ is a counter-example under $l = 2$. Indeed, one factorizes the polynomial $T(\sqrt{5}, X) = X^2 - 5$ into $(X + 1)^2$ over $\mathbb{Z}/2$, though $\mathbb{Q}(\sqrt{5})$ is a deg.-inert field. Meanwhile, the golden ratio $\gamma = (1 + \sqrt{5})/2$ generates $\mathbb{Q}(\sqrt{5})$ and the polynomial $T(\gamma, X) = X^2 - X - 1$ is irreducible over $\mathbb{Z}/2$.

Lemma 2.2. *The deg.-inert field K has a generator γ in O so that $T(\gamma, X)$ is irreducible over $\mathbb{Z}/p$.*

Proof. Let E be the residue class field $O/\mathfrak{p}$ of O at $\mathfrak{p}$, and F that of $\mathbb{Z}$ at p. Since the extension E/F is simple, one can take a generator ξ of E over F. For the natural projection $O \to E$ we choose a lift $\gamma \in O$ of ξ. Due to the genericity of ξ for E/F the polynomial $T(\gamma, X)$ is irreducible over F with degree n, and γ generates K. □

Let γ be a fixed one as in Lemma 2.2. The Galois closure field K^{gc} of K over $\mathbb{Q}$ is equal to the minimal splitting field of $T(\gamma, X)$ over $\mathbb{Q}$.

Lemma 2.3 (Dedekind, Cox [4, Thm 13.4.5])**.** *If $T(\gamma, X)$ is irreducible over $\mathbb{Z}/l$, then the group $G(K)$ of K contains a cycle with length n.*

Lemmas 2.2 and 2.3 with $l = p$ imply the assertion of Proposition 1.2 in §1.

Remark. The books [2], [3] and [4] deal with more general methods than those in Lemmas 2.1 and 2.3.

3. Determination of groups of deg.-inert fields

In this section, we will demonstrate Theorem 1.3 using known lists of the permutation groups and tables of number fields. We provide explicit irreducible polynomials $f(x)$ over $\mathbb{Q}$ with $G(\mathbb{Q}[x]/(f))$ giving the transitive groups in Theorem 1.3. The first half of the proof is to prepare a list of the groups satisfying the necessary condition in Proposition 1.2. Butler and McKay [1] present not only all the transitive subgroups of S_n with $n \leq 11$ but also their invariants. As used in [1] let $n\mathrm{T}k$ stand for the k-th transitive subgroup of S_n. The data in [1] provides Table 1 below with the three invariants of $n\mathrm{T}k$; the order, the signature and the number of n-cycles are listed in the 2nd to 4th columns at Table 1, headed ord, sgn and cyc, respectively. Here the signature is defined to be + if the n-th alternating group A_n contains $n\mathrm{T}k$, and − otherwise. Proposition 1.2 shows the non-existence of a deg.-inert field K whose group $G(K)$ has no n-cycles, and allows us to delete the cases in which the number at its 4th column cyc is equal to 0. Let t_n be the number of transitive subgroups in S_n, and d_n the number of the deleted ones. The numbers t_n and d_n are as follows:

n	2	3	4	5	6	7	8	9
t_n	1	2	5	5	16	7	50	34
d_n	0	0	2	0	7	0	32	16

Proposition 1.1 urges us to put $\emptyset$ in the 5th column at the row of 8T1, the cyclic octic group C_8.

The second half of the proof is to fill all the open boxes of the constructing table, that is, to discover a deg.-inert field K for each group whose 5th column is open. For the discovery we utilize the databases of number fields at the web cites of Klüners–Malle [7] and Jones [6]. We explain how to search the data at [7] for a deg.-inert field with the group 6T3 for example. At the page named Search in [7] specify the attributes so that the group degree and number are `=6` and `=3`, respectively, and the field discriminant factors are `2,3` with `none of these`. The button Search at the bottom yields many candidate polynomials $x^6 - x^5 - 8x^4 + 6x^3 + 16x^2 - 10x - 5, x^6 + 3x^5 - 4x^4 - 13x^3 + 3x^2 + 10x + 1, \ldots$ The above attribute for the field discriminant factors means that both 2 and 3 are unramified, which is a necessary condition to be deg.-inert. By using a software PARI/GP available freely at [9], we can check whether the fields defined by the candidates are deg.-inert or not. Faster simple examination due to Lemma 2.1 operates by the command `factormod(`f`,`p`)` on PARI/GP, which factors the polynomial f modulo the prime p. If the simple examination does not work, then detailed inspection accomplishes by the command `idealprimedec(nf,`p`)` on PARI/GP, which gives the prime ideal decomposition of the prime number p in the number field `nf`. Here we can make the data `nf` from the polynomial f with the command `nfinit(`f`)`. For the first polynomial $f(x) = x^6 - x^5 - 8x^4 + 6x^3 + 16x^2 - 10x - 5$ the simple result by `factormod(`f`,`p`)` is the factorizations $f(x) \equiv x^6 + x^5 + 1 \pmod 2$ and $f(x) \equiv (x+1)^2(x^2+x+2)(x^2+2x+2) \pmod 3$. This means that 2 remains prime and 3 decomposes into at least 3 prime ideals. The detailed one by `idealprimedec(nf,`p`)` tells us that $3O = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ with $[O/\mathfrak{p}_i : \mathbb{Z}/3] = 2$ for $i = 1, 2, 3$, and thus the field of the first $f(x)$ is not deg.-inert. For the second $f(x) = x^6 + 3x^5 - 4x^4 - 13x^3 + 3x^2 + 10x + 1$ the simple `factormod(`f`,`p`)` shows that $f(x)$ is irreducible not only over $\mathbb{Z}/2$ but also over $\mathbb{Z}/3$, that is, we now find a deg.-inert field. By the command `polgalois(`f`)` one can make sure that the Galois group of the second $f(x)$ is isomorphic to 6T3. Unfortunately, at the cite [7] up to degree nine we cannot find a deg.-inert field with the group 7T4. On the other hand, the cite [6] contains useful data for the case 7T4. At [6] specify the attributes so that Degree=7, Galois T-num=4 and p_1 =7 with c_1 =0. The attributes p_1 and c_1 mean that 7 is unramified. By the same examination as above we find a deg.-inert field for 7T4 described below.

Table 1: polynomials with degree less than 10

nTk	ord	sgn	cyc	polyn
2T1	2	$-$	1	$x^2 - 5$
3T1	3	$+$	2	$x^3 - x^2 - 2x + 1$
3T2	6	$-$	2	$x^3 - x^2 - 3x + 1$
4T1	4	$-$	2	$x^4 - x^3 - 4x^2 + 4x + 1$
4T3	8	$-$	2	$x^4 - x^3 - 3x^2 + x + 1$
4T5	24	$-$	6	$x^4 - 4x^2 - x + 1$
5T1	5	$+$	4	$x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1$
5T2	10	$+$	4	$x^5 - x^4 - 5x^3 + 4x^2 + 3x - 1$
5T3	20	$-$	4	$x^5 - 2x^4 - 12x^3 + 24x^2 + 8x - 23$
5T4	60	$+$	24	$x^5 - x^4 - 11x^3 + x^2 + 12x - 4$
5T5	120	$-$	24	$x^5 - 5x^3 + 4x - 1$
6T1	6	$-$	2	$x^6 - x^5 - 7x^4 + 2x^3 + 7x^2 - 2x - 1$
6T3	12	$-$	2	$x^6 - 3x^5 - 4x^4 + 13x^3 + 3x^2 - 10x + 1$
6T5	18	$-$	6	$x^6 - x^5 - 8x^4 + 5x^3 + 19x^2 - 4x - 11$
6T6	24	$-$	8	$x^6 - 2x^5 - 5x^4 + 11x^3 + 2x^2 - 9x + 1$
6T9	36	$-$	12	$x^6 - 3x^5 - 11x^4 + 24x^3 + 32x^2 - 11x - 1$
6T11	48	$-$	8	$x^6 - 3x^5 + x^4 + 4x^3 - 3x^2 - 2x + 1$
6T13	72	$-$	12	$x^6 - x^5 - 6x^4 + 4x^3 + 8x^2 - 1$

Table 1: polynomials with degree less than 10 (continued)

nTk	ord	sgn	cyc	polyn
6T14	120	$-$	20	$x^6 - x^5 - 287x^4 - 1058x^3 + 17939x^2 + 129814x + 231845$
6T16	6!	$-$	5!	$x^6 - x^5 - 5x^4 + 4x^3 + 5x^2 - 2x - 1$
7T1	7	$+$	6	$x^7 - x^6 - 12x^5 + 7x^4 + 28x^3 - 14x^2 - 9x - 1$
7T2	14	$-$	6	$x^7 - x^6 - 9x^5 + 2x^4 + 21x^3 + x^2 - 13x - 1$
7T3	21	$+$	6	$x^7 - 8x^5 - 2x^4 + 16x^3 + 6x^2 - 6x - 2$
7T4	42	$-$	6	$x^7 - 3x^6 + 6x^5 - 8x^4 + 12x^3 - 15x^2 + 10x - 9$
7T5	168	$+$	48	$x^7 - 8x^5 - 2x^4 + 15x^3 + 4x^2 - 6x - 2$
7T6	7!/2	$+$	6!	$x^7 - 2x^6 - 7x^5 + 11x^4 + 16x^3 - 14x^2 - 11x + 2$
7T7	7!	$-$	6!	$x^7 - 2x^6 - 5x^5 + 9x^4 + 7x^3 - 10x^2 - 2x + 1$
8T1	8	$-$	4	$\emptyset$
8T6	16	$-$	4	$x^8 - 3x^7 - 13x^6 + 47x^5 + 3x^4 - 109x^3 + 78x^2 - 16x + 1$
8T7	16	$-$	8	$x^8 - 20x^6 + 105x^4 + 15x^3 - 110x^2 + 45x - 5$
8T8	16	$-$	4	$x^8 - 14x^6 - 10x^5 + 31x^4 + 15x^3 - 14x^2 - 5x + 1$
8T15	32	$-$	8	$x^8 - x^7 - 11x^6 + 4x^5 + 21x^4 - 4x^3 - 11x^2 + x + 1$
8T16	32	$-$	16	$x^8 - 8x^7 + 8x^6 + 64x^5 - 125x^4 - 31x^3 + 68x^2 + 17x + 1$
8T17	32	$-$	8	$x^8 - 12x^6 - 2x^5 + 37x^4 + 17x^3 - 25x^2 - 16x - 1$
8T23	48	$-$	12	$x^8 - 4x^7 - 5x^6 + 29x^5 - 14x^4 - 25x^3 + 10x^2 + 8x + 1$
8T26	64	$-$	16	$x^8 + x^7 - 11x^6 - 8x^5 + 38x^4 + 21x^3 - 39x^2 - 23x + 1$
8T27	64	$-$	16	$x^8 - 4x^7 - x^6 + 17x^5 - 6x^4 - 21x^3 + 6x^2 + 8x + 1$
8T28	64	$-$	16	$x^8 - 4x^7 - 6x^6 + 19x^5 + 9x^4 - 19x^3 - 6x^2 + 4x + 1$
8T35	128	$-$	16	$x^8 - 3x^7 - 5x^6 + 14x^5 + 8x^4 - 16x^3 - 2x^2 + 5x - 1$
8T40	192	$-$	48	$x^8 - 18x^6 + 103x^4 - 5x^3 - 209x^2 + 26x + 93$
8T43	336	$-$	84	$x^8 - x^7 - 29x^6 + 111x^5 - 139x^4 + 37x^3 + 32x^2 - 10x - 1$
8T44	384	$-$	48	$x^8 - 4x^7 - x^6 + 15x^5 - 3x^4 - 16x^3 + 4x^2 + 4x - 1$
8T46	576	$-$	144	$x^8 - 21x^6 + 15x^5 + 121x^4 - 135x^3 - 126x^2 + 105x + 41$
8T47	1152	$-$	144	$x^8 - x^7 - 7x^6 + 5x^5 + 15x^4 - 7x^3 - 10x^2 + 2x + 1$
8T50	8!	$-$	7!	$x^8 - x^7 - 7x^6 + 4x^5 + 15x^4 - 3x^3 - 9x^2 + 1$
9T1	9	$+$	6	$x^9 - x^8 - 8x^7 + 7x^6 + 21x^5 - 15x^4 - 20x^3 + 10x^2 + 5x - 1$
9T3	18	$+$	6	$x^9 - 95x^7 + 430x^6 - 285x^5 - 1026x^4 + 1416x^3 - 152x^2 - 304x + 64$
9T6	27	$+$	18	$x^9 - 3x^8 - 10x^7 + 42x^6 - 28x^5 - 28x^4 + 28x^3 + 2x^2 - 6x + 1$

Table 1: polynomials with degree less than 10 (continued)

nTk	ord	sgn	cyc	polyn
9T10	54	$+$	18	$x^9 - 4x^8 - 14x^7 + 44x^6 + 62x^5 - 120x^4 - 92x^3 + 48x^2 + 12x - 4$
9T17	81	$+$	36	$x^9 - 2x^8 - 8x^7 + 18x^6 + 10x^5 - 36x^4 + 10x^3 + 12x^2 - 7x + 1$
9T20	162	$-$	36	$x^9 - 4x^8 - 4x^7 + 22x^6 - x^5 - 31x^4 + 4x^3 + 15x^2 - 1$
9T21	162	$+$	36	$x^9 + 3x^8 - 10x^7 - 37x^6 + 4x^5 + 81x^4 + 21x^3 - 50x^2 - 10x + 10$
9T22	162	$-$	36	$x^9 + 3x^8 + 5x^7 + 18x^6 + 34x^5 + 27x^4 + 5x^3 + 3x^2 - 6x + 1$
9T24	324	$-$	36	$x^9 - x^7 - 4x^6 + 3x^5 + 6x^4 + 3x^3 - 8x - 2$
9T25	324	$+$	144	$x^9 - 2x^8 - 18x^7 + 48x^6 + 68x^5 - 296x^4 + 164x^3 + 224x^2 - 229x + 41$
9T27	504	$+$	168	$x^9 - 3x^8 + 12x^6 - 14x^5 - 2x^4 + 12x^2 + x + 1$
9T28	648	$-$	144	$x^9 - x^8 - 8x^7 + 6x^6 + 20x^5 - 11x^4 - 18x^3 + 7x^2 + 4x - 1$
9T29	648	$-$	144	$x^9 - 3x^8 - 7x^7 + 21x^6 + 17x^5 - 42x^4 - 21x^3 + 22x^2 + 14x + 2$
9T30	648	$+$	144	$x^9 - 4x^8 - 10x^7 + 50x^6 + 6x^5 - 146x^4 + 44x^3 + 142x^2 - 50x - 34$
9T31	1296	$-$	144	$x^9 - 3x^8 - 5x^7 + 20x^6 - 31x^4 + 7x^3 + 15x^2 - 2x - 1$
9T32	1512	$+$	504	$x^9 + 2x^8 - 16176x^7 + 255808x^6 + 85921976x^5 - 2452969232x^4 - 147036710464x^3 + 4945237042432x^2 + 78691623510544x - 2853338165076832$
9T33	$9!/2$	$+$	$8!$	$x^9 + x^8 - 17x^7 - 35x^6 + 18x^5 + 69x^4 + 8x^3 - 33x^2 - 3x + 4$
9T34	$9!$	$-$	$8!$	$x^9 - 9x^7 - 2x^6 + 22x^5 + 5x^4 - 17x^3 - 4x^2 + 4x + 1$

By Table 1 we now finish the demonstration of Theorem 1.3.

4. Remarks on next cases

In this section, we will observe higher degree cases and discuss a variant of our degree-inert property. On higher degrees we have

Proposition 4.1. (1) *For $n = 10$ and $1 \le k \le 45$ the transitive subgroup $G = n\mathrm{T}k$ of S_n with n-cycles occurs as $G(K)$ of some deg.-inert field K, provided $k \ne 5, 17, 19, 27, 29, 33, 35$.*
(2) *For $n = 11$ and $1 \le k \le 8$ the transitive subgroup $G = n\mathrm{T}k$ of S_n with n-cycles occurs as $G(K)$ of some deg.-inert field K, provided $k \ne 3, 4$*

Remark. For all of the exceptions in Proposition 4.1, the groups $n\mathrm{T}k$ have n-cycles although they appear as none of the Galois groups of deg.-inert fields defined by polynomials at the cites [6] and [7].

The symbols in Table 2 below are the same as for Table 1 at § 3. For the exceptions described in Proposition 4.1 we denote ∃ ? at the 5th column polyn. On the numbers t_n of transitive subgroups in S_n and those d_n of the deleted ones from Table 2 due to Proposition 1.2, we have $t_{10} = 45$, $d_{10} = 22$, $t_{11} = 8$ and $d_{11} = 0$.

Table 2: polynomials with degree 10 or 11

$n\mathrm{T}k$	ord	sgn	cyc	polyn
10T1	10	–	4	$x^{10} + x^9 - 21x^8 - 21x^7 + 155x^6 + 155x^5 - 461x^4 - 461x^3 + 419x^2 + 419x + 67$
10T3	20	–	4	$x^{10} + 3x^9 - 18x^8 - 30x^7 + 110x^6 + 27x^5 - 205x^4 + 90x^3 + 75x^2 - 65x + 13$
10T5	40	–	4	∃ ?
10T6	50	–	20	$x^{10} - 3x^9 - 22x^8 + 77x^7 + 85x^6 - 439x^5 + 143x^4 + 427x^3 - 255x^2 - 20x + 9$
10T9	100	–	40	$x^{10} - 214x^8 + 733x^7 + 10695x^6 - 76150x^5 + 105427x^4 + 356178x^3 - 994203x^2 + 16649x + 1191167$
10T11	120	–	24	$x^{10} - 5x^9 - 15x^8 + 90x^7 + 53x^6 - 495x^5 + 52x^4 + 836x^3 - 324x^2 - 193x - 3$
10T14	160	–	64	$x^{10} - 2x^9 - 9x^8 + 15x^7 + 29x^6 - 31x^5 - 43x^4 + 14x^3 + 23x^2 + 3x - 1$
10T17	200	–	40	∃ ?
10T19	200	–	80	∃ ?
10T21	200	–	40	$x^{10} + 5x^9 - 2x^8 - 35x^7 - 16x^6 + 75x^5 + 49x^4 - 43x^3 - 39x^2 - 5x + 1$
10T22	240	–	24	$x^{10} - 3x^9 - 11x^8 + 35x^7 + 35x^6 - 130x^5 - 15x^4 + 159x^3 - 54x^2 - 37x + 17$
10T23	320	–	64	$x^{10} - 4x^9 - 8x^8 + 35x^7 + 14x^6 - 61x^5 - 14x^4 + 37x^3 + 7x^2 - 7x - 1$
10T27	400	–	80	∃ ?
10T29	640	–	64	∃ ?
10T30	720	–	144	$x^{10} - 3x^9 + x^8 + 5x^7 - 3x^6 + 2x^5 - x^4 - 5x^3 + 2x^2 + x + 1$
10T33	800	–	80	∃ ?
10T35	1440	–	144	∃ ?

Table 2: polynomials with degree 10 or 11 (continued)

nTk	ord	sgn	cyc	polyn
10T36	1920	$-$	384	$x^{10}-2x^9-21x^8+30x^7+128x^6-81x^5-223x^4$ $-34x^3+36x^2+12x+1$
10T39	3840	$-$	384	$x^{10}-2x^9-9x^8+12x^7+29x^6-22x^5-39x^4$ $+12x^3+19x^2+x-1$
10T40	7200	$-$	1440	$x^{10}-x^9-4x^8-4x^7+7x^6+12x^5+13x^4+5x^3$ $+2x^2-x+1$
10T41	14400	$-$	2880	$x^{10}-3x^9-2x^8+9x^7-4x^6-3x^5+14x^4-7x^3$ $-13x^2+2x+3$
10T43	28800	$-$	2880	$x^{10}-2x^7-3x^5+x^4+3x^2-1$
10T45	10!	$-$	9!	$x^{10}-x^9-9x^8+9x^7+25x^6-23x^5-24x^4+17x^3$ $+8x^2-3x-1$
11T1	11	$+$	10	$x^{11}+x^{10}-10x^9-9x^8+36x^7+28x^6-56x^5$ $-35x^4+35x^3+15x^2-6x-1$
11T2	22	$-$	10	$x^{11}+5x^{10}-18x^9-114x^8+37x^7+856x^6+746x^5$ $-2117x^4-3648x^3-135x^2+2677x+1301$
11T3	55	$+$	10	$\exists$?
11T4	110	$-$	10	$\exists$?
11T5	660	$+$	120	$x^{11}-20x^9+143x^7+4x^6-463x^5-54x^4+691x^3$ $+186x^2-392x-177$
11T6	7920	$+$	1440	$x^{11}+2x^{10}-5x^9+50x^8+70x^7-232x^6+796x^5$ $+1400x^4-5075x^3+10950x^2+2805x-90$
11T7	11!/2	$+$	10!	$x^{11}-3x^{10}-36x^9+60x^8+213x^7-9x^6-114x^5$ $-180x^4-108x^3+6x^2+36x+54$
11T8	11!	$-$	10!	$x^{11}-3x^{10}-8x^9+28x^8+15x^7-87x^6+16x^5$ $+95x^4-51x^3-17x^2+13x-1$

Finally, we shall add a remark on a variant of deg.-inert property. A number field K is called **weak deg.-inert**, if there exists a prime factor of the degree $[K:\mathbb{Q}]$ that is inert in K.

Lemma 4.2. *There exists no deg.-inert field K of degree 24 such that $G(K)$ is isomorphic to 24th cyclic group C_{24}. There exist infinitely many weak deg.-inert fields K of degree 24 with $G(K)\simeq C_{24}$.*

Proof. Suppose that a deg.-inert field K of degree 24 exists. Then K has a unique cyclic octic field K'. It follows from the definition that K' is a deg.-inert field K of degree 8 with $G(K)\simeq C_8$. It is contrary to Proposition 1.1. One can construct weak fields in the following. Let l be a prime number with $l \equiv 1 \pmod 8$ and $l \equiv 2 \pmod 3$. Then lth cyclotomic field $\mathbb{Q}(\zeta_l)$ has a unique cyclic octic field L and a unique quadratic field $L' = \mathbb{Q}(\sqrt{l})$. Since 3 is inert in L', so is in L. Let M be the maximal real subfield of 7th cyclotomic field $\mathbb{Q}(\zeta_7)$. Since 3 is inert in $\mathbb{Q}(\zeta_7)$, so is in M. Note that $[L : \mathbb{Q}] = 8$ and $[M : \mathbb{Q}] = 3$. Since the degrees of L and M are relatively prime, 3 is inert in the composite field $K = LM$. Hence K is weak deg.-inert of degree 24 with $G(K) \simeq C_{24}$. □

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Toru Komatsu
Department of Mathematics
Faculty of Science and Technology
Tokyo University of Science
Chiba, 278-8510 Japan
e-mail address: komatsu_toru@ma.noda.tus.ac.jp

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