Math. J. Okayama Univ. **58** (2016), 109–123

ANOTHER DESCRIPTION OF QUASI TERTIARY COMPOSITION

Hideaki Ōshima and Katsumi Ōshima

Abstract. We give another description of quasi tertiary composition in terms of horizontal and vertical compositions. As an application of the description and a modified result of Hardie-Kamps-Marcum-Oda, we see that any quasi tertiary composition has an indeterminacy.

1. Results

This paper is a supplement to our previous paper [**3**]. We use notations and results of [**3**] freely in this paper.

Let $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ be an admissible representative of an admissible null quadruple. That is, the following data are given:

$$X_0 \overset{a_1}{\longleftarrow} E^{n_1}X_1, \quad X_1 \overset{a_2}{\longleftarrow} E^{n_2}X_2, \quad X_2 \overset{a_3}{\leftarrow} X_3 \overset{a_4}{\leftarrow} X_4,$$
$$A_1 : a_1 \circ E^{n_1}a_2 \simeq *, \quad A_2 : a_2 \circ E^{n_2}a_3 \simeq *, \quad A_3 : a_3 \circ a_4 \simeq *$$

such that

$$[a_1, A_1, E^{n_1}a_2] \circ (E^{n_1}a_2, \widetilde{E}^{n_1}A_2, E^{n_1+n_2}a_3) \simeq *$$
$$(\text{equivalently } [a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1} \circ E^{n_1}(a_2, A_2, E^{n_2}a_3) \simeq *),$$
$$[a_2, A_2, E^{n_2}a_3] \circ (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4) \simeq *.$$

Take arbitrarily following five homotopies:

$$B_1 : [a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1} \circ E^{n_1}(a_2, A_2, E^{n_2}a_3) \simeq *,$$
$$B_2 : [a_2, A_2, E^{n_2}a_3] \circ (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4) \simeq *,$$
$$D_1 : a_1 \simeq [a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1} \circ E^{n_1}i_{a_2} = a_1,$$
$$D_2 : i_{a_2} \circ [a_2, A_2, E^{n_2}a_3] \simeq (a_2, A_2, E^{n_2}a_3) \circ q_{E^{n_2}a_3},$$
$$D_3 : q_{E^{n_2}a_3} \circ (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4) \simeq -E^{n_2+1}a_4$$

from which we can define two null homotopies:

$$\underline{B_1 \circ CE^{n_1}q_{E^{n_2}a_3}}_{(D_1,\widetilde{E}^{n_1}D_2)} : a_1 \circ E^{n_1}[a_2, A_2, E^{n_2}a_3] \simeq *, \tag{1.1}$$

$$\overline{i_{a_2} \circ B_2}^{(D_2,D_3)} : (a_2, A_2, E^{n_2}a_3) \circ (-E^{n_2+1}a_4) \simeq *. \tag{1.2}$$

Mathematics Subject Classification. primary 55Q05, secondary 55Q35.

Key words and phrases. Toda bracket, tertiary composition, quasi tertiary composition, horizontal composition, vertical composition.

Hence we have two maps from $E^{n_1+n_2+2}X_4$ to X_0:

$$\begin{aligned}(1.3)\qquad &\big[a_1, \underline{B_1 \circ CE^{n_1} q_{E^{n_2}a_3}}_{(D_1, \widetilde{E}^{n_1}D_2)}, E^{n_1}[a_2, A_2, E^{n_2}a_3]\big]\\ &\quad \circ \big(E^{n_1}[a_2, A_2, E^{n_2}a_3], \widetilde{E}^{n_1}B_2, E^{n_1}(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\big),\end{aligned}$$

$$\begin{aligned}(1.4)\qquad &\big[[a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1}, B_1, E^{n_1}(a_2, A_2, E^{n_2}a_3)\big]\\ &\quad \circ \big(E^{n_1}(a_2, A_2, E^{n_2}a_3), \widetilde{E}^{n_1}\overline{i_{a_2} \circ B_2}^{(D_2,D_3)}, E^{n_1}(-E^{n_2+1}a_4)\big)\end{aligned}$$

which are homotopic each other. As will be seen in the section 2, we can express (1.1), (1.2) and every element of any Toda bracket (for example (1.3) and (1.4)) in terms of horizontal composition $\bar{\circ}$ and vertical composition $\bullet$ [**2**, pp.272–275]. As a consequence we have the following main result.

Proposition 1.1. *Two maps (1.3) and (1.4) are homotopic to the map*

$$\begin{aligned}(1.5)\qquad &(1_{a_1} \,\bar{\circ}\, \widetilde{E}^{n_1}B_2)\\ &\bullet \big(1_{[a_1, A_1, E^{n_1}a_2]\circ(\psi^{n_1}_{a_2})^{-1}} \,\bar{\circ}\, (-\widetilde{E}^{n_1}D_2) \,\bar{\circ}\, 1_{E^{n_1}(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)}\big)\\ &\bullet \big((-B_1) \,\bar{\circ}\, 1_{E^{n_1}(q_{E^{n_2}a_3}\circ(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4))}\big).\end{aligned}$$

Note that (1.5) does not depend on D_1 and D_3, and recall that $1_f : X \times I \to Y$ denotes the constant homotopy $1_f(x,t) = f(x)$ for any map $f : X \to Y$. We should notice that (1.5) is a homotopy from the trivial map $* : E^{n_1+n_2+1}X_4 \to X_0$ to itself so that it is a map from $E^{n_1+n_2+2}X_4$ to X_0 by our convention (see the section 2). We readily have the following.

Corollary 1.2. *The quasi tertiary composition*

$$\{a_1, a_2, a_3, a_4; A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2}$$

is the set of homotopy classes of (1.5), where D_2 is fixed and all possible B_1, B_2 are taken.

This allows us to identify quasi tertiary compositions with modified 2-sided matrix Toda brackets (see the section 4). Indeed we have the following result which was suggested in the section 5 of Hardie-Kamps-Marcum-Oda [**1**] for the case $n_1 = n_2 = 0$.

Corollary 1.3. *We have*

$$\begin{aligned}&\{a_1, a_2, a_3, a_4; A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2}\\ &\quad = \left\{ \overline{a_1}' \quad \begin{array}{c} i_{a_2} \\ \widetilde{E^{n_2}a_3} \end{array} \quad \begin{array}{c} \overline{a_2} \\ q_{E^{n_2}a_3} \end{array} \quad \widetilde{E^{n_2}a_4} \quad ; D_2 \right\}_{n_1},\end{aligned}$$

where the right hand term is a modified 2-sided matrix Toda bracket and the following abbreviations are used

$$
\begin{aligned}
&\overline{a_1}' = [a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1}, \quad \overline{a_2} = [a_2, A_2, E^{n_2}a_3], \\
&\widetilde{E^{n_2}a_3} = (a_2, A_2, E^{n_2}a_3), \quad \widetilde{E^{n_2}a_4} = (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4).
\end{aligned} \tag{1.6}
$$

From Corollary 1.3 and a modified Proposition 4.8 of [**1**], we have the following in which we use abbreviations (1.6).

Corollary 1.4. *The quasi tertiary composition*

$$\{a_1, a_2, a_3, a_4; A_1, A_2, A_3\}^{(1)}_{n_1,n_2}$$

has the indeterminacy

$$\Gamma + [E^{n_1+n_2+2}X_3, X_0] \circ E^{n_1+n_2+2}a_4 + a_1 \circ E^{n_1}[E^{n_2+2}X_4, X_1],$$

where Γ is the subgroup of $[E^{n_1+n_2+2}X_4, X_0]$ which consists of homotopy classes of

$$
\begin{gathered}
(B_1 \mathbin{\bar{\circ}} 1_{E^{n_1}(q_{E^{n_2}a_3} \circ \widetilde{E^{n_2}a_4})}) \bullet (1_{\overline{a_1}'} \mathbin{\bar{\circ}} \widetilde{E}^{n_1}L \mathbin{\bar{\circ}} 1_{E^{n_1}\widetilde{E^{n_2}a_4}}) \\
\bullet((-B_1) \mathbin{\bar{\circ}} 1_{E^{n_1}(q_{E^{n_2}a_3} \circ \widetilde{E^{n_2}a_4})})
\end{gathered}
$$

for all $L : \widetilde{E^{n_2}a_3} \circ q_{E^{n_2}a_3} \simeq \widetilde{E^{n_2}a_3} \circ q_{E^{n_2}a_3}$ and a fixed but arbitrary $B_1 : \overline{a_1}' \circ E^{n_1}\widetilde{E^{n_2}a_3} \simeq *$, while Γ does not depend on a choice of B_1.

In the section 2, we recall from [**2**] definitions of $\bar{\circ}$ and $\bullet$ and mention their elementary properties. In the section 3, we prove Proposition 1.1. In the section 4, we modify 2-sided matrix Toda brackets and prove Corollary 1.3. In the section 5, we rewrite Proposition 4.8 of [**1**] as Proposition 5.1 for our purpose and prove Corollary 1.4.

2. Compositions

We work mainly in the category Top_* of spaces with base points. Given two maps $f, f' : X \to Y$, we denote by $H : f \Rightarrow f' : X \to Y$ or simply $H : f \simeq f'$ a homotopy $H : X \times I \to Y$ such that $H(x, 0) = f(x)$, $H(x, 1) = f'(x)$ and $H(*, t) = *$. It is pictured as the following.

$$
\begin{tikzcd}
Y & X \arrow[l, bend right, "f"'] \arrow[l, bend left, "f'"] \arrow[l, phantom, "\Downarrow H"]
\end{tikzcd}
$$

Let $-H : X \times I \to Y$ be defined by $(-H)(x, t) = H(x, 1 - t)$. Then $-H : f' \Rightarrow f : X \to Y$. As is easily seen, we have

$$\widetilde{E}^n(-H) = -\widetilde{E}^nH, \quad \widetilde{E}^n 1_f = 1_{E^nf}, \quad -1_f = 1_f. \tag{2.1}$$

Given two homotopies $H, H' : f \Rightarrow f' : X \to Y$, we write $H \simeq H'$, $H \simeq H' : f \Rightarrow f' : X \to Y$ or $F : H \simeq H'$ if there is a map $F : X \times I \times I \to Y$ such that

$$F(x,s,0) = H(x,s), \quad F(x,s,1) = H'(x,s), \quad F(*,s,t) = *,$$
$$F(x,0,t) = f(x), \quad F(x,1,t) = f'(x).$$

Then $\simeq$ is an equivalence relation on the set of homotopies from f to f'. We denote by $[H]$ the homotopy class of H. If $H \simeq H' : f \Rightarrow f'$, then $\widetilde{E}^n H \simeq \widetilde{E}^n H' : E^n f \Rightarrow E^n f'$ and $-H \simeq -H' : f' \Rightarrow f$.

Let $\mathrm{map}_*(EX, Y)$ denote the set of maps from EX to Y which preserve base points. We make the convention: *we always identify* $\mathrm{map}_*(EX, Y)$ *with* $\{H \mid H : * \Rightarrow * : X \to Y\}$ *by the bijection*

$$\mathrm{map}_*(EX,Y) \to \{H \mid H : * \Rightarrow * : X \to Y\}, \ f \mapsto f \circ q,$$

where $q : X \times I \to X \wedge (I/\{0,1\}) = EX$ is the quotient map. As is easily seen, it induces the identification $[EX, Y] = \{[H] \mid H : * \Rightarrow * : X \to Y\}$.

A **horizontal composition**

$$(K : g \Rightarrow g' : Y \to Z) \bar{\circ} (H : f \Rightarrow f' : X \to Y)$$

is defined to be $K \bar{\circ} H : g \circ f \Rightarrow g' \circ f' : X \to Z$, where

$$(K \bar{\circ} H)(x,t) = K(H(x,t),t).$$

It is the composite in the display.

$$\begin{tikzcd} Z & Y \arrow[l, bend right, "g"'] \arrow[l, bend left, "g'"] \arrow[l, phantom, "\Downarrow K"] & X \arrow[l, bend right, "f"'] \arrow[l, bend left, "f'"] \arrow[l, phantom, "\Downarrow H"] \end{tikzcd}$$

As is easily seen, we have

$$-(K \bar{\circ} H) = (-K) \bar{\circ} (-H), \quad \widetilde{E}^n(K \bar{\circ} H) = \widetilde{E}^n K \bar{\circ} \widetilde{E}^n H. \tag{2.2}$$

Lemma 2.1. (1) *The horizontal composition is associative.*

(2) *If* $g : Y \to Z$ *and* $f : X \to Y$, *then* $1_g \bar{\circ} 1_f = 1_{g \circ f}$.

(3) *If* $D : a \Rightarrow b : Y \to Z$ *and* $B : f \Rightarrow * : X \to Y$, *then*

$$D \bar{\circ} B \simeq 1_a \bar{\circ} B : a \circ f \Rightarrow * : X \to Z.$$

(4) *If* $B : * \Rightarrow b : Y \to Z$ *and* $D : f \Rightarrow g : X \to Y$, *then*

$$B \bar{\circ} D \simeq B \bar{\circ} 1_g : * \Rightarrow b \circ g : X \to Z.$$

(5) *If* $K \simeq K' : g \Rightarrow g' : Y \to Z$ *and* $H \simeq H' : f \Rightarrow f' : X \to Y$, *then*

$$K \bar{\circ} H \simeq K' \bar{\circ} H' : g \circ f \Rightarrow g' \circ f' : X \to Z.$$

Proof. (3) We define $u : I \times I \to I$ and $F : X \times I \times I \to Z$ by

$$u(s,t) = \begin{cases} 0 & s \le t \\ s-t & s \ge t \end{cases}, \quad F(x,s,t) = D(B(x,s), u(s,t)).$$

Then $F : D \,\bar{\circ}\, B \simeq 1_a \,\bar{\circ}\, B$.

(4) We define $u : I \times I \to I$ and $F : X \times I \times I \to Z$ by

$$u(s,t) = \begin{cases} s+t & s+t \le 1 \\ 1 & s+t \ge 1 \end{cases}, \quad F(x,s,t) = B(D(x, u(s,t)), s).$$

Then $F : B \,\bar{\circ}\, D \simeq B \,\bar{\circ}\, 1_g$.

Other assertions can be proved easily. □

A **vertical composition**

$$(K : f' \Rightarrow f'' : X \to Y) \bullet (H : f \Rightarrow f' : X \to Y)$$

is defined to be $K \bullet H : f \Rightarrow f'' : X \to Y$, where

$$(K \bullet H)(x,t) = \begin{cases} H(x, 2t) & 0 \le t \le \frac{1}{2} \\ K(x, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

It is the composite in the display.

$$\begin{tikzcd} Y & X \arrow[l, bend right=60, "f"'] \arrow[l, "f'" description] \arrow[l, bend left=60, "f''"] \end{tikzcd}$$

$$H: f \Rightarrow f', \quad K: f' \Rightarrow f''$$

While the vertical composition is not associative, it is associative up to homotopy as will be seen in Proposition 2.3(1) below.

Suppose that the following homotopies are given.

$$H_i : f_i \Rightarrow f_{i+1} : X \to Y \ (1 \le i \le k) \tag{2.3}$$

We define $H_k \bullet \cdots \bullet H_2 \bullet H_1 : f_1 \Rightarrow f_{k+1} : X \to Y$ by

$$(H_k \bullet \cdots \bullet H_2 \bullet H_1)(x,t) = \begin{cases} H_1(x, kt) & 0 \le t \le \frac{1}{k} \\ H_2(x, kt-1) & \frac{1}{k} \le t \le \frac{2}{k} \\ \quad \vdots & \quad \vdots \\ H_k(x, kt-(k-1)) & \frac{k-1}{k} \le t \le 1 \end{cases}.$$

As is easily seen, we have

$$-(H_k \bullet \cdots \bullet H_2 \bullet H_1) = (-H_1) \bullet (-H_2) \bullet \cdots \bullet (-H_k). \tag{2.4}$$

Lemma 2.2. (1) *If $k \geq 3$ and $1 \leq i \leq k-1$, then*

$$H_k \bullet \cdots \bullet H_2 \bullet H_1 \simeq H_k \bullet \cdots \bullet (H_{i+1} \bullet H_i) \bullet \cdots \bullet H_1.$$

(2) *If $H_i \simeq K_i : f_i \Rightarrow f_{i+1}$ $(1 \leq i \leq k)$, then $H_k \bullet \cdots \bullet H_1 \simeq K_k \bullet \cdots \bullet K_1$.*

Proof. (1) For every ℓ and i with $0 \leq \ell \leq k$ and $1 \leq i < k$, we define points of $I \times I \times I$ as follows:

$$P_\ell = (\frac{\ell}{k}, 0, 0), \quad P'_\ell = \begin{cases} P_\ell + (0,0,1) & \ell \equiv 1(2) \\ P_\ell & \ell \equiv 0(2) \end{cases},$$

$$Q_\ell(i) = \begin{cases} (\frac{\ell}{k-1}, 1, 0) & 0 \leq \ell \leq i-1 \\ (\frac{2i-1}{2k-2}, 1, 0) & \ell = i \\ (\frac{\ell-1}{k-1}, 1, 0) & i+1 \leq \ell \leq k \end{cases},$$

$$Q'_\ell(i) = \begin{cases} Q_\ell(i) + (0,0,1) & \ell \equiv 1(2) \\ Q_\ell(i) & \ell \equiv 0(2) \end{cases}.$$

Then

$$I \times I = I \times I \times \{0\} = \bigcup_{\ell=1}^{k} \{\triangle P_{\ell-1} Q_{\ell-1}(i) Q_\ell(i) \cup \triangle P_{\ell-1} Q_\ell(i) P_\ell\}.$$

Let $u_i : I \times I \to I$ be the map whose graph $\{(s,t,u_i(s,t)) \mid (s,t) \in I \times I\}$ is

$$\bigcup_{\ell=1}^{k} \{\triangle P'_{\ell-1} Q'_{\ell-1}(i) Q'_\ell(i) \cup \triangle P'_{\ell-1} Q'_\ell(i) P'_\ell\}.$$

Let $\Phi_i : X \times I \times I \to Y$ be the map which transfer (x,s,t) to

$$\begin{cases} H_{2\ell-1}(x, u_i(s,t)) & (s,t) \in \triangle P_{2\ell-2} Q_{2\ell-2}(i) Q_{2\ell-1}(i) \cup \triangle P_{2\ell-2} Q_{2\ell-1}(i) P_{2\ell-1} \\ H_{2\ell}(x, 1-u_i(s,t)) & (s,t) \in \triangle P_{2\ell-1} Q_{2\ell-1}(i) Q_{2\ell}(i) \cup \triangle P_{2\ell-1} Q_{2\ell}(i) P_{2\ell} \end{cases}.$$

Then $\Phi_i : H_k \bullet \cdots \bullet H_1 \simeq H_k \bullet \cdots \bullet (H_{i+1} \bullet H_i) \bullet \cdots \bullet H_1$.

(2) Let $F_i : H_i \simeq K_i$ $(1 \leq i \leq k)$ and define $F : X \times I \times I \to Y$ by

$$F(x,s,t) = \begin{cases} F_1(x, ks, t) & 0 \leq s \leq \frac{1}{k} \\ F_2(x, ks-1, t) & \frac{1}{k} \leq s \leq \frac{2}{k} \\ \quad \vdots & \quad \vdots \\ F_k(x, ks-k+1, t) & \frac{k-1}{k} \leq s \leq 1 \end{cases}.$$

Then $F : H_k \bullet \cdots \bullet H_1 \simeq K_k \bullet \cdots \bullet K_1$. $\square$

For (2.3), we consider all homotopies which are obtained by parenthesizing $H_k \bullet \cdots \bullet H_1$ without changing order of $H_k, \ldots, H_1$. Denote by φ_k or $\varphi(H_k, \ldots, H_1)$ any one of such homotopies. For example, if $k = 4$, then any one of the following eleven homotopies will be denoted by φ_4 or $\varphi(H_4,\dots,H_1)$.

$$\begin{aligned}
&H_4 \bullet H_3 \bullet H_2 \bullet H_1,\ (H_4 \bullet H_3 \bullet H_2) \bullet H_1,\ H_4 \bullet (H_3 \bullet H_2 \bullet H_1),\\
&(H_4 \bullet H_3) \bullet H_2 \bullet H_1,\ H_4 \bullet (H_3 \bullet H_2) \bullet H_1,\ H_4 \bullet H_3 \bullet (H_2 \bullet H_1),\\
&((H_4 \bullet H_3) \bullet H_2) \bullet H_1,\ (H_4 \bullet (H_3 \bullet H_2)) \bullet H_1,\ H_4 \bullet ((H_3 \bullet H_2) \bullet H_1),\\
&H_4 \bullet (H_3 \bullet (H_2 \bullet H_1)),\ (H_4 \bullet H_3) \bullet (H_2 \bullet H_1).
\end{aligned}$$

Proposition 2.3. (1) *$\varphi(H_k,\dots,H_1) \simeq H_k \bullet \cdots \bullet H_1$.*

(2) *The composition of homotopy classes $[K] \bullet [H] = [K \bullet H]$ is well defined when $K \bullet H$ can be defined, and it is associative.*

(3) *Under the identification $[EX, Y] = \{[H] \mid H : * \Rightarrow * : X \to Y\}$, we have $[H] + [K] = [K] \bullet [H]$.*

Proof. (1) We prove the assertion by the mathematical induction on $k \geq 1$. The assertion holds obviously for $k = 1, 2$. Assume that, given $n \geq 2$, the assertion holds for every $k \leq n$. Consider the case $k = n+1$. It suffices to assume $\varphi_{n+1} \neq H_{n+1} \bullet \cdots \bullet H_1$. Then φ_{n+1} contains $(H_{i+j} \bullet \cdots \bullet H_i)$ for some i and j with $n \geq i \geq 1$, $n > j \geq 1$ and $n+1 \geq i+j$. Take and fix an arbitrary pair of such i, j. If $j \geq 2$, then we have

$$\begin{aligned}
&\varphi_{n+1}\\
&\simeq H_{n+1} \bullet \cdots \bullet (H_{i+j} \bullet \cdots \bullet H_i) \bullet \cdots \bullet H_1 \quad \text{(by the inductive assumption)}\\
&\simeq H_{n+1} \bullet \cdots \bullet (H_{i+j} \bullet \cdots \bullet (H_{i+1} \bullet H_i)) \bullet \cdots \bullet H_1 \quad \text{(by Lemma 2.2)}\\
&\simeq H_{n+1} \bullet \cdots \bullet (H_{i+1} \bullet H_i) \bullet \cdots \bullet H_1 \quad \text{(by the inductive assumption)}\\
&\simeq H_{n+1} \bullet \cdots \bullet H_1 \quad \text{(by Lemma 2.2(1))}.
\end{aligned}$$

If $j = 1$, then, by deleting the second and third lines from the above proof, the desired result follows. This completes the induction.

(2) This follows from (1) and Lemma 2.2(2).

(3) Since $[H] + [K] = [H + K]$ in $[EX, Y]$ and $H + K = K \bullet H$ under the identification $\mathrm{map}_*(EX, Y) = \{H \mid H : * \Rightarrow * : X \to Y\}$, we have (3) from (2). □

The next result says that every Toda bracket can be representable by $\bar{\circ}$ and $\bullet$.

Proposition 2.4. *If $(a_1, a_2, a_3; A_1, A_2)_n$ is a representative of a null triple, then we have*

$$[a_1, A_1, E^n a_2] \circ (E^n a_2, \widetilde{E}^n A_2, E^n a_3) = (1_{a_1} \,\bar{\circ}\, \widetilde{E}^n A_2) \bullet ((-A_1) \,\bar{\circ}\, 1_{E^n a_3}).$$

Proof. This follows from definitions. □

Lemma 2.5. (1) *If $H : f \Rightarrow f' : X \to Y$, then*

$$H \bullet 1_f \simeq H \simeq 1_{f'} \bullet H, \tag{2.5}$$

$$(-H) \bullet H \simeq 1_f, \quad H \bullet (-H) \simeq 1_{f'}. \tag{2.6}$$

(2) *If $H : f \Rightarrow * : X \to Y$, then $1_* \bullet H \simeq H$.*
(3) *If $H : * \Rightarrow f : X \to Y$, then $H \bullet 1_* \simeq H$.*
(4) *If $H_i : f_i \Rightarrow f_{i+1} : X \to Y$ $(1 \leq i \leq k)$, $g : Y \to Z$ and $h : W \to X$ are given, then*

$$\widetilde{E}^n(H_k \bullet \cdots \bullet H_1) = \widetilde{E}^n H_k \bullet \cdots \bullet \widetilde{E}^n H_1 : E^n f_1 \Rightarrow E^n f_{k+1},$$

$$1_g \bar{\circ} (H_k \bullet \cdots \bullet H_1) = (1_g \bar{\circ} H_k) \bullet \cdots \bullet (1_g \bar{\circ} H_1) : g \circ f_1 \Rightarrow g \circ f_{k+1},$$

$$(H_k \bullet \cdots \bullet H_1) \bar{\circ} 1_h = (H_k \bar{\circ} 1_h) \bullet \cdots \bullet (H_1 \bar{\circ} 1_h) : f_1 \circ h \Rightarrow f_{k+1} \circ h.$$

If moreover $K_i : g_i \Rightarrow g_{i+1} : Y \to Z$ $(1 \leq i \leq k)$, then

$$(K_k \bullet \cdots \bullet K_1) \bar{\circ} (H_k \bullet \cdots \bullet H_1) = (K_k \bar{\circ} H_k) \bullet \cdots \bullet (K_1 \bar{\circ} H_1)$$
$$: g_1 \circ f_1 \Rightarrow g_{k+1} \circ f_{k+1} : X \to Z.$$

Proof. To prove (2.5), we divide $I \times I = K_1 \cup K_2 \cup K_3$ as follows: for $(s,t) \in I \times I$,

$$K_1 = \{(s,t) \,|\, t \leq -2s+1\}, \quad K_2 = \{(s,t) \,|\, -2s+1 \leq t \text{ and } 2s-1 \leq t\},$$
$$K_3 = \{(s,t) \,|\, t \leq 2s-1\}.$$

Let $u : I \times I \to I$ and $F : X \times I \times I \to Y$ be defined by

$$u(s,t) = \begin{cases} 0 & (s,t) \in K_1 \\ s + \frac{t}{2} - \frac{1}{2} & (s,t) \in K_2 \\ 2s-1 & (s,t) \in K_3 \end{cases}, \quad F(x,s,t) = H(x, u(s,t)).$$

Then $F : H \bullet 1_f \simeq H$. Similarly we can show $H \simeq 1_{f'} \bullet H$. This proves (2.5) and so, for example, (2) and (3) follow.

To prove (2.6), we divide $I \times I = K_1 \cup K_2 \cup K_3$ as follows: for $(s,t) \in I \times I$,

$$K_1 = \{(s,t) \,|\, s \leq t\}, \ K_2 = \{(s,t) \,|, 2s-1 \leq t \leq s\}, \ K_3 = \{(s,t) \,|\, t \leq 2s-1\}.$$

Let $u : I \times I \to I$ and $F : X \times I \times I \to Y$ be defined by

$$u(s,t) = \begin{cases} 0 & (s,t) \in K_1 \\ 2s-2t & (s,t) \in K_2 \\ -2s+2 & (s,t) \in K_3 \end{cases}, \quad F(x,s,t) = H(x, u(s,t)).$$

Then $F : (-H) \bullet H \simeq 1_f$. Similarly we can show $H \bullet (-H) \simeq 1_{f'}$. This proves (2.6) and completes the proof of (1).

The assertion (4) follows easily from definitions. Notice that the second and third equalities in (4) follow from the last equality in (4) and the equality $1_u \bullet \cdots \bullet 1_u = 1_u$ for any map $u : U \to V$. $\square$

Lemma 2.6. *If* $(h_0, h_1, h_2; D_1, D_2) : (b_1, b_2; B) \to (b'_1, b'_2; B')$ *is a quasi-map:*

$$\begin{tikzcd} Y_0 \arrow[d, "h_0"'] & Y_1 \arrow[l, "b_1"'] \arrow[d, "h_1"'] & Y_2 \arrow[l, "b_2"'] \arrow[d, "h_2"] \\ Y'_0 & Y'_1 \arrow[l, "b'_1"] & Y'_2 \arrow[l, "b'_2"] \end{tikzcd}$$

$$D_1 : h_0 \circ b_1 \Rightarrow b'_1 \circ h_1, \quad D_2 : h_1 \circ b_2 \Rightarrow b'_2 \circ h_2,$$
$$B : b_1 \circ b_2 \Rightarrow *, \quad B' : b'_1 \circ b'_2 \Rightarrow *,$$

then

$$\underline{B' \circ Ch_2}_{(D_1, D_2)} = (B' \mathbin{\bar{\circ}} 1_{h_2}) \bullet (1_{b'_1} \mathbin{\bar{\circ}} D_2) \bullet (D_1 \mathbin{\bar{\circ}} 1_{b_2}) \\ : h_0 \circ b_1 \circ b_2 \Rightarrow *, \tag{2.7}$$

$$\overline{h_0 \circ B}^{(D_1, D_2)} = (1_{h_0} \mathbin{\bar{\circ}} B) \bullet \big((-D_1) \mathbin{\bar{\circ}} 1_{b_2}\big) \bullet \big(1_{b'_1} \mathbin{\bar{\circ}} (-D_2)\big) \\ : b'_1 \circ b'_2 \circ h_2 \Rightarrow *. \tag{2.8}$$

Proof. This follows from definitions. □

3. Proof of Proposition 1.1

We use abbreviations (1.6). We have

$$\begin{aligned}
&\big[a_1, \underline{B_1 \circ CE^{n_1} q_{E^{n_2} a_3}}_{(D_1, \widetilde{E}^{n_1} D_2)}, E^{n_1}\overline{a_2}\big] \circ \big(E^{n_1}\overline{a_2}, \widetilde{E}^{n_1} B_2, E^{n_1}\widetilde{E^{n_2} a_4}\big) \\
&= (1_{a_1} \mathbin{\bar{\circ}} \widetilde{E}^{n_1} B_2) \bullet \big((-\underline{B_1 \circ CE^{n_1} q_{E^{n_2} a_3}}_{(D_1, \widetilde{E}^{n_1} D_2)}) \mathbin{\bar{\circ}} 1_{E^{n_1}\widetilde{E^{n_2} a_4}}\big) \\
&\qquad\qquad\qquad\qquad \text{(by Proposition 2.4)} \\
&= (1_{a_1} \mathbin{\bar{\circ}} \widetilde{E}^{n_1} B_2) \bullet \Big(\big(\big((-D_1) \mathbin{\bar{\circ}} 1_{E^{n_1}\overline{a_2}}\big) \bullet \big(1_{\overline{a_1}'} \mathbin{\bar{\circ}} (-\widetilde{E}^{n_1} D_2)\big) \\
&\qquad \bullet \big((-B_1) \mathbin{\bar{\circ}} 1_{E^{n_1} q_{E^{n_2} a_3}}\big)\big) \mathbin{\bar{\circ}} 1_{E^{n_1}\widetilde{E^{n_2} a_4})}\Big) \quad \text{(by (2.1), (2.2), (2.4), (2.7))} \\
&= (1_{a_1} \mathbin{\bar{\circ}} \widetilde{E}^{n_1} B_2) \bullet \Big(\big((-D_1) \mathbin{\bar{\circ}} 1_{E^{n_1}(\overline{a_2} \circ \widetilde{E^{n_2} a_4})}\big) \\
&\qquad \bullet \big(1_{\overline{a_1}'} \mathbin{\bar{\circ}} (-\widetilde{E}^{n_1} D_2) \mathbin{\bar{\circ}} 1_{E^{n_1}\widetilde{E^{n_2} a_4}}\big) \bullet \big((-B_1) \mathbin{\bar{\circ}} 1_{E^{n_1}(q_{E^{n_2} a_3} \circ \widetilde{E^{n_2} a_4})}\big)\Big) \\
&\qquad\qquad\qquad\qquad \text{(by Lemma 2.1(2) and Lemma 2.5(4))} \\
&\simeq \Big((1_{a_1} \mathbin{\bar{\circ}} \widetilde{E}^{n_1} B_2) \bullet \big((-D_1) \mathbin{\bar{\circ}} 1_{E^{n_1}(\overline{a_2} \circ \widetilde{E^{n_2} a_4})}\big)\Big) \\
&\qquad \bullet \big(1_{\overline{a_1}'} \mathbin{\bar{\circ}} (-\widetilde{E}^{n_1} D_2) \mathbin{\bar{\circ}} 1_{E^{n_1}\widetilde{E^{n_2} a_4}}\big) \bullet \big((-B_1) \mathbin{\bar{\circ}} 1_{E^{n_1}(q_{E^{n_2} a_3} \circ \widetilde{E^{n_2} a_4})}\big) \\
&\qquad\qquad\qquad\qquad \text{(by Proposition 2.3(1))}
\end{aligned}$$

and

$$\begin{aligned}
&(1_{a_1}\,\bar{\circ}\,\widetilde{E}^{n_1}B_2)\bullet\big((-D_1)\,\bar{\circ}\,1_{E^{n_1}(\overline{a_2}\circ\widetilde{E^{n_2}a_4})}\big)\\
&\simeq(D_1\,\bar{\circ}\,\widetilde{E}^{n_1}B_2)\bullet((-D_1)\,\bar{\circ}\,1_{E^{n_1}(\overline{a_2}\circ\widetilde{E^{n_2}a_4})})\quad\text{(by Lemma 2.1(3))}\\
&=(D_1\bullet(-D_1))\,\bar{\circ}\,(\widetilde{E}^{n_1}B_2\bullet 1_{E^{n_1}(\overline{a_2}\circ\widetilde{E^{n_2}a_4})})\quad\text{(by Lemma 2.5(4))}\\
&\simeq 1_{a_1}\,\bar{\circ}\,(\widetilde{E}^{n_1}B_2\bullet 1_{E^{n_1}(\overline{a_2}\circ\widetilde{E^{n_2}a_4})})\quad\text{(by (2.6))}\\
&=(1_{a_1}\,\bar{\circ}\,\widetilde{E}^{n_1}B_2)\bullet 1_{a_1\circ E^{n_1}(\overline{a_2}\circ\widetilde{E^{n_2}a_4})}\quad\text{(by Lemma 2.5(4))}\\
&\simeq 1_{a_1}\,\bar{\circ}\,\widetilde{E}^{n_1}B_2\quad\text{(by (2.5))}.
\end{aligned}$$

Hence (1.3) is homotopic to (1.5).

On the other hand, we have

$$\begin{aligned}
&[\overline{a_1}', B_1, E^{n_1}\widetilde{E^{n_2}a_3}]\circ(E^{n_1}\widetilde{E^{n_2}a_3}, \widetilde{E}^{n_1}\overline{i_{a_2}\circ B_2}^{(D_2,D_3)}, E^{n_1}(-E^{n_2+1}a_4))\\
&=(1_{\overline{a_1}'}\,\bar{\circ}\,\widetilde{E}^{n_1}\overline{i_{a_2}\circ B_2}^{(D_2,D_3)})\bullet((-B_1)\,\bar{\circ}\,1_{E^{n_1}(-E^{n_2+1}a_4)})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Proposition 2.4)}\\
&=\Big((1_{a_1}\,\bar{\circ}\,\widetilde{E}^{n_1}B_2)\bullet(1_{\overline{a_1}'}\,\bar{\circ}\,(-\widetilde{E}^{n_1}D_2)\,\bar{\circ}\,1_{E^{n_1}\widetilde{E^{n_2}a_4}})\\
&\qquad\bullet(1_{\overline{a_1}'\circ E^{n_1}\widetilde{E^{n_2}a_3}}\,\bar{\circ}\,(-\widetilde{E}^{n_1}D_3))\Big)\bullet((-B_1)\,\bar{\circ}\,1_{E^{n_1}(-E^{n_2+1}a_4)})\\
&\qquad\qquad\qquad\text{(by (2.1), (2.2), (2.8), Lemma 2.1(2) and Lemma 2.5(4))}\\
&\simeq(1_{a_1}\,\bar{\circ}\,\widetilde{E}^{n_1}B_2)\bullet(1_{\overline{a_1}'}\,\bar{\circ}\,(-\widetilde{E}^{n_1}D_2)\,\bar{\circ}\,1_{E^{n_1}\widetilde{E^{n_2}a_4}})\\
&\qquad\bullet\Big((1_{\overline{a_1}'\circ E^{n_1}\widetilde{E^{n_2}a_3}}\,\bar{\circ}\,(-\widetilde{E}^{n_1}D_3))\bullet((-B_1)\,\bar{\circ}\,1_{E^{n_1}(-E^{n_2+1}a_4)})\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Proposition 2.3(1))}
\end{aligned}$$

and

$$\begin{aligned}
&(1_{\overline{a_1}'\circ E^{n_1}\widetilde{E^{n_2}a_3}}\,\bar{\circ}\,(-\widetilde{E}^{n_1}D_3))\bullet((-B_1)\,\bar{\circ}\,1_{E^{n_1}(-E^{n_2+1}a_4)})\\
&\quad=(1_{\overline{a_1}'\circ E^{n_1}\widetilde{E^{n_2}a_3}}\bullet(-B_1))\,\bar{\circ}\,((-\widetilde{E}^{n_1}D_3)\bullet 1_{E^{n_1}(-E^{n_2+1}a_4)})\\
&\qquad\qquad\qquad\qquad\qquad\text{(by Lemma 2.5(4))}\\
&\quad\simeq(-B_1)\,\bar{\circ}\,(-\widetilde{E}^{n_1}D_3)\quad\text{(by (2.5))}\\
&\quad\simeq(-B_1)\,\bar{\circ}\,1_{E^{n_1}(q_{E^{n_2}a_3}\circ\widetilde{E^{n_2}a_4})}\quad\text{(by Lemma 2.1(4))}.
\end{aligned}$$

Hence (1.4) is homotopic to (1.5). This completes the proof of Proposition 1.1. □

4. 2-sided matrix Toda brackets

We modify the 2-sided matrix Toda bracket of [**1**, Definition 4.2]. Suppose that a non-negative integer n and six maps s, a, b, f, g, w are given as follows

$$(4.1)\qquad Y \xleftarrow{s} E^nX, \qquad \begin{tikzcd} & B \arrow[dl, "b"'] & \\ X & & C \arrow[ul, "g"'] \arrow[dl, "f"] & W \arrow[l, "w"'] \\ & A \arrow[ul, "a"] & \end{tikzcd}$$

such that $s \circ E^n a \simeq *$, $g \circ w \simeq *$ and $b \circ g \simeq a \circ f$. Given three homotopies $D_2 : b \circ g \simeq a \circ f$, $B_1 : s \circ E^n a \simeq *$ and $B_2 : g \circ w \simeq *$, we denote by

$$\left\{ \begin{matrix} s & \begin{matrix} b \\ a \end{matrix} & \begin{matrix} g \\ f \end{matrix} & w \end{matrix} \ ; D_2, B_1, B_2 \right\}_n \qquad \big(\in [E^{n+1}W, Y] \big)$$

the homotopy class of

$$(4.2)\qquad (1_{s\circ E^n b} \,\bar{\circ}\, \widetilde{E}^n B_2) \bullet (1_s \,\bar{\circ}\, (-\widetilde{E}^n D_2) \,\bar{\circ}\, 1_{E^n w}) \bullet ((-B_1) \,\bar{\circ}\, 1_{E^n(f\circ w)}).$$

We define **2-sided matrix Toda brackets** by

$$(4.3)\qquad \begin{aligned} &\left\{ \begin{matrix} s & \begin{matrix} b \\ a \end{matrix} & \begin{matrix} g \\ f \end{matrix} & w \end{matrix} \ ; D_2 \right\}_n \\ &\qquad = \bigcup_{B_1, B_2} \left\{ \begin{matrix} s & \begin{matrix} b \\ a \end{matrix} & \begin{matrix} g \\ f \end{matrix} & w \end{matrix} \ ; D_2, B_1, B_2 \right\}_n, \end{aligned}$$

$$(4.4)\qquad \left\{ \begin{matrix} s & \begin{matrix} b \\ a \end{matrix} & \begin{matrix} g \\ f \end{matrix} & w \end{matrix} \right\}_n = \bigcup_{D_2} \left\{ \begin{matrix} s & \begin{matrix} b \\ a \end{matrix} & \begin{matrix} g \\ f \end{matrix} & w \end{matrix} \ ; D_2 \right\}_n.$$

We have

$$(B_1 \,\bar{\circ}\, 1_{E^n f}) \bullet (1_s \,\bar{\circ}\, \widetilde{E}^n D_2) : s \circ E^n(b \circ g) \Rightarrow * : E^n C \to Y,$$
$$(1_b \,\bar{\circ}\, B_2) \bullet \big((-D_2) \,\bar{\circ}\, 1_w\big) : a \circ f \circ w \Rightarrow * : W \to X$$

and then

$$\begin{aligned} &[s \circ E^n b, (B_1 \,\bar{\circ}\, 1_{E^n f}) \bullet (1_s \,\bar{\circ}\, \widetilde{E}^n D_2), E^n g] \circ (E^n g, \widetilde{E}^n B_2, E^n w) \\ &\quad = (1_{s\circ E^n b} \,\bar{\circ}\, \widetilde{E}^n B_2) \bullet \Big(\big((-((B_1 \,\bar{\circ}\, 1_{E^n f}) \bullet (1_s \,\bar{\circ}\, \widetilde{E}^n D_2)))\big) \,\bar{\circ}\, 1_{E^n w}\Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Proposition 2.4)} \\ &\quad = (1_{s\circ E^n b} \,\bar{\circ}\, \widetilde{E}^n B_2) \bullet \Big(\big((1_s \,\bar{\circ}\, (-\widetilde{E}^n D_2)) \bullet ((-B_1) \,\bar{\circ}\, 1_{E^n f})\big) \,\bar{\circ}\, 1_{E^n w}\Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (2.1), (2.2) and (2.4))} \end{aligned}$$

$$
\begin{aligned}
&= (1_{s\circ E^n b}\,\bar{\circ}\,\widetilde{E}^n B_2)\bullet\Big((1_s\,\bar{\circ}\,(-\widetilde{E}^n D_2)\,\bar{\circ}\,1_{E^n w})\bullet((-B_1)\,\bar{\circ}\,1_{E^n f}\,\bar{\circ}\,1_{E^n w})\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Lemma 2.5(4))}\\
&\simeq (4.2)\quad\text{(by Proposition 2.3(1))},
\end{aligned}
$$

$$
\begin{aligned}
&[s, B_1, E^n a]\circ\big(E^n a, \widetilde{E}^n((1_b\,\bar{\circ}\,B_2)\bullet((-D_2)\,\bar{\circ}\,1_w)), E^n(f\circ w)\big)\\
&= \Big(1_s\,\bar{\circ}\,\widetilde{E}^n\big((1_b\,\bar{\circ}\,B_2)\bullet((-D_2)\,\bar{\circ}\,1_w)\big)\Big)\bullet\big((-B_1)\,\bar{\circ}\,1_{E^n(f\circ w)}\big)\\
&= \Big((1_s\,\bar{\circ}\,1_{E^n b}\,\bar{\circ}\,\widetilde{E}^n B_2)\bullet\big(1_s\,\bar{\circ}\,(-\widetilde{E}^n D_2)\,\bar{\circ}\,1_{E^n w}\big)\Big)\bullet((-B_1)\,\bar{\circ}\,1_{E^n(f\circ w)})\\
&\simeq (4.2)\quad\text{(by Proposition 2.3(1))}.
\end{aligned}
$$

Hence we obtain a generalization of Remarks 4.4(1) of [**1**]

$$
\begin{aligned}
\left\{\begin{matrix} s & \begin{matrix} b \\ a\end{matrix} & \begin{matrix} g \\ f\end{matrix} & w\end{matrix}\right\}_n &\subset \{s\circ E^n b, g, w\}_n \cap \{s, a, f\circ w\}_n\\
&\subset \{s, b\circ g, w\}_n = \{s, a\circ f, w\}_n \subset [E^{n+1}W, Y].
\end{aligned}
$$

Proof of Corollary 1.3. We take the following diagrams as (4.1).

$$
X_0 \xleftarrow{\overline{a_1}'} E^{n_1}(X_1\cup_{a_2} CE^{n_2}X_2)\ ,
$$

$$
\begin{tikzcd}
& X_1 \arrow[dl, "i_{a_2}"'] & & \\
X_1\cup_{a_2} CE^{n_2}X_2 & D_2 & E^{n_2}X_2\cup_{E^{n_2}a_3} CE^{n_2}X_3 \arrow[ul, "\overline{a_2}"'] \arrow[dl, "q_{E^{n_2}a_3}"] & E^{n_2+1}X_4 \arrow[l, "\widetilde{E^{n_2}a_4}"'] \ , \\
& E^{n_2+1}X_3 \arrow[ul, "\widetilde{E^{n_2}a_3}"] & &
\end{tikzcd}
$$

where we have used the abbreviations (1.6) and $D_2 : i_{a_2}\circ\overline{a_2}\simeq \widetilde{E^{n_2}a_3}\circ q_{E^{n_2}a_3}$. Then Corollary 1.3 follows from (4.2) and Proposition 1.1. □

5. Indeterminacy of quasi tertiary composition

The following proposition which modifies Proposition 4.8 of [**1**] for the 2-category Top_* says that the 2-sided matrix Toda brackets (4.3) and (4.4) have indeterminacies in a sense.

Proposition 5.1 (Proposition 4.8 of [**1**]). *Let*

$$
\theta = (1_{s\circ E^n b}\,\bar{\circ}\,\widetilde{E}^n B_2)\bullet(1_s\,\bar{\circ}\,(-\widetilde{E}^n D_2)\,\bar{\circ}\,1_{E^n w})\bullet((-B_1)\,\bar{\circ}\,1_{E^n(f\circ w)})
$$

represent a fixed but arbitrary element of $\left\{\begin{matrix} s & \begin{matrix} b \\ a\end{matrix} & \begin{matrix} g \\ f\end{matrix} & w\ ; D_2\end{matrix}\right\}_n$. *Let* $\mathit{\Gamma}_{B_1}$ *and* $\mathit{\Gamma}_{B_2}$ *be the subgroups of* $[E^{n+1}W, Y]$ *which consist of all homotopy* *classes of*

$$(B_1 \bar{\circ} 1_{E^n(f\circ w)}) \bullet (1_s \bar{\circ} \widetilde{E}^n L \bar{\circ} 1_{E^n w}) \bullet ((-B_1) \bar{\circ} 1_{E^n(f\circ w)}),$$
$$(1_{s\circ E^n b} \bar{\circ} \widetilde{E}^n B_2) \bullet (1_s \bar{\circ} \widetilde{E}^n M \bar{\circ} 1_{E^n w}) \bullet (1_{s\circ E^n b} \bar{\circ} (-\widetilde{E}^n B_2)),$$

respectively, where $L : a \circ f \simeq a \circ f$ *and* $M : b \circ g \simeq b \circ g$. *Then*

$$\left\{ \begin{matrix} s & \begin{matrix} b \\ a \end{matrix} & \begin{matrix} g \\ f \end{matrix} & w & ; D_2 \end{matrix} \right\}_n \tag{5.1}$$
$$= [E^{n+1}A, Y] \circ E^{n+1}(f \circ w) + [\theta] + (s \circ E^n b) \circ E^n[EW, B],$$

$$\left\{ \begin{matrix} s & \begin{matrix} b \\ a \end{matrix} & \begin{matrix} g \\ f \end{matrix} & w \end{matrix} \right\}_n \tag{5.2}$$
$$= [E^{n+1}A, Y] \circ E^{n+1}(f \circ w) + \mathit{\Gamma}_{B_1} + [\theta] + (s \circ E^n b) \circ E^n[EW, B]$$

and $\mathit{\Gamma}_{B_1} + [\theta] = [\theta] + \mathit{\Gamma}_{B_2}$. *If the group* $[E^{n+1}W, Y]$ *is abelian, then* $\mathit{\Gamma}_{B_1} = \mathit{\Gamma}_{B_2}$ *which do not depend on* θ, *and so* $\left\{ \begin{matrix} s & \begin{matrix} b \\ a \end{matrix} & \begin{matrix} g \\ f \end{matrix} & w & ; D_2 \end{matrix} \right\}_n$ *and* $\left\{ \begin{matrix} s & \begin{matrix} b \\ a \end{matrix} & \begin{matrix} g \\ f \end{matrix} & w \end{matrix} \right\}_n$ *have the indeterminacies*

$$[E^{n+1}A, Y] \circ E^{n+1}(f \circ w) + (s \circ E^n b) \circ E^n[EW, B],$$
$$\mathit{\Gamma} + [E^{n+1}A, Y] \circ E^{n+1}(f \circ w) + (s \circ E^n b) \circ E^n[EW, B],$$

respectively, where $\mathit{\Gamma} = \mathit{\Gamma}_{B_1} = \mathit{\Gamma}_{B_2}$.

Proof. Let $D_2' : b \circ g \Rightarrow a \circ f$, $B_1' : s \circ E^n a \Rightarrow *$ and $B_2' : g \circ w \Rightarrow *$ be arbitrary homotopies. We set

$$L = D_2' \bullet (-D_2) : a \circ f \Rightarrow a \circ f,\ M = (-D_2) \bullet D_2' : b \circ g \Rightarrow b \circ g,$$
$$J = B_2' \bullet (-B_2) : * \Rightarrow * : W \to B,\ N = B_1' \bullet (-B_1) : * \Rightarrow * : E^n A \to Y.$$

It follows from Proposition 2.3(1) and Lemma 2.5(1),(2),(3) that we have

$$L \bullet D_2 \simeq D_2' \simeq D_2 \bullet M, \quad B_1' \simeq N \bullet B_1, \quad B_2' \simeq J \bullet B_2, \tag{5.3}$$
$$M \bullet (-D_2) \simeq (-D_2) \bullet D_2' \bullet (-D_2) \simeq (-D_2) \bullet L. \tag{5.4}$$

We have

$$\begin{aligned}
&(1_{s\circ E^n b} \bar{\circ} \widetilde{E}^n B_2') \bullet (1_s \bar{\circ} (-\widetilde{E}^n D_2') \bar{\circ} 1_{E^n w}) \bullet ((-B_1') \bar{\circ} 1_{E^n(f\circ w)}) \\
&\simeq (1_{s\circ E^n b} \bar{\circ} \widetilde{E}^n (J \bullet B_2)) \bullet (1_s \bar{\circ} (-\widetilde{E}^n (L \bullet D_2)) \bar{\circ} 1_{E^n w}) \\
&\qquad \bullet ((-(N \bullet B_1)) \bar{\circ} 1_{E^n(f\circ w)}) \quad \text{(by (5.3))} \\
&\simeq (1_{s\circ E^n b} \bar{\circ} \widetilde{E}^n J) \bullet (1_{s\circ E^n b} \bar{\circ} \widetilde{E}^n B_2) \bullet (1_s \bar{\circ} (-\widetilde{E}^n D_2) \bar{\circ} 1_{E^n w}) \\
&\qquad \bullet (1_s \bar{\circ} (-\widetilde{E}^n L) \bar{\circ} 1_{E^n w}) \bullet ((-B_1) \bar{\circ} 1_{E^n(f\circ w)}) \bullet ((-N) \bar{\circ} 1_{E^n(f\circ w)}) \\
&\qquad \text{(by (2.4), Lemma 2.3(1) and Lemma 2.5(4))}
\end{aligned}$$

$$\begin{aligned}&\simeq (1_{s\circ E^n b}\,\bar\circ\,\widetilde{E}^n J)\bullet(1_{s\circ E^n b}\,\bar\circ\,\widetilde{E}^n B_2)\bullet(1_s\,\bar\circ\,(-\widetilde{E}^n D_2)\,\bar\circ\,1_{E^n w})\\&\qquad\bullet((-B_1)\,\bar\circ\,1_{E^n(f\circ w)})\bullet(B_1\,\bar\circ\,1_{E^n(f\circ w)})\bullet(1_s\,\bar\circ\,(-\widetilde{E}^n L)\,\bar\circ\,1_{E^n w})\\&\qquad\bullet((-B_1)\,\bar\circ\,1_{E^n(f\circ w)})\bullet((-N)\,\bar\circ\,1_{E^n(f\circ w)})\\&\simeq (1_{s\circ E^n b}\,\bar\circ\,\widetilde{E}^n J)\bullet\theta\bullet(B_1\,\bar\circ\,1_{E^n(f\circ w)})\bullet(1_s\,\bar\circ\,(-\widetilde{E}^n L)\,\bar\circ\,1_{E^n w})\\&\qquad\bullet((-B_1)\,\bar\circ\,1_{E^n(f\circ w)})\bullet((-N)\,\bar\circ\,1_{E^n(f\circ w)}).\end{aligned}$$

Since $1_{s\circ E^n b}\,\bar\circ\,\widetilde{E}^n J = s\circ E^n b\circ E^n J\circ(1_W\wedge\tau(\mathrm{S}^n,\mathrm{S}^1)) : EE^nW\to Y$ and since $\tau(\mathrm{S}^n,\mathrm{S}^1)$ is a self map of S^{n+1} of the degree $(-1)^n$ under the identification $\mathrm{S}^n\wedge\mathrm{S}^1=\mathrm{S}^{n+1}=\mathrm{S}^1\wedge\mathrm{S}^n$, it follows that the set of homotopy classes of $1_{s\circ E^n b}\,\bar\circ\,\widetilde{E}^n J$ is

$$\begin{aligned}&(s\circ E^n b)\circ E^n[EW,B]\circ(1_W\wedge\tau(\mathrm{S}^n,\mathrm{S}^1))\\&=(s\circ E^n b)\circ E^n[EW,B]\circ(-1)^n 1_{E^{n+1}W}=(-1)^n\big((s\circ E^n b)\circ E^n[EW,B]\big)\\&=(s\circ E^n b)\circ E^n[EW,B]\subset[E^{n+1}W,Y].\end{aligned}$$

Also $(-N)\,\bar\circ\,1_{E^n(f\circ w)}=(-N)\circ EE^n(f\circ w)$. Hence we have (5.2) and also (5.1) by taking $D_2'=D_2$. On the other hand, we have

$$\begin{aligned}&(1_{s\circ E^n b}\,\bar\circ\,\widetilde{E}^n B_2)\bullet(1_s\,\bar\circ\,\widetilde{E}^n M\,\bar\circ\,1_{E^n w})\bullet(1_{s\circ E^n b}\,\bar\circ\,(-\widetilde{E}^n B_2))\bullet\theta\\&\simeq(1_{s\circ E^n b}\,\bar\circ\,\widetilde{E}^n B_2)\bullet(1_s\,\bar\circ\,\widetilde{E}^n(M\bullet(-D_2))\,\bar\circ\,1_{E^n w})\bullet((-B_1)\,\bar\circ\,1_{E^n(f\circ w)})\\&\simeq(1_{s\circ E^n b}\,\bar\circ\,\widetilde{E}^n B_2)\bullet(1_s\,\bar\circ\,\widetilde{E}^n((-D_2)\bullet L)\,\bar\circ\,1_{E^n w})\bullet((-B_1)\,\bar\circ\,1_{E^n(f\circ w)})\\&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by (5.4))}\\&\simeq\theta\bullet(B_1\,\bar\circ\,1_{E^n(f\circ w)})\bullet(1_s\,\bar\circ\,\widetilde{E}^n L\,\bar\circ\,1_{E^n w})\bullet((-B_1)\,\bar\circ\,1_{E^n(f\circ w)}).\end{aligned}$$

Hence $[\theta]+\mathit{\Gamma}_{B_2}=\mathit{\Gamma}_{B_1}+[\theta]$.

Suppose that the group $[E^{n+1}W,Y]$ is abelian. Then $\mathit{\Gamma}_{B_2}=\mathit{\Gamma}_{B_1}$ by the above equality. We have

$$\begin{aligned}&(B_1'\,\bar\circ\,1_{E^n(f\circ w)})\bullet(1_s\,\bar\circ\,\widetilde{E}^n L\,\bar\circ\,1_{E^n w})\bullet((-B_1')\,\bar\circ\,1_{E^n(f\circ w)})\\&\simeq(N\,\bar\circ\,1_{E^n(f\circ w)})\bullet(B_1\,\bar\circ\,1_{E^n(f\circ w)})\bullet(1_s\,\bar\circ\,\widetilde{E}^n L\,\bar\circ\,1_{E^n w})\bullet((-B_1)\,\bar\circ\,1_{E^n(f\circ w)})\\&\qquad\bullet((-N)\,\bar\circ\,1_{E^n(f\circ w)})\qquad\qquad\qquad\text{(by (5.3))}\\&\simeq(B_1\,\bar\circ\,1_{E^n(f\circ w)})\bullet(1_s\,\bar\circ\,\widetilde{E}^n L\,\bar\circ\,1_{E^n w})\bullet((-B_1)\,\bar\circ\,1_{E^n(f\circ w)})\\&\qquad\qquad\qquad\qquad\text{(since }[E^{n+1}W,Y]\text{ is abelian)}\end{aligned}$$

and so $\mathit{\Gamma}_{B_1'}=\mathit{\Gamma}_{B_1}$. Similarly $\mathit{\Gamma}_{B_2'}=\mathit{\Gamma}_{B_2}$. Hence, if we set $\mathit{\Gamma}=\mathit{\Gamma}_{B_1}$, then $\mathit{\Gamma}$ does not depend on θ and (4.4) has the desired indeterminacy. This completes the proof. □

Proof of Corollary 1.4. Let θ be a fixed but arbitrary element of the form: $(1_{a_1}\,\bar{\circ}\,\widetilde{E}^{n_1}B_2)\bullet(1_{\overline{a_1}'}\,\bar{\circ}\,(-\widetilde{E}^{n_1}D_2)\,\bar{\circ}\,1_{E^{n_1}\widetilde{E^{n_2}a_4}})\bullet((-B_1)\,\bar{\circ}\,1_{E^{n_1}(\widetilde{q_{E^{n_2}a_3}\circ E^{n_2+1}a_4})})$.

Let Γ_{B_1} and Γ_{B_2} be the subgroups of $[E^{n_1+n_2+2}X_4, X_0]$ which consist of all homotopy classes of

$$\begin{aligned}&(B_1\,\bar{\circ}\,1_{E^{n_1}(-E^{n_2+1}a_4)})\bullet(1_{\overline{a_1}'}\,\bar{\circ}\,\widetilde{E}^{n_1}L\,\bar{\circ}\,1_{E^{n_1}\widetilde{E^{n_2}a_4}})\\ &\qquad\qquad\qquad\qquad\bullet((-B_1)\,\bar{\circ}\,1_{E^{n_1}(\widetilde{q_{E^{n_2}a_3}\circ E^{n_2+1}a_4})}),\\ &(1_{\overline{a_1}'\circ E^{n_1}i_{a_2}}\,\bar{\circ}\,\widetilde{E}^{n_1}B_2)\bullet(1_{\overline{a_1}'}\,\bar{\circ}\,\widetilde{E}^{n_1}M\,\bar{\circ}\,1_{E^{n_1}\widetilde{E^{n_2}a_4}})\\ &\qquad\qquad\qquad\qquad\bullet(1_{\overline{a_1}'\circ E^{n_1}i_{a_2}}\,\bar{\circ}\,(-\widetilde{E}^{n_1}B_2)),\end{aligned}$$

respectively, where $L:\widetilde{E^{n_2}a_3}\circ q_{E^{n_2}a_3}\simeq\widetilde{E^{n_2}a_3}\circ q_{E^{n_2}a_3}$ and $M: i_{a_2}\circ\overline{a_2}\simeq i_{a_2}\circ\overline{a_2}$. Since the group $[E^{n_1+n_2+2}X_4, X_0]$ is abelian, it follows from Proposition 5.1 and Corollary 1.3 that $\{a_1,a_2,a_3,a_4;A_1,A_2,A_3;D_2\}^{(1)}_{n_1,n_2}$ and $\{a_1,a_2,a_3,a_4;A_1,A_2,A_3\}^{(1)}_{n_1,n_2}$ have the indeterminacies

$$[E^{n_1+n_2+2}X_3,X_0]\circ E^{n_1+n_2+2}a_4+a_1\circ E^{n_1}[E^{n_2+2}X_4,X_1],$$
$$\Gamma+[E^{n_1+n_2+2}X_3,X_0]\circ E^{n_1+n_2+2}a_4+a_1\circ E^{n_1}[E^{n_2+2}X_4,X_1],$$

respectively, where $\Gamma=\Gamma_{B_1}=\Gamma_{B_2}$ which does not depend on θ and choices of B_1 and B_2. This completes the proof. □

References

[**1**] K.A. Hardie, K.H. Kamps, H.J. Marcum and N. Oda, Triple brackets and lax morphism categories, Applied categorical structures **12** (2004), 3–27.

[**2**] S. Mac Lane, Categories for the working mathematician, Second edition, GTM 5, Springer 1998.

[**3**] H. Ōshima and K. Ōshima, Quasi tertiary compositions and a Toda bracket in homotopy groups of SU(3), Math. J. Okayama Univ., **57** (2015), 13-78.

H. Ōshima
Emeritus Professor
Ibaraki University, Mito, Ibaraki 310-8512, Japan
e-mail address: ooshima@mx.ibaraki.ac.jp

K. Ōshima
1-4001-5 Ishikawa, Mito, Ibaraki 310-0905, Japan
e-mail address: k-oshima@mbr.nifty.com

(*Received June 5, 2013*)
(*Accepted October 25, 2013*)