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ZERO MEAN CURVATURE SURFACES IN LORENTZ-MINKOWSKI 3-SPACE AND 2-DIMENSIONAL FLUID MECHANICS

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Abstract. Space-like maximal surfaces and time-like minimal surfaces in Lorentz-Minkowski 3-space $\boldsymbol{R}^3_1$ are both characterized as zero mean curvature surfaces. We are interested in the case where the zero mean curvature surface changes type from space-like to time-like at a given non-degenerate null curve. We consider this phenomenon and its interesting connection to 2-dimensional fluid mechanics in this expository article.

1. Introduction

We denote by $\boldsymbol{R}^3_1 := \{(t,x,y)\,;\, t,x,y \in \boldsymbol{R}\}$ the Lorentz-Minkowski 3-space with the metric $\langle\ ,\ \rangle$ of signature $(-,+,+)$. Space-like maximal surfaces and time-like minimal surfaces in Lorentz-Minkowski 3-space $\boldsymbol{R}^3_1$ are both characterized as zero mean curvature surfaces. This is an expository article about type changes of zero mean curvature surfaces in $\boldsymbol{R}^3_1$. Klyachin [17] showed, under a sufficiently weak regularity assumption, that a zero mean curvature surface in $\boldsymbol{R}^3_1$ changes its causal type only on the following two subsets:

- null curves (i.e., regular curves whose velocity vector fields are light-like) which are non-degenerate (cf. Definition 2.1), or
- light-like lines, which are degenerate everywhere.

Recently, actual occurrence of the second case was shown in the authors' work [5]. So we now pay attention to the former possibilities: Given a non-degenerate null curve γ in $\boldsymbol{R}^3_1$, there exists a zero mean curvature surface which changes its causal type across this curve from a space-like maximal surface to a time-like minimal surface (cf. [10, 11, 12], [17], [16] and [14]). This construction can be accomplished using the Björling formula for the

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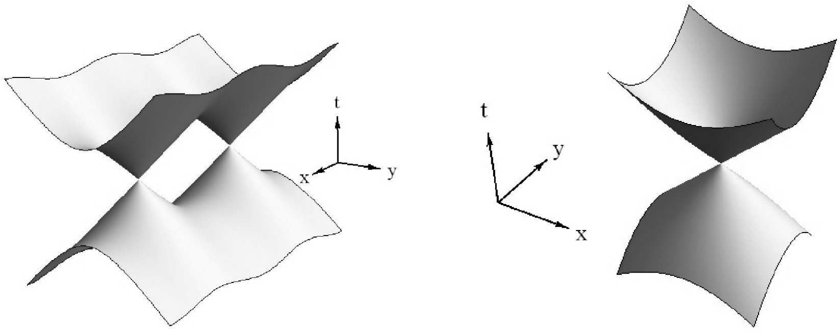

FIGURE 1. Hyperbolic catenoids $\mathcal{C}_+$, $\mathcal{C}_-$.

Weierstrass-type representation formula of maximal surfaces. By unifying the results of Gu [10], Klyachin [17], and [14], we explain the mechanism for how zero mean curvature surfaces change type across non-degenerate null curves, and give 'the fundamental theorem of type change for zero mean curvature surfaces' (cf. Theorem 2.19) in the second section of this paper. Locally, such a surface is the graph of a function $t = f(x, y)$ satisfying

$$(1 - f_y^2)f_{xx} + 2f_xf_yf_{xy} + (1 - f_x^2)f_{yy} = 0. \tag{1.1}$$

We call this and its graph the *zero mean curvature equation* and a *zero mean curvature surface* or a *zero mean curvature graph*, respectively.

As pointed out in [4], the space-like hyperbolic catenoid

$$\mathcal{C}_+ = \{(t, x, y) \in \boldsymbol{R}^3_1 \,;\, \sin^2 x + y^2 - t^2 = 0\} \tag{1.2}$$

and the time-like hyperbolic catenoid

$$\mathcal{C}_- = \{(t, x, y) \in \boldsymbol{R}^3_1 \,;\, \sinh^2 x + y^2 - t^2 = 0\} \tag{1.3}$$

are both typical examples of zero mean curvature surfaces containing singular light-like lines as subsets (cf. Figure 1). The space-like hyperbolic catenoid $\mathcal{C}_+$ is singly periodic.

Also, both the space-like Scherk surface (cf. [4])

$$\mathcal{S}_+ = \{(t, x, y) \in \boldsymbol{R}^3_1 \,;\, \cos t = \cos x \cos y\} \tag{1.4}$$

and the time-like Scherk surface of the first kind (cf. [4])

$$\mathcal{S}_- = \{(t, x, y) \in \boldsymbol{R}^3_1 \,;\, \cosh t = \cosh x \cosh y\} \tag{1.5}$$

contain singular light-like lines as subsets (cf. Figure 2). As seen in the left-hand side of Figure 2, $\mathcal{S}_+$ is triply periodic.

As an application of the results in Section 2, we show in Section 3 that $\mathcal{C}_+$ and $\mathcal{C}_-$ induce a common zero mean curvature graph (cf. Figure 3, left)

$$\mathcal{C}_0 = \{(t, x, y) \in \boldsymbol{R}^3_1 \,;\, t = y \tanh x\} \tag{1.6}$$

via their conjugate surfaces. The graph $\mathcal{C}_0$ changes type at two non-degenerate null curves. Similarly, we also show that the Scherk-type surfaces $\mathcal{S}_+$ and $\mathcal{S}_-$ induce a common zero mean curvature graph via their conjugate surfaces (cf. Figure 3, right)

$$\mathcal{S}_0 = \{(t,x,y) \in \boldsymbol{R}^3_1\,;\, e^t \cosh x = \cosh y\}, \tag{1.7}$$

which changes type at four non-degenerate null curves. These two phenomena were briefly commented upon in [4]. The entire zero-mean curvature graphs $\mathcal{C}_0$ and $\mathcal{S}_0$ were discovered by Osamu Kobayashi [18]. On the other hand, the above three examples (1.4), (1.5) and (1.7) are particular cases of the general families presented in Sergienko and Tkachev [20, Theorem 2]. Moreover, several doubly periodic mixed type zero mean curvature graphs with isolated singularities are given in [20]. Space-like maximal surfaces frequently have singularities. See the references [2], [21] and [9] for general treatment of these singularities.

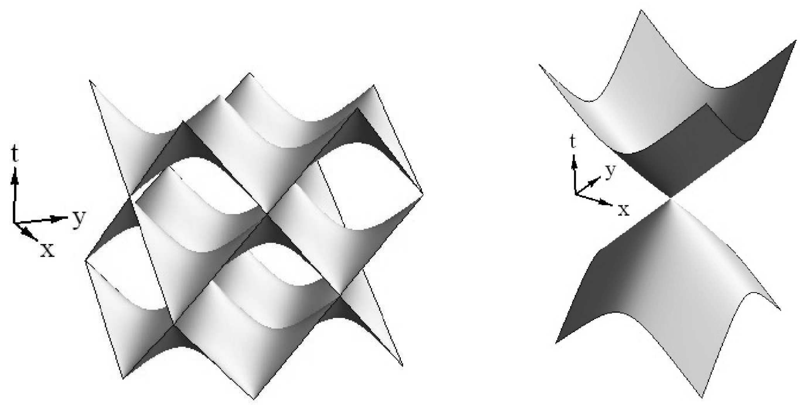

FIGURE 2. Scherk-type surfaces $\mathcal{S}_+$ and $\mathcal{S}_-$.

In Section 4, we remark on an interesting connection between zero mean curvature surfaces in $\boldsymbol{R}^3_1$ and irrotational two-dimensional barotropic steady flows, where the fluid is called *barotropic* if the pressure p is a function depending only on the density ρ. In fact, the stream function $\psi(x,y)$ satisfies (cf. [19], see also Proposition 4.1 in Section 4)

$$(\rho^2c^2 - \psi_y^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} + (\rho^2c^2 - \psi_x^2)\psi_{yy} = 0, \tag{1.8}$$

where c is the local speed of sound given by $c^2 = dp/d\rho$ (cf. (4.1)). We choose the units so that $\rho = 1$ and $c = 1$ when $\psi_x = \psi_y = 0$. Then the product ρc is equal to 1 if

$$p = p_0 - \frac{1}{\rho}, \tag{1.9}$$

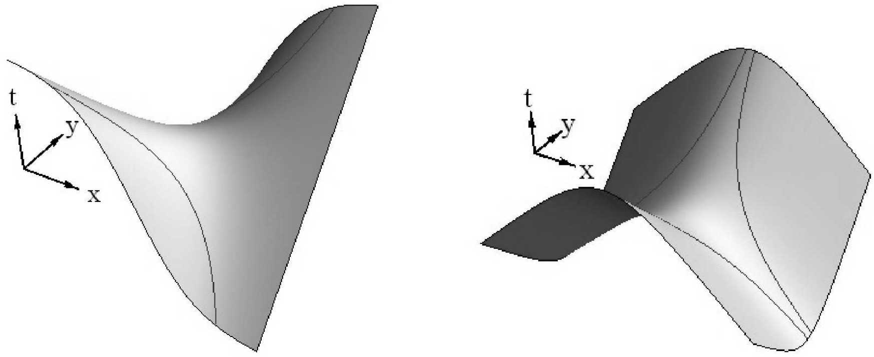

FIGURE 3. Zero mean curvature graphs $\mathcal{C}_0$ and $\mathcal{S}_0$ (the curves where the surfaces change type are also indicated).

for a constant p_0, which implies that the graphs of zero mean curvature surfaces can be interpreted as stream functions of a *virtual gas* with (1.9). (In fact, p is approximately proportional to $\rho^{1.4}$ for air.) As an application of the singular Björling formula for zero mean curvature surfaces (cf. Theorem 2.19), we can construct a family of stream functions which change from being subsonic to supersonic at a given locally convex curve in the xy-plane. The velocity vector fields of these gas flows diverge at the convex curve, although the streaming functions are real analytic.

2. Type change of zero mean curvature surfaces

In this section, we discuss type change for zero mean curvature surfaces, by unifying the results of Gu [10, 11, 12], Klyachin [17] and four of the authors here [14].

A regular curve $\gamma : (a,b) \to \boldsymbol{R}^3_1$ is called *null* or *isotropic* if $\gamma'(t) := d\gamma(t)/dt$ is a light-like vector for all $t \in (a,b)$.

Definition 2.1. A null curve $\gamma : (a,b) \to \boldsymbol{R}^3_1$ is called *degenerate* or *non-degenerate* at $t = c$ if $\gamma''(c)$ is or is not proportional to the velocity vector $\gamma'(c)$, respectively. If γ is non-degenerate at each $t \in (a,b)$, it is called a *non-degenerate* null curve.

We now give a characterization of zero mean curvature surfaces that change type across a real analytic non-degenerate null curve. Given an arbitrary real analytic null curve $\gamma : (a,b) \to \boldsymbol{R}^3_1$, we denote the unique complex analytic extension of it by γ again throughout this article, by a slight abuse of notation. We consider the two surfaces

$$\Phi(u,v) := \frac{\gamma(u+iv) + \gamma(u-iv)}{2},$$

and

$$\Psi(u,v) := \frac{\gamma(u+v)+\gamma(u-v)}{2},$$

which are defined for v sufficiently close to zero. We recall the following assertion:

Proposition 2.2 ([10, 11, 12], [17] and [14]). *Given a real analytic non-degenerate null curve $\gamma : (a,b) \to \boldsymbol{R}^3_1$, the union of the images of Φ and Ψ given as above are subsets of a real analytic immersion, and the intersection is γ. Moreover, Φ gives a space-like maximal surface and Ψ gives a time-like minimal surface if v is sufficiently close to zero. Furthermore, this analytic extension of the curve γ as a zero mean curvature surface does not depend upon the choice of the real analytic parametrization of the curve γ.*

Proof. We give here a proof for the sake of the readers' convenience. We have that

$$\Phi(u,v) = \sum_{n=0}^{\infty} \frac{(-1)^n \gamma^{(2n)}(u) v^{2n}}{(2n)!}, \qquad \Psi(u,v) = \sum_{n=0}^{\infty} \frac{\gamma^{(2n)}(u) v^{2n}}{(2n)!}$$

near $v = 0$, where $\gamma^{(j)} = d^j\gamma/dt^j$. In particular, if we set

$$F(u,v) := \sum_{n=0}^{\infty} \frac{\gamma^{(2n)}(u) v^{n}}{(2n)!},$$

then it gives a germ of a real analytic function satisfying

$$F(u,-v^2) = \Phi(u,v), \qquad F(u,v^2) = \Psi(u,v),$$

which prove that the images of Φ and Ψ lie on a common real analytic surface. Since γ is non-degenerate, the two vectors

$$F_u(u,0) = \gamma'(u), \qquad F_v(u,0) = \frac{\gamma''(u)}{2}$$

are linearly independent, and F gives an immersion which contains γ.

Moreover, it can be easily checked that Φ gives a space-like maximal surface (cf. Lemma 2.16) and Ψ gives a time-like minimal surface.

We now show the last assertion: Since the surface is real analytic, it is sufficient to show that given an arbitrary real analytic diffeomorphism μ from (a,b) onto its image in $\boldsymbol{R}$,

$$\Psi(u,v) = \frac{\gamma(u+iv)+\gamma(u-iv)}{2} \quad \text{and} \quad \tilde{\Psi}(u,v) = \frac{\tilde{\gamma}(u+iv)+\tilde{\gamma}(u-iv)}{2}$$

induce the same surface as their graphs, where $\tilde{\gamma}(t) := \gamma(\mu(t))$. We define A, B by

$$A = (\mu(u+v)+\mu(u-v))/2, \qquad B = (\mu(u+v)-\mu(u-v))/2.$$

Thus it is sufficient to show that the map

$$(u, v) \mapsto (A, B)$$

is an immersion at $(u, 0)$. In fact, the Jacobian of the map is given by

$$J = \det \begin{pmatrix} \mu'(u) & 0 \\ 0 & \mu'(u) \end{pmatrix} \neq 0.$$

□

Definition 2.3. Let Ω^2 be a domain in $\boldsymbol{R}^2$ and $f : \Omega^2 \to \boldsymbol{R}$ a C^∞-function satisfying (1.1). We set

$$B := 1 - f_x^2 - f_y^2.$$

A point p on Ω^2 is called a *non-degenerate point of type change*[1] with respect to f if

$$B(p) = 0, \qquad \nabla B(p) \neq 0,$$

where $\nabla B := (B_x, B_y)$ is the gradient vector of the function B.

Since ∇B does not vanish at p, the function f actually changes type at the non-degenerate point p.

Proposition 2.4 ([11, 12])**.** *Under the assumption that $B(p)$ vanishes, the following two assertions are equivalent.*

(1) *p is a non-degenerate point of type change.*
(2) *p is a dually regular point in the sense of* [11]*, that is, $f_{xx}f_{yy} - (f_{xy})^2$ does not vanish at p.*

Proof. Note that $(\nabla B)^T = -2H(\nabla f)^T$, where ^{T} is the transpose and $H := \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$. Note also that $B(p) = 0$ implies that $\nabla f(p) \neq 0$.

Now suppose that (2) holds. Then $\det H(p) \neq 0$, which with $\nabla f(p) \neq 0$ implies that $H(p)(\nabla f(p))^T \neq 0$, that is, (1) holds.

Suppose on the contrary that (2) does not hold. By a suitable linear coordinate change of (x, y), we may assume without loss of generality that $f_{xy}(p) = 0$. Then either $f_{xx}(p) = 0$ or $f_{yy}(p) = 0$. Also, the zero mean curvature equation

$$0 = (1 - f_y^2)f_{xx} + 2f_xf_yf_{xy} + (1 - f_x^2)f_{yy}$$

with $B(p) = 0$ and $f_{xy}(p) = 0$ imply that

$$f_x(p)^2 f_{xx}(p) + f_y(p)^2 f_{yy}(p) = 0.$$

[1]In Gu [11], 'dual regularity' for points of type change is equivalent to our notion. Klyachin [17] did not define this particular notion, but used it in an essential way.

This with $f_{xx}(p) = 0$ or $f_{yy}(p) = 0$ implies that

$$H(p)(\nabla f(p))^T = \begin{pmatrix} f_x(p) f_{xx}(p) \\ f_y(p) f_{yy}(p) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so (1) does not hold. □

Moreover, the following assertion holds:

Proposition 2.5 ([11, 12], [17]). *Let γ be a real analytic non-degenerate null curve, and let f_γ be the real analytic function induced by γ as in Proposition 2.2, which satisfies* (1.1). *Then the image of γ consists of non-degenerate points of type change with respect to f_γ.*

Note that the conclusion is stronger than that of Proposition 2.2.

Proof. Let γ be a non-degenerate null curve. Without loss of generality, we may take the time-component t as the parameter of γ. Then we have the expression

$$\gamma(t) = (t, x(t), y(t)) \qquad (a < t < b)$$

such that

$$x'(t)^2 + y'(t)^2 = 1. \tag{2.1}$$

Since γ is non-degenerate, it holds that

$$0 \neq \gamma''(t) = (0, x''(t), y''(t)). \tag{2.2}$$

Differentiating the relation $t = f(x(t), y(t))$, we have that

$$x'(t) f_x(x(t), y(t)) + y'(t) f_y(x(t), y(t)) = 1. \tag{2.3}$$

On the other hand, the relation $B = 0$ implies that

$$f_x(x(t), y(t))^2 + f_y(x(t), y(t))^2 = 1. \tag{2.4}$$

Then by (2.1), (2.3) and (2.4), it holds that

$$x'(t) = f_x, \qquad y'(t) = f_y.$$

Thus we have that

$$\begin{aligned} (x'', y'') &= \frac{d}{dt}\Big(f_x(x(t), y(t)), f_y(x(t), y(t)) \Big) \\ &= (x' f_{xx} + y' f_{xy}, x' f_{xy} + y' f_{yy}) \\ &= (f_x f_{xx} + f_y f_{xy}, f_x f_{xy} + f_y f_{yy}) = -\frac{1}{2} \nabla B. \end{aligned} \tag{2.5}$$

By (2.2), we get the assertion. □

Conversely, we can prove the following.

Proposition 2.6 ([11, 12], [17, Lemma 2]). *Let $f : \Omega^2 \to \boldsymbol{R}$ be a C^∞-function satisfying the zero mean curvature equation* (1.1), *and let $p = (x_0, y_0) \in \Omega^2$ be a non-degenerate point of type change. Then there exists a non-degenerate C^∞-regular null curve in $\boldsymbol{R}^3_1$ with image passing through $(f(x_0, y_0), x_0, y_0)$ and contained in the graph of f.*

Proof. By the implicit function theorem, there exists a unique C^∞-regular curve $\sigma(t) = (x(t), y(t))$ in the xy-plane with $p = \sigma(0)$ so that $B = 0$ along the curve. Since $B = 0$ on σ, the velocity vector σ' is perpendicular to ∇B. Since ∇f is also perpendicular to ∇B, we can conclude that ∇f is proportional to σ'. In fact

$$\begin{aligned} -\frac{1}{2}\nabla f \cdot \nabla B &= (f_x, f_y)\begin{pmatrix} f_x f_{xx} + f_y f_{xy} \\ f_x f_{xy} + f_y f_{yy} \end{pmatrix} = f_x^2 f_{xx} + 2 f_x f_y f_{xy} + f_y^2 f_{yy} \\ &= (1 - f_y^2) f_{xx} + 2 f_x f_y f_{xy} + (1 - f_x^2) f_{yy} \\ &\qquad\qquad\qquad - (1 - f_x^2 - f_y^2)(f_{xx} + f_{yy}) \\ &= 0. \end{aligned}$$

Since $f_x^2 + f_y^2 = 1$, by taking an arclength parameter of σ, we may set

$$x' = f_x, \qquad y' = f_y,$$

and then

$$B = 1 - f_x^2 - f_y^2 = 1 - (x')^2 - (y')^2 = 0$$

holds along σ, which implies that $t \mapsto (t, x(t), y(t))$ is a null curve. Since

$$\frac{d}{dt} f(x(t), y(t)) = x' f_x + y' f_y = f_x^2 + f_y^2 = 1,$$

there exists a constant c such that $f(x(t), y(t)) = t + c$. By translating the graph vertically if necessary, we may assume that

$$f(x(t), y(t)) = t$$

holds for each t. Then we obtain the identity (2.5) in this situation, which implies that $\nabla B(p) = (x''(0), y''(0)) \neq 0$, namely,

$$(a, b) \ni t \mapsto (f(x(t), y(t)), x(t), y(t)) = (t, x(t), y(t)) \in \boldsymbol{R}^3_1$$

gives a non-degenerate null curve near $t = 0$ lying in the graph of f. $\square$

Definition 2.7 ([3]). Let Σ^2 be a Riemann surface. A C^∞-map $\varphi : \Sigma^2 \to \boldsymbol{R}^3_1$ is called a *generalized maximal surface* if there exists an open dense subset W of Σ^2 such that the restriction $\varphi|_W$ of φ to W gives a conformal (space-like) immersion of zero mean curvature. A *singular point* of φ is a point at which φ is not an immersion. A singular point p satisfying $d\varphi(p) = 0$ is called a *branch point* of φ. Moreover, φ is called a *maxface* if φ does not have any branch points. (A maxface may have singular points in general).

Remark 2.8. The above definition of maxfaces is given in [3], which is simpler than the definition given in [21] and [9]. However, this new definition is equivalent to the previous one, as we now explain. Suppose that $\varphi|_W$ is a conformal (space-like) immersion of zero mean curvature. Then $\partial\varphi = \varphi_z\, dz$ is a $\boldsymbol{C}^3$-valued holomorphic 1-form on W, where z is a complex coordinate of Σ^2. Since φ is a C^∞-map on Σ^2, $\partial\varphi$ can be holomorphically extended to Σ^2. Then the line integral $\Phi(z) = \int_{z_0}^{z} \partial\varphi$ with respect to a base point $z_0 \in \Sigma^2$ gives a holomorphic map defined on the universal cover of Σ^2 whose real part coincides with $\varphi(z) - \varphi(z_0)$. The condition that φ does not have any branch point implies that Φ is an immersion. Moreover, since φ is conformal on W, the map Φ satisfies

$$-(d\Phi_0)^2 + (d\Phi_1)^2 + (d\Phi_2)^2 = 0 \qquad \big(\Phi = (\Phi_0, \Phi_1, \Phi_2)\big),$$

namely, Φ is a null immersion. So φ satisfies the definition of maxface as in [21] and [9]. We call Φ the *holomorphic lift* of the maxface φ.

Remark 2.9. By the above definition, maxfaces are orientable. However, there are non-orientable maximal surfaces, as shown in [6]. The definition of non-orientable maxfaces is given in [6, Def. 2.1]. In this paper, we work only with orientable maximal surfaces. It should be remarked that non-orientable maxfaces will be orientable when taking double coverings.

Let $\varphi : \Sigma^2 \to \boldsymbol{R}^3_1$ be a maxface with Weierstrass data (G, η) (see [21] for the definition of Weierstrass data). Using the data (G, η), the maxface φ has the expression

$$\varphi = \mathrm{Re}(\Phi), \quad \Phi = \int_{z_0}^{z} (-2G, 1 + G^2, i(1 - G^2))\eta. \tag{2.6}$$

The imaginary part

$$\varphi^* := \mathrm{Im}(\Phi) : \tilde{\Sigma}^2 \longrightarrow \boldsymbol{R}^3_1 \tag{2.7}$$

also gives a maxface called the *conjugate surface* of φ, which is defined on the universal cover $\tilde{\Sigma}^2$ of Σ^2. The following fact is known:

Fact 2.10 ([21, 9]). *A point p of Σ^2 is a singular point of φ if and only if $|G(p)| = 1$.*

Definition 2.11. A singular point p of φ is called *non-degenerate* if dG does not vanish at p.

Fact 2.12 ([21, 9]). *If a singular point p of φ is non-degenerate, then there exists a neighborhood U of p and a regular curve $\gamma(t)$ in U so that $\gamma(0) = p$ and the singular set of φ in U coincides with the image of the curve γ.*

This curve γ is called the *singular curve* at the non-degenerate singular point p.

Definition 2.13. A regular curve γ on Σ^2 is called a *non-degenerate fold singularity* if it consists of non-degenerate singular points such that the real part of the meromorphic function $dG/(G^2\eta)$ vanishes identically along the singular curve γ. Each point on the non-degenerate fold singularity is called a *fold singular point*.

A singular point of a C^∞-map $\varphi : \Sigma^2 \to \boldsymbol{R}^3$ has a *fold singularity* at p if there exists a local coordinate system (u, v) centered at p such that $\varphi(u, v) = \varphi(u, -v)$. Later, we show that a non-degenerate fold singularity is actually a fold singularity (cf. Lemma 2.17).

Suppose that p is a non-degenerate fold singular point of φ. The following duality between fold singularities and generalized cone-like singularities (cf. [7, Definition 2.1]) holds:

Proposition 2.14 ([16])**.** *Let $\varphi : \Sigma^2 \to \boldsymbol{R}^3_1$ be a maxface and φ^* the conjugate maxface. Then p is a non-degenerate fold singular point of φ if and only if it is a generalized cone-like singular point of φ^*.*

Proof. This assertion is immediate from comparison of the above definition of non-degenerate fold singularities and the definition of generalized cone-like singular points as in [7, Definition 2.1 and Lemma 2.3]. □

We now show the following assertion, which characterizes the non-degenerate fold singularities on maxfaces.

Theorem 2.15. *Let $\varphi : \Sigma^2 \to \boldsymbol{R}^3_1$ be a maxface which has non-degenerate fold singularities along a singular curve $\gamma : (a, b) \to \Sigma^2$. Then $\hat{\gamma} := \varphi \circ \gamma$ is a non-degenerate null curve, and the image of the map*

$$\tilde{\varphi}(u, v) := \frac{\hat{\gamma}(u + v) + \hat{\gamma}(u - v)}{2} \tag{2.8}$$

is real analytically connected to the image of φ along γ as a time-like minimal immersion. Conversely, any real analytic zero mean curvature immersion which changes type across a non-degenerate null curve is obtained as a real analytic extension of non-degenerate fold singularities of a maxface.

This assertion follows immediately from Fact 2.12 and the following Lemmas 2.16 and 2.17.

Lemma 2.16. *Let $\gamma : (a, b) \to \boldsymbol{R}^3_1$ be a real analytic non-degenerate null curve. Then*

$$\varphi(u + iv) := \frac{\gamma(u + iv) + \gamma(u - iv)}{2}$$

gives a maxface with non-degenerate fold singularities on the real axis.

Proof. We set $z = u + iv$. Then it holds that

$$\varphi_z = \frac{1}{2}(\varphi_u - i\varphi_v) = \frac{1}{2}\gamma'(u+iv),$$

where $\gamma'(t) := d\gamma(t)/dt$. Since γ is a regular real analytic curve, the map

$$\Phi(u+iv) := \gamma(u+iv) \tag{2.9}$$

gives a null holomorphic immersion if v is sufficiently small. Thus $\varphi = \mathrm{Re}(\Phi)$ gives a maxface.

Since γ is a null curve, it holds that

$$\gamma_0'(t)^2 = \gamma_1'(t)^2 + \gamma_2'(t)^2, \tag{2.10}$$

where we set $\gamma = (\gamma_0, \gamma_1, \gamma_2)$. Moreover, since γ is a regular curve, (2.10) implies

$$\gamma_0'(t) \neq 0 \qquad (a < t < b). \tag{2.11}$$

It can be easily checked that the maxface φ has the Weierstrass data

$$\eta := \frac{1}{2}(d\Phi_1 - id\Phi_2) = \frac{\gamma_1'(z) - i\gamma_2'(z)}{2}dz, \tag{2.12}$$

$$G := -\frac{d\Phi_0}{2\eta} = -\frac{\gamma_0'(z)}{\gamma_1'(z) - i\gamma_2'(z)} = -\frac{\gamma_1'(z) + i\gamma_2'(z)}{\gamma_0'(z)}, \tag{2.13}$$

where we set $\Phi = (\Phi_0, \Phi_1, \Phi_2)$ and use the identity

$$(\gamma_1' - i\gamma_2')(\gamma_1' + i\gamma_2') = (\gamma_1')^2 + (\gamma_2')^2 = (\gamma_0')^2.$$

In particular, (2.13) implies that $|G| = 1$ holds on the u-axis, which implies that the u-axis consists of singular points. By (2.13), dG vanishes if and only if

$$\Delta := (\gamma_1' + i\gamma_2')'\gamma_0' - (\gamma_1' + i\gamma_2')\gamma_0'' = (\gamma_0'\gamma_1'' - \gamma_1'\gamma_0'') + i(\gamma_0'\gamma_2'' - \gamma_2'\gamma_0'')$$

vanishes. In other words, $\Delta = 0$ if and only if (γ_0'', γ_j'') is proportional to (γ_0', γ_j') for $j = 1, 2$, namely γ'' is proportional to γ'. Since γ is non-degenerate, this is impossible. So the image of the curve γ consists of non-degenerate singular points (cf. Definition 2.11).

By (2.11), (2.12) and (2.13), we have that

$$\frac{dz}{G^2\eta} = \frac{2(\gamma_1' - i\gamma_2')}{(\gamma_0')^2}.$$

Thus the u-axis consists of non-degenerate fold singular points if and only if the real part of $\Delta_1 := (\gamma_1' - i\gamma_2')\Delta$ vanishes. Here

$$\mathrm{Re}(\Delta_1) = \gamma_0'(\gamma_1'\gamma_1'' + \gamma_2'\gamma_2'') - \gamma_0''((\gamma_1')^2 + (\gamma_2')^2) = \gamma_0'(\gamma_0'\gamma_0'') - \gamma_0''(\gamma_0')^2 = 0,$$

where we used the identity (2.10) and its derivative. This implies that γ consists of non-degenerate fold singularities. $\square$

Finally, we prove the converse assertion:

Lemma 2.17. *Let $\varphi : \Sigma^2 \to \boldsymbol{R}^3_1$ be a maxface which has non-degenerate fold singularities along a singular curve $\gamma : (a,b) \to \Sigma^2$. Then, the space curve $\hat\gamma(t) := \varphi \circ \gamma(t)$ is a non-degenerate real analytic null curve such that*

$$\hat\varphi(u,v) := \frac{1}{2}\Bigl(\hat\gamma(u+iv) + \hat\gamma(u-iv)\Bigr)$$

coincides with the original maxface φ. In particular, $\hat\varphi$ satisfies the identity $\hat\varphi(u,v) = \hat\varphi(u,-v)$.

Proof. The singular set of φ can be characterized by the set $|G| = 1$, where (G,η) is the Weierstrass data as in (2.6). Let T be a Möbius transformation on $S^2 = \boldsymbol{C} \cup \{\infty\}$ which maps the unit circle $\{\zeta \in \boldsymbol{C}\,;\, |\zeta| = 1\}$ to the real axis. Then $T \circ G$ maps the image of the singular curve γ to the real axis. Since $dG \neq 0$ (cf. Definition 2.11), we can choose $T \circ G$ as a local complex coordinate. We denote it by

$$z = u + iv.$$

Then the image of γ coincides with the real axis $\{v = 0\}$. Let Φ be the holomorphic lift of φ. Since the real axis consists of non-degenerate fold singularities, Proposition 2.14 implies that $\mathrm{Im}(\Phi)$ is constant on the real axis. Since Φ has an ambiguity of translations by pure imaginary vectors, we may assume without loss of generality that

$$\mathrm{Im}(\Phi) = 0 \qquad \text{on the real axis.} \tag{2.14}$$

Thus, the curve $\hat\gamma$ is expressed by (cf. (2.9))

$$\hat\gamma(u) = \varphi(u,0) = \mathrm{Re}\bigl(\Phi(u,0)\bigr) = \Phi(u,0), \tag{2.15}$$

namely, the two $\boldsymbol{C}^3$-valued holomorphic functions $\Phi(u+iv)$ and $\hat\gamma(u+iv)$ take the same values on the real axis. Hence $\Phi(z) = \hat\gamma(u+iv)$ and thus

$$\varphi(z) = \frac{\hat\gamma(u+iv) + \hat\gamma(u-iv)}{2} = \hat\varphi(u,v).$$

So it is sufficient to show that $\hat\gamma(t)$ is a non-degenerate null curve. Since $|G| = 1$ on the real axis, there exists a real-valued function $t = t(u)$ such that

$$G(u) = e^{it(u)} \qquad (u \in \boldsymbol{R}). \tag{2.16}$$

Differentiating this along the real axis, we have $G_u(u) = ie^{it}(dt/du)$. Here, dt/du does not vanish because $dG \neq 0$ on the real axis. Since γ consists of non-degenerate fold singularities,

$$i\frac{dG}{G^2\eta} = -\frac{e^{-it(u)}}{w(u)}\frac{dt}{du}$$

must be real valued (cf. Definition 2.13), where $\eta = w(z)\,dz$. Since Φ is an immersion, G must have a pole at $z = u$ if $w(u) = 0$ (cf. (2.6)), but this contradicts the fact that $|G| = 1$ along γ. Thus we have $w(u) \neq 0$. It then follows that

$$\xi(u) := e^{it(u)}w(u) = G(u)w(u)$$

is a non-vanishing real valued analytic function. Now, if we write $\hat{\gamma} = (\hat{\gamma}_0, \hat{\gamma}_1, \hat{\gamma}_2)$, (2.6) yields

$$\hat{\gamma}_0'(u) = \mathrm{Re}\Big(-2G(u)w(u)\Big) = -2\xi(u),$$

and

$$\hat{\gamma}_1'(u) = \mathrm{Re}\Big((1 + G(u)^2)w(u)\Big) = 2\xi(u)\cos t(u),$$
$$\hat{\gamma}_2'(u) = \mathrm{Re}\Big(i(1 - G(u)^2)w(u)\Big) = 2\xi(u)\sin t(u).$$

This implies that $\hat{\gamma}(u)$ is a regular real analytic null curve. Since $dt/du \neq 0$, the acceleration vector

$$\hat{\gamma}''(u) = \big(\log \xi(u)\big)'\hat{\gamma}'(u) + 2\xi(u)\Big(0, -\sin t(u), \cos t(u)\Big)\frac{dt}{du}$$

is not proportional to $\hat{\gamma}'(u)$. Thus, $\hat{\gamma}(u)$ is non-degenerate. □

Corollary 2.18. *Let $\varphi : \Sigma^2 \to \boldsymbol{R}^3_1$ be a maxface. Then a singular point $p \in \Sigma^2$ lies on a non-degenerate fold singularity if and only if there exists a local complex coordinate $z = u + iv$ with $p = (0, 0)$ satisfying the following two properties:*

(1) $\varphi(u, v) = \varphi(u, -v)$,
(2) $\varphi_u(0, 0)$ *and* $\varphi_{uu}(0, 0)$ *are linearly independent.*

Proof. Suppose p is a non-degenerate fold singularity of φ. Then (1) follows from Lemma 2.17, and (2) follows from the fact that the null curve parameterizing fold singularities is non-degenerate.

Conversely, suppose there is a coordinate system around a singular point p with $p = (0, 0)$ which satisfies (1) and (2). Differentiating (1), we have that $\varphi_v(u, 0) = 0$. Let Φ be a holomorphic lift of φ. Since Φ is a null holomorphic map, the relation $\varphi_v(u, 0) = 0$ implies that

$$0 = \langle \Phi_z(u, 0), \Phi_z(u, 0)\rangle = 4\langle \varphi_z(u, 0), \varphi_z(u, 0)\rangle = \langle \varphi_u(u, 0), \varphi_u(u, 0)\rangle,$$

where $\langle , \rangle$ is the canonical inner product of $\boldsymbol{R}^3_1$. This implies $\gamma : u \mapsto \varphi(u, 0)$ is a null curve. By the condition (2), this null curve is non-degenerate. Then by Lemma 2.16,

$$\varphi_\gamma(u+iv) := \frac{\gamma(u+iv)+\gamma(u-iv)}{2}$$

is a maxface such that γ parametrizes a non-degenerate fold singularity of φ_γ. Moreover, $\Phi_\gamma := \gamma(u+iv)$ gives a holomorphic lift of φ_γ (cf. (2.9)). Since

$$\Phi_z(u,0) = 2\varphi_z(u,0) = \gamma'(u) = (\Phi_\gamma)_z(u,0),$$

the holomorphicity of Φ and Φ_γ yields that $\Phi_z(z)$ coincides with $(\Phi_\gamma)_z(z)$. Thus Φ coincides with Φ_γ up to a constant. Then φ_γ coincides with φ, and we can conclude that γ parametrizes a non-degenerate fold singularity of φ. □

So far, we have looked at the singular curves of maxfaces. Now we turn our attention to the singular curves of zero mean curvature surfaces and prove the following assertion, which can be considered as the fundamental theorem of type change for zero mean curvature surfaces:

Theorem 2.19 (Gu [10, 11, 12] and Klyachin [17]). *Let $\gamma : (a,b) \to \boldsymbol{R}^3_1$ be a non-degenerate real analytic null curve. We set*

$$\hat\varphi_\gamma(u,v) := \begin{cases} \dfrac{\gamma(u+i\sqrt{v})+\gamma(u-i\sqrt{v})}{2} & (v \geq 0), \\[2ex] \dfrac{\gamma(u+\sqrt{|v|})+\gamma(u-\sqrt{|v|})}{2} & (v < 0), \end{cases}$$

for sufficiently small $|v|$. Then $\hat\varphi_\gamma$ gives a real analytic zero mean curvature immersion such that the image of γ consists of non-degenerate points of type change with respect to $\hat\varphi_\gamma$.

Conversely, let $f : \Omega^2 \to \boldsymbol{R}$ be a C^∞-function satisfying the zero mean curvature equation (1.1)*, and let $p = (x_0, y_0)$ be a non-degenerate point of type change with respect to f, where Ω^2 is a domain in the xy-plane. Then there exists a real analytic non-degenerate null curve γ in $\boldsymbol{R}^3_1$ through $(f(x_0,y_0), x_0, y_0)$ with $\hat\varphi_\gamma$ coinciding with the graph of f in a small neighborhood of p.*

This assertion was proved by Gu [10, 11, 12]. Later, Klyachin [17] analyzed type-changes of zero mean curvature surfaces not only at non-degenerate points of type changes but also degenerate cases as mentioned in the introduction, and got the same assertion as a corollary. Note that the conclusion for regularity of the converse statement is stronger than that of Proposition 2.6. We remark that Gu [12] gave a generalization of Theorem 2.19 for 2-dimensional zero mean curvature surfaces in $\boldsymbol{R}^{n+1}_1$ $(n \geq 2)$.

Proof. We have already proved the first assertion. (In fact, Proposition 2.5 implies that γ consists of non-degenerate points of type change with respect to $\hat{\varphi}_\gamma$.) Then it is sufficient to show the converse assertion, which is proved in [10, 11, 12] and [17]. Here referring to [17], we give only a sketch of the proof: Let $f : \Omega^2 \to \boldsymbol{R}$ be a C^∞-function satisfying the zero mean curvature equation (1.1), and let $p = (x_0, y_0) \in \Omega^2$ be a non-degenerate point of type change with respect to f. (As pointed out in Gu [11, 12] and Klyachin [17], one can prove the real analyticity of f at p, assuming only C^3-regularity of f, using the same argument as below.) By Proposition 2.6, there exists a C^∞-regular curve $\sigma(u)$ ($|u| < \delta$) such that

$$\gamma(u) := (f \circ \sigma(u), \sigma(u))$$

is a non-degenerate null curve passing through $(f(x_0, y_0), x_0, y_0)$, where δ is a sufficiently small positive number. We set $B := 1 - f_x^2 - f_y^2$. Let Ω^+ be a simply connected domain such that $B > 0$, and suppose that σ lies on the boundary of Ω^+. We set

$$t = t(x, y) := f(x, y), \qquad s = s(x, y) := \int_{q_0}^{q} \frac{-f_y dx + f_x dy}{\sqrt{B}},$$

where $q_0 \in \Omega^+$ is a base point and $q := (x, y)$. Since $\alpha := (-f_y dx + f_x dy)/\sqrt{B}$ is a closed 1-form, its (line) integral $\displaystyle\int_{q_0}^{q} \alpha$ does not depend on the choice of path. Let $\tau(v) = (a(v), b(v))$ $(0 \le v \le \epsilon)$ be a path starting from p and going into Ω^+ which is transversal to the curve σ. Since $B(p) = 0$ and $\nabla B(p) \neq 0$, there exists a constant $C > 0$ such that $B \circ \tau(v) = Cv + O(v^2)$, where $O(v^2)$ denotes the higher order terms. Then there exists a constant m such that

$$\left| \frac{-f_y \circ \tau(v) \dfrac{da}{dv}(v) + f_x \circ \tau(v) \dfrac{db}{dv}(v)}{\sqrt{B \circ \tau(v)}} \right| < \frac{m}{\sqrt{v}} \qquad \text{for } 0 < v \le \epsilon,$$

hence

$$\lambda := \int_\tau |\alpha| < \int_0^\epsilon \frac{m}{\sqrt{v}} dv < \infty,$$

which is just the case (1) of [17, Lemma 6], and (t, s) gives an isothermal coordinate system of Ω^+ with respect to the immersion

$$\varphi : (t, s) \mapsto (f(x(t, s), y(t, s)), x(t, s), y(t, s)) = (t, x(t, s), y(t, s)) \qquad (s > 0).$$

Moreover, the function $s(x, y)$ can be continuously extended to the image of the curve σ. Since σ is an integral curve of ∇f, we may assume that σ parametrizes the level set $s = 0$, where we have used the fact that $s(x, y)$ is constant along each integral curve of ∇f. In particular, φ satisfies $\varphi_{tt} +$

$\varphi_{ss} = 0$. Then $\varphi(t,s)$ can be extended to a harmonic $\boldsymbol{R}^3$-valued function for $s < 0$ satisfying $\varphi(t,s) = \varphi(t,-s)$ via the symmetry principle (see the proof of [17, Theorem 6]). In particular, f is a real analytic function whose graph coincides with the image of φ on Ω^+ near p. Moreover $t \mapsto \varphi(t,0)$ parametrizes the curve γ (cf. [17, Page 219]). By Corollary (2.18), γ can be considered as a non-degenerate fold singularity of the maxface $(t,s) \mapsto \varphi(t,s)$. (In fact, the condition (2) of Corollary (2.18) corresponds to the fact that γ is a non-degenerate curve near p.) Then Theorem 2.15 implies that $\hat{\varphi}_\gamma$ coincides with the graph of f on a sufficiently small neighborhood of p. □

As an application of Theorem 2.19, embedded triply periodic zero mean curvature surfaces of mixed type in $\boldsymbol{R}^3_1$ with the same topology as the Schwarz D surface in the Euclidean 3-space $\boldsymbol{R}^3$ have been constructed, in [8].

3. The conjugates of hyperbolic catenoids and Scherk-type surfaces

The two entire graphs of n variables

$$f_1(x_1, \ldots, x_n) := x_1 \tanh(x_2),$$
$$f_2(x_1, \ldots, x_n) := (\log \cosh x_1) - (\log \cosh x_2),$$

given by Osamu Kobayashi [18], are zero mean curvature hypersurfaces in $\boldsymbol{R}^{n+1}_1$ which change type from space-like to time-like. When $n = 2$, the image of f_1 is congruent to $\mathcal{C}_0$ and the image of f_2 is congruent to $\mathcal{S}_0$. On the other hand, the space-like catenoid $\mathcal{C}_+$ (resp. the space-like Scherk surface $\mathcal{S}_+$) and the time-like catenoid $\mathcal{C}_-$ (resp. the time-like Scherk surface $\mathcal{S}_-$) are typical examples of zero mean curvature surfaces which contain singular light-like lines. Moreover, they are closely related to $\mathcal{C}_0$ (resp. $\mathcal{S}_0$) by taking their conjugate surfaces as follows:

Fact 3.1 ([18], [14] and [4]). *The conjugate space-like maximal surface of the space-like hyperbolic catenoid $\mathcal{C}_+$ and the conjugate time-like minimal surface of the time-like hyperbolic catenoid $\mathcal{C}_-$ are both congruent to subsets of the entire graph $\mathcal{C}_0$.*

The space-like hyperbolic catenoid $\mathcal{C}_+$ was originally given by Kobayashi [18] as the catenoid of 2nd kind, and he also pointed out that the space-like part $\mathcal{C}_0^+$ of $\mathcal{C}_0$ is the conjugate surface of $\mathcal{C}_+$ (see [18]). $\mathcal{C}_0^+$ is connected and is called the *space-like hyperbolic helicoid*. The time-like part $\mathcal{C}_0^-$ of $\mathcal{C}_0$ splits into two connected components, each of which is congruent to the *time-like hyperbolic helicoid* (see Figure 3, left). The entire assertion, including the case of the time-like part, has been pointed out in [14, Lemma 2.11 (3)] and the caption of Figure 1 in [4] without proof.

Proof. We give a proof here as an application of the results in the previous section. A subset of the space-like hyperbolic catenoid $\mathcal{C}_+$ can be parametrized by

$$\begin{aligned}(3.1)\qquad &\varphi_1(u,v) = (\cosh u \sin v, v, \sinh u \sin v) = -\operatorname{Re} i(\sinh z, z, \cosh z),\\ &\psi_1(u,v) = (-\cosh u \sin v, v, -\sinh u \sin v),\end{aligned}$$

where $z = u + iv$. In fact, $\mathcal{C}_+$ is the union of the closure of the images of φ_1 and ψ_1. The surface φ_1 has generalized conical singularities at $(u, n\pi)$ for any $u \in \boldsymbol{R}$ and $n \in \boldsymbol{Z}$, as pointed out in [4]. Using this, one can easily compute that the conjugate of φ_1 is congruent to the following surface

$$\begin{aligned}(3.2)\qquad \varphi_1^*(u,v) :&= -\operatorname{Im} i(\sinh z, z, \cosh z) = -\operatorname{Re}(\sinh z, z, \cosh z)\\ &= -(\sinh u \cos v, u, \cosh u \cos v).\end{aligned}$$

By Proposition 2.14, the conjugate surface has non-degenerate fold singularities. Then by Theorem 2.15 one can get an analytic continuation of φ_1^* as a zero mean curvature surface in $\boldsymbol{R}^3_1$ which changes type across the fold singularities. We can get an explicit description of such an extension of φ_1^* as follows: We set

$$(t, x, y) = \varphi_1^*(u,v) = -(\sinh u \cos v, u, \cosh u \cos v).$$

Then the surface has fold singularities at $(u, n\pi)$ for any $u \in \boldsymbol{R}$ and $n \in \boldsymbol{Z}$. Then it holds that

$$\frac{t}{y} = \tanh u = -\tanh x$$

and the image of $-\varphi_1^*$ is contained in the surface $\mathcal{C}_0$.

On the other hand, a subset of the time-like hyperbolic catenoid $\mathcal{C}_-$ has a parametrization

$$(3.3)\quad \varphi_2(u,v) := \frac{1}{2}(\sinh u + \sinh v, u + v, \cosh u - \cosh v) = \frac{\alpha(u) + \beta(v)}{2},$$

where

$$(3.4)\qquad \alpha(u) := (\sinh u, u, \cosh u), \qquad \beta(v) := (\sinh v, v, -\cosh v).$$

Also

$$(3.5)\qquad \psi_2(u,v) := \frac{1}{2}(-\sinh u - \sinh v, u + v, -\cosh u + \cosh v)$$

gives a parametrization of a subset of $\mathcal{C}_-$. More precisely, $\mathcal{C}_-$ is the union of the closure of the images of φ_2 and ψ_2. We get the following parametrization of the conjugate surface φ_2^* of φ_2

$$\varphi_2^*(u,v) := \frac{1}{2}\bigl(\alpha(u)-\beta(v)\bigr) = \frac{1}{2}(\sinh u - \sinh v, u-v, \cosh u + \cosh v), \tag{3.6}$$

where α and β are as in (3.4). (See [13] for the definition of the conjugate surfaces of time-like minimal surfaces.) To find the implicit function representation of the image of φ_2^*, take a new coordinate system (ξ,ζ) as

$$u = \xi + \zeta, \qquad v = \xi - \zeta.$$

Then

$$\varphi_2^*(\xi,\zeta) = (\cosh\xi\sinh\zeta, \zeta, \cosh\xi\cosh\zeta),$$

which implies that the image is a subset of $\mathcal{C}_0$. The entire graph $\mathcal{C}_0$ changes type across two disjoint real analytic null curves $\{y = \pm\cosh x\}$. $\square$

Next, we prove a similar assertion for the Scherk surfaces, which is also briefly mentioned in the caption of Figure 1 in [4]:

Theorem 3.2. *The conjugate space-like maximal surface of the space-like Scherk surface $\mathcal{S}_+$ and the conjugate time-like minimal surface of the time-like Scherk surface $\mathcal{S}_-$ of the first kind are both congruent to subsets of the entire graph $\mathcal{S}_0$.*

Proof. Using the identities

$$\cos\arg z = \mathrm{Re}\Bigl(\frac{z}{|z|}\Bigr), \qquad \sin\arg z = \mathrm{Im}\Bigl(\frac{z}{|z|}\Bigr),$$

one can prove that a subset of the space-like Scherk surface $\mathcal{S}_+$ is parametrized by the complex variable z as

$$\begin{aligned}\varphi_1(z) &= -\,\mathrm{Re}\, i\left(\log\frac{1+z^2}{1-z^2},\ \log\frac{1-z}{1+z},\ \log\frac{1-iz}{1+iz}\right) + \frac{\pi}{2}(1,1,1) \\ &= \left(\arg\frac{1+z^2}{1-z^2},\ \arg\frac{1-z}{1+z},\ \arg\frac{1-iz}{1+iz}\right) + \frac{\pi}{2}(1,1,1).\end{aligned} \tag{3.7}$$

In fact, $\varphi_1(z)$ is a multi-valued $\boldsymbol{R}^3_1$-valued function, but can be considered as a single-valued function on the universal cover of $\boldsymbol{C}\cup\{\infty\}\setminus\{\pm1,\pm i\}$. We now set

$$\varphi_1(z) = \bigl(t(z), x(z), y(z)\bigr)$$

and

$$\psi_1(z) := \bigl(\pi - t(z), x(z), \pi - y(z)\bigr).$$

Then $\mathcal{S}_+$ is the union of the closure of the images of φ_1 and ψ_1. The conjugate ψ_1^* of the space-like Scherk surface ψ_1 is obtained by

$$\begin{aligned}\psi_1^*(z) &= \operatorname{Im} i\left(\log\frac{1+z^2}{1-z^2},\ \log\frac{1-z}{1+z},\ \log\frac{1-iz}{1+iz}\right)\\ &= \left(\log\left|\frac{1+z^2}{1-z^2}\right|,\ \log\left|\frac{1-z}{1+z}\right|,\ \log\left|\frac{1-iz}{1+iz}\right|\right).\end{aligned}\tag{3.8}$$

Since ψ_1 admits only generalized cone-like singularities (cf. Proposition 2.14), ψ_1^* admits only fold singularities, and has a real analytical extension across the fold singularities to a time-like minimal surface in $\boldsymbol{R}^3_1$ (cf. Theorem 2.15). More precisely, the image of the conjugate ψ_1^* is contained in the graph $\mathcal{S}_0$, shown as follows: The singular sets of ψ_1 and ψ_1^* are both parametrized as $\{z = e^{iu}\}$. Then the image of a connected component of singular curve by ψ_1^* as in (3.8) is parametrized as

$$\gamma(u) = \frac{1}{2}\left(2\log\cot u,\ \log\frac{1-\cos u}{1+\cos u},\ \log\frac{1+\sin u}{1-\sin u}\right) \qquad \left(0 < u < \frac{\pi}{2}\right).\tag{3.9}$$

By the singular Björling formula (2.8) in Section 2, we have the following analytic extension of γ

$$\hat{\psi}_2(u,v) := \frac{\gamma(u)+\gamma(v)}{2}.\tag{3.10}$$

Now, we check that the conjugate $\psi_2 := \hat{\psi}_2^*$ of $\hat{\psi}_2$ as in (3.10) coincides with the time-like Scherk surface $\mathcal{S}_-$. By (3.10), the conjugate ψ_2 of $\hat{\psi}_2$ is parametrized by (see [13] for the definition of the conjugate surfaces of time-like minimal surfaces)

$$\begin{aligned}\psi_2(u,v) &= \frac{1}{2}\big(\gamma(u)-\gamma(v)\big)\\ &= \Bigg(\frac{1}{2}\big(\log(\cot u)-\log(\cot v)\big), \frac{1}{4}\left(\log\frac{1-\cos u}{1+\cos u}-\log\frac{1-\cos v}{1+\cos v}\right),\\ &\qquad \frac{1}{4}\left(\log\frac{1+\sin u}{1-\sin u}-\log\frac{1+\sin v}{1-\sin v}\right)\Bigg).\end{aligned}\tag{3.11}$$

We set $(t,x,y)=\psi_2(u,v)$, and will show that (t,x,y) lies in $\mathcal{S}_-$: In fact, by (3.11), we have

$$e^{2t} = \frac{\cot u}{\cot v} = \frac{\cos u\sin v}{\sin u\cos v},$$

which implies

$$\cosh t = \frac{1}{2}\left(\sqrt{\frac{\cos u \sin v}{\sin u \cos v}} + \sqrt{\frac{\sin u \cos v}{\cos u \sin v}}\right) = \frac{\sin(u+v)}{\sqrt{\sin 2u \sin 2v}}.$$

Using

$$e^{4x} = \frac{1-\cos u}{1+\cos u} \times \frac{1+\cos v}{1-\cos v} = \left(\tan\frac{u}{2}\cot\frac{v}{2}\right)^2,$$

we have that

$$\cosh x = \frac{1}{\sqrt{\sin u \sin v}} \sin\frac{u+v}{2}.$$

Similarly,

$$\cosh y = \frac{1}{\sqrt{\cos u \cos v}} \cos\frac{u+v}{2}$$

holds. Hence the analytic extension of the conjugate of $\mathcal{S}_+$ coincides with $\mathcal{S}_0$. As pointed out in [14], the space-like part of $\mathcal{S}_0$ is connected, and the time-like part of $\mathcal{S}_0$ consists of four connected components, each of which is congruent to the image of ψ_1 (see Figure 3, right). $\square$

In the introduction, we saw that the conjugate surfaces of $\mathcal{C}_+$ and $\mathcal{C}_-$ (resp. $\mathcal{S}_+$ and $\mathcal{S}_-$) are both subsets of the same zero mean curvature surface $\mathcal{C}_0$ (resp. $\mathcal{S}_0$). As pointed out in [14], a similar phenomenon also holds between elliptic catenoids and parabolic catenoids: The helicoid $x \sin t = y \cos t$ is well known as a ruled minimal surface in the Euclidean 3-space $\boldsymbol{R}^3$, and also gives a zero-mean curvature in $\boldsymbol{R}^3_1$. The *space-like elliptic catenoid*

$$\varphi^E_+(u,v) := (v, \cos u \sinh v, \sin u \sinh v)$$

and the *time-like elliptic catenoid*

$$\varphi^E_-(u,v) := (v, \cosh u \sinh v, \sinh u \sinh v)$$

induce their conjugate surfaces, both of which are subsets of the helicoids.

The *space-like parabolic catenoid*, on the other hand,

$$\varphi^P_+(u,v) := \left(v - \frac{v^3}{3} + u^2 v, v + \frac{v^3}{3} - u^2 v, 2uv\right)$$

is given by Kobayashi [18] as an Enneper surface of the 2nd kind. Consider the ruled zero-mean curvature surface, which we call the *parabolic helicoid*

$$\varphi^P_0(u,v) := \gamma(u) + v(u,-u,1), \qquad \left(\gamma(u) := \left(-u - \frac{u^3}{3}, -u + \frac{u^3}{3}, -u^2\right)\right),$$

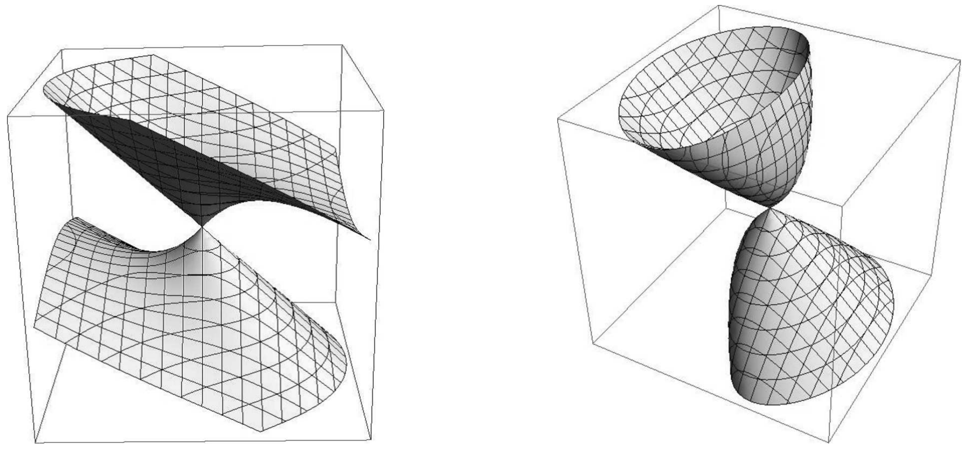

FIGURE 4. The completions of parabolic catenoids φ_+^P (left) and φ_-^P (right).

where $\gamma(u)$ is a non-degenerate null curve at which the surface changes type. If $v > 0$, φ_0^P gives the conjugate surface of φ_+^P. If $v < 0$, φ_0^P gives the conjugate surface of the *time-like parabolic catenoid* [2] given by

$$\varphi_-^P(u,v) := \left(-u - \frac{u^3}{3} - uv^2, -u + \frac{u^3}{3} + uv^2, -u^2 - v^2\right).$$

The image of the parabolic catenoid $\varphi_+^P(u,v)$ (resp. $\varphi_-^P(u,v)$) is a subset of (cf. Figure 4)

(3.12)

$$12(x^2 - t^2) = (x+t)^4 - 12y^2 \qquad \left(\text{resp. } 12(x^2 - t^2) = -(x+t)^4 - 12y^2\right).$$

We call it *the completion of the space-like* (*resp. time-like*) *parabolic catenoid*, which contains a light-like line

$$L := \{y = x + t = 0\}.$$

The initial parametrization $\varphi_\pm^P$ does not include this line L. Kobayashi [18, Example 2.3] noticed the line L in the surface and drew a hand-drawn figure of it that coincides with the left-hand side of Figure 4.

In [4], the zero mean curvature surfaces containing singular light-like lines are categorized into the following six classes

$$\alpha^+, \quad \alpha_I^0, \quad \alpha_{II}^0, \quad \alpha_I^-, \quad \alpha_{II}^-, \quad \alpha_{III}^-. \tag{3.13}$$

The surfaces belonging to α^+ (resp. α_I^-, α_{II}^-, α_{III}^-) are space-like (resp. time-like). On the other hand, the causalities of surfaces in α_I^0 and α_{II}^0 are not

[2] In [14, Examples 2.8 and 2.9], these are called spacelike (timelike) parabolic helicoids.

unique. In fact, the light cone and the hyperbolic catenoids $\mathcal{C}_\pm$ are examples of surfaces[3] of type α^0_{II}. We get the following assertion:

Proposition 3.3. *The completion of space-like (resp. time-like) parabolic catenoids gives an example of surfaces of type α^0_{II} at each point $(-c, c, 0)$ $(c \neq 0)$ on the light-like line L. In other words, both space-like surfaces and time-like surfaces exist in the class α^0_{II} of zero mean curvature surfaces containing light-like lines.*

Proof. We set

$$F_\pm := (x+t)\{12(x-t) \mp (x+t)^3\} + 12y^2.$$

Then (3.12) is rewritten as $F_\pm = 0$. We fix a point $(-c, c, 0)$ $(c \neq 0)$ on the set $\{F_\pm = 0\}$. Since

$$\frac{\partial F}{\partial t}(-c, c, 0) = -24t \mp 4(x+t)^3\big|_{(t,x)=(-c,c)} = 24c(\neq 0),$$

the implicit function theorem yields that there exists a C^∞-function $t = t_\pm(x, y)$ such that the set $\{F_\pm = 0\}$ is parametrized by the graph $t = t_\pm(x, y)$ around the point $(-c, c, 0)$. By [4], we know that $t = t_\pm(x, y)$ has the following expression

$$t_\pm(c, y) = -c - \frac{\alpha(c)}{2}y^2 + \beta(c, y)y^3,$$

where $\alpha = \alpha(x)$ and $\beta = \beta(x, y)$ are C^∞-functions. Differentiating the equation $F_\pm(t_\pm(x, y), x, y) = 0$ with respect to y, we have that

$$-24t(x, y)t_y(x, y) - 4(x + t(x, y))^3 t_y(x, y) + 24y = 0, \tag{3.14}$$

where $t_y = \partial t/\partial y$. Substituting $(t, x, y) = (-c, c, 0)$, we get

$$-ct_y(c, 0) = t(c, 0)t_y(c, 0) = 0,$$

which implies that

$$t_y(c, 0) = 0. \tag{3.15}$$

Differentiating (3.14) with respect to y again, we have

$$-24t_y^2 - 24t\, t_{yy} \mp 12(x+t)^2 t_y^2 \mp 4(x+t)^3 t_{yy} + 24 = 0.$$

Substituting $(t, x, y) = (-c, c, 0)$ and (3.15), we get

$$-24ct_{yy} + 24 = 0,$$

namely $\alpha(c) = 2/c$. This implies that $t = t_\pm(x, y)$ is of type α^0_{II} at $(c, 0)$. □

[3] In [4], we wrote that the hyperbolic catenoids $\mathcal{C}_\pm$ are examples of surfaces of type α^0_I, but this is a typographical error. Also, in [4], we wrote that the time-like Scherk surface of the first kind (resp. of the second kind) is of type α^-_I (resp. of type α^-_{II}), however this is again a typographical error, and it is, in fact, of type α^-_{II} (resp. of type α^-_I).

In the authors' previous work [5], surfaces of type α_I^0 changing type across a light-like line have been constructed. The only other possibility for the existence of surfaces changing type across a light-like line must be of type α_{II}^0 (cf. [4]). So the following question is of interest:

Problem. *Do there exist zero-mean curvature surfaces of type α_{II}^0 which change type across a light-like line?*

Also, the existence of space-like maximal surfaces of type α_I^0 is unknown (the time-like surfaces given in [4, Example 1] are of types α_I^0 and α_{III}^-).

4. A relationship to fluid mechanics

As mentioned in the introduction, we give an application of Theorem 2.19 to fluid mechanics: Consider a two-dimensional flow on the xy-plane with velocity vector field $\boldsymbol{v} = (u, v)$, and with density ρ and pressure p. We assume the following:

(i) The fluid is *barotropic*, that is, there exists a strictly increasing function $p(s)$ $(s > 0)$ such that the pressure p is expressed by $p = p(\rho)$. A positive function c defined by

$$c^2 = \frac{dp}{d\rho} = p'(\rho) \tag{4.1}$$

is called the *local speed of sound*, cf. [1, pages 5–6].

(ii) The flow is steady, that is, $\boldsymbol{v}$, p and ρ do not depend on time.

(iii) There are no external forces,

(iv) and the flow is irrotational, that is, $\operatorname{rot}\boldsymbol{v}(= v_x - u_y) = 0$.

By the assumption (ii), the equation of continuity is reduced to

$$\operatorname{div}(\rho\boldsymbol{v}) = (\rho u)_x + (\rho v)_y = 0.$$

Hence there exists locally a smooth function $\psi = \psi(x, y)$ such that

$$\psi_x = -\rho v, \qquad \psi_y = \rho u, \tag{4.2}$$

which is called the *stream function* of the flow. The following assertion is well-known:

Fact 4.1. *The stream function ψ of a two-dimensional flow under the conditions* (i)–(iv) *satisfies*

$$(\rho^2 c^2 - \psi_y^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} + (\rho^2 c^2 - \psi_x^2)\psi_{yy} = 0. \tag{4.3}$$

Proof. By the assumptions (ii), (iii) and (iv), Euler's equation of motion

$$\frac{\partial\boldsymbol{v}}{\partial t} + u\frac{\partial\boldsymbol{v}}{\partial x} + v\frac{\partial\boldsymbol{v}}{\partial y} + \frac{1}{\rho}\operatorname{grad} p = 0$$

is reduced to

$$uu_x + vv_x + \frac{p_x}{\rho} = 0, \qquad uu_y + vv_y + \frac{p_y}{\rho} = 0, \tag{4.4}$$

that is,

$$dp + \rho q\, dq = 0 \qquad (q = |\boldsymbol{v}| = \sqrt{u^2 + v^2}). \tag{4.5}$$

Here, by the barotropicity (i), we have

$$p_x = \frac{\partial}{\partial x} p(\rho) = p'(\rho) \frac{\partial \rho}{\partial x} = c^2 \rho_x, \qquad p_y = c^2 \rho_y. \tag{4.6}$$

Substituting these into the equation of motion (4.4), we have

$$\rho_x = \frac{p_x}{c^2} = -\frac{\rho}{c^2}(uu_x + vv_x), \qquad \rho_y = -\frac{\rho}{c^2}(uu_y + vv_y), \tag{4.7}$$

and hence

$$\begin{aligned}
(\rho v)_x &= \rho_x v + \rho v_x = -\frac{\rho}{c^2}(uu_x + vv_x)v + \rho v_x, \\
(\rho v)_y &= \rho_y v + \rho v_y = -\frac{\rho}{c^2}(uu_y + vv_y)v + \rho v_y, \\
(\rho u)_x &= \rho_x u + \rho u_x = -\frac{\rho}{c^2}(uu_x + vv_x)u + \rho u_x, \\
(\rho u)_y &= \rho_y u + \rho u_y = -\frac{\rho}{c^2}(uu_y + vv_y)u + \rho u_y
\end{aligned}$$

hold. Thus, we have

$$\begin{aligned}
(\rho^2 c^2 - \psi_y^2)\psi_{xx} &= (\rho^2 c^2 - \rho^2 u^2)(-\rho v)_x \\
&= -\rho^2 (c^2 - u^2) \left(-\frac{\rho}{c^2}(uu_x + vv_x)v + \rho v_x \right), \\
\psi_x \psi_y \psi_{xy} &= -\rho^2 uv(-\rho v)_y = \rho^2 uv \left(-\frac{\rho}{c^2}(uu_y + vv_y)v + \rho v_y \right), \\
\psi_x \psi_y \psi_{yx} &= -\rho^2 uv(\rho u)_x = -\rho^2 uv \left(-\frac{\rho}{c^2}(uu_x + vv_x)u + \rho u_x \right), \\
(\rho^2 c^2 - \psi_x^2)\psi_{yy} &= (\rho^2 c^2 - \rho^2 v^2)(\rho u)_y \\
&= \rho^2 (c^2 - v^2) \left(-\frac{\rho}{c^2}(uu_y + vv_y)u + \rho u_y \right).
\end{aligned}$$

Summing these up, it holds that

$$(\rho^2 c^2 - \psi_y^2)\psi_{xx} + 2\psi_x \psi_y \psi_{xy} + (\rho^2 c^2 - \psi_x^2)\psi_{yy} = \rho^3 (u^2 + v^2 - c^2)(v_x - u_y). \tag{4.8}$$

Here, by the assumption (iv), we have $v_x = u_y$. Then we have the conclusion. $\square$

When $\rho c = 1$, the equation (4.3) coincides with the zero mean curvature equation (1.1). In this case, (4.1) yields that

$$p = p_0 - \frac{1}{\rho}, \tag{4.9}$$

where p_0 is a positive constant. This means that the zero mean curvature equation induces a virtual gas. For actual gas, the pressure p is proportional to ρ^γ, where γ is a constant (> 1, $\gamma \approx 1.4$ for air).

Differentiating (4.9), we have that

$$dp = \frac{d\rho}{\rho^2}. \tag{4.10}$$

Substituting (4.10) into (4.5), we have that

$$d\left(-\frac{1}{\rho^2} + q^2\right) = 0,$$

that is, there exists a constant k such that

$$-\frac{1}{\rho^2} + q^2 = k. \tag{4.11}$$

Here, by (4.2), it holds that

$$(u, v) = \frac{1}{\rho}(\psi_y, -\psi_x). \tag{4.12}$$

Thus, (4.11) is equivalent to

$$1 - \psi_x^2 - \psi_y^2 = -k\rho^2.$$

We suppose that $1 - \psi_x^2 - \psi_y^2$ does not vanish identically. Then $k \neq 0$ and we may set $\sqrt{|k|} = 1/\rho_0$, where ρ_0 is a positive constant. Then we have

$$\rho = \rho_0 |1 - \psi_x^2 - \psi_y^2|^{1/2}.$$

Note that by (4.12) and the assumption $c\rho = 1$, the speed $|\boldsymbol{v}| = q$ is greater than (resp. less than) the speed of sound c, that is, the flow is *supersonic* (resp. *subsonic*), if and only if $1 - \psi_x^2 - \psi_y^2 < 0$ (resp. > 0).

Suppose now that the flow changes from being subsonic to supersonic at a curve

$$\sigma(t) := \big(x(t), y(t)\big) \qquad (a \leq t \leq b).$$

Without loss of generality, we may assume that t is an arclength parameter of the curve σ. In particular,

$$\rho = \rho_0 |1 - \psi_x^2 - \psi_y^2|^{1/2} = 0$$

holds on the curve σ. Since the local speed of sound is given by

$$c = \rho^{-1} = \rho_0^{-1} |1 - \psi_x^2 - \psi_y^2|^{-1/2},$$

the curve σ is a singularity of the flow, although the stream function itself is real analytic near σ. Moreover, we suppose that each point of the curve $\sigma(t)$ is a non-degenerate point of type change with respect to ψ. Then, as seen in the proof of Proposition 2.6, we can take the parameter t of the curve σ such that $t \mapsto (t, \sigma(t))$ gives a non-degenerate null curve in $\boldsymbol{R}^3_1$. Moreover, it holds that

$$x'(t) = \psi_x\bigl(x(t), y(t)\bigr), \qquad y'(t) = \psi_y\bigl(x(t), y(t)\bigr)$$

and $x'(t)^2 + y'(t)^2 = 1$. Since $\nabla B \neq 0$ at a non-degenerate point of type change, the proof of Proposition 2.5 yields that $\sigma''(t) \neq 0$. Since $x'(t)^2 + y'(t)^2 = 1$, this implies that $\sigma(t)$ is a locally convex curve. Consequently, we get the following assertion:

Theorem 4.2. *Let $\sigma(t) := \bigl(x(t), y(t)\bigr)$ be a locally convex curve on the xy-plane with an arc-length parameter t. Then the graph $t = \psi(x, y)$ of the zero mean curvature surface $\hat{\varphi}_{\tilde{\sigma}}$ as in Theorem 2.19 associated to the null curve $\tilde{\sigma} := \bigl(t, x(t), y(t)\bigr)$ gives a real analytic stream function satisfying* (4.3) *with* (4.9) (*i.e.* $c\rho = 1$) *which changes from being subsonic to supersonic at the curve σ. Moreover,*

$$(u, v) := \frac{1}{\rho}(\psi_y, -\psi_x)$$

gives the velocity vector of the flow such that

(1) *$u^2 + v^2$ diverges to ∞ as (x, y) approaches the image of σ.*
(2) *The flow changes from being subsonic to being supersonic across σ.*
(3) *The acceleration vector $\sigma''(t)$ points to the supersonic region.*

Proof. Since σ is locally convex, its lift $\tilde{\sigma}$ is a non-degenerate null curve in $\boldsymbol{R}^3_1$. Then we can apply Theorem 2.19 for the curve $\tilde{\sigma}$, and get a graph $\psi(x, y)$ of a zero mean curvature surface which changes type at $\tilde{\sigma}$. Then ψ can be considered as a stream function which changes from being subsonic to supersonic at the curve σ. We set

$$P := \frac{\sigma(t+s) + \sigma(t-s)}{2} \approx \sigma(t) + \frac{\sigma''(t)}{2} s^2,$$

which is the midpoint of the two points $\sigma(t+s), \sigma(t-s)$ of the curve σ. By Theorem 2.19, the flow is supersonic at the point P, which proves the last assertion. □

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References

[1] L. Bers, Mathematical aspects of subsonic and transonic gas dynamics, John Wiley & Sons (1958).

[2] F. J. M. Estudillo and A. Romero, *Generalized maximal surfaces in Lorentz-Minkowski space L^3*, Math. Proc. Camb. Phil. Soc., **111** (1992), 515–524.

[3] S. Fujimori, Y. Kawakami, M. Kokubu, W. Rossman, M. Umehara, K. Yamada, *Hyperbolic Metrics on Riemann surfaces and Space-like CMC-1 surfaces in de Sitter 3-space*, M. Sánchez et al (eds), Recent Trends in Lorentzian Geometry, Springer Proceedings in Mathematics & Statistics 26, 1–47 (2012).

[4] S. Fujimori, Y. W. Kim, S.-E. Koh, W. Rossman, H. Shin, H. Takahashi, M. Umehara, K. Yamada and S.-D. Yang, *Zero mean curvature surfaces in $\mathbf{L}^3$ containing a light-like line*, C.R. Acad. Sci. Paris. Ser. I. **350** (2012) 975–978.

[5] S. Fujimori, Y. W. Kim, S.-E. Koh, W. Rossman, H. Shin, M. Umehara, K. Yamada and S.-D. Yang, *Zero mean curvature surfaces in Lorenz-Minkowski* 3*-space which change type across a light-like line*, to appear in Osaka J. Math.

[6] S. Fujimori and F. J. Lopez, *Nonorientable maximal surfaces in the Lorentz-Minkowski* 3*-space*, Tohoku Math. J., **62** (2010), 311–328.

[7] S. Fujimori, W. Rossman, M. Umehara, K. Yamada and S.-D. Yang, *New maximal surfaces in Minkowski* 3*-space with arbitrary genus and their cousins in de Sitter* 3*-space*, Results in Math., **56** (2009), 41–82.

[8] S. Fujimori, W. Rossman, M. Umehara, K. Yamada and S.-D. Yang, *Embedded triply periodic zero mean curvature surfaces of mixed type in Lorentz-Minkowski 3-space*, Michigan Math. J. **63** (2014), 189–207.

[9] S. Fujimori, K. Saji, M. Umehara and K. Yamada, *Singularities of maximal surfaces*, Math. Z., **259** (2008), 827–848.

[10] C. Gu, *The extremal surfaces in the* 3*-dimensional Minkowski space*, Acta. Math. Sinica. **1** (1985), 173–180.

[11] C. Gu, *A global study of extremal surfaces in* 3*-dimensional Minkowski space*, Differential geometry and differential equations (Shanghai, 1985), Lecture Notes in Math., 1255, Springer, Berlin, (1987), 26–33.

[12] C. Gu, *Extremal surfaces of mixed type in Minkowski space $\mathbf{R}^{n+1}$*, Variational methods (Paris, 1988), Progr. Nonlinear Differential Equations Appl. **4**, Birkhauser Boston, Boston, MA, (1990) 283–296.

[13] J. Inoguchi and M. Toda, *Timelike minimal surfaces via loop groups*, Acta. Appl. Math. **83** (2004), 313–335.

[14] Y. W. Kim, S.-E. Koh, H. Shin and S.-D. Yang, *Spacelike maximal surfaces, timelike minimal surfaces, and Björling representation formulae*, Journal of Korean Math. Soc. **48** (2011), 1083–1100.

[15] Y. W. Kim and S.-D. Yang, *A family of maximal surfaces in Lorentz-Minkowski three-space*, Proc. Amer. Math. Soc. **134** (2006), 3379–3390.

[16] Y. W. Kim and S.-D. Yang, *Prescribing singularities of maximal surfaces via a singular Björling representation formula*, J. Geom. Phys., **57** (2007), 2167–2177.

[17] V. A. Klyachin, *Zero mean curvature surfaces of mixed type in Minkowski space*, Izv. Math., **67** (2003), 209–224.

[18] O. Kobayashi, *Maximal surfaces in the 3-dimensional Minkowski space* L^3, Tokyo J. Math., **6** (1983), 297–309.

[19] J. M. Rassias, *Mixed Type Partial Differential Equations With Initial and Boundary Values in Fluid Mechanics*, International Journal of Applied Mathematics & Statistics, Int. J. Appl. Math. Stat.; Vol. 13; No. J08; June 2008; 77-107 ISSN 0973-1377.

[20] V. Sergienko and V.G. Tkachev, *Doubly periodic maximal surfaces with singularities*, Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999), 571–584, Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000.

[21] M. Umehara and K. Yamada, *Maximal surfaces with singularities in Minkowski space*, Hokkaido Math. J., **35** (2006), 13–40.

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