Math. J. Okayama Univ. **57** (2015), 123–128

THE EQUIVARIANT SIMPLICIAL DE RHAM COMPLEX AND THE CLASSIFYING SPACE OF A SEMI-DIRECT PRODUCT GROUP

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Abstract. We show that the cohomology group of the total complex of the equivariant simplicial de Rham complex is isomorphic to the cohomology group of the classifying space of a semi-direct product group.

1. Introduction

In [8], Weinstein introduced the equivariant version of the simplicial de Rham complex. That is a double complex whose components are equivariant differential forms which is called the Cartan model([1]). Weinstein expected that the cohomology group of its total complex is isomorphic to $H^*(B(G \rtimes H))$. Here $B(G \rtimes H)$ is the classifying space of a semi-direct product group. In this paper, we show this conjecture is true.

2. Review of the simplicial de Rham complex

In this section we recall the relation between the simplicial manifold NG and the classifying space BG. We also recall the notion of the equivariant version of the simplicial de Rham complex.

2.1. The double complex on simplicial manifold.

For any Lie group G, we have simplicial manifolds NG, $N\bar{G}$ and simplicial G-bundle $\gamma : N\bar{G} \to NG$ as follows:

$NG(q) = \overbrace{G \times \cdots \times G}^{q-times} \ni (g_1, \cdots, g_q)$:
face operators $\varepsilon_i : NG(q) \to NG(q-1)$

$$\varepsilon_i(g_1, \cdots, g_q) = \begin{cases} (g_2, \cdots, g_q) & i = 0 \\ (g_1, \cdots, g_i g_{i+1}, \cdots, g_q) & i = 1, \cdots, q-1 \\ (g_1, \cdots, g_{q-1}) & i = q \end{cases}$$

Mathematics Subject Classification. Primary 55R35; Secondary 55R91.
Key words and phrases. simplicial de Rham complex, classifying space.

$N\bar{G}(q) = \overbrace{G \times \cdots \times G}^{q+1-times} \ni (\bar{g}_1, \cdots, \bar{g}_{q+1})$:
face operators $\bar{\varepsilon}_i : N\bar{G}(q) \to N\bar{G}(q-1)$

$$\bar{\varepsilon}_i(\bar{g}_1, \cdots, \bar{g}_{q+1}) = (\bar{g}_1, \cdots, \bar{g}_i, \bar{g}_{i+2}, \cdots, \bar{g}_{q+1}) \qquad i = 0, 1, \cdots, q$$

We define $\gamma : N\bar{G} \to NG$ as $\gamma(\bar{g}_1, \cdots, \bar{g}_{q+1}) = (\bar{g}_1\bar{g}_2^{-1}, \cdots, \bar{g}_q\bar{g}_{q+1}^{-1})$.

(The standard definition also involves degeneracy operators but we do not need them here).

Remark 2.1. Here we use the notation $\bar{g}_i$ to distinguish elements in NG from elements in $N\bar{G}$. It does not mean the complex conjugate.

For any simplicial manifold $\{X_*\}$, we can associate a topological space $\| X_* \|$ called the fat realization defined as follows.

$$\| X_* \| \stackrel{\text{def}}{=} \coprod_n \Delta^n \times X_n / \ (\varepsilon^i t, x) \sim (t, \varepsilon_i x).$$

Here Δ^n is the standard n-simplex and ε^i is a face map of it. It is well-known that $\| \gamma \| : \| N\bar{G} \| \to \| NG \|$ is the universal bundle $EG \to BG$ (see [4] [6] [7], for instance).

Now we introduce a double complex associated to a simplicial manifold.

Definition 2.1. For any simplicial manifold $\{X_*\}$ with face operators $\{\varepsilon_*\}$, we have a double complex $\Omega^{p,q}(X) \stackrel{\text{def}}{=} \Omega^q(X_p)$ with derivatives as follows:

$$d' = \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*, \qquad d'' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p).$$

□

For NG and $N\bar{G}$ the following holds.

Theorem 2.1 ([2] [4] [6]). *There exist ring isomorphisms*

$$H^*(\Omega^*(NG)) \cong H^*(BG), \qquad H^*(\Omega^*(N\bar{G})) \cong H^*(EG).$$

Here $\Omega^*(NG)$ and $\Omega^*(N\bar{G})$ means the total complexes.

□

2.2. **Equivariant version.** When a Lie group H acts on a manifold M, there is the complex of equivariant differential forms $\Omega^*_H(M) := (\Omega^*(M) \otimes S(\mathcal{H}^*))^H$ with suitable differential d_H ([1] [3]). Here $\mathcal{H}$ is the Lie algebra of H and $S(\mathcal{H}^*)$ is the algebra of polynomial functions on $\mathcal{H}$. This is called the Cartan Model. When M is a Lie group G, we can define the double complex $\Omega^*_H(NG(*))$ in the same way as in Definition 2.1. This double complex is originally introduced by Weinstein in [8].

3. The triple complex on bisimplicial manifold

In this section we construct a triple complex on a bisimplicial manifold.

A bisimplicial manifold is a sequence of manifolds with horizontal and vertical face and degeneracy operators which commute with each other. A bisimplicial map is a sequence of maps commuting with horizontal and vertical face and degeneracy operators. Let H be a subgroup of G. We define a bisimplicial manifold $NG(*) \rtimes NH(*)$ as follows;

$$NG(p) \rtimes NH(q) := \overbrace{G \times \cdots \times G}^{p-times} \times \overbrace{H \times \cdots \times H}^{q-times}.$$

Horizontal face operators $\varepsilon^G_i : NG(p) \rtimes NH(q) \to NG(p-1) \rtimes NH(q)$ are the same as the face operators of $NG(p)$. Vertical face operators ε^H_i : $NG(p) \rtimes NH(q) \to NG(p) \rtimes NH(q-1)$ are

$$\varepsilon^H_i(\vec{g}, h_1, \cdots, h_q) = \begin{cases} (\vec{g}, h_2, \cdots, h_q) & i = 0 \\ (\vec{g}, h_1, \cdots, h_i h_{i+1}, \cdots, h_q) & i = 1, \cdots, q-1 \\ (h_q \vec{g} h_q^{-1}, h_1, \cdots, h_{q-1}) & i = q. \end{cases}$$

Here $\vec{g} = (g_1, \cdots, g_p)$.

We define a bisimplicial map $\gamma_\rtimes : N\bar{G}(p) \times N\bar{H}(q) \to NG(p) \rtimes NH(q)$ as $\gamma_\rtimes(\vec{\bar{g}}, \bar{h}_1, \cdots, \bar{h}_{q+1}) = (\bar{h}_{q+1}\gamma(\vec{\bar{g}})\bar{h}^{-1}_{q+1}, \gamma(\bar{h}_1, \cdots, \bar{h}_{q+1}))$. Now we fix a semi-direct product operator $\cdot_\rtimes$of $G \rtimes H$ as $(g, h) \cdot_\rtimes (g', h') := (ghg'h^{-1}, hh')$, then $G \rtimes H$ acts $N\bar{G}(p) \times N\bar{H}(q)$ by right as $(\vec{\bar{g}}, \vec{\bar{h}}) \cdot (g, h) = (h^{-1}\vec{\bar{g}}gh, \vec{\bar{h}}h)$. Since $\gamma_\rtimes(\vec{\bar{g}}, \vec{\bar{h}}) = \gamma_\rtimes((\vec{\bar{g}}, \vec{\bar{h}}) \cdot (g, h))$, one can see that $\gamma_\rtimes$ is a principal $(G \rtimes H)$-bundle. $\parallel N\bar{G}(*) \times N\bar{H}(*) \parallel$ is $EG \times EH$ so its homotopy groups are trivial in any dimension and $\parallel NG(*) \rtimes NH(*) \parallel$ is homeomorphic to $(EG \times EH)/(G \rtimes H)$. We can also check that $EG \times EH \to (EG \times EH)/(G \rtimes H)$ is a principal $(G \rtimes H)$-bundle since $(G \rtimes H)$ is an absolute neighborhood retract (see for example [4] P.73). Hence $\parallel NG(*) \rtimes NH(*) \parallel$ is a model of $B(G \rtimes H)$.

Definition 3.1. For a bisimplicial manifold $NG(*) \rtimes NH(*)$, we have a triple complex as follows:

$$\Omega^{p,q,r}(NG(*) \rtimes NH(*)) \stackrel{\text{def}}{=} \Omega^r(NG(p) \rtimes NH(q))$$

Derivatives are:

$$d' = \sum_{i=0}^{p+1}(-1)^i(\varepsilon_i^G)^*, \qquad d'' = \sum_{i=0}^{q+1}(-1)^i(\varepsilon_i^H)^* \times (-1)^p$$

$$d''' = (-1)^{p+q} \times \text{the exterior differential on } \Omega^*(NG(p) \rtimes NH(q)).$$

□

Let $C^*(X)$ denote the set of singular cochains of a topological space X. We can also define the triple complex $C^{p,q,r}(NG(*) \rtimes NH(*))$ in the same way. Applying the de Rham theorem and the lemma below twice, we can see that the total complex $\Omega^*(NG \rtimes NH)$ of the triple complex in the Definition 3.1 is quasi-isomorphic to the total complex of $C^{p,q,r}(NG(*) \rtimes NH(*))$.

Lemma 3.1 ([4], lemma 1.19). *Let $K_1^{p,q}$ and $K_2^{p,q}$ be 1.quadrant double complexes, i.e. $K_1^{p,q} = K_2^{p,q} = 0$ if either $p < 0$ or $q < 0$. Suppose $f : K_1^{*,*} \to K_2^{*,*}$ is a homomorphism of double complexes and suppose $f^{p,q} : H^p(K_1^{*,q}, d_1') \to H^p(K_2^{*,q}, d_2')$ is an isomorphism. Then also $f^* : H^*(K_1, d_1) \to H^*(K_2, d_2)$ is an isomorphism.* □

Remark 3.1. Let $C_*(X)$ denote the set of singular chains of a topological space X. We can also define the triple complex $C_{p,q,r}(NG(*) \rtimes NH(*)) := C_r(NG(p) \rtimes NH(q))$ of the singular chains in the same way.

4. Main theorem

Theorem 4.1. *If H is compact, there exists an isomorphism*

$$H(\Omega_H^*(NG)) \cong H(\Omega^*(NG \rtimes NH)) \cong H^*(B(G \rtimes H)).$$

Here $\Omega_H^*(NG)$ means the total complex in subsection 2.2.

Proof. At first we recall the Getzler's result in [5]. When a Lie group H acts on a manifold M by left, there is a simplicial manifold $\{M \rtimes NH(q)\}$ with face operators:

$$\varepsilon_i(u, h_1, \cdots, h_q) = \begin{cases} (u, h_2, \cdots, h_q) & i = 0 \\ (u, h_1, \cdots, h_i h_{i+1}, \cdots, h_q) & i = 1, \cdots, q-1 \\ (h_q u, h_1, \cdots, h_{q-1}) & i = q. \end{cases}$$

We need the following theorem for the proof.

Theorem 4.2 ([5]). *If H is compact, there is a cochain map between the total complex of the double complex $\Omega^*(M \rtimes NH(*))$ and $(\Omega_H^*(M), d_H)$ which induces an isomorphism in cohomology.*

As a corollary of this theorem, we obtain the following statement.

Corollary 4.1. *For any fixed p, the total complex of the double complex* $\Omega^*(NG(p) \rtimes NH(*))$ *is quasi-isomorphic to* $(\Omega^*_H(G^p), (-1)^p d_H)$

Hence using the Lemma 3.1, we can see that $H^*(\Omega^*_H(NG))$ is isomorphic to $H^*(\Omega^*(NG \rtimes NH))$.

Now we prove the existence of the another isomorphism. Let $S_*(X)$ denote the set of singular simplexes of a topological space X. For a triple simplicial set $S_r(NG(p) \rtimes NH(q))$, we have the fat realization

$$\coprod_{r,p,q\geq 0} \Delta^p \times \Delta^q \times \Delta^r \times S_r(NG(p) \rtimes NH(q))/\sim .$$

with suitable identifications. This is a CW complex and the set of n-cells are in one-to-one correspondence with $\coprod_{r+p+q=n} S_r(NG(p) \rtimes NH(q))$. Its homology group coincides with the homology group of the total complex of the triple complex $C_{p,q,r}(NG \rtimes NH)$.

So we need to show the cohomology group of this CW complex is isomorphic to $H^*(\| NG \rtimes NH \|)$. We recall that the map $\rho : \Delta^r \times S_r(X) \to X$ which is defined as $\rho(t, \sigma_r) := \sigma_r(t)$ induces an isomorphism $H_*(\coprod_r \Delta^r \times S_r(X)/\sim) \cong H_*(X)$ (see for instance [4] P.82). Hence for any fixed p, q, the following map $\rho_{p,q}$ which is same as ρ induces an isomorphism in homology.

$$\rho_{p,q} : \coprod_r \Delta^r \times S_r(NG(p) \rtimes NH(q))/\sim \;\to NG(p) \rtimes NH(q).$$

We also use the following lemma.

Lemma 4.1 ([4], Lemma 5.16)**.** *Let* $f : \{X_*\} \to \{X'_*\}$ *be a simplicial map of simplicial spaces such that* $f_p : X_p \to X'_p$ *induces an isomorphism in homology with coefficients in a ring* λ *for all p. Then* $\| f \| : \| X_* \| \to \| X'_* \|$ *also induces an isomorphism in homology and cohomology with coefficients in* λ.

By applying the Lemma 4.1, we see that for any fixed p, $\| \rho_{p,*} \| : \coprod_q \Delta^q \times (\coprod_r \Delta^r \times S_r(NG(p) \rtimes NH(q))/\sim)/\sim \;\to \coprod_q \Delta^q \times NG(p) \rtimes NH(q)/\sim$ induces an isomorphism in homology.

Hence again applying the Lemma 4.1 we can see that

$$\| \rho_{*,*} \| : \coprod_p \Delta^p \times (\coprod_q \Delta^q \times (\coprod_r \Delta^r \times S_r(NG(p) \rtimes NH(q))/\sim)/\sim)/\sim$$

$$\to \coprod_p \Delta^p \times (\coprod_q \Delta^q \times NG(p) \rtimes NH(q)/\sim)/\sim .$$

induces an isomorphism in cohomology. This completes the proof of Theorem 4.1. □

Acknowledgments.
The author is indebted to his supervisor, Professor H. Moriyoshi for helpful discussion and good advice. The author would like to thank the referee for his/her several suggestions to improve the present paper.

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(*Received March 11, 2013*)
(*Revised December 20, 2013*)