**SUPPLEMENTED MORPHISMS**

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**Abstract.** In the present paper, left $R$-modules $M$ and $N$ are studied under the assumptions that $\text{Hom}_R(M, N)$ is supplemented. It is shown that $\text{Hom}(M, N)$ is $(\oplus, \mathcal{G}^*, \text{amply})$-supplemented if and only if $N$ is $(\oplus, \mathcal{G}^*, \text{amply})$-supplemented. Some applications to cosemisimple modules, refinable modules and UCC-modules are presented. Finally, the relationship between the Jacobson radical $J[M, N]$ of $\text{Hom}_R(M, N)$ and $\text{Hom}_R(M, N)$ is supplemented are investigated. Let $M$ be a finitely generated, self-projective left $R$-module and $N \in \text{Gen}(M)$. We show that if $\text{Hom}(M, N)$ is supplemented and $N$ has GD2 then $\text{Hom}(M, N)/J(M, N)$ is semisimple as a left $E_M$-module.

1. **Introduction**

Throughout this article, all rings are associative with unity, and all modules are unital left modules. Let $R$ be a ring. If $R_M$ and $R_N$ are modules, we use the following notations: $E_M = \text{End}(M_R)$. If $N \subseteq M$, then $N \leq M$, $N \ll M$, $N \leq_d M$ and $\text{Rad}(M)$ denote $N$ is a submodule of $M$, $N$ is a small submodule of $M$, $N$ is a direct summand of $M$ and the Jacobson radical of $M$, respectively.

We recall the fundamental terminology for our paper. Let $U$ be a submodule of an $R$-module $M$. A submodule $V$ of $M$ is called *supplement* of $U$ in $M$ if $V$ is a minimal element in the set of submodules $L$ of $M$ with $U + L = M$. $V$ is a supplement of $U$ if and only if $U + V = M$ and $U \cap V$ is small in $V$. An $R$-module $M$ is *supplemented* if every submodule of $M$ has a supplement in $M$. The module $M$ is *amply supplemented* if, for any submodules $A$ and $B$ of $M$ with $M = A + B$, there exists a supplement $P$ of $A$ such that $P \leq B$.

For the other definitions in this note, we refer to [1], [14] and [17].

In the present paper, we establish an order-preserving bijective correspondence between the sets of coclosed left $R$-submodules of $N$ and coclosed left $E_M$-submodules of $\text{Hom}_R(M, N)$. This concept is extremely useful in analyzing the structure of the endomorphism ring of a supplemented module. For instance, by definitions of supplemented modules, one easily checks that there is no any direct implication between the notions supplemented modules and when $\text{Hom}_R(M, N)$ is supplemented. But we prove that if.

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$M$ is a finitely generated, self-projective left $R$-module and $N \in \text{Gen}(M)$, then $\text{Hom}(M, N)$ is $(\oplus, \text{amply}, G^*)$-supplemented if and only if $N$ is $(\oplus, \text{amply}, G^*)$-supplemented.

Beidar and Kasch [2] defined and studied substructures, the singular ideal $\Delta(M, N)$ and the co-singular ideal $\nabla(M, N)$, of $\text{Hom}_R(M, N)$ such as:

\[
\Delta(M, N) = \{ f \in \text{Hom}_R(M, N) : \ker(f) \leq^e M \}
\]

\[
\nabla(M, N) = \{ f \in \text{Hom}_R(M, N) : \im(f) \ll N \}.
\]

The other substructure, radical, of $\text{Hom}(M, N)$ was introduced and studied by Kasch-Mader [11] and Nicholson-Zhou [15]. They have shown that:

\[
J(M, N) = \{ \alpha \in \text{Hom}_R(M, N) : 1_M - \alpha \beta \in \text{Aut}(M), \forall \beta \in \text{Hom}_R(N, M) \} = \{ \alpha \in \text{Hom}_R(M, N) : \alpha \beta \in J(E_M), \forall \beta \in \text{Hom}_R(N, M) \} = \{ \alpha \in \text{Hom}_R(M, N) : \beta \alpha \in J(E_N), \forall \beta \in \text{Hom}_R(N, M) \}.
\]

Thus, we have $J[M, M] = J(E_M)$, which is similar to well known notion $J[R, R] = J(R)$. For the other new properties of these substructures, we refer to [12], [13] and [18]. Let $M$ be a finitely generated, self-projective left $R$-module and $N \in \text{Gen}(M)$. We show that if $\text{Hom}_R(M, N)$ is supplemented, then $\text{Hom}_R(M, N)/\nabla(M, N)$ is semisimple as a left $E_M$-module.

2. Results.

Let $M$ and $N$ be $R$-modules. If there is an epimorphism $f : M^{(\Lambda)} \longrightarrow N$ for some set $\Lambda$, then $N$ is said to be an $M$-generated module, denoted by $N \in \text{Gen}(M)$, (see [17]). We denote

\[
N_M = M\text{Hom}(M, N) = \{ \sum_{i=1}^k m_i f_i : m_i \in M, f_i \in \text{Hom}(M, N) \}.
\]

Clearly, if $N_M = N$ then $N$ is $M$-generated.

**Lemma 2.1.** (1) Let $N$ and $M$ be two left $R$-modules. Then $N$ is an $M$-generated $R$-module if and only if, for all non-zero $R$-homomorphism $f : N \to K$, there exists $h : M \to N$ such that $hf \neq 0$.

(2) If $N_1$ and $N_2$ are $M$-generated modules with $N = N_1 + N_2$, then $N$ is also $M$-generated.

**Proof.** Clear. \qed

The class of supplemented modules under $\text{Hom}$ need not closed under taking factor modules, in general.

**Proposition 2.2.** Let $M$ be a $P$-projective module and $P \in \text{Gen}(M)$. If $\text{Hom}(M, P)$ is supplemented then every homomorphic image of $P$ is again supplemented under $\text{Hom}$.
Proof. Let $X$ be a submodule of $P$. We will prove that $\text{Hom}(M, P/X)$ is a supplemented $E_M$-module. Let $A$ be a submodule of $\text{Hom}(M, P/X)$. For every element $f \in A$, there exists $g \in \text{Hom}(M, P)$ such that $gs = f$, where $s : P \to P/X$ is the canonical projection. Let $B$ be the set of all $h \in \text{Hom}(M, P)$ such that $h$ extends an elements in $A$. It is a simple matter to prove that $B$ is a submodule of $\text{Hom}(M, P)$. Since $\text{Hom}(M, P)$ is supplemented, there exists a submodule $C$ of $\text{Hom}(M, P)$ such that $C$ is minimal for the property $\text{Hom}(M, P) = B + C$. Let $D = \{fs \mid f \in C\}$. It is clear that $D$ is a submodule of $\text{Hom}(M, P/X)$ and $\text{Hom}(M, P/X) = A + D$. Let $E$ be a submodule of $\text{Hom}(M, P/X)$ contained in $D$ such that $\text{Hom}(M, P/X) = A + E$. Therefore

$\text{Hom}(M, P) = \text{Hom}(M, X) + B + F$,

where $F = \{f \in C \mid fs \in E\}$ and it is a submodule of $C$. But $\text{Hom}(M, X) \leq B$. Then $\text{Hom}(M, P) = B + F$. Since $F \leq C$, we have $F = C$. Consequently, $D$ is a supplement of $A$ in $\text{Hom}(M, P/X)$. Hence $\text{Hom}(M, P/X)$ is a supplemented $E_M$-module. □

Let $K \subset L \subset M$. Recall that $K$ is said to be cosmall of $L$ in $M$ if $L/K \ll M/K$ and we denote it by $K \overset{cs}{\rightarrow} L$. A submodule $L$ of the module $M$ is called co-closed in $M$ if $K \overset{cs}{\rightarrow} L$ implies $K = L$.

Lemma 2.3. Let $K \subset L \subset M$. Then $K \overset{cs}{\rightarrow} L$ if and only if, for any submodule $X$ of $M$, $M = L + X$ implies $M = K + X$.

Proof. It is well known. □

Let $M$ be an $R$-module and $X, Y \leq M$. In [3], the notion of $\beta^*$ relation on submodules $X, Y$ of $M$, denoted by $X \beta^* Y$, is defined such as $(X + Y)/Y \ll M/Y$ and $(X + Y)/X \ll M/X$. We notice that $\beta^*$ is an equivalence relation by [3, Lemma 2.2].

Lemma 2.4. Let $M$ be an $R$-module and $X, Y \leq M$. Then $X \beta^* Y$ if and only if for each $A \leq M$ such that $M = X + Y + A$ then $M = X + A$ and $M = Y + A$.

Proof. See [3, Theorem 2.3]. □

Proposition 2.5. Let $M$ be a finitely generated self-projective $R$-module and $N \in \text{Gen}(M)$. Then the following conditions hold.

1. For every $K, L \leq N$, $\text{Hom}(M, K + L) = \text{Hom}(M, K) + \text{Hom}(M, L)$.
2. For every $I \leq \text{Hom}(M, N)$, $I = \text{Hom}(M, MI)$.
3. If $K \leq N$, then $K_M \beta^* K$ and $K_M \overset{cs}{\rightarrow} K$ in $N$.
4. Let $K \leq L \leq N$.
   a) If $K \overset{cs}{\rightarrow} L$ in $N$, then $\text{Hom}(M, K) \overset{cs}{\rightarrow} \text{Hom}(M, L)$ in $\text{Hom}(M, N)$. 

(1) For every $K, L \leq N$, $\text{Hom}(M, K + L) = \text{Hom}(M, K) + \text{Hom}(M, L)$.
(2) For every $I \leq \text{Hom}(M, N)$, $I = \text{Hom}(M, MI)$.
(3) If $K \leq N$, then $K_M \beta^* K$ and $K_M \overset{cs}{\rightarrow} K$ in $N$.
(4) Let $K \leq L \leq N$.
   a) If $K \overset{cs}{\rightarrow} L$ in $N$, then $\text{Hom}(M, K) \overset{cs}{\rightarrow} \text{Hom}(M, L)$ in $\text{Hom}(M, N)$.
b) If $K\beta^*L$, then $\text{Hom}(M, K)\beta^*\text{Hom}(M, L)$.

(5) Let $A, B \leq \text{Hom}(M, N)$.
   a) If $A \overset{cs}{\hookrightarrow} B$ in $\text{Hom}(M, N)$, then $MA \overset{cs}{\hookrightarrow} MB$ in $N$.
   b) If $A\beta^*B$, then $(MA)\beta^*(MB)$.

(6) If $K \leq \text{cs} N$, then $K \in \text{Gen}(M)$.

Proof. (1) and (2) have been shown in [17, 18.4].

(3) For two submodules $X$ and $Y$ of $M$ with $X \leq Y$, it is clear that $X\beta^*Y$ and $X \overset{cs}{\hookrightarrow} Y$ in $M$ are equivalent. Let $K \leq N$ and $N = K + L$ for some $L \leq N$. Since $N \in \text{Gen}(M)$, we can obtain that

$$N = M\text{Hom}(M, N) = M\text{Hom}(M, K) + M\text{Hom}(M, L) \subseteq K_M + L$$

by (1). It follows that $N = K_M + L$. By Lemma 2.3, we can obtain that $K_M \overset{cs}{\hookrightarrow} K$ in $N$.

(4) Let $K \overset{cs}{\hookrightarrow} L$ in $N$ (or $K\beta^*L$, respectively) and $\text{Hom}(M, N) = \text{Hom}(M, L) + A$ for some $A \leq \text{Hom}(M, N)$. Since $N \in \text{Gen}(M)$, we can obtain that

$$N = M\text{Hom}(M, N) = M\text{Hom}(M, L) + MA \subseteq L + MA$$

by (1). We note that $MA \leq N$. Then $N = L + MA$.

(a) Since $K \overset{cs}{\hookrightarrow} L$ in $N$, by Lemma 2.3, we have $N = K + MA$. By (1) and (2),

$$\text{Hom}(M, N) = \text{Hom}(M, K) + \text{Hom}(M, MA) \ldots \text{(*)}$$

$$= \text{Hom}(M, K) + A.$$

(b) By Lemma 2.4, $N = L + MA$ implies that $N = K + MA$. By (1), (2) and the equation (*), we can obtain that

$$\text{Hom}(M, N) = \text{Hom}(M, K) + \text{Hom}(M, MA)$$

$$= \text{Hom}(M, K) + A.$$

Next, we assume that $\text{Hom}(M, N) = \text{Hom}(M, K) + H$ for some $H \leq \text{Hom}(M, N)$. Since $N \in \text{Gen}(M)$, we can obtain that

$$N = M\text{Hom}(M, N) = M\text{Hom}(M, K) + MH.$$

By Lemma 2.4, we have $N = L + MH$. Then, by (1) and (2),

$$\text{Hom}(M, N) = \text{Hom}(M, L) + \text{Hom}(M, MH)$$

$$= \text{Hom}(M, L) + H.$$

They imply that $\text{Hom}(M, K)\beta^*\text{Hom}(M, L)$ by [3, Theorem 2.3].

(5) We only give a proof of (a). Let $A \overset{cs}{\hookrightarrow} B$ in $\text{Hom}(M, N)$ and let $N =$
MB + L for some L ≤ N. By (1) and (2),
\[ \text{Hom}(M, N) = \text{Hom}(M, MB) + \text{Hom}(M, L) = B + \text{Hom}(M, L). \]

Since \( A \xrightarrow{cs} B \) in Hom\((M, N)\) and \( N \in \text{Gen}(M) \), we can obtain that \( \text{Hom}(M, N) = A + \text{Hom}(M, L) \) and so
\[ N = M\text{Hom}(M, N) = M(A + \text{Hom}(M, L)) \subseteq MA + L \subseteq MA + L. \]

Thus \( N = MA + L \). By Lemma 2.3, \( MA \xrightarrow{cs} MB \) in \( N \).

(6) Assume that \( K \leq_{cc} N \). Since \( K_M \xrightarrow{cs} K \) in \( N \) by (3), we obtain \( K = K_M \), that is \( K \in \text{Gen}(M) \). □

The proof of the following theorem can be seen also from [5, Corollary 4.2], and we give the proof for the sake of completeness.

**Theorem 2.6.** Let \( M \) be a finitely generated self-projective \( R \)-module and \( N \in \text{Gen}(M) \). If \( K \) is a co-closed submodule of \( N \), then \( \text{Hom}(M, K) \) is coclosed in \( \text{Hom}(M, N) \), and, conversely, if \( A \) is a co-closed submodule of \( \text{Hom}(M, K) \) in \( \text{Hom}(M, N) \) then \( MA \) is coclosed in \( N \). Furthermore, there exists a bijection between the direct summands of \( N \) and the direct summands of \( \text{Hom}(M, N) \).

**Proof.** Let \( K \) be a coclosed submodule of \( N \) and \( A \xrightarrow{cs} \text{Hom}(M, K) \) in \( \text{Hom}(M, N) \). By Proposition 2.5 (5), \( MA \xrightarrow{cs} M\text{Hom}(M, K) = K_M \) in \( N = M\text{Hom}(M, N) \). By Proposition 2.5 (3) and [4, 3.2], we can obtain that \( MA \xrightarrow{cs} K \) in \( N \). Since \( K \) is a coclosed submodule of \( N \), \( MA = K_M = K \) and so \( \text{Hom}(M, K) = \text{Hom}(M, MA) = A \) by Proposition 2.5 (3). This implies that \( \text{Hom}(M, K) \) is a coclosed submodule of \( \text{Hom}(M, N) \).

For converse, let \( A \) be a coclosed submodule of \( \text{Hom}(M, N) \) and \( L \xrightarrow{cs} MA \) in \( N \). By Proposition 2.5 (3) and (4), \( \text{Hom}(M, L) \xrightarrow{cs} A = \text{Hom}(M, MA) \) in \( \text{Hom}(M, N) \). Since \( A \) is a coclosed submodule of \( \text{Hom}(M, N) \), we can obtain that \( A = \text{Hom}(M, L) \) and so \( MA = M\text{Hom}(M, L) = L_M \subseteq L \). It follows that \( L = MA \). Hence \( MA \) is a coclosed submodule of \( N \). □

**Corollary 2.7.** Let \( M \) be a finitely generated, self-projective left \( R \)-module and \( N \in \text{Gen}(M) \).

1. \( \text{Hom}(M, N) \) is supplemented if and only if \( N \) is supplemented.
2. \( \text{Hom}(M, N) \) is \( \oplus \)-supplemented if and only if \( N \) is \( \oplus \)-supplemented.
3. \( \text{Hom}(M, N) \) is amply supplemented if and only if \( N \) is amply supplemented.
Proof. (1) It follows from [5, Corollary 4.1(ii)]. (2) This is similar to (3).
(3) Assume that $N$ is an amply supplemented module and let $I \leq \text{Hom}(M, N)$.
By [10, Proposition 1.5], let $K$ be a coclosure of $MI$ in $N$, i.e., $K \overset{cs}{\rightarrow} MI$ in $N$ and $K$ is coclosed in $N$. By hierarchy, there exists a supplemented submodule $X$ of $K$ such that $X \overset{cs}{\rightarrow} K$ in $N$. It follows that $X = K$, and so $K$ is supplemented and $K \in \text{Gen}(M)$ by Proposition 2.5(6). By (1), $\text{Hom}(M, K)$ is supplemented. Now, by Proposition 2.5(2) and (4), we can obtain that $\text{Hom}(M, K) \overset{cs}{\rightarrow} \text{Hom}(M, MI) = I$ in $\text{Hom}(M, N)$. This implies that $\text{Hom}(M, N)$ is amply supplemented.

For converse, assume that $\text{Hom}(M, N)$ is amply supplemented. Let $L \leq N$. Then $\text{Hom}(M, L) \leq \text{Hom}(M, N)$ and we assume that $I$ is a coclosure of $\text{Hom}(M, L)$ in $\text{Hom}(M, N)$ by [10, Proposition 1.5]. Hence $I \overset{cs}{\rightarrow} \text{Hom}(M, L)$ in $\text{Hom}(M, N)$ and $I$ is coclosed in $\text{Hom}(M, N)$. By Theorem 2.6, $NI$ is coclosed in $N$. Since $MI \in \text{Gen}(M)$ by Proposition 2.5(6), we can obtain that there is a supplemented submodule $I'$ of $I$ such that $I' \overset{cs}{\rightarrow} I$ in $\text{Hom}(M, N)$. Then $I' = I$ and $I = \text{Hom}(M, MI)$ is supplemented. By Theorem 2.6, $MI$ is supplemented. On the other hand, we can obtain that $MI \overset{cs}{\rightarrow} M\text{Hom}(M, L) = L_M$ in $N$ by Proposition 2.5(5). But we know that $L_M \overset{cs}{\rightarrow} L$ in $N$ by [4, 3.2]. This implies that $N$ is amply supplemented. □

We have the following corollary.

**Corollary 2.8.** Let $H$ be a hollow projective module and $K$ be a finitely $H$-generated module. Then $K$ is a supplemented module and $\text{Hom}(H, K)$ is supplemented.

In [3], the authors used the $\beta^*$ equivalence relation to define the class of $G^*$-lifting modules and the class of $G^*$-supplemented modules. $M$ is called $G^*$-lifting if, for each $X$ of $M$, there exists a direct summand $D$ of $M$ such that $X\beta^*D$, and $M$ is $G^*$-supplemented if, for each $X$ submodule of $M$, there exists a supplement $S$ of $M$ such that $X\beta^*S$.

By [3, Theorem 3.6], we have the following hierarchy:

lifting $\Rightarrow$ $G^*$ - lifting $\iff H$ - supplemented $\Rightarrow$ $G^*$ - supplemented $\Rightarrow$ supplemented.

**Theorem 2.9.** Let $M$ be a finitely generated self-projective $R$-module and $N \in \text{Gen}(M)$. Then;

(a) $N$ is $G^*$-lifting ($H$-supplemented) if and only if $\text{Hom}_R(M, N)$ is $G^*$-lifting ($H$-supplemented) as an $E_M$-module.

(b) $N$ is $G^*$-supplemented if and only if $\text{Hom}_R(M, N)$ is $G^*$-supplemented as an $E_M$-module.
Proof. (a) This is similar to (b).

(b) Assume that $N$ is a $G^*$-supplemented module. Let $I \subseteq \text{Hom}_R(M, N)$ be an $E_M$-submodule. Then $MI$ is a submodule of $N$. Since $N$ is $G^*$-supplemented, there exists a supplement submodule, say $A$, in $N$ such that $A^*(MI)$. Hence there exists $W \leq N$ such that $N = A + W$ and $A$ is minimal with respect to this property. We show that $\text{Hom}(M, A)$ is a supplement of $\text{Hom}(M, W)$ in $\text{Hom}(M, N)$ and $\text{Hom}(M, A)^* I$. By Proposition 2.5,$$
\text{Hom}(M, N) = \text{Hom}(M, A + W) = \text{Hom}(M, A) + \text{Hom}(M, W).
$$Let $\text{Hom}(M, N) = S + \text{Hom}(M, W)$ for $S \subseteq \text{Hom}(M, A)$. Then$$N = M\text{Hom}(M, N) = M(S + \text{Hom}(M, W)) = MS + M\text{Hom}(M, W) \subseteq MS + W_M \subseteq N$$by Proposition 2.5. Minimality of $A$ implies that $A = MS$. By Proposition 2.5, we obtain that $\text{Hom}(M, A) = S$. Hence $\text{Hom}(M, A)$ is a supplement of $\text{Hom}(M, W)$ in $\text{Hom}(M, N)$. On the other hand, $A^*(MI)$ implies that $\text{Hom}(M, A)^* \text{Hom}(M, MI) = I$ by Proposition 2.5. Hence $\text{Hom}_R(M, N)$ is $G^*$-supplemented as an $E_M$-module.

Conversely, assume that $\text{Hom}_R(M, N)$ is $G^*$-supplemented as an $E_M$-module. Let $X \leq N$. Then $\text{Hom}(M, X) \subseteq \text{Hom}(M, N)$. Since $\text{Hom}_R(M, N)$ is $G^*$-supplemented as an $E_M$-module, there exists a supplement submodule, say $I$, in $\text{Hom}(M, N)$ such that $I^* \text{Hom}(M, X)$. Hence there exists $Y \leq \text{Hom}(M, N)$ such that $\text{Hom}(M, N) = I + Y$ and $I$ is minimal with respect to this property. We show that $MI$ is a supplement of $MY$ and $(MI)^* X$. By Proposition 2.5,$$\begin{align*}
N &= M\text{Hom}(M, N) = M(I + Y) \\
&= MI + MY.
\end{align*}$$Let $N = K + MY$ for $K \subseteq MI$. Then$$\begin{align*}
N &= M\text{Hom}(M, N) = M\text{Hom}(M, K + MY) \\
&= M\text{Hom}(M, K) + M\text{Hom}(M, MY) \\
&\subseteq K_M + MY \subseteq K + MY \subseteq N
\end{align*}$$by Proposition 2.5. Minimality of $MI$ implies that $K = MI$. By Proposition 2.5, we can also obtain that $(MI)^* X$. Hence $N$ is $G^*$-supplemented. □

Recall that an $R$-module $M$ is said to be cosemisimple if all simple modules are $M$-injective. By [4, 3.8], $M$ is a cosemisimple module iff every submodule of $M$ is coclosed in $M$.

**Theorem 2.10.** Let $M$ be a a finitely generated, self-projective $R$-module. Then the following cases are equivalent for the module $N \in \text{Gen}(M)$.
An $R$-module $M$ is said to be refinable if, for any submodules $U, V$ of $M$ with $M = U + V$, there exists a direct summand $D$ of $M$ with $D \subseteq U$ and $M = D + V$ ([4]). A ring $R$ is called left refinable if $RR$ is a refinable module.

**Theorem 2.11.** Let $M$ be a finitely generated, self-projective left $R$-module and $N \in \text{Gen}(M)$. Then:

1. $\text{Hom}(M, N)$ is refinable if and only if $N$ is refinable.
2. If $N$ is a refinable module, then the following are equivalent.
   - (i) $N$ is $\oplus$-supplemented
   - (ii) $\text{Hom}(M, N)$ is $\oplus$-supplemented
   - (iii) $N$ is supplemented.
   - (iv) $\text{Hom}(M, N)$ is supplemented.

**Proof.** We only prove (1). The rest is clear.

(1) $(\Rightarrow)$ Let $U, V \leq N$ with $N = U + V$. Then

\[
\text{Hom}(M, N) = \text{Hom}(M, U + V) = \text{Hom}(M, U) + \text{Hom}(M, V)
\]

by Proposition 2.5. Since $\text{Hom}(M, N)$ is refinable as a left $E_M$-module, there exists a direct summand $D$ of $\text{Hom}(M, N)$ such that $\text{Hom}(M, N) = D + \text{Hom}(M, V)$. By Theorem 2.6, we can obtain that $MD$ is a direct summand of $N$. Now, it is easy to see that $N = MD + V$.

$(\Leftarrow)$ Assume that $N$ is a refinable module. Let $SI_S J \subseteq \text{Hom}(M, N)$ with $\text{Hom}(M, N) = I + J$. By Proposition 2.5, we have $I = \text{Hom}(M, MI)$ and $J = \text{Hom}(M, MJ)$ . We also note that $MI$ and $MJ$ are submodule of $N$ and $N = MI + MJ$. Since $N$ is a refinable module, there exists a direct summand $D$ of $N$ such that $N = D + MJ$. By Theorem 2.6, we can obtain that $\text{Hom}(M, D)$ is a direct summand of $\text{Hom}(M, N)$. Now, it is easy to see that $\text{Hom}(M, N) = \text{Hom}(M, D) + J$. □

As a consequence, we have the following result (see [4, 11.28]):

**Corollary 2.12.** Let $M$ be a finitely generated, self-projective left $R$-module. Then the following cases are equivalent.

1. $E_M$ is left refinable.
2. $M$ is refinable.

Recall that an $R$-module $M$ is said to be a unique coclosure module, denoted by UCC, if every submodule of $M$ has a unique coclosure in $M$ (see [7]). By [4, 21.3], $M$ is a UCC module if and only if, given $N \subseteq M$, there exists a coclosure $N'$ of $N$ such that $N' \subseteq L$ whenever $L \subseteq N$ in $M$. 
Theorem 2.13. Let $M$ be a finitely generated, self-projective left $R$-module and $N \in \text{Gen}(M)$. Then the following cases are equivalent.

(1) $\text{Hom}(M, N)$ is a UCC module.

(2) $N$ is a UCC module.

Proof. (1) $\Rightarrow$ (2) Let $A \leq N$. Then $\text{Hom}(M, A) \leq \text{Hom}(M, N)$. Since $\text{Hom}(M, N)$ is a UCC module, there exists a coclosure, say $K$, of $\text{Hom}(M, A)$ such that $K \subseteq L$ whenever $L \overset{cs}{\rightarrow} \text{Hom}(M, A)$ in $\text{Hom}(M, N)$, i.e. $K \overset{cs}{\rightarrow} \text{Hom}(M, A)$ in $\text{Hom}(M, N)$ and $K$ is coclosed in $\text{Hom}(M, N)$. By Proposition 2.5 (5) and Theorem 2.6, we can obtain that $MK \overset{cs}{\rightarrow} A$ in $N$ and $MK$ is coclosed in $N$. It implies that $MK$ is a coclosure of $A$ in $N$.

(2) $\Rightarrow$ (1) Let $S = \text{Hom}(M, MI) \subset \text{Hom}(M, N)$. Then, by Proposition 2.5 (2), $I = \text{Hom}(M, MI)$ and $MI$ is a submodule of $N$. Since $N$ is a UCC-module, there exists a coclosure $K$ of $MI$ in $N$ such that $K \subseteq L$ whenever $L \overset{cs}{\rightarrow} MI$ in $N$. Since $K$ is a coclosure of $MI$ in $N$, we have $K \overset{cs}{\rightarrow} MI$ in $N$ and $K$ is coclosed in $N$. By Proposition 2.5 (4) and Theorem 2.6, we can obtain that $MK \overset{cs}{\rightarrow} \text{Hom}(M, MI) = I$ in $\text{Hom}(M, N)$ and $\text{Hom}(M, K)$ is coclosed in $\text{Hom}(M, N)$. They imply that $\text{Hom}(M, K)$ is a coclosure of $I$ in $\text{Hom}(M, N)$.

3. The Substructure $\nabla(M, N)$

In this section, we study the concept of the substructure $\nabla(M, N)$.

Theorem 3.1. Let $M$ be a finitely generated, self-projective left $R$-module and $N \in \text{Gen}(M)$. If $\text{Hom}(M, N)$ is supplemented as a left $E_M$-module, then $\text{Hom}(M, N)/\nabla(M, N)$ is semisimple as a left $E_M$-module.

Proof. Let $\overline{A} = A/\nabla(M, N) \leq \text{Hom}(M, N)/\nabla(M, N)$. There exists $B \leq \text{Hom}(M, N)$ such that $\text{Hom}(M, N) = A + B$ and $A \cap B \ll B$. Then

$$\text{Hom}(M, N)/\nabla(M, N) = A/\nabla(M, N) + (B + \nabla(M, N))/\nabla(M, N).$$

For any $f \in A \cap B$, we note that $E_Mf \leq A \cap B$ and so $E_Mf \ll \text{Hom}(M, N)$. Now we show that $f \in \nabla(M, N)$. Let $H \leq N$ with $N = \text{Im}f + H$. By Proposition 2.5 (1), we can obtain that

$$\text{Hom}(M, N) = \text{Hom}(M, f(M)) + \text{Hom}(M, H).$$
It follows that $E_M f + \text{Hom}(M, H) = \text{Hom}(M, N)$ and hence $\text{Hom}(M, H) = \text{Hom}(M, N)$. On the other hand, $N = M\text{Hom}(M, N) = M\text{Hom}(M, H) \leq H$ since $N \in \text{Gen}(M)$. Therefore $N = H$, i.e. $\text{Im} f \ll N$. Hence $f \in \nabla(M, N)$.

Thus

$$\text{Hom}(M, N) / \nabla(M, N) = A / \nabla(M, N) \oplus (B + \nabla(M, N)) / \nabla(M, N),$$

as desired. \qed

Recall that;

(D2) For any submodule $A$ of $M$ for which $M/A$ is isomorphic to a direct summand of $M$, then $A$ is a direct summand of $M$.

(GD2) For any submodule $A$ of $M$ for which $M/A$ is isomorphic to $M$, then $A$ is a direct summand of $M$.

A module $M$ is called discrete (respectively, generalized discrete) if $M$ satisfies (D1) and (D2) (respectively, (D1) and (GD2)). Let $p$ be a prime number. Then $M_{\mathbb{Z}} = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ is generalized discrete but not discrete.

**Lemma 3.2.** ([9, Lemma 3.1]) Let $M$ and $N$ be $R$-modules. If $N$ satisfies (GD2), then $\nabla(M, N) \subset J[M, N]$.

**Proof.** Let $\beta \in \nabla(M, N)$ and $f \in \text{Hom}(N, M)$. Then

$$\text{Im} \beta + \text{Im}(1_N - \beta f) = N.$$

Let $\eta := 1_N - \beta f$. Since $\text{Im} \beta \ll N$, we have $\text{Im}(\eta) = N$. It follows that $N \cong N / \text{Ker}(\eta)$. By (GD2), we have $\text{Ker}(\eta)$ is a direct summand of $N$. Since $\text{Ker}(\eta) \leq \text{Im}(\beta)$, we can obtain that $\text{Ker}(\eta) \ll N$. Hence $\text{Ker}(\eta) = 0$. Now $\eta$ is an isomorphism. Thus $\beta \in J[M, N]$. \qed

The next result extends Mohammed and Müller [14, Theorem 5.4].

**Corollary 3.3.** Let $M$ be a finitely generated, self-projective left $R$-module and $N \in \text{Gen}(M)$. If $\text{Hom}(M, N)$ is supplemented and $N$ satisfies GD2 then $\text{Hom}(M, N) / J[M, N]$ is semisimple as a left $E_M$-module.

**Proof.** It is clear from Theorem 3.1 and Lemma 3.2. \qed

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