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STEENROD-ČECH HOMOLOGY-COHOMOLOGY THEORIES ASSOCIATED WITH BIVARIANT FUNCTORS

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Abstract. Let $\mathbf{NG}_0$ denote the category of all pointed numerically generated spaces and continuous maps preserving base-points. In [SYH], we described a passage from bivariant functors $\mathbf{NG}_0^{\mathrm{op}} \times \mathbf{NG}_0 \to \mathbf{NG}_0$ to generalized homology and cohomology theories. In this paper, we construct a bivariant functor such that the associated cohomology is the Čech cohomology and the homology is the Steenrod homology (at least for compact metric spaces).

1. introduction

According to [Du], a topological space X is said to be Δ-generated if it has the final topology with respect to its singular simplexes. CW-complexes are typical examples of such Δ-generated spaces. In [SYH], we showed that the category of Δ-generated spaces is equivalent to the subcategory of the category **Diff** of diffeological spaces consisting of those special type of objects which we call numerically generated spaces. Throughout this pager, we use term "numerically generated" instead of "Δ-generated". Let $\mathbf{NG}_0$ be the category of pointed numerically generated spaces and pointed continuous maps. In [SYH], we showed that $\mathbf{NG}_0$ is a symmetric monoidal closed category with respect to the smash product, and that every bilinear enriched functor $F : \mathbf{NG}_0^{\mathrm{op}} \times \mathbf{NG}_0 \to \mathbf{NG}_0$ gives rise to a pair of generalized homology and cohomology theories, denoted by $h_\bullet(-, F)$ and $h^\bullet(-, F)$ respectively, such that

$$h_n(X, F) \cong \pi_0 F(S^{n+k}, \Sigma^k X), \quad h^n(X, F) \cong \pi_0 F(\Sigma^k X, S^{n+k})$$

hold whenever k and $n + k$ are non-negative.

As an example, consider the bilinear enriched functor F which assigns to (X, Y) the mapping space from X to the topological free abelian group $AG(Y)$ generated by the points of Y modulo the relation $* \sim 0$. The Dold-Thom theorem says that if X is a CW-complex then the groups $h_n(X, F)$ and $h^n(X, F)$ are, respectively, isomorphic to the singular homology and cohomology groups of X. But this is not the case for general X; there exists a space X such that $h_n(X, F)$ (resp. $h^n(X, F)$) is not isomorphic to the singular homology (resp. cohomology) group of X.

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The aim of this paper is to construct a bilinear enriched functor such that for any space X the associated cohomology groups are isomorphic to the Čech cohomology groups of X. Interestingly, it turns out that the corresponding homology groups are isomorphic to the Steenrod homology groups for any compact metrizable space X. Thus we obtain a bibariant theory which ties together the Čech cohomology and the Steenrod homology theories.

Let $\mathbf{NGC}_0$ be the full subcategory of $\mathbf{NG}_0$ consisting of compact metric spaces. For given a linear enriched functor $T : \mathbf{NG}_0 \to \mathbf{NG}_0$, let

$$\check{\mathrm{F}} : \mathbf{NG}_0^{\mathrm{op}} \times \mathbf{NGC}_0 \to \mathbf{NG}_0$$

be a bifunctor which maps (X, Y) to the space $\varinjlim_\lambda \mathrm{map}_0(X_\lambda, \underleftarrow{\mathrm{holim}}_{\mu_i} T(Y_{\mu_i}^{\check{\mathrm{C}}}))$. Here λ runs through coverings of X, and X_λ is the Vietoris nerve corresponding to λ ([P]). The main results of the paper can be stated as follows.

Theorem 1.1. *The functor $\check{\mathrm{F}}$ is a bilinear enriched functor.*

Theorem 1.2. *Let X be a compact metraizable space. Then $h_n(X, \check{\mathrm{F}}) = H_n^{st}(X, \mathbb{S})$ is the Steenrod homology group with coefficients in the spectrum $\mathbb{S} = \{T(S^k)\}$.*

In particular, let T be the functor which assigns to every X the topological abelian group $AG(X)$, and let

$$\check{\mathrm{C}} : \mathbf{NG}_0^{\mathrm{op}} \times \mathbf{NGC}_0 \to \mathbf{NG}_0$$

be the corresponding bifunctor.

Theorem 1.3. *For any pointed space X, $h^n(X, \check{\mathrm{C}})$ is the Čech cohomology group of X, and $h_n(X, \check{\mathrm{C}})$ is the Steenrod homology group of X if X is a compact metralizable space.*

Recall that the Steenrod homology group is related to the Čech homology group of X by the exact sequence

$$0 \longrightarrow \varprojlim_{\lambda_i}^1 \tilde{H}_{n+1}(X_{\lambda_i}^{\check{\mathrm{C}}}) \longrightarrow H_n^{st}(X) \longrightarrow \tilde{H}_n(X) \longrightarrow 0.$$

According to [KKS], if X is a movable compactum then we have $\varprojlim_{\lambda_i}^1 \tilde{H}_{n+1}(X_{\lambda_i}^{\check{\mathrm{C}}}) = 0$, and hence the following corollary follows.

Corollary 1.4. *Let X be a movable compactum. Then $h_n(X, \check{\mathrm{C}})$ is the Čech homology group of X.*

The paper is organized as follows. In Section 2 we recall from [SYH] the category $\mathbf{NG}_0$ and the passage from bilinear enriched functors to generalized homology and cohomology theories. We also recall the definition of Čech cohomology and Steenrod homology group, and Vietoris and Čech nerves; In Section 3 we prove Theorem 1.1; Finally, in Section 4 we prove Theorems 1.2 and 1.3.

2. Preliminaries

2.1. **Homology and cohomology theories via bifunctors.** Let $\mathbf{NG}_0$ be the category of pointed numerically generated topological spaces and pointed continuous maps. In [SYH] we showed that $\mathbf{NG}_0$ satisfies the following properties:

(1) It contains pointed CW-complexes;
(2) It is complete and cocomplete;
(3) It is monoidally closed in the sense that there is an internal hom Z^Y satisfying a natural bijection $\hom_{\mathbf{NG}_0}(X \wedge Y, Z) \cong \hom_{\mathbf{NG}_0}(X, Z^Y)$;
(4) There is a coreflector $\nu\colon \mathrm{Top}_0 \to \mathbf{NG}_0$ such that the coreflection arrow $\nu X \to X$ is a weak equivalence;
(5) The internal hom Z^Y is weakly equivalent to the space of pointed maps from Y to Z equipped with the compact-open topology.

Throughout the paper, we write $\mathrm{map}_0(Y, Z) = Z^Y$ for any $Y,\ Z \in \mathbf{NG}_0$.

A map $f\colon X \to Y$ between topological spaces is said to be numerically continuous if the composite $f \circ \sigma\colon \Delta^n \to Y$ is continuous for every singular simplex $\sigma\colon \Delta^n \to X$. We have the following.

Proposition 2.1. ([SYH]) *Let $f\colon X \to Y$ be a map between numerically generated spaces. Then f is numerically continuous if and only if f is continuous.*

From now on, we assume that $\mathbf{C_0}$ satisfies the following conditions: (i) $\mathbf{C_0}$ contains all finite CW-complexes. (ii) $\mathbf{C_0}$ is closed under finite wedge sum. (iii) If $A \subset X$ is an inclusion of objects in $\mathbf{C_0}$ then its cofiber $X \cup CA$ belongs to $\mathbf{C_0}$; in particular, $\mathbf{C_0}$ is closed under the suspension functor $X \mapsto \Sigma X$.

Definition 2.2. Let $\mathbf{C_0}$ be a full subcategory of $\mathbf{NG}_0$. A functor $T\colon \mathbf{C_0} \to \mathbf{NG_0}$ is called *enriched* (*or continuous*) if the map

$$T : \mathrm{map}_0(X, X') \to \mathrm{map}_0(T(X), T(X')),$$

which assigns $T(f)$ to every f, is a pointed continuous map.

Note that if f is constant, then so is $T(f)$.

Definition 2.3. An enriched functor T is called *linear* if for any pair of a pointed space X, a sequence

$$T(A) \to T(X) \to T(X \cup CA)$$

induced by the cofibration sequence $A \to X \to X \cup CA$, is a homotopy fibration sequence.

Example 2.4. Let $AG : \mathrm{CW}_0 \to \mathbf{NG}_0$ be the functor which assigns to a pointed CW-complex (X, x_0) the topological abelian group $AG(X)$ generated by the points of X modulo the relation $x_0 \sim 0$. Then AG is a linear enriched functor. (see [SYH])

Theorem 2.5. ([SYH, Th 6.4]) *A linear enriched functor T defines a generalized homology $\{h_n(X,T)\}$ satisfying*

$$h_n(X,T) = \begin{cases} \pi_n T(X), & n \geq 0 \\ \pi_0 T(\Sigma^{-n} X), & n < 0. \end{cases}$$

Next we introduce the notion of a bilinear enriched functor, and describe a passage from a bilinear enriched functor to generalized cohomology and generalized homology theories. We assume that $\mathbf{C_0'}$ satisfies the same conditions of $\mathbf{C_0}$.

Definition 2.6. Let $\mathbf{C_0}$ and $\mathbf{C_0'}$ be full subcategories of $\mathbf{NG}_0$. A *bifunctor* $F\colon \mathbf{C_0^{op}} \times \mathbf{C_0'} \to \mathbf{NG_0}$ is a function which

(1) to each objects $X \in \mathbf{C_0}$ and $Y \in \mathbf{C_0'}$ assigns an object $F(X,Y) \in \mathbf{NG}_0$;
(2) to each $f \in \mathrm{map}_0(X,X')$, $g \in \mathrm{map}_0(Y,Y')$ assigns a continuous map $F(f,g) \in \mathrm{map}_0(F(X',Y),\ F(X,Y'))$.

F is required to satisfy the following equalities:
(a) $F(1_X, 1_Y) = 1_{F(X,Y)}$;
(b) $F(f,g) = F(1_X, g) \circ F(f, 1_Y) = F(f, 1_{Y'}) \circ F(1_{X'}, g)$;
(c) $F(f' \circ f, 1_Y) = F(f, 1_Y) \circ F(f', 1_Y)$, $F(1_X, g' \circ g) = F(1_X, g') \circ F(1_X, g)$.

Definition 2.7. A bifunctor $F\colon \mathbf{C_0^{op}} \times \mathbf{C_0} \to \mathbf{NG_0}$ is called *enriched* if the map

$$F : \mathrm{map}_0(X,X') \times \mathrm{map}_0(Y,Y') \to \mathrm{map}_0(F(X',Y), F(X,Y')),$$

which assigns $F(f,g)$ to every pair (f,g), is a pointed continuous map.

Note that if either f or g is constant, then so is $F(f,g)$.

Definition 2.8. For any pairs of pointed spaces (X,A) and (Y,B), F is *bilinear* if the sequences

(1) $F(X \cup CA, Y) \to F(X,Y) \to F(A,Y)$
(2) $F(X,B) \to F(X,Y) \to F(X, Y \cup CB)$,

induced by the cofibration sequences $A \to X \to X \cup CA$ and $B \to Y \to Y \cup CB$, are homotopy fibration sequences.

Example 2.9. Let $T : \mathbf{NG}_0 \to \mathbf{NG}_0$ be a linear enriched functor, and let $F(X,Y) = \mathrm{map}_0(X, T(Y))$ for $X, Y \in \mathbf{NG}_0$. Then $F : \mathbf{NG}_0^{\mathrm{op}} \times \mathbf{NG}_0 \to \mathbf{NG}_0$ is a bilinear enriched functor.

Theorem 2.10. ([SYH, Th 7.4]) *A bilinear enriched functor F defines a generalized cohomology $\{h^n(-,F)\}$ and a generalized homology $\{h_n(-,F)\}$ such that*

$$h_n(Y,F) = \begin{cases} \pi_0 F(S^n, Y) & n \geq 0 \\ \pi_0 F(S^0, \Sigma^{-n} Y) & n < 0, \end{cases} \quad h^n(X,F) = \begin{cases} \pi_0 F(X, S^n) & n \geq 0 \\ \pi_{-n} F(X, S^0) & n < 0, \end{cases}$$

hold for any $X \in \mathbf{C_0}$ and $Y \in \mathbf{C'_0}$.

Proposition 2.11. ([SYH]) *If X is a CW-complex, we have $h_n(X,F) = H_n(X,\mathbb{S})$ and $h^n(X,F) = H^n(X,\mathbb{S})$, the generalized homology and cohomology groups with coefficients in the spectrum $\mathbb{S} = \{F(S^0,S^n) \mid n \geq 0\}$.*

2.2. **Čech cohomology and Steenrod homorogy groups.** We recall that the Čech cohomology group of X with coefficients group G is defined to be the colimit of the singlular cohomology groups

$$\check{H}^n(X,G) = \varinjlim_{\lambda} H^n(X^{\check{C}}_{\lambda}, G),$$

where λ runs through coverings of X and $X^{\check{C}}_{\lambda}$ is the Čech nerve corresponding to λ., i.e. $v \in X^{\check{C}}_{\lambda}$ is a vertex of $X^{\check{C}}_{\lambda}$ corresponding to an open set $V \in \lambda$. On the other hand, the Steenrod homology group of a compact metric space X is defined as follows. As X is a compact metric space, there is a sequence $\{\lambda_i\}_{i \geq 0}$ of finite open covers of X such that $\lambda_0 = \{X\}$, λ_i is a refinement of λ_{i-1}, and X is the inverse limit $\varprojlim_i X^{\check{C}}_{\lambda_i}$. According to [F], the Steenrod homology group of X with coefficients in the spectrum $\mathbb{S}$ is defined to be the group

$$H^{st}_n(X,\mathbb{S}) = \pi_n \underleftarrow{\mathrm{holim}}_{\lambda_i} (X^{\check{C}}_{\lambda_i} \wedge \mathbb{S})$$

where $\underleftarrow{\mathrm{holim}}$ denotes the homotopy inverse limit. (See also [KKS] for the definition without using subdivisions.)

2.3. **Vietoris and Čech nerves.** For each $X \in \mathbf{NG}_0$, let λ be an open covering of X. According to [P], the Vietoris nerve of λ is a simplicial set in which an n-simplex is an ordered $(n+1)$-tuple $(x_0, x_1, \cdots, x_n)$ of points contained in an open set $U \in \lambda$. Face and degeneracy operators are respectively given by

$$d_i(x_0, \cdots, x_n) = (x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

and

$$s_i(x_0, x_1, \cdots x_n) = (x_0, x_1, \cdots x_{i-1}, x_i, x_i, x_{i+1}, \cdots, x_n), \ 0 \leq i \leq n.$$

We denote the realization of the Vietoris nerve of λ by X_λ. If λ is a refinement of μ, then there is a canonical map $\pi^{\lambda}_{\mu} : X_\lambda \to X_\mu$ induced by the identity map of X.

The relation between the Vietoris and the Čech nerves is given by the following Proposition due to Dowker.

Proposition 2.12. ([Do]) *The Čech nerve $X_\lambda^{\check{C}}$ and the Vietoris nerve X_λ have the same homotopy type.*

According to [Do], for arbitrary topological space, the Vietoris and Čech homology groups are isomorphic and the Alexander-Spanier and Čech cohomology groups are isomorphic.

3. Proof of Theorem 1.1

Let T be a linear enriched functor. We define a bifunctor $\check{\mathrm{F}} : \mathbf{NG}_0^{\mathrm{op}} \times \mathbf{NGC}_0 \to \mathbf{NG}_0$ as follows. For $X \in \mathbf{NG}_0$ and $Y \in \mathbf{NGC}_0$, we put

$$\check{\mathrm{F}}(X,Y) = \varinjlim_\lambda \mathrm{map}_0(X_\lambda,\ \underleftarrow{\mathrm{holim}}_{\mu_i} T(Y_{\mu_i}^{\check{C}})),$$

where λ is an open covering of X and $\{\mu_i\}_{i\geq 0}$ is a set of finite open covers of Y such that $\mu_0 = \{Y\}$, μ_i is a refinement of μ_{i-1}, and Y is the inverse limit $\varprojlim_i Y_{\mu_i}^{\check{C}}$.

Given based maps $f : X \to X'$ and $g : Y \to Y'$, we define a map

$$\check{\mathrm{F}}(f,g) \in \mathrm{map}_0(\check{\mathrm{F}}(X',Y), \check{\mathrm{F}}(X,Y'))$$

as follows. Let ν and γ be open covering of X' and Y' respectively, and let $f^\#\nu = \{f^{-1}(U) \mid U \in \nu\}$ and $g^\#\gamma = \{g^{-1}(V) \mid V \in \gamma\}$. Then $f^\#\nu$ and $g^\#\gamma$ are open coverings of X and Y respectively. By the definition of the nerve, there are natural maps $f_\nu : X_{f^\#\nu} \to X'_\nu$ and $g_\gamma : Y_{g^\#\gamma}^{\check{C}} \to (Y')_\gamma^{\check{C}}$. Hence we have the map

$$T(g_\gamma)^{f_\nu} : T(Y_{g^\#\gamma}^{\check{C}})^{X'_\nu} \to T((Y')_\gamma^{\check{C}})^{X_{f^\#\nu}}$$

induced by f_ν and g_γ. Thus we can define

$$\check{\mathrm{F}}(f,g) = \varinjlim_\nu \underleftarrow{\mathrm{holim}}_\gamma T(g_\gamma)^{f_\nu} : \check{\mathrm{F}}(X',Y) \to \check{\mathrm{F}}(X,Y').$$

Theorem 1.1. The functor $\check{\mathrm{F}}$ is a bilinear enriched functor.

First we prove that the sequence

$$\check{\mathrm{F}}(X \cup CA, Z) \to \check{\mathrm{F}}(X,Z) \to \check{\mathrm{F}}(A,Z)$$

induced by the sequence $A \to X \to X \cup CA$, is a homotopy fibration sequence. Let λ be an open covering of $X \cup CA$, and let λ_X, λ_{CA} and λ_A be the coverings of X, CA and A consisting of those $U \in \lambda$ such that U intersects with X, CA, and A, respectively. We need the following lemma.

Lemma 3.1. *We have a homotopy equivalence*

$$(X \cup CA)_\lambda^{\check{C}} \simeq X_{\lambda_X}^{\check{C}} \cup C(A_{\lambda_A}^{\check{C}}).$$

Proof. By the definition of the Čceh nerve, we have $(X \cup CA)^{\check{C}}_{\lambda} = X^{\check{C}}_{\lambda_X} \cup (CA)^{\check{C}}_{\lambda_{CA}}$. Since

$$X^{\check{C}}_{\lambda_X} \cup (CA)^{\check{C}}_{\lambda_{CA}} \simeq X^{\check{C}}_{\lambda_X} \cup A^{\check{C}}_{\lambda_A} \times I \cup (CA)^{\check{C}}_{\lambda_{CA}},$$

and since $(CA)^{\check{C}}_{\lambda_{CA}} \simeq *$, we have

$$X^{\check{C}}_{\lambda_X} \cup (CA)^{\check{C}}_{\lambda_{CA}} \simeq X^{\check{C}}_{\lambda_X} \cup C(A^{\check{C}}_{\lambda_A}).$$

Hence we have $(X \cup CA)_{\lambda} \simeq X^{\check{C}}_{\lambda_X} \cup C(A^{\check{C}}_{\lambda_A})$. □

By Proposition 2.12 and Lemma 3.1, the sequence

$$A_{\lambda_A} \to X_{\lambda_X} \to (X \cup CA)_{\lambda}$$

is a homotopy cofibration sequence. Hence the sequence

$$[(X \cup CA)_{\lambda}, Z] \to [X_{\lambda_X}, Z] \to [A_{\lambda_A}, Z]$$

is an exact sequence for any λ. Since the nerves of the form λ_X (resp. λ_A) are cofinal in the set of nerves of X (resp. A), we conclude that the sequence

$$\check{\mathrm{F}}(X \cup CA, Z) \to \check{\mathrm{F}}(X, Z) \to \check{\mathrm{F}}(A, Z)$$

is a homotopy fibration sequence.

Now we show that the sequence $\check{\mathrm{F}}(Z, A) \to \check{\mathrm{F}}(Z, X) \to \check{\mathrm{F}}(Z, X \cup CA)$ is a homotopy fibration sequence. By the linearity of T, the sequence

$$T(A^{\check{C}}_{\lambda_A}) \to T(X^{\check{C}}_{\lambda_X}) \to T((X \cup CA)^{\check{C}}_{\lambda})$$

is a homotopy fibration sequence. Since the fibre $T(A^{\check{C}}_{\lambda_A})$ is homeomorphic to the inverse limit

$$\varprojlim(* \to T((X \cup CA)^{\check{C}}_{\lambda}) \leftarrow T(X^{\check{C}}_{\lambda_X})),$$

we have

$$\begin{aligned}
&\varprojlim(* \to \underleftarrow{\mathrm{holim}}_{\lambda} T((X \cup CA)^{\check{C}}_{\lambda}) \leftarrow \underleftarrow{\mathrm{holim}}_{\lambda_X} T(X^{\check{C}}_{\lambda_X})) \\
&\simeq \varprojlim \underleftarrow{\mathrm{holim}}_{\lambda}(* \to T((X \cup CA)^{\check{C}}_{\lambda}) \leftarrow T(X^{\check{C}}_{\lambda_X})) \\
&\simeq \underleftarrow{\mathrm{holim}}_{\lambda} \varprojlim(* \to T((X \cup CA)^{\check{C}}_{\lambda}) \leftarrow T(X^{\check{C}}_{\lambda_X})) \\
&\simeq \underleftarrow{\mathrm{holim}}_{\lambda} T(A^{\check{C}}_{\lambda_A}).
\end{aligned}$$

This implies that the sequence

$$\underleftarrow{\mathrm{holim}}_{\lambda_A} T(A^{\check{C}}_{\lambda_A}) \to \underleftarrow{\mathrm{holim}}_{\lambda_X} T(X^{\check{C}}_{\lambda_X}) \to \underleftarrow{\mathrm{holim}}_{\lambda} T((X \cup CA)^{\check{C}}_{\lambda})$$

is a homotopy fibration sequence, hence so is $\check{\mathrm{F}}(Z, A) \to \check{\mathrm{F}}(Z, X) \to \check{\mathrm{F}}(Z, X \cup CA)$.

Next we prove the continuity of $\check{\mathrm{F}}$. Let $F(X,Y) = \mathrm{map}_0(X, \underleftarrow{\mathrm{holim}}_{\mu_i} T(Y^{\check{\mathrm{C}}}_{\mu_i}))$, so that we have $\check{\mathrm{F}}(X,Y) = \varinjlim_\lambda F(X_\lambda, Y)$. We need the following lemma.

Lemma 3.2. *The functor F is an enriched bifunctor.*

Proof. Let $F_1(Y) = \underleftarrow{\mathrm{holim}}_{\mu_i} T(Y^{\check{\mathrm{C}}}_{\mu_i})$ and $F_2(X,Z) = \mathrm{map}_0(X,Z)$, so that we have $F(X,Y) = F_2(X, F_1(Y))$. Clearly F_2 is continuous.

Let G_1 be the functor which maps Y to $\underleftarrow{\mathrm{holim}}_{\mu_i} Y^{\check{\mathrm{C}}}_{\mu_i}$. Since T is enriched, F_1 is continuous if so is G_1. It suffices to show that the map $G'_1\colon \mathrm{map}_0(Y,Y') \times \underleftarrow{\mathrm{holim}}_{\mu_i} Y^{\check{\mathrm{C}}}_{\mu_i} \to \underleftarrow{\mathrm{holim}}_{\lambda_j} (Y')^{\check{\mathrm{C}}}_{\lambda_j}$, adjoint to G_1, is continuous for any Y and Y'. Given an open covering λ of Y', let p^n_λ be the natural map $\underleftarrow{\mathrm{holim}}_\lambda (Y')^{\check{\mathrm{C}}}_\lambda \to \mathrm{map}_0(\Delta^n, (Y')^{\check{\mathrm{C}}}_\lambda)$. Then G'_1 is continuous if so is the composite

$$p^n_\lambda \circ G'_1\colon \mathrm{map}_0(Y,Y') \times \underleftarrow{\mathrm{holim}}_{\mu_i} Y^{\check{\mathrm{C}}}_{\mu_i} \to \mathrm{map}_0(\Delta^n, (Y')^{\check{\mathrm{C}}}_\lambda)$$

for every $\lambda \in \mathrm{Cov}(Y')$ and every n. Here we may assume by [SYH, Proposition 4.3] that $\mathrm{map}_0(\Delta^n, (Y')^{\check{\mathrm{C}}}_\lambda)$ is equipped with the compact open topology. Let $(g,\alpha) \in \mathrm{map}_0(Y,Y') \times \underleftarrow{\mathrm{holim}}_{\mu_i} Y^{\check{\mathrm{C}}}_{\mu_i}$, and let $W_{K,U} \subset \mathrm{map}_0(\Delta^n, (Y')^{\check{\mathrm{C}}}_\lambda)$ be an open neighborhood of $p^n_\lambda(G'_1(g,\alpha))$, where K is a compact set of Δ^n and U is an open set of $(Y')^{\check{\mathrm{C}}}_\lambda$.

Let us choose simplices σ of $Y^{\check{\mathrm{C}}}_{g^\sharp\lambda}$ with vertices $g^{-1}(U(\sigma,k))$, where $U(\sigma,k) \in \lambda$ for $0 \le k \le \dim\sigma$. Let

$$O(\sigma) = \textstyle\bigcap_{0\le k\le \dim\sigma} U(\sigma,k) \subset Y'.$$

Let us choose a point $y_\sigma \in \bigcap_{0\le k\le\dim\sigma} g^{-1}(U(\sigma,k))$, then $g(y_\sigma) \in O(\sigma)$. Let W_1 be the intersection of all $W_{y_\sigma, O(\sigma)}$.

There is an integer l such that

$$\mu_l > \overline{\mu}_l > g^\# \lambda$$

where $\overline{\mu_l}$ is a closed covering $\{\overline{V} | V \in \mu_l\}$ of Y. Thus for any $U \in \mu_l$, there is an open set $V_U \in g^\#\lambda$ such that $\overline{U} \subset g^{-1}(V_U)$. Since Y is a compact set, $\overline{U}$ is compact. Let W_2 be the intersection of $W_{\overline{U}, V_U}$, and let $W = W_1 \cap W_2$.

Since $\mu_l > g^\#\lambda$, we have

$$p^n_\lambda(G'_1(g,\alpha)) = (g_\lambda)_*(\pi^{\mu_l}_{g^\#\lambda})_* p^n_{\mu_l}\alpha.$$

where $(g_\lambda)_*$ and $(\pi^{\mu_l}_{g^\#\lambda})_*$ are induced by $g_\lambda : Y^{\check{\mathrm{C}}}_{g^\#\lambda} \to (Y')^{\check{\mathrm{C}}}_\lambda$ and $\pi^{\mu_l}_{g^\#\lambda} : Y^{\check{\mathrm{C}}}_{\mu_l} \to Y^{\check{\mathrm{C}}}_{g^\#\lambda}$, respectively. Let

$$W' = (p^n_{\mu_l})^{-1}(W_{K, (\pi^{\mu_l}_{g^\#\lambda})^{-1}(g_\lambda)^{-1}(U)}).$$

Then $W \times W'$ is a neighborhood of (g,α) in $\mathrm{map}_0(Y,Y') \times \underleftarrow{\mathrm{holim}}_{\mu_i} Y_{\mu_i}$. To see that $p_\lambda \circ G_1'$ is continuous at (g,α), we need only show that $W \times W'$ is contained in $(p_\lambda \circ G_1')^{-1}(U)$. Suppose (h,β) belongs to $W \times W'$. Since W is contained in W_1, we have

$$y_\sigma \in h^{-1}(O(\sigma)) \subset \bigcap_{0\le k\le \dim\sigma} h^{-1}(U(\sigma,k)).$$

This means that the vertices $h^{-1}(U(\sigma,k)) \in h^\sharp\lambda$, $0 \le k \le \dim\sigma$, determine simplices σ' of $Y_{h^\sharp\lambda}$ each corresponding to each $\sigma \subset Y_{g^\sharp\lambda}$. Thus we have an isomorphism

$$\begin{gathered} s : Y^{\check{\mathrm{C}}}_{h^\sharp\lambda} \to Y^{\check{\mathrm{C}}}_{g^\sharp\lambda}, \\ h^{-1}(U(\sigma,k)) \mapsto g^{-1}(U(\sigma,k)). \end{gathered}$$

Moreover since W is contained in W_2, we have $\overline{\mu_l} > h^\#\lambda$.

Since the commutative diagram

$$\begin{tikzcd} Y^{\check{\mathrm{C}}}_{\mu_l} \arrow[r] \arrow[dr] & Y^{\check{\mathrm{C}}}_{g^\#\lambda} \arrow[r, "g_\lambda"] & (Y')^{\check{\mathrm{C}}}_\lambda \\ & Y^{\check{\mathrm{C}}}_{h^\#\lambda} \arrow[u, "s"] \arrow[ur, "h_\lambda"'] & \end{tikzcd}$$

is commutative, we have the equation

$$p^n_\lambda \circ G_1'(h,\beta)(K) = h_\lambda \pi^{\mu_l}_{h^\#\lambda}(\beta)(K) = g_\lambda \pi^{\mu_l}_{g^\#\lambda}(\beta)(K)$$

Since $g_\lambda \pi^{\mu_l}_{g^\#\lambda}(\beta)(K)$ is continued in U, so is $p^n_\lambda \circ G_1'(h,\beta)(K)$.

Thus $p^n_\lambda \circ G_1'$ is continuous for all $\lambda \in \mathrm{Cov}(Y')$, and hence so is

$$G_1' \colon \mathrm{map}_0(Y,Y') \times \underleftarrow{\mathrm{holim}}_{\mu_i} Y^{\check{\mathrm{C}}}_{\mu_i} \to \underleftarrow{\mathrm{holim}}_{\lambda_j} (Y')^{\check{\mathrm{C}}}_{\lambda_j}.$$

□

We are now ready to prove Theorem 1.1. For given pointed spaces X, Y and a covering μ of X, let i_μ denote the natural map $F(X_\mu, Y) \to \varinjlim_\mu F(X_\mu, Y)$. To prove the theorem, it suffices to show that the map

$$\begin{aligned} \check{\mathrm{F}}' \circ (1 \times i_\lambda) \colon \mathrm{map}_0(X,X') \times F(X'_\lambda, Y) &\to \mathrm{map}_0(X,X') \times \varinjlim_\lambda F(X'_\lambda, Y) \\ &\to \varinjlim_\mu F(X_\mu, Y) \end{aligned}$$

which maps (f,α) to $i_{f^\sharp\lambda}(F(f_\lambda, 1_Y)(\alpha))$, is continuous for every covering λ of X.

Let

$$R_\lambda \colon \mathrm{map}_0(X,X') \to \varinjlim_\mu \mathrm{map}_0(X_\mu, X'_\lambda)$$

be the map which assigns to $f : X \to X'$ the image of $\mathrm{map}_0(X, X')$, $f_\lambda \in \mathrm{map}_0(X_{f^\sharp\lambda}, X'_\lambda)$ in $\varinjlim_\mu \mathrm{map}_0(X_\mu, X'_\lambda)$, and let Q_λ be the map

$$\varinjlim_\mu \mathrm{map}_0(X_\mu, X'_\lambda) \times F(X'_\lambda, Y) \to \varinjlim_\mu F(X_\mu, Y),$$

$$[f, \alpha] \mapsto i_{f^\sharp\lambda} f_\lambda \circ \alpha = i_{f^\sharp\lambda}(F(f_\lambda, 1_Y)(\alpha)).$$

Since we have $\check{\mathrm{F}}' \circ (1 \times i_\lambda) = Q_\lambda \circ (R_\lambda \times 1)$, we need only show the continuity of Q_λ and R_λ. Since Q_λ is induced by the maps $\mathrm{map}_0(X_\mu, X'_\lambda) \times F(X'_\lambda, Y) \to F(X_\mu, Y)$, Q_λ is continuous.

To see that R_λ is continuous, let $W_{K^f,U}$ be a neighborhood of f_λ in $\mathrm{map}_0(X_{f^\sharp\lambda}, X'_\lambda)$, where K^f is a compact subset of $X_{f^\sharp\lambda}$ and U is an open subset of X'_λ. Since K^f is compact, there is a finite subcomplex S^f of $X_{f^\sharp\lambda}$ such that $K^f \subset S^f$. Let τ^f_i, $0 \le i \le m$, be simplexes of S^f. By taking a suitable subdivision of $X_{f^\sharp\lambda}$, we may assume that there is a simplicial neighborhood $N_{\tau^f_i}$ of each τ^f_i, $1 \le i \le m$, such that $K^f \subset S^f \subset \cup_i N_{\tau^f_i} \subset f_\lambda^{-1}(U)$.

Let $\{x^i_k\}$ be the set of vertices of τ^f_i and let W be the intersection of all $W_{\{x^i_k\},U_{(\tau^f_i)'}}$ where $U_{(\tau^f_i)'}$ is an open set of X'_λ containing the set $\{f(x^i_k)\}$. Then W is a neighborhood of f. We need only show that $R_\lambda(W) \subset i_{f\#\lambda}(W_{K^f,U})$. Suppose that g belongs to W. Since $\{x^i_k\}$ is contained in $g^{-1}(U_{(\tau^f_i)'})$ for any i, a simplex τ^g_i spanned by the vertices is contained in $X_{g^\sharp\lambda}$. Let S^g be the finite subcomplex of $X_{g^\sharp\lambda}$ consists of simplexes τ^g_i. By the construction, S^f and S^g are isomorphic. Moreover there is a compact subset K^g of $X_{g^\sharp\lambda}$ such that K^g and K^f are homeomorphic. On the other hand, since $g(\{x^i_k\}) \subset U_{(\tau^f_i)'}$, there is a simplex of X'_λ having $g_\lambda(\tau^g_i)$ and $(\tau^f_i)'$ as its faces. This means that $g_\lambda(\tau^g_i) \subset f_\lambda(\cup_i N_{\tau^f_i})$. Thus we have $g_\lambda(K^g) = \cup_i g_\lambda(\tau^g_i) \subset f_\lambda(\cup_i N_{\tau^f_i})$.

Let $f^\sharp\lambda \cap g^\sharp\lambda$ be an open covering

$$\{f^{-1}(U) \cap g^{-1}(V) \mid U, V \in \lambda\}$$

of X. We regard $X_{f^\sharp\lambda}$ and $X_{g^\sharp\lambda}$ as a subcomplex of $X_{f^\sharp\lambda\cap g^\sharp\lambda}$. Since $g_\lambda|X_{f^\sharp\lambda\cap g^\sharp\lambda}$ is contiguous to $f_\lambda|X_{f^\sharp\lambda\cap g^\sharp\lambda}$, we have a homotopy equivalence $g_\lambda|X_{f^\sharp\lambda\cap g^\sharp\lambda} \simeq f_\lambda|X_{f^\sharp\lambda\cap g^\sharp\lambda}$. By the homotopy extension property of $g_\lambda|X_{f^\sharp\lambda\cap g^\sharp\lambda} : X_{f^\sharp\lambda\cap g^\sharp\lambda} \to X'_\lambda$ and $f_\lambda : X_{f^\sharp\lambda} \to X'_\lambda$, $g_\lambda|X_{f^\sharp\lambda\cap g^\sharp\lambda}$ extends to map $G : X_{f^\sharp\lambda} \to X'_\lambda$.

We have the relation $G \sim \pi^{f^\sharp\lambda\cap g^\sharp\lambda}_{f^\sharp\lambda} G = g_\lambda|X_{f^\sharp\lambda\cap g^\sharp\lambda} = \pi^{f^\sharp\lambda\cap g^\sharp\lambda}_{g^\sharp\lambda} g_\lambda \sim g_\lambda$, where $\sim$ is the relation of the direct limit. Moreover by $G(K^f) \subset$

$f_\lambda(\cup_i N_{\tau_i^f}) \subset U$, we have $[g_\lambda] = [G] \in i_{f\#\lambda}(W_{K^f,U})$. Hence R_λ is continuous, and so is $\check{\mathrm{F}}'$.

4. Proofs of Theorems 1.2 and 1.3

To prove Theorems 1.2 and 1.3, we need several lemmas.

Lemma 4.1. *There exists a sequence $\lambda_1^n < \lambda_2^n < \cdots < \lambda_m^n < \cdots$ of open coverings of S^n such that :*

(1) *For each open covering μ of S^n, there is an $m \in \mathbb{N}$ such that λ_m^n is a refinement of μ:*
(2) *For any m, $S^n_{\lambda_m}$ is homotopy equivalent to S^n.*

Proof. We prove by induction on n. For $n = 1$, we define an open covering λ_m^1 of S^1 as follows. For any i with $0 \leq i < 4m$, we put

$$U(i,m) = \left\{ (\cos\theta,\ \sin\theta) \mid \frac{(4i-3)\pi}{8m} < \theta < \frac{(4i+5)\pi}{8m} \right\}.$$

Let $\lambda_m^1 = \{U(i,m) | \ 0 \leq i < 4m\}$. Then the set λ_m^1 is an open covering of S^1 and is a refinement of λ_{m-1}^1. Clearly $(S^1)^{\check{\mathrm{C}}}_{\lambda_m^1}$ is homeomorphic to S^1, hence $S^1_{\lambda_m^1}$ is homotopy equivalent to S^1. Moreover for any open covering μ of S^1, there exists an m such that λ_m^1 is a refinement of μ. Hence the lemma is true for $n = 1$. Assume now that the lemma is true for $1 \leq k \leq n-1$. Let λ'^n_m be the open covering $\lambda_m^{n-1} \times \lambda_m^1$ of $S^{n-1} \times S^1$ and let λ_m^n be the open covering of S^n induced by the natural map $p : S^{n-1} \times S^1 \to S^{n-1} \times S^1/S^{n-1} \vee S^1$. Since $S^{n-1}_{\lambda_m^{n-1}}$ is a homotopy equivalence of S^{n-1}, we have

$$S^n_{\lambda_m^n} \approx (S^{n-1} \times S^1/S^{n-1} \vee S^1)_{\lambda_m^n} \approx (S^{n-1}_{\lambda_m^{n-1}} \times S^1_{\lambda_m})/(S^{n-1}_{\lambda_m^{n-1}} \vee S^1_{\lambda_m}) \approx S^n.$$

Thus the sequence $\lambda_1^n < \lambda_2^n < \cdots < \lambda_m^n < \cdots$ satisfies the required conditions. □

Lemma 4.2. $h_n(X, \check{\mathrm{F}}) \cong \pi_n \underleftarrow{\operatorname{holim}}_\mu T(X^{\check{\mathrm{C}}}_\mu)$ *for $n \geq 0$.*

Proof. By Lemma 4.1, we have an isomorphism

$$\underrightarrow{\lim}_\lambda [S^n_\lambda, \underleftarrow{\operatorname{holim}}_\mu T(X^{\check{\mathrm{C}}}_\mu)] \cong [S^n, \underleftarrow{\operatorname{holim}}_\mu T(X^{\check{\mathrm{C}}}_\mu)].$$

Thus we have

$$\begin{aligned} h_n(X, \check{\mathrm{F}}) &= \pi_0\, \check{\mathrm{F}}(S^n, X) \\ &= \pi_0 \varinjlim_\lambda \operatorname{map}_0(S^n_\lambda, \operatorname{\underleftarrow{holim}}_\mu T(X^{\check{\mathrm{C}}}_\mu)) \\ &\cong \varinjlim_\lambda [S^0, \operatorname{map}_0(S^n_\lambda, \operatorname{\underleftarrow{holim}}_\mu T(X^{\check{\mathrm{C}}}_\mu)] \\ &\cong \varinjlim_\lambda [S^n_\lambda, \operatorname{\underleftarrow{holim}}_\mu T(X^{\check{\mathrm{C}}}_\mu)] \\ &\cong [S^n, \operatorname{\underleftarrow{holim}}_\mu T(X^{\check{\mathrm{C}}}_\mu)] \\ &\cong \pi_n \operatorname{\underleftarrow{holim}}_\mu T(X^{\check{\mathrm{C}}}_\mu). \end{aligned}$$

□

Now we are ready to prove Theorem 1.2. Let X be a compact metric space and let $\mathbb{S} = \{T(S^k) \mid k \geq 0\}$. Since X is a compact metric space, there is a sequence $\{\mu_i\}_{i\geq 0}$ of finite open covers of X with $\mu_0 = X$ and μ_i refining μ_{i-1} such that $X = \varprojlim_i X^{\check{\mathrm{C}}}_{\mu_i}$ holds. Let us denote $X^{\check{\mathrm{C}}}_{\mu_i} = X^{\check{\mathrm{C}}}_i$ and $X_{\mu_i} = X_i$ if there is no possibility of confusion. According to [F], there is a short exact sequence

$$0 \longrightarrow \varprojlim\nolimits^1_i H_{n+1}(X^{\check{\mathrm{C}}}_i, \mathbb{S}) \longrightarrow H^{st}_n(X, \mathbb{S}) \longrightarrow \varprojlim_i H_n(X^{\check{\mathrm{C}}}_i, \mathbb{S}) \longrightarrow 0$$

where $H_n(X, \mathbb{S})$ is the homology group of X with coefficients in the spectrum $\mathbb{S}$. (This is a special case of the Milnor exact sequence [MI].) On the other hand, by [BK], we have the following.

Lemma 4.3. ([BK]) *There is a natural short exact sequence*

$$0 \longrightarrow \varprojlim\nolimits^1_i \pi_{n+1} T(X^{\check{\mathrm{C}}}_i) \longrightarrow \pi_n \operatorname{\underleftarrow{holim}}_i T(X_i) \longrightarrow \varprojlim_i \pi_n T(X^{\check{\mathrm{C}}}_i) \longrightarrow 0.$$

By Proposition 2.11, we have a diagram

(4.1)

$$\begin{tikzcd} 0 \arrow[r] & \varprojlim^1_i H_{n+1}(X^{\check{\mathrm{C}}}_i, \mathbb{S}) \arrow[r] \arrow[d, "\cong"] & H^{st}_n(X, \mathbb{S}) \arrow[r] & \varprojlim_i H_n(X^{\check{\mathrm{C}}}_i, \mathbb{S}) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \varprojlim^1_i \pi_{n+1}(T(X^{\check{\mathrm{C}}}_i)) \arrow[r] & \pi_n(\operatorname{\underleftarrow{holim}}_i T(X^{\check{\mathrm{C}}}_i)) \arrow[r] & \varprojlim_i \pi_n(T(X^{\check{\mathrm{C}}}_i)) \arrow[r] & 0. \end{tikzcd}$$

Hence it suffices to construct a natural homomorphism

$$H^{st}_n(X, \mathbb{S}) \to \pi_n(\operatorname{\underleftarrow{holim}}_i T(X^{\check{\mathrm{C}}}_i))$$

making the diagram (4.1) commutative.

Since T is continuous, the identity map $X \wedge S^k \to X \wedge S^k$ induces a continuous map $i' : X \wedge T(S^k) \to T(X \wedge S^k)$. Hence we have the composite homomorphism

$$\begin{array}{rcl} H_n^{st}(X, \mathbb{S}) & = & \pi_n \underleftarrow{\mathrm{holim}}_i (X_i^{\check{\mathrm{C}}} \wedge \mathbb{S}) \\ & \cong & \underrightarrow{\lim}_k \pi_{n+k} (\underleftarrow{\mathrm{holim}}_i (X_i^{\check{\mathrm{C}}} \wedge T(S^k)) \\ & \xrightarrow{I} & \underrightarrow{\lim}_k \pi_{n+k} (\underleftarrow{\mathrm{holim}}_i T(X_i^{\check{\mathrm{C}}} \wedge S^k)) \\ & \cong & \pi_n (\underleftarrow{\mathrm{holim}}_i T(X_i^{\check{\mathrm{C}}})) \end{array}$$

in which $I = \underrightarrow{\lim}_k {i'}_*^k$ is induced by the homomorphisms

$${i'}_*^k : \pi_{n+k}(\underleftarrow{\mathrm{holim}}_i (X_i^{\check{\mathrm{C}}} \wedge T(S^k)) \to \pi_{n+k}(\underleftarrow{\mathrm{holim}}_i T(X_i^{\check{\mathrm{C}}} \wedge S^k)).$$

Clearly resulting the homomorphism $H_n^{st}(X, \mathbb{S}) \to \pi_n(\underleftarrow{\mathrm{holim}}_i T(X_i^{\check{\mathrm{C}}}))$ makes the diagram (4.1) commutative. Thus $h_n(X, \check{\mathrm{F}})$ is isomorphic to the Steenrod homology group coefficients in the spectrum $\mathbb{S}$.

Finally, to prove Theorem 1.3 it suffices to show that $h^n(X, \check{\mathrm{C}})$ is isomorphic to the Čech cohomology group of X.

By Lemma 4.1, we have a homotopy commutative diagram

$$\begin{tikzcd} \cdots \arrow[r, "="] & AG(S^n) \arrow[r, "="] \arrow[d, "\simeq"] & AG(S^n) \arrow[r, "="] \arrow[d, "\simeq"] & \cdots \\ \cdots \arrow[r] & AG(S^n_{\lambda^n_{m-1}}) \arrow[r, "\simeq"] & AG(S^n_{\lambda^n_m}) \arrow[r] & \cdots . \end{tikzcd}$$

Hence we have $AG(S^n) \simeq \underleftarrow{\mathrm{holim}}_i AG(S^n_{\lambda^n_i})$.

Thus we have

$$\begin{array}{rcl} h^n(X, \check{C}) & = & \pi_0 \check{C}(X, S^n) \\ & = & \pi_0 \underrightarrow{\lim}_\lambda \mathrm{map}_0(X_\lambda, \underleftarrow{\mathrm{holim}}_\mu AG((S^n)_\mu^{\check{\mathrm{C}}})) \\ & \cong & [S^0, \underrightarrow{\lim}_\lambda \mathrm{map}_0(X_\lambda, AG(S^n)] \\ & \cong & \underrightarrow{\lim}_\lambda [S^0, \mathrm{map}_0(X_\lambda, AG(S^n)] \\ & \cong & \underrightarrow{\lim}_\lambda [S^0 \wedge X_\lambda, AG(S^n)] \\ & \cong & \underrightarrow{\lim}_\lambda [X_\lambda, AG(S^n)]. \end{array}$$

Hence $h^n(X, \check{\mathrm{C}})$ is isomorphic to the Čech cohomology group of X.

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