ON MODEL STRUCTURE FOR COREFLECTIVE SUBCATEGORIES OF A MODEL CATEGORY

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1. INTRODUCTION

Let **C** be a coreflective subcategory of a cofibrantly generated model category **D**. In this paper we show that under suitable conditions **C** admits a cofibrantly generated model structure which is left Quillen adjunct to the model structure on **D**. As an application, we prove that well-known convenient categories of topological spaces, such as k-spaces, compactly generated spaces, and Δ -generated spaces [3] (called numerically generated in [12]) admit a finitely generated model structure which is Quillen equivalent to the standard model structure on the category **Top** of topological spaces.

2. Coreflective subcategories of a model category

Let **D** be a cofibrantly generated model category [7, 2.1.17] with generating cofibrations I, generating trivial cofibrations J and the class of weak equivalences $W_{\mathbf{D}}$. If the domains and codomains of I and J are finite relative to I-cell [7, 2.1.4], then **D** is said to be finitely generated.

Recall that a subcategory \mathbf{C} of \mathbf{D} is said to be coreflective if the inclusion functor $i: \mathbf{C} \to \mathbf{D}$ has a right adjoint $G: \mathbf{D} \to \mathbf{C}$, so that there is a natural isomorphism $\varphi: \operatorname{Hom}_{\mathbf{D}}(X, Y) \to \operatorname{Hom}_{\mathbf{C}}(X, GY)$. The counit of this adjunction $\epsilon: GY \to Y \ (Y \in \mathbf{D})$ is called the coreflection arrow.

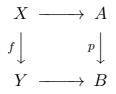
Theorem 2.1. Let \mathbb{C} be a coreflective subcategory of a cofibrantly generated model category \mathbb{D} which is complete and cocomplete. Suppose that the unit of the adjunction $\eta: X \to GX$ is a natural isomorphism, and that the classes Iand J of cofibrations and trivial cofibrations in \mathbb{D} are contained in \mathbb{C} . Then \mathbb{C} has a cofibrantly generated model structure with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $W_{\mathbb{C}}$ as the class of weak equivalences, where $W_{\mathbb{C}}$ is the class of all weak equivalences contained in \mathbb{C} . If \mathbb{D} is finitely generated, then so is \mathbb{C} . Moreover, the adjunction $(i, G, \varphi): \mathbb{C} \to \mathbb{D}$ is a Quillen adjunction in the sense of [7, 1.3.1].

Proof. It suffices to show that **C** satisfies the six conditions of [7, 2.1.19] with respect to I, J and $W_{\mathbf{C}}$. Clearly, the first condition holds because

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 $W_{\mathbf{C}}$ satisfies the two out of three property and is closed under retracts. To see that the second and the third conditions hold, let $I_{\mathbf{C}}$ -cell and $J_{\mathbf{C}}$ -cell be the collections of relative *I*-cell and *J*-cell complexes contained in \mathbf{C} , respectively. Since $I_{\mathbf{C}}$ -cell and $J_{\mathbf{C}}$ -cell are subcollections of the collections of relative *I*-cell and *J*-cell complexes in \mathbf{D} , respectively, the domains of *I* and *J* are small relative to $I_{\mathbf{C}}$ -cell and $J_{\mathbf{C}}$ -cell, respectively. The rest of the conditions are verified as follows. Let $f: X \to Y$ be a map in \mathbf{C} . Since $\eta: X \to GX$ is isomorphic for $X \in \mathbf{D}$, f is *I*-injective in \mathbf{C} if and only if it is *I*-injective in \mathbf{D} . Similarly, f is *J*-injective in \mathbf{C} if and only if it is *J*-injective in \mathbf{D} . Let f be an *I*-cofibration in \mathbf{D} . Then it has the left lifting property with respect to all *I*-injective maps in \mathbf{C} . Suppose we are given a commutative diagram



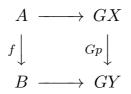
where p is I-injective in **D**. Then there is a relative I-cell complex $g: X \to Z$ [7, 2.1.9] such that f is a retract of g by [7, 2.1.15]. Since g is an I-cofibration in **D**, there is a lift $Z \to A$ of g with respect to p. Then the composite $Y \to Z \to A$ is a lift of f with respect to p. Therefore f is an I-cofibration in **D**. Similarly, f is a J-cofibration in **C** if and only if it is a J-cofibration in **D**. Thus we have the desired inclusions

- $J_{\mathbf{C}}$ -cell $\subseteq W_{\mathbf{C}} \cap I_{\mathbf{C}}$ -cof,
- $I_{\mathbf{C}}$ -inj $\subseteq W_{\mathbf{C}} \cap J_{\mathbf{C}}$ -inj, and
- either $W_{\mathbf{C}} \cap I_{\mathbf{C}}$ -cof $\subseteq J_{\mathbf{C}}$ -cof or $W_{\mathbf{C}} \cap J_{\mathbf{C}}$ -inj $\subseteq I_{\mathbf{C}}$ -inj.

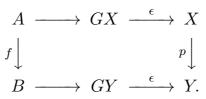
Here $I_{\mathbf{C}}$ -inj and $I_{\mathbf{C}}$ -cof denote, respectively, the classes of *I*-injective maps and *I*-cofibrations in \mathbf{C} , and similarly for $J_{\mathbf{C}}$ -inj and $J_{\mathbf{C}}$ -cof. Therefore \mathbf{C} is a cofibrantly generated model category by [7, 2.1.19].

It is clear, by the definition, that \mathbf{C} is finitely generated if so is \mathbf{C} .

Finally, to prove that (i, G, φ) is a Quillen adjunction, it suffices to show that $G: \mathbf{D} \to \mathbf{C}$ is a right Quillen functor, or equivalently, G preserves J-injective maps in \mathbf{D} by [7, 1.3.4] and [7, 2.1.17]. Let $p: X \to Y$ be a J-injective map in \mathbf{D} . Suppose there is a commutative diagram



where $f \in J$. Then we have a commutative diagram



Since p is J-injective in \mathbf{D} , there is a lift $h: B \to X$ of f. Thus we have a lift $Gh \circ \eta: B \cong GB \to GX$ of f with respect to Gp. Therefore $Gp: GX \to GY$ is J-injective in \mathbf{C} . Similarly, we can show that G preserves I-injective maps in \mathbf{C} , and so G preserves trivial fibrations in \mathbf{C} . Hence (i, G, φ) is a Quillen adjunction.

We turn to the case of pointed categories [7, p.4]. Let \mathbf{D}_* be the pointed category associated with \mathbf{D} , and let $U: \mathbf{D}_* \to \mathbf{D}$ be the forgetful functor. We denote by I_+ and J_+ the classes of those maps $f: X \to Y$ in \mathbf{D}_* such that $Uf: UX \to UY$ belongs to I and J, respectively. Then we have the following. (Compare [7, 1.1.8], [7, 1.3.5], and [7, 2.1.21].)

Theorem 2.2. Let **D** be a cofibrantly (resp. finitely) generated model category, and let **C** be a coreflective subcategory satisfying the conditions of Theorem 2.1. Then the pointed category \mathbf{C}_* has a cofibrantly (resp. finitely) generated model structure, with generating cofibrations I_+ and generating trivial cofibrations J_+ , such that the induced adjunction $(i_*, G_*, \varphi_*): \mathbf{C}_* \to \mathbf{D}_*$ is a Quillen adjunction.

We also have the following Proposition.

Proposition 2.3. Suppose \mathbf{C} and \mathbf{D} satisfy the conditions of Theorem 2.1. Suppose, further, that the coreflection arrow $\epsilon \colon GY \to Y$ is a weak equivalence for any fibrant object Y in \mathbf{D} . Then the adjunctions $(i, G, \varphi) \colon \mathbf{C} \to \mathbf{D}$ and $(i_*, G_*, \varphi_*) \colon \mathbf{C}_* \to \mathbf{D}_*$ are Quillen equivalences.

Proof. Let X be a cofibrant object in **C** and Y a fibrant object in **D**. Let $f: X \to Y$ be a map in **D**. Then we have $\varphi f = Gf \circ \eta: X \cong GX \to GY$. Since f coincides with the composite $X \xrightarrow{\varphi f} GY \xrightarrow{\epsilon} Y$ and ϵ is a weak equivalence in **D**, φf is a weak equivalence in **C** if and only if f is a weak equivalence in **D**. It follows by [7, 1.3.17] that that the induced adjunction (i_*, G_*, φ_*) is a Quillen equivalence.

3. On a model structure of the category \mathbf{NG}

In [12] we introduced the notion of numerically generated spaces which turns out to be the same notion as Δ -generated spaces introduced by Jeff Smith (cf. [3]). Let X be a topological space. A subset U of X is numerically open if for every continuous map $P: V \to X$, where V is an open subset of Euclidean space, $P^{-1}(U)$ is open in V. Similarly, U is numerically closed if for every such map P, $P^{-1}(U)$ is closed in V. A space X is called a numerically generated space if every numerically open subset is open in X.

Let **NG** denote the full subcategory of **Top** consisting of numerically generated spaces. Then the category **NG** is cartesian closed [12, 4.6]. To any X we can associate the numerically generated space topology, denoted νX , by letting U open in νX if and only if U is numerically open in X. Therefore we have a functor $\nu: \mathbf{Top} \to \mathbf{NG}$ which takes X to νX . Clearly, the identity map $\nu X \to X$ is continuous. By the results of [7, §3] the following holds.

Proposition 3.1. The functor ν : Top \rightarrow NG is a right adjoint to the inclusion functor i: NG \rightarrow Top, so that NG is a coreflective subcategory of Top.

A continuous map $f: X \to Y$ between topological spaces is called a weak homotopy equivalence in **Top** if it induces an isomorphism of homotopy groups

$$f_* \colon \pi_n(X, x) \to \pi_n(Y, f(x))$$

for all n > 0 and $x \in X$. Let I be the set of boundary inclusions $S^{n-1} \to D^n$, $n \ge 0$, J the set of inclusions $D^n \times \{0\} \to D^n \times I$, and W_{Top} the class of weak homotopy equivalences. The standard model structure on **Top** can be described as follows.

Theorem 3.2 ([7, 2.4.19]). There is a finitely generated model structure on **Top** with I as the set of generating cofibraitons, J as the set of generating trivial cofibrations, and W_{Top} as the class of weak equivalences.

The category **NG** is complete and cocomplete by [12, 3.4]. A space X is numerically generated if and only if $\nu X = X$ holds. Thus the unit of the adjunction $\eta: X \to \nu X$ is a natural homeomorphism. Moreover, since CW-complexes are numerically generated spaces by [12, 4.4], the classes I and J are contained in **NG**. Let $W_{\mathbf{NG}}$ be the class of maps $f: X \to Y$ in **NG** which is a weak equivalence in **Top**. Since the coreflection arrow $\nu Y \to Y$, given by the identity of $Y \in \mathbf{Top}$, is a weak equivalence (cf. [12, 5.4]), we have the following by Theorem 2.1 and Proposition 2.3.

Theorem 3.3. The category NG has a finitely generated model structure with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and W_{NG} as the class of weak equivalences. Moreover the adjunction $(i, \nu, \varphi): NG \to Top$ is a Quillen equivalence.

We turn to the case of pointed spaces. Let \mathbf{Top}_* be the category of pointed topological spaces. By [7, 2.4.20], there is a finitely generated model structure on the category \mathbf{Top}_* , with generating cofibrations I_+ and generating

trivial cofibrations J_+ . Then we have the following by Theorem 2.2 and Proposition 2.3.

Corollary 3.4. There is a finitely generated model structure on the category \mathbf{NG}_* of pointed numerically generated spaces, with generating cofibrations I_+ and generating trivial cofibrations J_+ . Moreover, the inclusion functor $i_*: \mathbf{NG}_* \to \mathbf{Top}_*$ is a Quilen equivalence.

Remark. (1) The argument of Theorem 3.3 can be applied to the subcategories **K** of k-spaces and **T** of compactly generated spaces. Similarly, the argument of Corollary 3.4 can be applied to the pointed categories \mathbf{K}_* and \mathbf{T}_* . Compare [2.4.28], [2.4.25], [2.4.26] of [7].

(2) Let **Diff** be the category of diffeological spaces (cf. [8]). In [12] we introduced a pair of functors $T : \mathbf{Diff} \to \mathbf{Top}$ and $D : \mathbf{Diff} \to \mathbf{Top}$, where T is a left adjoint to D, and showed that the composite TD coincides with $\nu : \mathbf{Top} \to \mathbf{NG}$. Thus \mathbf{NG} can be embedded as a full subcategory into \mathbf{Diff} . It is natural to ask whether \mathbf{Diff} has a model category structure with respect to which the pair (T, D) gives a Quillen adjuntion between \mathbf{Top} and \mathbf{Diff} .

Let I be the unit interval, and let $\lambda \colon \mathbf{R} \to I$ be the smashing function, that is, a smooth function such that $\lambda(t) = 0$ for $t \leq 0$ while $\lambda(t) = 1$ for $t \geq 1$. Let \tilde{I} denote the unit interval equipped with the quotient diffeology $\lambda_*(D_{\mathbf{R}})$, where $D_{\mathbf{R}}$ is the standard diffeology of \mathbf{R} . In [5] we introduce a finitely generated model category structure on **Diff** with the boundary inclusions $\partial \tilde{I}^{n-1} \to \tilde{I}^n$ as generating cofibrations, and with the inclusions $\partial \tilde{I}^{n-1} \times \tilde{I} \cup \tilde{I}^n \times \{0\} \to \tilde{I}^n \times \tilde{I}$ as generating trivial cofibrations. Its class of weak equivalences consists of those smooth maps $f \colon X \to Y$ inducing an isomorphism $f_* \colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for every $n \geq 0$ and $x_0 \in$ X. Here, the homotopy set $\pi_n(X, x_0)$ is defined to be the set of smooth homotopy classes of smooth maps $(\tilde{I}^n, \partial \tilde{I}^n) \to (X, x_0)$.

It is expected that with respect to the model structure on **Diff** described above, the pair (T, D) induces a Quillen adjunction between **Top** and **Diff**.

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