Math. J. Okayama Univ. **57** (2015), 13–78

QUASI TERTIARY COMPOSITIONS AND A TODA BRACKET IN HOMOTOPY GROUPS OF SU(3)

Hideaki Ōshima and Katsumi Ōshima

Abstract. We revise the theories of tertiary compositions studied by Ôguchi and Mimura. As a byproduct, we determine a Toda bracket in homotopy groups of SU(3) which solves an ambiguity in a previous paper of Maruyama and the first author.

1. Introduction

Since secondary compositions (Toda brackets) are powerful tools for computing homotopy groups of spaces, one has expected that higher Toda brackets are also useful if they exist. Hence several authors have tried to define higher Toda brackets. First of all Toda suggested the existence of tertiary compositions in [**17**] and then in [**19**] constructed elements $\mu_3 \in \pi_{12}(\mathrm{S}^3)$ and $\kappa_7 \in \pi_{21}(\mathrm{S}^7)$ by essentially tertiary compositions (see 5.9, 5.10, 6.1 below). These works stimulated Ôguchi [**13**] and Mimura [**10**] to research on tertiary compositions. But in [**13**, **10**] there are a few gaps and errors. On the other hand, J. Cohen [**3**] defined k-fold Toda bracket for every $k \geq 3$ (see Appendix B). His 3-fold Toda bracket is bigger than the usual Toda bracket in general (see B.4 and B.5) and it seems that his k-fold Toda brackets are useful not in unstable homotopy but in stable homotopy. So we resume studying unstable tertiary compositions by revising theories of Ôguchi [**13**] and Mimura [**10**].

The main parts of this paper are the sections 4, 5 and 6. Suppose that the following data are given: two non-negative integers n_1, n_2, four maps

$$X_0 \xleftarrow{a_1} E^{n_1}X_1, \quad X_1 \xleftarrow{a_2} E^{n_2}X_2, \quad X_2 \xleftarrow{a_3} X_3 \xleftarrow{a_4} X_4$$

and three null homotopies

$$A_1 : a_1 \circ E^{n_1}a_2 \simeq *, \quad A_2 : a_2 \circ E^{n_2}a_3 \simeq *, \quad A_3 : a_3 \circ a_4 \simeq *$$

such that $(A_1, A_2, A_3)_{n_1,n_2}$ is admissible (see the section 4 for definitions), where E^n is the n-fold suspension. We define a number of subsets of $[E^{n_1+n_2+2}X_4, X_0]$: *quasi tertiary compositions*

$$\{A_1, A_2, A_3\}^{(0)}_{n_1,n_2} \subset \{A_1, A_2, A_3\}^{(1)}_{n_1,n_2} \subset \{A_1, A_2, A_3\}^{(2)}_{n_1,n_2} \subset \{A_1, A_2, A_3\}^{(3)}_{n_1,n_2}$$

Mathematics Subject Classification. primary 55Q05, secondary 55Q35.

Key words and phrases. Toda bracket, tertiary composition, quasi tertiary composition, homotopy group, special unitary group, Samelson product.

in the section 4, and *tertiary compositions*

$$\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}^{(0)}_{n_1,n_2} \subset \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}^{(1)}_{n_1,n_2} \subset \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}^{(2)}_{n_1,n_2} \subset \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}^{(3)}_{n_1,n_2}$$

in the section 6, such that $\{A_1,A_2,A_3\}^{(k)}_{n_1,n_2} \subset \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}^{(k)}_{n_1,n_2}$, where α_i is the homotopy class of a_i. In case of $n_1 = n_2 = 0$, $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}^{(3)}_{0,0}$ is a revised version of Mimura's tertiary composition [**10**], $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}^{(2)}_{0,0}$ is a subset of Cohen's 4-fold Toda bracket $\{a_1,a_2,a_3,a_4\}^C$ (see B.6), and $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}^{(1)}_{0,0}$ is written as $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ by Ôguchi [**13**].

In the section 5, we prove elementary properties of quasi tertiary compositions. In the section 6, we give revisions of results in [**13**, **10**].

In the section 7, we give applications of quasi tertiary compositions to homotopy groups of SU(3). One of them is the following proposition (see the section 7 for notations).

Proposition 7.5. The Toda bracket $\{[2\iota_5]\eta_5, 4\nu_5, \eta_8\}$ consists of a single element $[\nu_5\eta_8^2]$.

In the section 8, we prove Hamanaka-Kono's results [**4**, Theorem 2.5, Theorem 2.3] as a corollary to Proposition 7.5 so that we can solve an ambiguity in [**9**, Theorem 7.1] (see 8.3).

We recall the definitions of extension and coextension [**13**, **19**] in the section 2 and Toda bracket [**19**] in the section 3. Many results in sections 2 and 3 are well-known or folklore. In Appendix A, we give a counterexample to Proposition (6.5) of [**13**]. In Appendix B, we study some properties of Cohen's k-fold Toda brackets.

2. EXTENSIONS AND COEXTENSIONS

In this paper all spaces have the base point and all maps and homotopies preserve the base point. We denote the base point of the space X by x_0 or $*$. We denote by $1_X : X \to X$ and $* : X \to Y$ the identity map of X and the constant map to y_0, respectively. In particular we denote by $*_\ell : \mathrm{S}^{\ell+1} \to \mathrm{S}^\ell$ and $*^m_\ell : \mathrm{S}^{\ell+m} \to \mathrm{S}^\ell$ the trivial maps. We denote the homotopy classes of $1_X, 1_{\mathrm{S}^n}, *, *_\ell, *^m_\ell$ by $\iota_X, \iota_n, 0, 0_\ell, 0^m_\ell$, respectively. Frequently we do not distinguish in notation between a map and its homotopy class.

For spaces X and Y, we denote by $[X,Y]$ the set of homotopy classes of maps from X into Y. Let $X \vee Y = X \times \{*\} \cup \{*\} \times Y \subset X \times Y$ be the one point union of X and Y. We denote the quotient space $(X \times Y)/(X \vee Y)$ by $X \wedge Y$, and $x \wedge y \in X \wedge Y$ is the point represented by $(x,y) \in X \times Y$. Maps $f : X \to X'$ and $g : Y \to Y'$ induce maps $f \times g : X \times Y \to X' \times Y'$,

$f \vee g : X \vee Y \to X' \vee Y'$ and $f \wedge g : X \wedge Y \to X' \wedge Y'$ by $(f \times g)(x,y) = (f(x), g(y))$, $f \vee g = f \times g|_{X \vee Y}$ and $(f \wedge g)(x \wedge y) = f(x) \wedge g(y)$.

Let I be the unit interval $[0,1]$ whose base point is 1. We use identifications $I/\{0,1\} = \mathrm{S}^1$ and $\mathrm{S}^m \wedge \mathrm{S}^n = \mathrm{S}^{m+n}$ as in [**19**, pp.5-6]. Saying rough, $\underbrace{\mathrm{S}^1 \wedge \cdots \wedge \mathrm{S}^1}_{n} = \mathrm{S}^n$ and so $\mathrm{S}^m \wedge \mathrm{S}^n = \mathrm{S}^{m+n} = \mathrm{S}^n \wedge \mathrm{S}^m$ by $(x_1 \wedge \cdots \wedge x_m) \wedge (x_{m+1} \wedge \cdots \wedge x_{m+n}) = x_1 \wedge \cdots \wedge x_{m+n} = (x_1 \wedge \cdots \wedge x_n) \wedge (x_{n+1} \wedge \cdots \wedge x_{m+n})$, where $x_i \in \mathrm{S}^1$. For a space X, its cone CX, suspensions EX and E^nX $(n \geq 0)$ are defined as follows: $CX = X \wedge I$, $EX = (X \wedge I)/(X \wedge \{0,1\})$ and $E^nX = X \wedge \mathrm{S}^n$. We identify EX with E^1X by the canonical homeomorphism. We write $x \wedge t \in CX$ and $x \wedge \bar{t} \in EX$ which are represented by $(x,t) \in X \times I$. We regard X as a subspace of CX by the identification $x = x \wedge 0$. For a map $f : X \to Y$, let $E^nf = f \wedge 1_{\mathrm{S}^n} : E^nX \to E^nY$ and $Cf = f \wedge 1_I : CX \to CY$. Also let $C_f = Y \cup_f CX$ denote the mapping cone of f, that is, it is obtained from the disjoint union of Y and CX by identifying $x \wedge 0 \in CX$ with $f(x) \in Y$. The image of $x \wedge t \in CX$ in $Y \cup_f CX$ is also denoted by $x \wedge t$ for simplicity. We regard Y as a subspace of $Y \cup_f CX$ by the canonical embedding $i_f : Y \to Y \cup_f CX$. We denote by $q_f : Y \cup_f CX \to EX$ the quotient map.

In case of that Z is a locally compact Hausdorff space or X and Y have closed base points, we identify: $(X \vee Y) \wedge Z = (X \wedge Z) \vee (Y \wedge Z)$ and $Z \wedge (X \vee Y) = (Z \wedge X) \vee (Z \wedge Y)$ by the canonical homeomorphisms. Hence $C(X \vee Y) = CX \vee CY$ and $E^n(X \vee Y) = E^nX \vee E^nY$. We define $\theta_{\mathrm{S}^1} : \mathrm{S}^1 \to \mathrm{S}^1 \vee \mathrm{S}^1$ and $\theta_{EX} : EX \to EX \vee EX$ by

$$\theta_{\mathrm{S}^1}(\bar{t}) = \begin{cases} (\overline{2t}, *) & 0 \leq t \leq \frac{1}{2} \\ (*, \overline{2t-1}) & \frac{1}{2} \leq t \leq 1 \end{cases}, \qquad \theta_{EX} = 1_X \wedge \theta_{\mathrm{S}^1}.$$

These two maps are comultiplications. Since S^2 has the unique comultiplication up to homotopy, and since $\theta_{\mathrm{S}^1} \wedge 1_{\mathrm{S}^1}$ and $1_{\mathrm{S}^1} \wedge \theta_{\mathrm{S}^1}$ are comultiplications on $\mathrm{S}^1 \wedge \mathrm{S}^1 = \mathrm{S}^2$, we have $\theta_{\mathrm{S}^1} \wedge 1_{\mathrm{S}^1} \simeq 1_{\mathrm{S}^1} \wedge \theta_{\mathrm{S}^1}$. Therefore

$$E\theta_{EX} \simeq \theta_{E^2X}. \tag{2.1}$$

Let $\nabla_X : X \vee X \to X$ be the folding map. For two maps $a_1, a_2 : EX \to Y$, we define $a_1 + a_2 = \nabla_Y \circ (a_1 \vee a_2) \circ \theta_{EX} : EX \to Y$. This induces a group operation $+$ in $[EX, Y]$.

For two maps $b_i : Y_i \to Z$ $(i = 1, 2)$, we abbreviate $\nabla_Z \circ (b_1 \vee b_2)$ to $b_1 \underline{\vee} b_2$. We easily have

Lemma 2.1. *Given four maps $a_i : EX \to Y_i$ and $b_i : Y_i \to Z$ $(i = 1, 2)$, we have $(b_1 \underline{\vee} b_2) \circ (a_1 \vee a_2) \circ \theta_{EX} = b_1 \circ a_1 + b_2 \circ a_2 : EX \to Z$.*

If $H : X \times I \to Y$ is a homotopy from f to g, that is, if $H(x,0) = f(x)$, $H(x,1) = g(x)$ and $H(*,t) = *$, then we write $f \underset{H}{\simeq} g : X \to Y$ or simply $H : f \simeq g$, and define $-H : X \times I \to Y$ by $(-H)(x,t) = H(x,1-t)$. In particular if moreover $g = *$, then H induces a map $CX \to Y$, $x \wedge t \mapsto H(x,t)$, which is denoted by the same letter H for simplicity.

Toda [**19**] introduced the notions of extension and coextension. We use notations of [**13**] for them. Given maps $a_i : X_i \to X_{i-1}$ $(i = 1,2)$ and $H : a_1 \circ a_2 \simeq *$, we define

$$[a_1, H, a_2] : X_1 \cup_{a_2} CX_2 \to X_0, \quad \text{an } \textit{extension} \text{ of } a_1 \text{ with respect to } a_2,$$
$$[a_1, H, a_2](x_1) = a_1(x_1), \quad [a_1, H, a_2](x_2 \wedge t) = H(x_2 \wedge t);$$
$$(a_1, H, a_2) : EX_2 \to X_0 \cup_{a_1} CX_1, \quad \text{a } \textit{coextension} \text{ of } a_2 \text{ with respect to } a_1,$$
$$(a_1, H, a_2)(x_2 \wedge \bar{t}) = \begin{cases} a_2(x_2) \wedge (1-2t) & 0 \le t \le \frac{1}{2} \\ H(x_2 \wedge (2t-1)) & \frac{1}{2} \le t \le 1 \end{cases}.$$

Notice that our coextension is different in sign to one given in [**12**, **13**]. We have

$$[a_1, H, a_2] \circ i_{a_2} = a_1, \qquad q_{a_1} \circ (a_1, H, a_2) \simeq -Ea_2. \tag{2.2}$$

Let $\mathrm{Ext}_{a_2}(a_1)$ and $\mathrm{Coext}_{a_1}(a_2)$ be respectively the sets of homotopy classes of $[a_1, H, a_2]$ and (a_1, H, a_2), where we take all possible H. Since $\mathrm{Ext}_{a_2}(a_1)$ and $\mathrm{Coext}_{a_1}(a_2)$ depend on the homotopy classes of a_1 and a_2 respectively, we denote them by $\mathrm{Ext}_{a_2}(\alpha_1)$ and $\mathrm{Coext}_{a_1}(\alpha_2)$ respectively, where α_i is the homotopy class of a_i. Elements of $\mathrm{Ext}_{a_2}(\alpha_1)$ and $\mathrm{Coext}_{a_1}(\alpha_2)$ are frequently written as $\overline{\alpha_1}$ and $\widetilde{\alpha_2}$, respectively.

The following two lemmas are obtained easily.

Lemma 2.2. *Let four maps* $X_0 \xleftarrow{a_1} X_1 \xleftarrow{a_2} X_2 \xleftarrow{a_3} X_3 \xleftarrow{a_4} X_4$ *be given.*

(1) *If* $H : a_1 \circ a_2 \simeq *$, *then*

$$[a_1, H \circ Ca_3, a_2 \circ a_3] = [a_1, H, a_2] \circ (1_{X_1} \cup Ca_3),$$
$$(a_1, H \circ Ca_3, a_2 \circ a_3) = (a_1, H, a_2) \circ Ea_3.$$

(2) *If* $H : a_2 \circ a_3 \simeq *$, *then*

$$[a_1 \circ a_2, a_1 \circ H, a_3] = a_1 \circ [a_2, H, a_3],$$
$$(a_1 \circ a_2, a_1 \circ H, a_3) = (a_1 \cup 1_{CX_2}) \circ (a_2, H, a_3).$$

(3) *If* $H : a_1 \circ a_2 \circ a_3 \simeq *$, *then*

$$[a_1 \circ a_2, H, a_3] = [a_1, H, a_2 \circ a_3] \circ (a_2 \cup 1_{CX_3}),$$
$$(a_1, H, a_2 \circ a_3) = (1_{X_0} \cup Ca_2) \circ (a_1 \circ a_2, H, a_3).$$

(4) *If* $H : a_1 \circ a_2 \simeq *$ *and* $K : a_2 \circ a_3 \circ a_4 \simeq *$, *then*

$$[a_1, H \circ Ca_3, a_2 \circ a_3] \circ (a_2 \circ a_3, K, a_4) = [a_1, H, a_2] \circ (a_2, K, a_3 \circ a_4).$$

(5) *If* $H : a_1 \circ a_2 \circ a_3 \simeq *$ *and* $K : a_3 \circ a_4 \simeq *$, *then*

$$[a_1 \circ a_2, H, a_3] \circ (a_3, K, a_4) = [a_1, H, a_2 \circ a_3] \circ (a_2 \circ a_3, a_2 \circ K, a_4).$$

Lemma 2.3 (Lemma 2.10 of [**10**])**.** *Suppose the following data are given:*

$$a_i \in \alpha_i \in [X_i, X_{i-1}]\,(i = 1, 2), \quad \alpha_1 \circ \alpha_2 = 0, \quad \beta \in [X_0, V], \quad \gamma \in [U, X_2].$$

Then

$$\beta \circ \mathrm{Ext}_{a_2}(\alpha_1) \subset \mathrm{Ext}_{a_2}(\beta \circ \alpha_1) \subset [X_1 \cup_{a_2} CX_2, V],$$
$$\mathrm{Coext}_{a_1}(\alpha_2) \circ E\gamma \subset \mathrm{Coext}_{a_1}(\alpha_2 \circ \gamma) \subset [EU, X_0 \cup_{a_1} CX_1].$$

We denote by $\tau(X, Y) : X \wedge Y \to Y \wedge X$ the switching map, that is, $\tau(X, Y)(x \wedge y) = y \wedge x$. Given a map $f : X \to Y$, the "canonical" homeomorphism [**19**, (1.16)]

$$\psi^n_{(Y,f,X)} : E^nY \cup_{E^nf} CE^nX \underset{\approx}{\longrightarrow} E^n(Y \cup_f CX)$$

is defined by $\psi^n_{(Y,f,X)}(y \wedge s_n) = y \wedge s_n$ and $\psi^n_{(Y,f,X)}(x \wedge s_n \wedge t) = x \wedge t \wedge s_n$, where $y \in Y$, $s_n \in \mathrm{S}^n$, $x \in X$, $t \in I$. Sometimes we abbreviate $\psi^n_{(Y,f,X)}$ to ψ^n_f. If $0 \leq m \leq n$, then

$$(2.3) \quad E^m\big(\psi^{n-m}_f\big) \circ \psi^m_{E^{n-m}f} = \psi^n_f \quad \textit{i.e.} \quad \psi^m_{E^{n-m}f} = E^m\big(\psi^{n-m}_f\big)^{-1} \circ \psi^n_f.$$

We have

$$\psi^n_{(X,1_X,X)} = 1_X \wedge \tau(\mathrm{S}^n, I) : CE^nX \to E^nCX,$$
$$\psi^n_{(\{*\},*,X)} = 1_X \wedge \tau(\mathrm{S}^n, \mathrm{S}^1) : EE^nX \to E^nEX.$$

Since the degree of $\tau(\mathrm{S}^n, \mathrm{S}^1) : \mathrm{S}^{n+1} \to \mathrm{S}^{n+1}$ is $(-1)^n$, we have $\psi^n_{(\{*\},*,X)} \simeq (-1)^n 1_{E^{n+1}X}$. Given a map $H : X \times I \to Y$ with $H(\{*\} \times I) = *$ (i.e. a homotopy), we define

$$\widetilde{E}^nH : E^nX \times I \to E^nY, \quad (x \wedge s_n, t) \mapsto H(x, t) \wedge s_n.$$

As is easily shown, $\widetilde{E}^m\widetilde{E}^nH = \widetilde{E}^{m+n}H$. If $H : CX \to Y$, then we have

$$\widetilde{E}^nH = E^nH \circ (1_X \wedge \tau(\mathrm{S}^n, I)) : CE^nX \to E^nY.$$

The following lemma is obvious from definitions.

Lemma 2.4. *We have* $E^n[a_1, H, a_2] = [E^na_1, \widetilde{E}^nH, E^na_2] \circ (\psi^n_{a_2})^{-1}$ *and*

$$E^n(a_1, H, a_2) = \psi^n_{a_1} \circ (E^na_1, \widetilde{E}^nH, E^na_2) \circ (1_{X_2} \wedge \tau(\mathrm{S}^1, \mathrm{S}^n)).$$

For a map $f : X \to Y$, the *co-operation* [**5**]

$$\theta = \theta_f : Y \cup_f CX \to (Y \cup_f CX) \vee EX$$

is defined by

$$\theta(y) = (y, *), \quad \theta(x \wedge t) = \begin{cases} (x \wedge (2t), *) & 0 \leq t \leq \frac{1}{2} \\ (*, x \wedge \overline{2t-1}) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

When $Y = \{*\}$, we have $\theta = \theta_{EX}$. For maps $g : Y \cup_f CX \to Z$ and $h : EX \to Z$, we define $g \dotplus h = (g \underline{\vee} h) \circ \theta_f$ which is the composite of

$$Y \cup_f CX \xrightarrow{\theta_f} (Y \cup_f CX) \vee EX \xrightarrow{g \vee h} Z \vee Z \xrightarrow{\nabla} Z.$$

This defines an action $\dotplus : [Y \cup_f CX, Z] \times [EX, Z] \to [Y \cup_f CX, Z]$. We easily have

Lemma 2.5. *Given maps $f : X \to Y$, $g : Y \cup_f CX \to Z$ and $h : EX \to Z$, we have*

$$\begin{aligned} &E^n(g \dotplus h) \circ \psi_f^n = (E^n g \circ \psi_f^n) \dotplus (E^n h \circ \psi_{(\{*\},*,X)}^n) \\ &\simeq (E^n g \circ \psi_f^n) \dotplus \big((-1)^n E^n h\big) \ rel\ E^n Y : E^n Y \cup_{E^n f} CE^n X \to E^n Z. \end{aligned}$$

In particular, if moreover $f = 1_X$, then

$$\widetilde{E}^n(g \dotplus h) \simeq \widetilde{E}^n g \dotplus (-1)^n E^n h \ rel\ E^n X : CE^n X \to E^n Z.$$

Proposition 2.6 (Chapter 15 of [**5**]). *If $\alpha, \beta \in [Y \cup_f CX, Z]$ and $\lambda, \mu \in [EX, Z]$, then*

1. $\alpha \dotplus (\lambda + \mu) = (\alpha \dotplus \lambda) \dotplus \mu$,
2. *if $a \in \alpha$, then $a \dotplus * \simeq a$ rel Y,*
3. $q_f^*(\lambda) \dotplus \mu = q_f^*(\lambda + \mu)$,
4. *${i_f}^*(\alpha) = {i_f}^*(\beta)$ if and only if $\beta = \alpha \dotplus \lambda$ for some λ.*

We easily have

Lemma 2.7. *If $Y \xleftarrow{f} X \xleftarrow{g} W$ are maps and $A : f \circ g \simeq *$, then*

$$\begin{aligned} j \circ \theta_f \circ (f, A, g) \simeq j \circ \big(-(f, A, g) \vee Eg\big) \circ \theta_{EW} \circ (-1_{EW}) \\ : EW \to (Y \cup_f CX) \times EX, \end{aligned}$$

where $j : (Y \cup_f CX) \vee EX \to (Y \cup_f CX) \times EX$ is the inclusion map. If moreover $j_* : [EW, (Y \cup_f CX) \vee EX] \to [EW, (Y \cup_f CX) \times EX]$ is injective, then

$$\theta_f \circ (f, A, g) \simeq \big(-(f, A, g) \vee Eg\big) \circ \theta_{EW} \circ (-1_{EW}) : EW \to (Y \cup_f CX) \vee EX.$$

For maps $A, B : CX \to Y$ with $A|_X = B|_X$, we define

$$d(A,B) : EX \to Y, \quad x \wedge \bar{t} \mapsto \begin{cases} A(x \wedge (1-2t)) & 0 \leq t \leq \frac{1}{2} \\ B(x \wedge (2t-1)) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and denote its homotopy class by $\delta(A,B) \in [EX, Y]$. It is a generalization of "separation element" in [**6**], while our $d(A,B)$ is written as $d(B,A)$ in [**13**, **18**].

Lemma 2.8. (1) *Let $A, B, D : CX \to Y$ satisfy $A|_X = B|_X = D|_X$.*

(a) $d(A,B) = -d(B,A)$.

(b) $\delta(A,B) + \delta(B,D) = \delta(A,D)$.

(c) $\delta(A,A) = 0$.

(d) $d(A, A \dotplus h) \simeq h$ *for every map* $h : EX \to Y$.

(e) $A \dotplus d(A,B) \simeq B$ *rel* X.

(f) $\delta(A,B) = 0$ *if and only if* $A \simeq B$ *rel* X.

(g) *For a fixed* A, $[EX, Y]$ *is the set of* $\delta(A,B)$ *with* $A|_X = B|_X$.

(h) *For maps* $f : W \to X$ *and* $g : Y \to Z$, *we have*

$$d(A,B) \circ Ef = d(A \circ Cf, B \circ Cf), \quad g \circ d(A,B) = d(g \circ A, g \circ B).$$

(i) *We have* $E^n d(A,B) = d(\widetilde{E}^n A, \widetilde{E}^n B) \circ (1_X \wedge \tau(\mathrm{S}^1, \mathrm{S}^n))$ *which is homotopic to* $(-1)^n d(\widetilde{E}^n A, \widetilde{E}^n B)$.

(2) *If* $Z \xleftarrow{a} Y \xleftarrow{b} X$ *and* $h : EX \to Z$ *are maps and* $H : a \circ b \simeq *$, *then*

$$[a, H \dotplus h, b] = [a, H, b] \dotplus h : Y \cup_b CX \to Z, \tag{2.4}$$

$$(a, H \dotplus h, b) \simeq (a, H, b) + i_a \circ h : EX \to Z \cup_a CY. \tag{2.5}$$

If moreover $H' : a \circ b \simeq *$ *and* $H \simeq H'$ *rel* X, *then*

$$[a, H, b] \simeq [a, H', b] \ rel\ Y, \quad (a, H, b) \simeq (a, H', b). \tag{2.6}$$

Proof. (1) is easy, and (2) is proved by giving homotopies explicitly. ☐

If $f \underset{F}{\simeq} g : X \to Y$, then we define $\varphi_F : Y \cup_f CX \to Y \cup_g CX$ by

$$\varphi_F(y) = y, \quad \varphi_F(x \wedge t) = \begin{cases} F(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ x \wedge (2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

For every $n \geq 0$, we have

$$\psi^n_g \circ \varphi_{\widetilde{E}^n F} = E^n \varphi_F \circ \psi^n_f. \tag{2.7}$$

We denote by $1_f : X \times I \to Y$ the constant homotopy of f, that is, $1_f(x,t) = f(x)$. Then

$$\varphi_{1_f} \simeq 1_{Y \cup_f CX} \ rel\ Y. \tag{2.8}$$

The following lemma can be proved by giving homotopies explicitly.

Lemma 2.9. *Under the above notations, φ_F is a homotopy equivalence whose homotopy inverse is $\varphi_{-F} : Y \cup_g CX \to Y \cup_f CX$, and the following diagram is homotopy commutative:*

$$\begin{tikzcd}
Y \cup_f CX \arrow[rr, "\varphi_F"] \arrow[dr, "q_f"'] & & Y \cup_g CX \arrow[dl, "q_g"] \\
& EX &
\end{tikzcd}$$

Given homotopies $F, A : X \times I \to Y$ and $G : Y \times I \to Z$ such that $F(x,1) = A(x,0)$ for all $x \in X$, we define $A \bullet F : X \times I \to Y$ and $G \,\bar{\circ}\, F : X \times I \to Z$ by

$$(A \bullet F)(x,t) = \begin{cases} F(x,2t) & 0 \le t \le \frac{1}{2} \\ A(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}, \quad (G \,\bar{\circ}\, F)(x,t) = G(F(x,t),t).$$

Note that $\bullet$ and $\bar{\circ}$ are called in [**8**, pp.272-273] vertical composition and horizontal composition, respectively.

The following lemma can be proved by giving homotopies explicitly.

Lemma 2.10 ([**13**]). *If $a_1 \underset{H}{\simeq} a_1' : E^n X_1 \to X_0$, $a_2 \underset{K}{\simeq} a_2' : X_2 \to X_1$ and $A : a_1 \circ E^n a_2 \simeq *$, then*

$$[a_1, A, E^n a_2] \simeq [a_1', A', E^n a_2'] \circ \varphi_{\widetilde{E}^n K}, \ \varphi_H \circ (a_1, A, E^n a_2) \simeq (a_1', A', E^n a_2'),$$

where $A' = A \bullet \big((-H) \,\bar{\circ}\, \widetilde{E}^n(-K)\big) : CE^n X_2 \to X_0$.

Proposition 2.11. *Suppose the following data are given: $a_1 \underset{H}{\simeq} a_1' : E^n X_1 \to X_0$, $a_2 \underset{K}{\simeq} a_2' : X_2 \to X_1$, $a_3 \underset{L}{\simeq} a_3' : X_3 \to X_2$, $A_1 : a_1 \circ E^n a_2 \simeq *$, $A_2 : a_2 \circ a_3 \simeq *$. Then*

$$\begin{aligned} d(A_1 \circ CE^n a_3,\, a_1 \circ \widetilde{E}^n A_2) &= [a_1, A_1, E^n a_2] \circ (E^n a_2, \widetilde{E}^n A_2, E^n a_3) \\ &\simeq [a_1', A_1', E^n a_2'] \circ (E^n a_2', \widetilde{E}^n A_2', E^n a_3') = d(A_1' \circ CE^n a_3', a_1' \circ \widetilde{E}^n A_2'), \end{aligned}$$

where

$$A_1' = A_1 \bullet \big((-H) \,\bar{\circ}\, \widetilde{E}^n(-K)\big) : a_1' \circ E^n a_2' \simeq *,$$
$$A_2' = A_2 \bullet \big((-K) \,\bar{\circ}\, (-L)\big) : a_2' \circ a_3' \simeq *.$$

Proof. By definitions, two equalities are obvious. By Lemma 2.10, we have

$$[a_1, A_1, E^n a_2] \simeq [a_1', A_1', E^n a_2'] \circ \varphi_{\widetilde{E}^n K},$$
$$\varphi_{\widetilde{E}^n K} \circ (E^n a_2, \widetilde{E}^n A_2, E^n a_3) \simeq (E^n a_2', \widetilde{E}^n A_2', E^n a_3').$$

Hence we obtain the result. □

Proposition 2.12. *Suppose that the following data are given:* $a_k : X_k \to X_{k-1}$ $(k = 1, 2, 3)$, $A_\ell : a_\ell \circ a_{\ell+1} \simeq *$ $(\ell = 1, 2)$, $h : X_1 \cup_{a_2} CX_2 \to Z$, $f : EX_2 \to Z$ *and* $g : EX_3 \to X_1$. *Then*

$$(h \dotplus f) \circ (a_2, A_2, a_3) \simeq f \circ (-Ea_3) + h \circ (a_2, A_2, a_3), \tag{2.9}$$

$$h \circ (a_2, A_2 \dotplus g, a_3) \simeq h \circ (a_2, A_2, a_3) + h \circ i_{a_2} \circ g. \tag{2.10}$$

If moreover $Z = X_0$, *then*

$$\begin{aligned} &[a_1, A_1 \dotplus f, a_2] \circ (a_2, A_2 \dotplus g, a_3) \\ &\qquad \simeq f \circ (-Ea_3) + [a_1, A_1, a_2] \circ (a_2, A_2, a_3) + a_1 \circ g. \end{aligned} \tag{2.11}$$

Proof. We have (2.10) from (2.5). In order to prove (2.9), consider the decomposition: $I \times I = K_1 \cup \cdots \cup K_5$, where $(s, t) \in I \times I$ and

$$K_1 = \{(s,t) \mid t \geq 2s\}, \quad K_2 = \{(s,t) \mid 4s - 1 \leq t \leq 2s\},$$
$$K_3 = \{(s,t) \mid 4s - 2 \leq t \leq 4s - 1\},$$
$$K_4 = \{(s,t) \mid 2s - 1 \leq t \leq 4s - 2\}, \quad K_5 = \{(s,t) \mid t \leq 2s - 1\}.$$

We define $\phi : I \times I \to I$ and $\Phi : X_3 \times I \times I \to Z$ by

$$\phi(s,t) = \begin{cases} 2s & (s,t) \in K_1 \\ 4s - t & (s,t) \in K_2 \\ -4s + t + 2 & (s,t) \in K_3 \\ 4s - t - 2 & (s,t) \in K_4 \\ 2s - 1 & (s,t) \in K_5 \end{cases},$$

$$\Phi(x_3, s, t) = \begin{cases} (-f)\big(a_3(x_3) \wedge \phi(s,t)\big) & (s,t) \in K_1 \cup K_2 \\ h\big(a_3(x_3) \wedge \phi(s,t)\big) & (s,t) \in K_3 \\ h\big(A_2(x_3 \wedge \phi(s,t))\big) & (s,t) \in K_4 \cup K_5 \end{cases}.$$

Let $\widetilde{\Phi} : EX_3 \times I \to Z$ be defined by $\widetilde{\Phi}(x_3 \wedge \overline{s}, t) = \Phi(x_3, s, t)$. Then $\widetilde{\Phi}$ is a desired homotopy of (2.9). We have (2.11) by (2.4), (2.9) and (2.10). This completes the proof. □

From Lemma 2.5, (2.6) and (2.11), we have

Corollary 2.13. *Suppose the following data are given:*

$$X_0 \xleftarrow{a_1} E^n X_1, \ X_1 \xleftarrow{a_2} X_2 \xleftarrow{a_3} X_3, \ A_1 : a_1 \circ E^n a_2 \simeq *, \ A_2 : a_2 \circ a_3 \simeq *,$$
$$f : E^{n+1} X_2 \to X_0, \quad g : EX_3 \to X_1.$$

Then

$$\begin{aligned} &[a_1, \, A_1 \dotplus f, E^n a_2] \circ (E^n a_2, \widetilde{E}^n (A_2 \dotplus g), E^n a_3) \\ &\quad \simeq f \circ (-E^{n+1} a_3) + [a_1, A_1, E^n a_2] \circ (E^n a_2, \widetilde{E}^n A_2, E^n a_3) + a_1 \circ (-1)^n E^n g. \end{aligned}$$

3. Toda brackets

If G is an abelian group and α is a coset of a subgroup H of G, then H is called the *indeterminacy* of α and we write $\operatorname{Indet}\alpha = H$.

We use notations of Toda [**19**] for elements of homotopy groups of spheres. Let $\mathbb{Z}_m\{\alpha\}$ denote the cyclic group of order m whose generator is α, and let $\mathbb{Z}_m^n$ denote the direct sum of n copies of $\mathbb{Z}_m$. For example, $\pi_n(\mathrm{S}^n) = \mathbb{Z}\{\iota_n\}$ $(n \geq 1)$, $\pi_3(\mathrm{S}^2) = \mathbb{Z}\{\eta_2\}$, $\pi_{n+1}(\mathrm{S}^n) = \mathbb{Z}_2\{\eta_n\}$ $(n \geq 3)$, $\pi_{n+2}(\mathrm{S}^n) = \mathbb{Z}_2\{\eta_n^2\}$ $(n \geq 2)$, where $\eta_n^2 = \eta_n\eta_{n+1}$, $\pi_{n+3}(\mathrm{S}^n) = \mathbb{Z}_8\{\nu_n\} \oplus \mathbb{Z}_3$ $(n \geq 5)$, and $\pi_9(\mathrm{S}^5) = \mathbb{Z}_2\{\nu_5\eta_8\}$.

Suppose that a non-negative integer n and the following *null triple* [**13**] are given

$$\begin{aligned} \alpha_1 \in [E^nX_1, X_0],\ \alpha_k \in [X_k, X_{k-1}]\ (k = 2, 3), \\ \alpha_1 \circ E^n\alpha_2 = 0,\ \alpha_2 \circ \alpha_3 = 0. \end{aligned} \tag{3.1}$$

We abbreviate it to $(\alpha_1, \alpha_2, \alpha_3)_n$. A *representative* of (3.1) is a 6-tuple $(a_1, a_2, a_3; A_1, A_2)_n$ such that $a_k \in \alpha_k$ $(k = 1, 2, 3)$, $A_1 : a_1 \circ E^na_2 \simeq *$ and $A_2 : a_2 \circ a_3 \simeq *$. Sometimes we write $(a_1, a_2, a_3)_n$ instead of $(\alpha_1, \alpha_2, \alpha_3)_n$. Denote by $\{a_1, a_2, a_3\}_n$ the set of homotopy classes of

$$[a_1, A_1, E^na_2] \circ (E^na_2, \widetilde{E}^nA_2, E^na_3)$$

for all A_1, A_2 such that $(a_1, a_2, a_3; A_1, A_2)_n$ is a representative of (3.1). Then $\{a_1, a_2, a_3\}_n$ depends only on α_k $(k = 1, 2, 3)$ by Proposition 2.11. Therefore we denote $\{a_1, a_2, a_3\}_n$ by $\{\alpha_1, \alpha_2, \alpha_3\}_n$ which is called the *Toda bracket* or the *secondary composition* [**16**, **19**]. This is different only in sign to one given in [**18**, **12**]. By Corollary 2.13, the Toda bracket $\{\alpha_1, \alpha_2, \alpha_3\}_n$ is a double coset of the subgroups $[E^{n+1}X_2, X_0] \circ E^{n+1}\alpha_3$ and $\alpha_1 \circ E^n[EX_3, X_1]$ of the group $[E^{n+1}X_3, X_0]$, that is, an element of

$$[E^{n+1}X_2, X_0] \circ E^{n+1}\alpha_3 \backslash [E^{n+1}X_3, X_0] / \alpha_1 \circ E^n[EX_3, X_1].$$

If $[E^{n+1}X_3, X_0]$ is abelian, then

$$\operatorname{Indet}\{\alpha_1, \alpha_2, \alpha_3\}_n = [E^{n+1}X_2, X_0] \circ E^{n+1}\alpha_3 + \alpha_1 \circ E^n[EX_3, X_1].$$

As is easily seen, we have $\{\alpha_1, \alpha_2, \alpha_3\}_n \subset \{\alpha_1, E^{n-m}\alpha_2, E^{n-m}\alpha_3\}_m$ for $0 \leq m \leq n$, and $-E\{\alpha_1, \alpha_2, \alpha_3\}_n \subset \{E\alpha_1, \alpha_2, \alpha_3\}_{n+1}$. As in [**19**], we abbreviate $\{\alpha_1, \alpha_2, \alpha_3\}_0$ to $\{\alpha_1, \alpha_2, \alpha_3\}$.

Cohen [**3**] defines k-fold Toda brackets for every $k \geq 3$ (see B.3). If $(a_1, a_2, a_3)_0$ is a null triple, then his 3-fold Toda bracket $\{a_1, a_2, a_3\}^C$ contains the Toda bracket $\{a_1, a_2, a_3\}$ (see B.4) and they are generally not the same (see B.5).

Remark 3.1. *The original notation* [**19**] *for* $\{\alpha_1, \alpha_2, \alpha_3\}_n$ *is*

$$\{\alpha_1, E^n\alpha_2, E^n\alpha_3\}_n.$$

The original one may cause a misunderstanding that it depends on $E^n\alpha_i$ $(i=2,3)$. *For example,* $\{\iota_3,\eta_2\circ\nu',\nu_6\}_1=\varepsilon_3\neq 0=\{\iota_3,0_2^4,\nu_6\}_1$, *while* $E(\eta_2\circ\nu')=E0_2^4$.

Lemma 3.2. *Let* $EW\overset{q}{\leftarrow}X\cup_f CW\overset{i_f}{\leftarrow}X\overset{f}{\leftarrow}W$ *be a cofibre sequence. Then* $\{q,\,i_f,\,f\}\ni 1_{EW}$.

Proof. Let $A=*:CX\to EW$ and $B:CW\to X\cup_f CW$ the canonical map. Then

$$[q,A,i_f]\circ(i_f,B,f):EW\to EW,\quad w\wedge\overline{t}\mapsto\begin{cases}w\wedge\overline{0}=* & 0\le t\le\frac{1}{2}\\ w\wedge\overline{2t-1} & \frac{1}{2}\le t\le 1\end{cases}.$$

Hence $[q,A,i_f]\circ(i_f,B,f)\simeq 1_{EW}$. □

Lemma 3.3. *Suppose* $\{\alpha_1,\,\alpha_2,\,\alpha_3\}_n\ni 0$.

(1) *If* $\alpha_1\circ E^n[EX_3,X_1]\supset[E^{n+1}X_2,X_0]\circ E^{n+1}\alpha_3$, *then for any* $A_1:a_1\circ E^na_2\simeq *$ *there exists* $A_2:a_2\circ a_3\simeq *$ *such that* $[a_1,A_1,E^na_2]\circ(E^na_2,\widetilde{E}^nA_2,E^na_3)\simeq *$.

(2) *If* $\alpha_1\circ E^n[EX_3,X_1]\subset[E^{n+1}X_2,X_0]\circ E^{n+1}\alpha_3$, *then for any* $A_2:a_2\circ a_3\simeq *$ *there exists* $A_1:a_1\circ E^na_2\simeq *$ *such that* $[a_1,A_1,E^na_2]\circ(E^na_2,\widetilde{E}^nA_2,E^na_3)\simeq *$.

Proof. Since $\{\alpha_1,\,\alpha_2,\,\alpha_3\}_n\ni 0$, there exist $A_1':a_1\circ E^na_2\simeq *$ and $A_2':a_2\circ a_3\simeq *$ such that $[a_1,A_1',E^na_2]\circ(E^na_2,\widetilde{E}^nA_2',E^na_3)\simeq *$. Let $A_1:a_1\circ E^na_2\simeq *$ and $A_2:a_2\circ a_3\simeq *$. By Lemma 2.8 and Corollary 2.13, we have

$$\begin{aligned}[a_1,&A_1,E^na_2]\circ(E^na_2,\widetilde{E}^nA_2,E^na_3)\\ &\simeq[a_1,A_1'\dotplus d(A_1',A_1),E^na_2]\circ(E^na_2,\widetilde{E}^nA_2'\dotplus d(\widetilde{E}^nA_2',\widetilde{E}^nA_2),E^na_3)\\ &\simeq d(A_1',A_1)\circ(-E^{n+1}a_3)+a_1\circ(-1)^nE^nd(A_2',A_2).\end{aligned}$$

Then the assertions follow from Lemma 2.8(1)(g). □

For (3.1), we define

$$\begin{aligned}G_1'&=E^{-n}\circ(\alpha_{1*})^{-1}\circ\big(E^{n+1}\alpha_3\big)^*[E^{n+1}X_2,X_0]\;\subset\;[EX_3,X_1],\\ G_2'&=\big(E^{n+1}\alpha_3^*\big)^{-1}\circ\alpha_{1*}\circ E^n[EX_3,X_1]\;\subset\;[E^{n+1}X_2,X_0].\end{aligned}$$

Lemma 3.4. *Suppose that (3.1) has a representative* $(a_1,a_2,a_3;A_1,A_2)_n$ *such that* $[a_1,A_1,E^na_2]\circ(E^na_2,\widetilde{E}^nA_2,E^na_3)\simeq *$.

(1) *If* $A_2':a_2\circ a_3\simeq *$, *then there exists* $A_1':a_1\circ E^na_2\simeq *$ *such that* $[a_1,A_1',E^na_2]\circ(E^na_2,\widetilde{E}^nA_2',E^na_3)\simeq *$ *if and only if* $\delta(A_2,A_2')\in G_1'$.

(2) *If* $A_1':a_1\circ E^na_2\simeq *$, *then there exists* $A_2':a_2\circ a_3\simeq *$ *such that* $[a_1,A_1',E^na_2]\circ(E^na_2,\widetilde{E}^nA_2',E^na_3)\simeq *$ *if and only if* $\delta(A_1,A_1')\in G_2'$.

Proof. (1) Let $\lambda : E^{n+1}X_2 \to X_0$. By Lemma 2.8 and Corollary 2.13, we easily see

$$[a_1, A_1 \dotplus \lambda, E^n a_2] \circ (E^n a_2, \widetilde{E}^n A_2', E^n a_3) \\ \simeq \lambda \circ (-E^{n+1}a_3) + a_1 \circ (-1)^n E^n d(A_2, A_2').$$

Hence $[a_1, A_1 \dotplus \lambda, E^n a_2] \circ (E^n a_2, \widetilde{E}^n A_2', E^n a_3) \simeq *$ if and only if $\lambda \circ E^{n+1}a_3 \simeq a_1 \circ (-1)^n E^n d(A_2, A_2')$. Therefore following three statements are equivalent: (i) there exists A_1' with $[a_1, A_1', E^n a_2] \circ (E^n a_2, \widetilde{E}^n A_2', E^n a_3) \simeq *$; (ii) $\alpha_1 \circ (-1)^n E^n \delta(A_2, A_2') \in [E^{n+1}X_2, X_0] \circ E^{n+1}\alpha_3$; (iii) $\delta(A_2, A_2') \in G_1'$.

Similarly we can prove (2). We omit details. □

Lemma 3.5. *If maps* $Y_0 \overset{b_1}{\leftarrow} E^n Y_1$, $Y_1 \overset{b_2}{\leftarrow} Y_2 \overset{b_3}{\leftarrow} EY_3$ *and* $Y_1 \xleftarrow{b_2'} Y_2' \xleftarrow{b_3'} EY_3$ *satisfy* $b_1 \circ E^n b_2 \simeq *$, $b_1 \circ E^n b_2' \simeq *$, $b_2 \circ b_3 \simeq *$ *and* $b_2' \circ b_3' \simeq *$, *then*

$$\{b_1, b_2 \underline{\vee} b_2', (b_3 \vee b_3') \circ \theta_{EY_3}\}_n = \{b_1, b_2, b_3\}_n + \{b_1, b_2', b_3'\}_n.$$

Proof. We have $\{b_1, b_2 \underline{\vee} b_2', (b_3 \vee b_3') \circ \theta_{EY_3}\}_n \supset \{b_1, b_2 \underline{\vee} b_2', b_3 \vee b_3'\}_n \circ \theta_{E^{n+2}Y_3}$ by [**19**, Proposition 1.2(i)] and (2.1). Every null homotopy of $b_1 \circ E^n(b_2 \underline{\vee} b_2')$ has a form

$$A_1 \underline{\vee} A_1' : CE^n Y_2 \vee CE^n Y_2' = CE^n(Y_2 \vee Y_2') \to Y_0,$$

where $A_1 : b_1 \circ E^n b_2 \simeq *$ and $A_1' : b_1 \circ E^n b_2' \simeq *$, and every null homotopy of $(b_2 \underline{\vee} b_2') \circ (b_3 \vee b_3') = (b_2 \circ b_3) \underline{\vee} (b_2' \circ b_3')$ has a form

$$A_2 \underline{\vee} A_2' : CEY_3 \vee CEY_3 = C(EY_3 \vee EY_3) \to Y_1,$$

where $A_2 : b_2 \circ b_3 \simeq *$ and $A_2' : b_2' \circ b_3' \simeq *$. By routine calculations, we have

$$[b_1, A_1 \underline{\vee} A_1', E^n(b_2 \underline{\vee} b_2')] \circ \big(E^n(b_2 \underline{\vee} b_2'), \widetilde{E}^n(A_2 \underline{\vee} A_2'), E^n(b_3 \vee b_3')\big) \circ \theta_{E^{n+2}Y_3} \\ = [b_1, A_1, E^n b_2] \circ (E^n b_2, \widetilde{E}^n A_2, E^n b_3) \\ + [b_1, A_1', E^n b_2'] \circ (E^n b_2', \widetilde{E}^n A_2', E^n b_3').$$

Hence $\{b_1, b_2 \underline{\vee} b_2', b_3 \vee b_3'\}_n \circ \theta_{E^{n+2}Y_3} \subset \{b_1, b_2, b_3\}_n + \{b_1, b_2', b_3'\}_n$. We have

$$\begin{aligned} &\mathrm{Indet}\{b_1, b_2 \underline{\vee} b_2', (b_3 \vee b_3') \circ \theta_{EY_3}\}_n \\ &= [E^{n+1}(Y_2 \vee Y_2'), Y_0] \circ E^{n+1}\big((b_3 \vee b_3') \circ \theta_{EY_3}\big) + b_1 \circ E^n[E^2Y_3, Y_1] \\ &= [E^{n+1}(Y_2 \vee Y_2'), Y_0] \circ E^{n+1}(b_3 \vee b_3') \circ \theta_{E^{n+2}Y_3} + b_1 \circ E^n[E^2Y_3, Y_1] \\ &\qquad \text{(by (2.1))} \\ &= [E^{n+1}Y_2, Y_0] \circ E^{n+1}b_3 + [E^{n+1}Y_2', Y_0] \circ E^{n+1}b_3' + b_1 \circ E^n[E^2Y_3, Y_1] \\ &= \mathrm{Indet}\{b_1, b_2, b_3\}_n + \mathrm{Indet}\{b_1, b_2', b_3'\}_n. \end{aligned}$$

Hence the equality in the assertion is obtained. □

Lemma 3.6 (Proposition (5.11) of [**13**]). *If $Z \overset{a}{\leftarrow} Y \overset{b}{\leftarrow} X$ are maps and $H : a \circ b \simeq *$, then the following square is homotopy commutative:*

$$\begin{tikzcd}
Z \arrow[d, "i_a"'] & Y \cup_b CX \arrow[l, "{[a,H,b]}"'] \arrow[d, "q_b"] \\
Z \cup_a CY & EX \arrow[l, "{(a,H,b)}"]
\end{tikzcd}$$

Proof. We define $\xi : I \times I \to I$ and $\widetilde{G} : (Y \cup_b CX) \times I \to Z \cup_a CY$ as follows:

$$\xi(s,t) = \begin{cases} s & s \geq t \\ 2s - t & 2s \geq t \geq s \ , \\ -2s + t & 2s \leq t \end{cases} \tag{3.2}$$
$$\widetilde{G}(y,t) = y \wedge t, \quad \widetilde{G}(x \wedge s, t) = \begin{cases} b(x) \wedge \xi(s,t) & 2s \leq t \\ H(x \wedge \xi(s,t)) & 2s \geq t \end{cases},$$

where $y \in Y$ and $x \in X$. Then $\widetilde{G} : i_a \circ [a,H,b] \simeq (a,H,b) \circ q_b$. □

We call the above $\widetilde{G}$ the *typical homotopy* for $(a,b;H)$.

Remark 3.7. *Even if $H, H' : a \circ b \simeq *$, the following square is not necessarily homotopy commutative.*

$$\begin{tikzcd}
Z \arrow[d, "i_a"'] & Y \cup_b CX \arrow[l, "{[a,H,b]}"'] \arrow[d, "q_b"] \\
Z \cup_a CY & EX \arrow[l, "{(a,H',b)}"]
\end{tikzcd}$$

For example, if $Z = Y = \mathrm{S}^6, X = \mathrm{S}^7, a = 2\iota_6, b = *,\ H = \eta_6^2 \circ \pi : CX \to Z$, where $\pi : CX \to EX$ is the quotient map, and $H' = * : CX \to Z$, then $[a,H,b] = 2\iota_6 \underline{\vee} \eta_6^2$ and $(a,H',b) = *$ so that $i_a \circ [a,H,b] \simeq * \underline{\vee} i_{2\iota_6 *} \eta_6^2 \not\simeq *$ and $(a,H',b) \circ q_b = *$.

Proposition 3.8. *If $(a_1, a_2, a_3; A_1, A_2)_n$ is a representative of (3.1), then*

$$\begin{aligned} &a_1 \circ E^n[a_2, A_2, a_3] \\ &\quad \simeq [a_1, A_1, E^n a_2] \circ (E^n a_2, \widetilde{E}^n A_2, E^n a_3) \circ (1_{X_3} \wedge \tau(\mathrm{S}^1, \mathrm{S}^n)) \circ E^n q_{a_3}. \end{aligned}$$

Proof. We have

$$\begin{aligned} &a_1 \circ E^n[a_2, A_2, a_3] = [a_1, A_1, E^n a_2] \circ i_{E^n a_2} \circ E^n[a_2, A_2, a_3] \\ &= [a_1, A_1, E^n a_2] \circ \left(\psi^n_{a_2}\right)^{-1} \circ E^n i_{a_2} \circ E^n[a_2, A_2, a_3] \end{aligned}$$

$$\simeq [a_1, A_1, E^n a_2] \circ \big(\psi^n_{a_2}\big)^{-1} \circ E^n(a_2, A_2, a_3) \circ E^n q_{a_3} \quad \text{(by 3.6)}$$
$$= [a_1, A_1, E^n a_2] \circ (E^n a_2, \widetilde{E}^n A_2, E^n a_3) \circ (1_{X_3} \wedge \tau(\mathrm{S}^1, \mathrm{S}^n)) \circ E^n q_{a_3} \text{ (by 2.4).}$$

□

4. Quasi tertiary compositions

A *null quadruple* [**13**] is a set of two non-negative integers n_1, n_2 and four homotopy classes $\alpha_k \in [E^{n_k} X_k, X_{k-1}]$ $(k = 1, 2)$, $\alpha_\ell \in [X_\ell, X_{\ell-1}]$ $(\ell = 3, 4)$ such that $\alpha_k \circ E^{n_k}\alpha_{k+1} = 0$ $(k = 1, 2)$ and $\alpha_3 \circ \alpha_4 = 0$. This is abbreviated to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_{n_1,n_2}$ and expressed as

$$\tag{4.1} \begin{gathered} X_0 \xleftarrow{\alpha_1} E^{n_1} X_1,\ X_1 \xleftarrow{\alpha_2} E^{n_2} X_2,\ X_2 \xleftarrow{\alpha_3} X_3 \xleftarrow{\alpha_4} X_4, \\ \alpha_1 \circ E^{n_1}\alpha_2 = 0, \quad \alpha_2 \circ E^{n_2}\alpha_3 = 0, \quad \alpha_3 \circ \alpha_4 = 0. \end{gathered}$$

A *representative* of (4.1) is a 9-tuple $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ or shortly 5-tuple $(A_1, A_2, A_3)_{n_1,n_2}$ such that

$$a_k \in \alpha_k\ (k = 1, 2, 3, 4),\ A_k : a_k \circ E^{n_k} a_{k+1} \simeq *\ (k = 1, 2),\ A_3 : a_3 \circ a_4 \simeq *,$$

and it is called *admissible* if

$$[a_1, A_1, E^{n_1} a_2] \circ (E^{n_1} a_2, \widetilde{E}^{n_1} A_2, E^{n_1} E^{n_2} a_3) \simeq *,$$
$$[a_2, A_2, E^{n_2} a_3] \circ (E^{n_2} a_3, \widetilde{E}^{n_2} A_3, E^{n_2} a_4) \simeq *.$$

A null quadruple is called admissible if it has an admissible representative.

If (4.1) has an admissible representative $(A_1, A_2, A_3)_{n_1,n_2}$ and $0 \le m_i \le n_i$ $(i = 1, 2)$, then $(A_1, \widetilde{E}^{n_1-m_1} A_2, \widetilde{E}^{n_1+n_2-m_1-m_2} A_3)_{m_1,m_2}$ is an admissible representative of the null quadruple

$$(\alpha_1, E^{n_1-m_1}\alpha_2, E^{n_1+n_2-m_1-m_2}\alpha_3, E^{n_1+n_2-m_1-m_2}\alpha_4)_{m_1,m_2}$$

by Lemma 2.4.

When $n_2 = 0$ or $n_1 = n_2 = 0$, we usually omit the subscript n_2 or n_1, n_2 from the above notations respectively. For example, we abbreviate $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{0,0}$ to $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)$ or (A_1, A_2, A_3).

It is obvious that if (4.1) is admissible then $\{\alpha_1, \alpha_2, E^{n_2}\alpha_3\}_{n_1} \ni 0$ and $\{\alpha_2, \alpha_3, \alpha_4\}_{n_2} \ni 0$. A sufficient condition that (4.1) is admissible was essentially given by Ôguchi [**13**, Proposition (6.3)] as follows.

Proposition 4.1. *If*

$$\{\alpha_1, \alpha_2, E^{n_2}\alpha_3\}_{n_1} \ni 0, \quad \{\alpha_2, \alpha_3, \alpha_4\}_{n_2} \ni 0, \quad G_1 + G_2 = [E^{n_2+1} X_3, X_1],$$

then (4.1) is admissible, where G_1 and G_2 are defined by

$$G_1 = (E^{n_1})^{-1}\Big((\alpha_{1*})^{-1}\big((E^{n_1+n_2+1}\alpha_3)^*[E^{n_1+n_2+1} X_2, X_0]\big)\Big),$$
$$G_2 = (E^{n_2+1}{\alpha_4}^*)^{-1}\big(\alpha_{2*}(E^{n_2}[EX_4, X_2])\big).$$

Proof. Let $a_i \in \alpha_i$ $(1 \leq i \leq 4)$. By assumptions and Proposition 2.11, there exist null homotopies $A_1 : a_1 \circ E^{n_1} a_2 \simeq *$, $A_2, A_2' : a_2 \circ E^{n_2} a_3 \simeq *$, $A_3' : a_3 \circ a_4 \simeq *$ such that $[a_1, A_1, E^{n_1} a_2] \circ (E^{n_1} a_2, \widetilde{E}^{n_1} A_2, E^{n_1} E^{n_2} a_3) \simeq *$ and $[a_2, A_2', E^{n_2} a_3] \circ (E^{n_2} a_3, \widetilde{E}^{n_2} A_3', E^{n_2} a_4) \simeq *$. By the assumption on $G_1 + G_2$, we can write $\delta(A_2, A_2') = \gamma_1 + \gamma_2$ with $\gamma_i \in G_i$ $(i = 1, 2)$. Let $c_1 \in \gamma_1$. Then $\delta(A_2', A_2 \dotplus c_1) = \delta(A_2', A_2) + \delta(A_2, A_2 \dotplus c_1) = -\gamma_2 \in G_2$ by Lemma 2.8. Since G_2 is G_2' for $(\alpha_2, \alpha_3, \alpha_4)_{n_2}$, it follows from Lemma 3.4(2) that there exists $A_3 : a_3 \circ a_4 \simeq *$ such that $[a_2, A_2 \dotplus c_1, E^{n_2} a_3] \circ (E^{n_2} a_3, \widetilde{E}^{n_2} A_3, E^{n_2} a_4) \simeq *$. By the definition of G_1, there exists $\beta \in [E^{n_1+n_2+1} X_2, X_0]$ such that $\beta \circ E^{n_1+n_2+1} \alpha_3 = \alpha_1 \circ E^{n_1} (-1)^{n_1} \gamma_1$. Let $b \in \beta$. Then, by Corollary 2.13, we have

$$\begin{aligned} &[a_1, A_1 \dotplus b, E^{n_1} a_2] \circ (E^{n_1} a_2, \widetilde{E}^{n_1} (A_2 \dotplus c_1), E^{n_1} E^{n_2} a_3) \\ &\simeq b \circ (-E^{n_1+n_2+1} a_3) + [a_1, A_1, E^{n_1} a_2] \circ (E^{n_1} a_2, \widetilde{E}^{n_1} A_2, E^{n_1+n_2} a_3) \\ &\qquad\qquad + a_1 \circ E^{n_1} (-1)^{n_1} c_1 \simeq *. \end{aligned}$$

Hence $(a_1, a_2, a_3, a_4; A_1 \dotplus b, A_2 \dotplus c_1, A_3)_{n_1, n_2}$ is an admissible representative of (4.1). □

Remark 4.2. *There is an admissible null quadruple such that* $G_1 + G_2 \subsetneqq [E^{n_2+1} X_3, X_1]$. *For example, the following null quadruple is admissible and* $G_1 + G_2 = \{0\} \subset \pi_{n_2+4}(\mathrm{S}^{n_2+3}) = \mathbb{Z}_2\{\eta_{n_2+3}\}$.

$$\mathrm{S}^{n_1+n_2+2} \xleftarrow{\eta_{n_1+n_2+2}} E^{n_1}\, \mathrm{S}^{n_2+3},$$

$$\mathrm{S}^{n_2+3} \xleftarrow{0^0_{n_2+3}} E^{n_2}\, \mathrm{S}^3, \quad \mathrm{S}^3 \xleftarrow{0^0_3} \mathrm{S}^3 \xleftarrow{\eta_3} \mathrm{S}^4 .$$

In fact, $(*^2_{n_1+n_2+2} \circ p_{n_1+n_2+3}, *^1_{n_2+3} \circ p_{n_2+3}, *^2_3 \circ p_4)_{n_1, n_2}$ *is admissible, where* $p_m : C\,\mathrm{S}^m \to E\,\mathrm{S}^m = \mathrm{S}^{m+1}$ *is the quotient map, and* $G_1 = G_2 = \{0\}$, $\{\eta_{n_1+n_2+2}, 0^0_{n_2+3}, E^{n_2} 0^0_3\}_{n_1} = \mathbb{Z}_2\{\eta^2_{n_1+n_2+2}\}$, $\{0^0_{n_2+3}, 0^0_3, \eta_3\}_{n_2} = \mathbb{Z}_2\{\eta^2_{n_2+3}\}$.

Lemma 4.3. $[E^{n_2+1} X_3, X_1]$ *is* G_1 *or* G_2 *according as* $\{\alpha_1, \alpha_2, E^{n_2} \alpha_3\}_{n_1} = \{0\}$ *or* $\{\alpha_2, \alpha_3, \alpha_4\}_{n_2} = \{0\}$.

Proof. This is obvious from definitions. □

Mimura [**10**] considered the following conditions on (4.1).

(i) $\{\alpha_1, \alpha_2, E^{n_2} \alpha_3\}_{n_1} = \{0\}$ and $\{\alpha_2, \alpha_3, \alpha_4\}_{n_2} \ni 0$.
(ii) $\{\alpha_1, \alpha_2, E^{n_2} \alpha_3\}_{n_1} \ni 0$ and $\{\alpha_2, \alpha_3, \alpha_4\}_{n_2} = \{0\}$.

Proposition 4.4. *If* (i) *or* (ii) *holds, then the hypotheses of Proposition 4.1 are satisfied so that (4.1) is admissible.*

Proof. This follows from definitions (or Proposition 4.1 and Lemma 4.3). □

Example 4.5. *A null quadruple* $(2\iota_3, \eta_3^2, 2\iota_5, \eta_5)$ *is admissible,* $G_1 + G_2 = [E^{n_2+1}X_3, X_1](= \pi_6(\mathrm{S}^3) \cong \mathbb{Z}_{12})$ *and satisfies neither* (i) *nor* (ii)*.*

Proof. We have $\{2\iota_3, \eta_3^2, 2\iota_5\} = 2\pi_6(\mathrm{S}^3) \cong \mathbb{Z}_6$ and $\{\eta_3^2, 2\iota_5, \eta_5\} = \pi_7(\mathrm{S}^3) \cong \mathbb{Z}_2$ by [**19**], and $G_1 = [E^{n_2+1}X_3, X_1]$. ☐

Proposition 4.6. *Let* $a_k \in \alpha_k$ $(1 \leq k \leq 4)$*.*

(1) *If* (i) *holds, then there exist* $A_2 : a_2 \circ E^{n_2}a_3 \simeq *$ *and* $A_3 : a_3 \circ a_4 \simeq *$ *such that* $(A_1, A_2, A_3)_{n_1,n_2}$ *is admissible for every* $A_1 : a_1 \circ E^{n_1}a_2 \simeq *$*.*

(2) *If* (ii) *holds, then there exist* $A_1 : a_1 \circ E^{n_1}a_2 \simeq *$ *and* $A_2 : a_2 \circ E^{n_2}a_3 \simeq *$ *such that* $(A_1, A_2, A_3)_{n_1,n_2}$ *is admissible for every* $A_3 : a_3 \circ a_4 \simeq *$*.*

Proof. These are obvious from definitions. ☐

Corollary 4.7. *Let* $a_k \in \alpha_k$ $(1 \leq k \leq 4)$*.*

(1) *If* (i) *holds and* $\alpha_2 \circ E^{n_2}[EX_4, X_2] \supset [E^{n_2+1}X_3, X_1] \circ E^{n_2+1}\alpha_4$*, then for any* $A_1 : a_1 \circ E^{n_1}a_2 \simeq *$ *and* $A_2 : a_2 \circ E^{n_2}a_3 \simeq *$ *there exists* $A_3 : a_3 \circ a_4 \simeq *$ *such that* $(A_1, A_2, A_3)_{n_1,n_2}$ *is an admissible representative of (4.1).*

(2) *If* (ii) *holds and* $\alpha_1 \circ E^{n_1}[EX_3, X_1] \subset [E^{n_1+1}X_2, X_0] \circ E^{n_1+1}\alpha_3$*, then for any* $A_2 : a_2 \circ E^{n_2}a_3 \simeq *$ *and* $A_3 : a_3 \circ a_4 \simeq *$ *there exists* $A_1 : a_1 \circ E^{n_1}a_2 \simeq *$ *such that* $(A_1, A_2, A_3)_{n_1,n_2}$ *is an admissible representative of (4.1).*

Proof. These follow immediately from Lemma 3.3 and Proposition 4.6. ☐

Proposition 4.8. *If* $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ *is an admissible representative of (4.1), then the following diagrams are homotopy commutative.*

$$\begin{tikzcd}
X_0 \arrow[d, equal] & E^{n_1}X_1 \arrow[l, "a_1"'] \arrow[d, "E^{n_1}i_{a_2}"] \\
X_0 & E^{n_1}(X_1 \cup_{a_2} CE^{n_2}X_2) \arrow[l, "\overline{a_1}'"']
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \arrow[d, "i_{a_2}"] & E^{n_2}X_2 \cup_{E^{n_2}a_3} CE^{n_2}X_3 \arrow[l, "\overline{a_2}"] \arrow[d, "q_{E^{n_2}a_3}"] & EE^{n_2}X_4 \arrow[l, "\widetilde{E^{n_2}a_4}"] \arrow[d, equal] \\
X_1 \cup_{a_2} CE^{n_2}X_2 & EE^{n_2}X_3 \arrow[l, "\widetilde{E^{n_2}a_3}"'] & EE^{n_2}X_4 \arrow[l, "-EE^{n_2}a_4"']
\end{tikzcd}$$

In the above diagrams we have used the following abbreviations:

$$(4.2)\quad \begin{aligned} &\overline{a_1}' = [a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1}, \quad \overline{a_2} = [a_2, A_2, E^{n_2}a_3], \\ &\widetilde{E^{n_2}a_3} = (a_2, A_2, E^{n_2}a_3), \quad \widetilde{E^{n_2}a_4} = (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4). \end{aligned}$$

Proof. By definitions, the first square is commutative and the third square is homotopy commutative. Let $\widetilde{G}$ be the typical homotopy for $(a_2, E^{n_2}a_3; A_2)$. Then $\widetilde{G} : (E^{n_2}X_2 \cup_{E^{n_2}a_3} CE^{n_2}X_3) \times I \to X_1 \cup_{a_2} CE^{n_2}X_2$ is a homotopy from $i_{a_2} \circ \overline{a_2}$ to $\widetilde{E^{n_2}a_3} \circ q_{E^{n_2}a_3}$. ☐

Theorem 4.9. *If* $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ *is an admissible representative of (4.1), then*

$$\begin{aligned}
&\{a_1,\ [a_2, A_2, E^{n_2}a_3], (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\}_{n_1}\\
&\subset \{[a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1}, i_{a_2} \circ [a_2, A_2, E^{n_2}a_3], (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\}_{n_1}\\
&= \{[a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1}, (a_2, A_2, E^{n_2}a_3) \circ q_{E^{n_2}a_3},\\
&\qquad\qquad\qquad\qquad\qquad\qquad (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\}_{n_1}\\
&\supset \{[a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1}, (a_2, A_2, E^{n_2}a_3), -E^{n_2+1}a_4\}_{n_1},
\end{aligned}$$

where the first bracket and the last bracket have a common element.

The relations $\subset$, $=$, $\supset$ in the above theorem follow from [**19**, Proposition 1.2] and the homotopy commutative diagrams of Proposition 4.8. To prove the underlined part which is the main part of the theorem, we need preparations. Indeed the proof will be completed before Definition 4.12.

While we can take another way, we shall go on Ôguchi's way.

For a homotopy commutative square and a homotopy

$$\begin{tikzcd}
X_0 \arrow[d, "h_0"'] & X_1 \arrow[l, "f"'] \arrow[d, "h_1"] \\
Y_0 & Y_1 \arrow[l, "g"]
\end{tikzcd} \qquad h_0 \circ f \underset{H}{\simeq} g \circ h_1$$

we define $h_0 \cup_H h_1 : X_0 \cup_f CX_1 \to Y_0 \cup_g CY_1$ to be the composite of the following maps:

$$X_0 \cup_f CX_1 \xrightarrow{h_0 \cup 1} Y_0 \cup_{h_0 \circ f} CX_1 \xrightarrow[\simeq]{\varphi_H} Y_0 \cup_{g \circ h_1} CX_1 \xrightarrow{1 \cup Ch_1} Y_0 \cup_g CY_1.$$

A *null couple* (β_1, β_2) consists of $\beta_1 \in [Y_1, Y_0]$ and $\beta_2 \in [Y_2, Y_1]$ such that $\beta_1 \circ \beta_2 = 0$. A *representative* of (β_1, β_2) is a triple $(b_1, b_2; B)$, where $b_k \in \beta_k$ $(k = 1, 2)$ and $B : b_1 \circ b_2 \simeq *$. A *quasi-map* $(h_0, h_1, h_2; D_1, D_2) : (b_1, b_2; B) \to (b'_1, b'_2; B')$ between representatives of null couples consists of a homotopy commutative diagram and four homotopies:

$$\begin{tikzcd}
Y_0 \arrow[d, "h_0"'] & Y_1 \arrow[l, "b_1"'] \arrow[d, "h_1"'] & Y_2 \arrow[l, "b_2"'] \arrow[d, "h_2"] \\
Y'_0 & Y'_1 \arrow[l, "b'_1"] & Y'_2 \arrow[l, "b'_2"]
\end{tikzcd}$$

$$D_1 : h_0 \circ b_1 \simeq b'_1 \circ h_1, \quad D_2 : h_1 \circ b_2 \simeq b'_2 \circ h_2,$$

$$B : b_1 \circ b_2 \simeq *, \quad B' : b'_1 \circ b'_2 \simeq *.$$

For a quasi-map $(h_0,h_1,h_2;D_1,D_2):(b_1,b_2;B)\to(b_1',b_2';B')$, we define two null homotopies $\underline{B'\circ Ch_2}_{(D_1,D_2)}$, $\overline{h_0\circ B}^{(D_1,D_2)}:CY_2\to Y_0'$ by

$$\underline{B'\circ Ch_2}_{(D_1,D_2)}(y_2\wedge t)=\begin{cases} D_1(b_2(y_2),3t) & 0\le t\le \frac{1}{3}\\ b_1'\big(D_2(y_2,3t-1)\big) & \frac{1}{3}\le t\le\frac{2}{3}\\ B'(h_2(y_2),3t-2) & \frac{2}{3}\le t\le 1\end{cases},$$

$$\overline{h_0\circ B}^{(D_1,D_2)}(y_2\wedge t)=\begin{cases} b_1'\big((-D_2)(y_2,3t)\big) & 0\le t\le \frac{1}{3}\\ (-D_1)(b_2(y_2),3t-1) & \frac{1}{3}\le t\le\frac{2}{3}\\ h_0(B(y_2,3t-2)) & \frac{2}{3}\le t\le 1\end{cases}. \tag{4.3}$$

Then $\underline{B'\circ Ch_2}_{(D_1,D_2)}:h_0\circ b_1\circ b_2\simeq *$ and $\overline{h_0\circ B}^{(D_1,D_2)}:b_1'\circ b_2'\circ h_2\simeq *$.

Lemma 4.10. *Under the above conditions, we have the following properties.*

1. $d(h_0\circ B,\underline{B'\circ Ch_2}_{(D_1,D_2)})\simeq d(\overline{h_0\circ B}^{(D_1,D_2)},B'\circ Ch_2):EY_2\to Y_0'$.
2. $h_0\circ B\simeq\underline{B'\circ Ch_2}_{(D_1,D_2)}$ *rel* Y_2 *if and only if* $\overline{h_0\circ B}^{(D_1,D_2)}\simeq B'\circ Ch_2$ *rel* Y_2.
3. *If* $h_0\circ B\simeq\underline{B'\circ Ch_2}_{(D_1,D_2)}$ *rel* Y_2, *then the following two squares are homotopy commutative.*

$$\begin{tikzcd}
Y_0 \arrow[d, "h_0"'] & Y_1\cup_{b_2}CY_2 \arrow[l, "{[b_1,B,b_2]}"'] \arrow[d, "h_1\cup_{D_2}h_2"'] & Y_0\cup_{b_1}CY_1 \arrow[d, "h_0\cup_{D_1}h_1"] & EY_2 \arrow[l, "{(b_1,B,b_2)}"'] \arrow[d, "Eh_2"] \\
Y_0' & Y_1'\cup_{b_2'}CY_2' \arrow[l, "{[b_1',B',b_2']}"] & Y_0'\cup_{b_1'}CY_1' & EY_2' \arrow[l, "{(b_1',B',b_2')}"]
\end{tikzcd}$$

Proof. We have (1) by giving a homotopy so that (2) follows from (1)(f) of Lemma 2.8.

(3) We prove $h_0\circ[b_1,B,b_2]\simeq[b_1',B',b_2']\circ(h_1\cup_{D_2}h_2)$ as follows. By assumptions, we have

$$h_0\circ[b_1,B,b_2]=[h_0\circ b_1,h_0\circ B,b_2]\simeq[h_0\circ b_1,\underline{B'\circ Ch_2}_{(D_1,D_2)},b_2].$$

Hence it suffices to show $[h_0\circ b_1,\underline{B'\circ Ch_2}_{(D_1,D_2)},b_2]\simeq[b_1',B',b_2']\circ(h_1\cup_{D_2}h_2)$. Decompose $I\times I=K_1\cup\cdots\cup K_5$ as follows: let $(s,t)\in I\times I$ and

$$K_1=\{(s,t)\,|\,t\le -3s+1\},\quad K_2=\{(s,t)\,|\,t\ge -3s+1 \text{ and } t\ge 6s-2\},$$
$$K_3=\{(s,t)\,|\,t\le 6s-2 \text{ and } t\le -6s+4\},$$
$$K_4=\{(s,t)\,|\,t\ge -6s+4 \text{ and } t\ge 3s-2\},\quad K_5=\{(s,t)\,|\,t\le 3s-2\}.$$

Define $u : I \times I \to I$, $\Psi' : Y_1 \times I \to Y_0'$ and $\Psi'' : CY_2 \times I \to Y_0'$ by

$$u(s,t) = \begin{cases} 3s+t & (s,t) \in K_1 \\ -2s - \frac{2}{3}t + \frac{5}{3} & (s,t) \in K_2 \\ -3s - \frac{1}{2}t + 2 & (s,t) \in K_3 \\ 2s + \frac{1}{3}t - \frac{4}{3} & (s,t) \in K_4 \\ 3s - 2 & (s,t) \in K_5 \end{cases}, \qquad \Psi'(y_1, t) = D_1(y_1, t),$$

$$\Psi''(y_2 \wedge s, t) = \begin{cases} D_1(b_2(y_2), u(s,t)) & (s,t) \in K_1 \\ b_1' \circ (-D_2)(y_2, u(s,t)) & (s,t) \in K_2 \cup K_3 \\ B'(h_2(y_2), u(s,t)) & (s,t) \in K_4 \cup K_5 \end{cases}.$$

Then Ψ', Ψ'' define $\Psi : [h_0 \circ b_1, \underline{B' \circ Ch_2}_{(D_1,D_2)}, b_2] \simeq [b_1', B', b_2'] \circ (h_1 \cup_{D_2} h_2)$.

Next we prove $(h_0 \cup_{D_1} h_1) \circ (b_1, B, b_2) \simeq (b_1', B', b_2') \circ Eh_2$. We define two null homotopies

$$(h_0 \circ B)' = (h_0 \circ B) \bullet \big((-D_1) \bar{\circ} (-1_{b_2})\big) : b_1' \circ h_1 \circ b_2 \simeq *,$$
$$(h_0 \circ B)'' = (h_0 \circ B)' \bullet \big((-1_{b_1'}) \bar{\circ} (-D_2)\big) : b_1' \circ b_2' \circ h_2 \simeq *,$$

that is, they are maps from CY_2 to Y_0' and

$$(h_0 \circ B)'(y_2 \wedge t) = \begin{cases} D_1(b_2(y_2), 1-2t) & 0 \le t \le \frac{1}{2} \\ h_0 \circ B(y_2, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases},$$

$$(h_0 \circ B)''(y_2 \wedge t) = \begin{cases} b_1' \circ D_2(y_2, 1-2t) & 0 \le t \le \frac{1}{2} \\ D_1(b_2(y_2), 3-4t) & \frac{1}{2} \le t \le \frac{3}{4} \\ h_0 \circ B(y_2, 4t-3) & \frac{3}{4} \le t \le 1 \end{cases}. \tag{4.4}$$

Consider the following diagram.

$$\begin{tikzcd}
EY_2 \arrow[r, "="] \arrow[d, "{(b_1,B,b_2)}"'] & EY_2 \arrow[r, "="] \arrow[d, "{(h_0\circ b_1,h_0\circ B,b_2)}"'] & EY_2 \arrow[r, "="] \arrow[d, "{(b_1' h_1,(h_0\circ B)',b_2)}"'] & EY_2 \arrow[d, "{(b_1',(h_0\circ B)',h_1\circ b_2)}"'] \\
Y_0 \cup_{b_1} CY_1 \arrow[r, "h_0\cup 1"'] & Y_0' \cup_{h_0\circ b_1} CY_1 \arrow[r, "\varphi_{D_1}"'] & Y_0' \cup_{b_1'\circ h_1} CY_1 \arrow[r, "1\cup Ch_1"'] & Y_0' \cup_{b_1'} CY_1'
\end{tikzcd}$$

The second square is homotopy commutative by Lemma 2.10 and other two squares are commutative by Lemma 2.2. Hence $(h_0 \cup_{D_1} h_1) \circ (b_1, B, b_2) \simeq (b_1', (h_0 \circ B)', h_1 \circ b_2)$, where the latter is homotopic to $(b_1', (h_0 \circ B)'', b_2' \circ h_2)$ by (2.8) and Lemma 2.10. On the other hand, by assumptions, we have

$$(b_1', B', b_2') \circ Eh_2 = (b_1', B' \circ Ch_2, b_2' \circ h_2) \simeq \big(b_1', \overline{h_0 \circ B}^{(D_1,D_2)}, b_2' \circ h_2\big).$$

Thus, by (2.6), it suffices to prove $(h_0\circ B)''\simeq \overline{h_0\circ B}^{(D_1,D_2)}$ *rel* Y_2. We do it as follows. We divide $I\times I=K_1\cup\cdots\cup K_6$: let $(s,t)\in I\times I$ and

$$K_1=\{(s,t)\,|\,t\geq 3s\},\quad K_2=\{(s,t)\,|\,t\leq 3s \text{ and } t\leq -6s+3\},$$
$$K_3=\{(s,t)\,|\,t\geq -6s+3 \text{ and } t\geq 6s-3\},$$
$$K_4=\{(s,t)\,|\,t\leq 6s-3 \text{ and } t\leq -12s+9\},$$
$$K_5=\{(s,t)\,|\,t\geq -12s+9 \text{ and } t\geq 4s-3\},\quad K_6=\{(s,t)\,|\,t\leq 4s-3\}.$$

We define $u:K_1\cup K_2\to I$, $v:K_3\cup\cdots\cup K_6\to I$ and $\Phi:CY_2\times I\to Y'_0$ by

$$u(s,t)=\begin{cases}-3s+1 & (s,t)\in K_1\\ -2s-\frac{t}{3}+1 & (s,t)\in K_2\end{cases},$$

$$v(s,t)=\begin{cases}-3s-\frac{t}{2}+\frac{5}{2} & (s,t)\in K_3\\ -4s-\frac{t}{3}+3 & (s,t)\in K_4\\ 3s+\frac{t}{4}-\frac{9}{4} & (s,t)\in K_5\\ 4s-3 & (s,t)\in K_6\end{cases},$$

$$\Phi(y_2\wedge s,t)=\begin{cases}b'_1\circ D_2(y_2,u(s,t)) & (s,t)\in K_1\cup K_2\\ D_1(b_2(y_2),v(s,t)) & (s,t)\in K_3\cup K_4\\ h_0\circ B(y_2,v(s,t)) & (s,t)\in K_5\cup K_6\end{cases}.$$

Then $\Phi:(h_0\circ B)''\simeq \overline{h_0\circ B}^{(D_1,D_2)}$ *rel* Y_2 by (4.3) and (4.4). □

A *quasi-map*

$$(h_0,h_1,h_2,h_3;D_1,D_2,D_3):(b_1,b_2,b_3;B_1,B_2)_n\to(b'_1,b'_2,b'_3;B'_1,B'_2)_n$$

between representatives of null triples is defined similarly:

$$\begin{tikzcd}
Y_0 \arrow[d, "h_0"'] & E^nY_1 \arrow[l, "b_1"'] \arrow[d, "E^nh_1"'] & & Y_1 \arrow[d, "h_1"] & Y_2 \arrow[l, "b_2"'] \arrow[d, "h_2"'] & Y_3 \arrow[l, "b_3"'] \arrow[d, "h_3"] \\
Y'_0 & E^nY'_1 \arrow[l, "b'_1"] & & Y'_1 & Y'_2 \arrow[l, "b'_2"] & Y'_3 \arrow[l, "b'_3"]
\end{tikzcd}$$

$$D_1:h_0\circ b_1\simeq b'_1\circ E^nh_1,\quad D_2:h_1\circ b_2\simeq b'_2\circ h_2,\quad D_3:h_2\circ b_3\simeq b'_3\circ h_3,$$
$$B_1:b_1\circ E^nb_2\simeq *,\quad B_2:b_2\circ b_3\simeq *,$$
$$B'_1:b'_1\circ E^nb'_2\simeq *,\quad B'_2:b'_2\circ b'_3\simeq *.$$

The quasi-map is called a *map* if both of the following relations hold

$$h_0\circ B_1\simeq \underline{B'_1\circ CE^nh_2}_{(D_1,\widetilde{E}^nD_2)}\ rel\ E^nY_2:CE^nY_2\to Y'_0,$$
$$h_1\circ B_2\simeq \underline{B'_2\circ Ch_3}_{(D_2,D_3)}\ rel\ Y_3:CY_3\to Y'_1. \tag{4.5}$$

Proposition 4.11 (Lemma (5.5) of [**13**]). *Under the above notations, if*

$$(h_0,h_1,h_2,h_3;D_1,D_2,D_3):(b_1,b_2,b_3;B_1,B_2)_n\to(b_1',b_2',b_3';B_1',B_2')_n$$

is a map between representatives of null triples, then the following diagram is homotopy commutative

$$\begin{tikzcd}
Y_0 \arrow[d, "h_0"'] & E^nY_1\cup_{E^nb_2}CE^nY_2 \arrow[l, "{[b_1,B_1,E^nb_2]}"'] \arrow[d, "E^nh_1\cup_{\widetilde{E}^nD_2}E^nh_2"] & EE^nY_3 \arrow[l, "{(E^nb_2,\widetilde{E}^nB_2,E^nb_3)}"'] \arrow[d, "EE^nh_3"] \\
Y_0' & E^nY_1'\cup_{E^nb_2'}CE^nY_2' \arrow[l, "{[b_1',B_1',E^nb_2']}"] & EE^nY_3' \arrow[l, "{(E^nb_2',\widetilde{E}^nB_2',E^nb_3')}"]
\end{tikzcd}$$

and hence

$$\begin{aligned}
h_0\circ[b_1,B_1,E^nb_2]&\circ(E^nb_2,\widetilde{E}^nB_2,E^nb_3)\\
&\simeq[b_1',B_1',E^nb_2']\circ(E^nb_2',\widetilde{E}^nB_2',E^nb_3')\circ EE^nh_3.
\end{aligned}$$

Proof. By (4.5), we can easily show

$$E^nh_1\circ\widetilde{E}^nB_2\simeq\underline{\widetilde{E}^nB_2'\circ CE^nh_3}_{(\widetilde{E}^nD_2,\widetilde{E}^nD_3)}\ rel\ E^nY_3.$$

Then we have the assertion from Lemma 4.10. □

Proof of Theorem 4.9. We use the abbreviations (4.2) of Proposition 4.8. We shall prove the underlined part of Theorem 4.9 which says that

$\{a_1,\overline{a_2},\widetilde{E^{n_2}a_4}\}_{n_1}$ and $\{\overline{a_1}',\widetilde{E^{n_2}a_3},-E^{n_2+1}a_4\}_{n_1}$ have a common element.

We take following five homotopies arbitrarily

$$(4.6)\quad \begin{cases}B_1:\overline{a_1}'\circ E^{n_1}\widetilde{E^{n_2}a_3}\simeq *, \quad B_2:\overline{a_2}\circ\widetilde{E^{n_2}a_4}\simeq *,\\ D_1:a_1\simeq\overline{a_1}'\circ E^{n_1}i_{a_2}=a_1,\quad D_2:i_{a_2}\circ\overline{a_2}\simeq\widetilde{E^{n_2}a_3}\circ q_{E^{n_2}a_3},\\ D_3:q_{E^{n_2}a_3}\circ\widetilde{E^{n_2}a_4}\simeq -EE^{n_2}a_4.\end{cases}$$

Consider the following two diagrams from Proposition 4.8.

$$\begin{tikzcd}
X_0 \arrow[d, "="'] \arrow[dr, phantom, "D_1"] & E^{n_1}X_1 \arrow[l, "a_1"'] \arrow[d, "E^{n_1}i_{a_2}"] \arrow[dr, phantom, "\widetilde{E}^{n_1}D_2"] & E^{n_1}(E^{n_2}X_2\cup_{E^{n_2}a_3}CE^{n_2}X_3) \arrow[l, "E^{n_1}\overline{a_2}"'] \arrow[d, "E^{n_1}q_{E^{n_2}a_3}"] \\
X_0 & E^{n_1}(X_1\cup_{a_2}CE^{n_2}X_2) \arrow[l, "\overline{a_1}'"'] & E^{n_1}EE^{n_2}X_3 \arrow[l, "E^{n_1}\widetilde{E^{n_2}a_3}"']
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \arrow[d, "i_{a_2}"] \arrow[dr, phantom, "D_2"] & E^{n_2}X_2\cup_{E^{n_2}a_3}CE^{n_2}X_3 \arrow[l, "\overline{a_2}"'] \arrow[d, "q_{E^{n_2}a_3}"] \arrow[dr, phantom, "D_3"] & EE^{n_2}X_4 \arrow[l, "\widetilde{E^{n_2}a_4}"'] \arrow[d, "="] \\
X_1\cup_{a_2}CE^{n_2}X_2 & EE^{n_2}X_3 \arrow[l, "\widetilde{E^{n_2}a_3}"'] & EE^{n_2}X_4 \arrow[l, "-EE^{n_2}a_4"']
\end{tikzcd}$$

Then, from these diagrams, we have

$$\underline{B_1 \circ CE^{n_1} q_{E^{n_2}a_3}}_{(D_1,\widetilde{E}^{n_1}D_2)} : a_1 \circ E^{n_1}\overline{a_2} \simeq *,$$
$$\overline{i_{a_2} \circ B_2}^{(D_2,D_3)} : \widetilde{E^{n_2}a_3} \circ (-EE^{n_2}a_4) \simeq *.$$

It follows that $\{a_1, \overline{a_2}, \widetilde{E^{n_2}a_4}\}_{n_1}$ contains the homotopy class of

$$(4.7)\quad \big[a_1, \underline{B_1 \circ CE^{n_1} q_{E^{n_2}a_3}}_{(D_1,\widetilde{E}^{n_1}D_2)}, E^{n_1}\overline{a_2}\big] \circ \big(E^{n_1}\overline{a_2}, \widetilde{E}^{n_1}B_2, E^{n_1}\widetilde{E^{n_2}a_4}\big)$$

and that $\{\overline{a_1}', \widetilde{E^{n_2}a_3}, -E^{n_2+1}a_4\}_{n_1}$ contains the homotopy class of

$$(4.8)\quad \begin{aligned} &\big[\overline{a_1}', B_1, E^{n_1}\widetilde{E^{n_2}a_3}\big] \\ &\qquad \circ \big(E^{n_1}\widetilde{E^{n_2}a_3}, \widetilde{E}^{n_1}\overline{i_{a_2} \circ B_2}^{(D_2,D_3)}, E^{n_1}(-EE^{n_2}a_4)\big). \end{aligned}$$

Since $\overline{i_{a_2} \circ B_2}^{(D_2,D_3)} = \overline{i_{a_2} \circ B_2}^{(D_2,D_3)}$, it follows from Lemma 4.10(2) that

$$i_{a_2} \circ B_2 \simeq \underline{\overline{i_{a_2} \circ B_2}^{(D_2,D_3)}}_{(D_2,D_3)} \ rel\ EE^{n_2}X_4$$

so that the quasi-map

$$\begin{aligned} &(1_{X_0}, i_{a_2}, q_{E^{n_2}a_3}, 1_{EE^{n_2}X_4}; D_1, D_2, D_3) : \\ &\qquad \big(a_1, \overline{a_2}, \widetilde{E^{n_2}a_4}; \underline{B_1 \circ CE^{n_1} q_{E^{n_2}a_3}}_{(D_1,\widetilde{E}^{n_1}D_2)}, B_2\big)_{n_1} \\ &\qquad\qquad \longrightarrow \big(\overline{a_1}', \widetilde{E^{n_2}a_3}, -EE^{n_2}a_4; B_1, \overline{i_{a_2} \circ B_2}^{(D_2,D_3)}\big)_{n_1} \end{aligned}$$

is a map between representatives of null triples. Hence (4.7) is homotopic to (4.8) by Proposition 4.11. Therefore we obtain the underlined part of Theorem 4.9. □

Notice that the homotopy classes of (4.7) and (4.8) do not depend on D_1 and D_3 so that we take usually 1_{a_1} as D_1.

Definition 4.12. *If* $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ *is an admissible representative of (4.1) and* $D_2 : i_{a_2} \circ [a_2, A_2, E^{n_2}a_3] \simeq (a_2, A_2, E^{n_2}a_3) \circ q_{E^{n_2}a_3}$, *then we define*

$$\{a_1, a_2, a_3, a_4; A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2}$$

to be the set of homotopy classes of (4.7) hence of (4.8) for all possible B_1, B_2, D_1, D_3 *in (4.6), and define*

$$\{a_1, a_2, a_3, a_4; A_1, A_2, A_3\}^{(0)}_{n_1,n_2} = \{a_1, a_2, a_3, a_4; A_1, A_2, A_3; \widetilde{G}\}^{(1)}_{n_1,n_2},$$

where $\widetilde{G}$ *is the typical homotopy for* $(a_2, E^{n_2}a_3; A_2)$, *and define*

$$\{a_1, a_2, a_3, a_4; A_1, A_2, A_3\}^{(1)}_{n_1,n_2} = \bigcup_{D_2} \{a_1, a_2, a_3, a_4; A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2},$$

$$\begin{aligned}&\{a_1,a_2,a_3,a_4;A_1,A_2,A_3\}^{(2)}_{n_1,n_2}\\&=\big\{a_1,[a_2,A_2,E^{n_2}a_3],(E^{n_2}a_3,\widetilde{E}^{n_2}A_3,E^{n_2}a_4)\big\}_{n_1}\\&\qquad\qquad\cap\big\{[a_1,A_1,E^{n_1}a_2]\circ\big(\psi^{n_1}_{a_2}\big)^{-1},(a_2,A_2,E^{n_2}a_3),-E^{n_2+1}a_4\big\}_{n_1},\\&\{a_1,a_2,a_3,a_4;A_1,A_2,A_3\}^{(3)}_{n_1,n_2}\\&=\big\{[a_1,A_1,E^{n_1}a_2]\circ(\psi^{n_1}_{a_2})^{-1},i_{a_2}\circ[a_2,A_2,E^{n_2}a_3],(E^{n_2}a_3,\\&\qquad\qquad\qquad\qquad\qquad\qquad\widetilde{E}^{n_2}A_3,E^{n_2}a_4)\big\}_{n_1}.\end{aligned}$$

We call these five subsets of $[E^{n_1+n_2+2}X_4,X_0]$ *quasi tertiary compositions and abbreviate them to*

$$\{A_1,A_2,A_3;D_2\}^{(1)}_{n_1,n_2},\ \{A_1,A_2,A_3\}^{(k)}_{n_1,n_2}\ (k=0,1,2,3).$$

We have $\operatorname{Indet}\{A_1,A_2,A_3\}^{(3)}_{n_1,n_2}=\Phi_1+\Phi_2$, where

$$\begin{aligned}\Phi_1&=[E^{n_1+1}(E^{n_2}X_2\cup_{E^{n_2}a_3}CE^{n_2}X_3),X_0]\circ E^{n_1+1}(E^{n_2}a_3,\widetilde{E}^{n_2}A_3,E^{n_2}a_4),\\\Phi_2&=[a_1,A_1,E^{n_1}a_2]\circ\big(\psi^{n_1}_{a_2}\big)^{-1}\circ E^{n_1}[E^{n_2+2}X_4,X_1\cup_{a_2}CE^{n_2}X_2],\end{aligned}$$

$$\left.\begin{aligned}\Phi_1&\supset[E^{n_1+n_2+2}X_3,X_0]\circ E^{n_1+n_2+2}a_4\\\Phi_2&\supset a_1\circ E^{n_1}[E^{n_2+2}X_4,X_1]\end{aligned}\right.\quad\text{(by (2.2)).}$$

We have

$$\begin{aligned}&\operatorname{Indet}\big\{a_1,[a_2,A_2,E^{n_2}a_3],(E^{n_2}a_3,\widetilde{E}^{n_2}A_3,E^{n_2}a_4)\big\}_{n_1}\\&\qquad=\Phi_1+a_1\circ E^{n_1}[E^{n_2+2}X_4,X_1],\\&\operatorname{Indet}\big\{[a_1,A_1,E^{n_1}a_2]\circ\big(\psi^{n_1}_{a_2}\big)^{-1},\ (a_2,A_2,E^{n_2}a_3),-E^{n_2+1}a_4\big\}_{n_1}\\&\qquad=[E^{n_1+n_2+2}X_3,X_0]\circ E^{n_1+n_2+2}a_4+\Phi_2.\end{aligned}$$

The intersection of the last two indeterminacies is $\operatorname{Indet}\{A_1,A_2,A_3\}^{(2)}_{n_1,n_2}$. As will be seen in Proposition 5.6,

$$\begin{aligned}&\operatorname{Indet}\{A_1,A_2,A_3;D_2\}^{(1)}_{n_1,n_2}\\&\qquad=[E^{n_1+n_2+2}X_3,X_0]\circ E^{n_1+n_2+2}a_4+a_1\circ E^{n_1}[E^{n_2+2}X_4,X_1],\end{aligned}$$

but we do not know if $\{A_1,A_2,A_3\}^{(1)}_{n_1,n_2}$ has an indeterminacy (cf. Corollary 5.7(1)).

When $n_2=0$ or $n_1=n_2=0$, we usually omit the subscript n_2 or n_1,n_2 from notations. For example, we abbreviate $\{A_1,A_2,A_3;D_2\}^{(1)}_{n_1,0}$ to $\{A_1,A_2,A_3;D_2\}^{(1)}_{n_1}$.

Ôguchi [**13**, p.48] denoted $\{A_1,A_2,A_3;D_2\}^{(1)}_{0,0}$ by $\gamma(A_1,A_2,A_3)$ to which we prefer $\gamma(A_1,A_2,A_3;D_2)$. He asserted that if (A_1,A_2,A_3) and (A_1,A_2,A_3')

are admissible, then $\gamma(A_1, A_2, A_3; D_2) = \gamma(A_1, A_2, A'_3; D_2)$. But, as will be seen in (5.7) and (5.8) below, it is not true. As a consequence, Proposition (6.5) of [**13**] does not hold (see Example A.1 in Appendix A). Also there are gaps in proofs of several assertions in [**13**, pp.49-52].

5. Properties of quasi tertiary compositions

Proposition 5.1. *If $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ is an admissible representative of (4.1) and $D_2 : i_{a_2} \circ [a_2, A_2, E^{n_2}a_3] \simeq (a_2, A_2, E^{n_2}a_3) \circ q_{E^{n_2}a_3}$, then*

$$\begin{aligned}\{A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2} &\subset \{A_1, A_2, A_3\}^{(1)}_{n_1,n_2}\\ &\subset \{A_1, A_2, A_3\}^{(2)}_{n_1,n_2} \subset \{A_1, A_2, A_3\}^{(3)}_{n_1,n_2} \subset [E^{n_1+n_2+2}X_4, X_0],\end{aligned}$$

where containments are proper in general.

Proof. Containments are obvious from definitions, and the last assertion will be obtained from Example 5.2 below. □

Example 5.2. *Consider the next null quadruple:*

$$\begin{tikzcd} \mathrm{S}^3 & \mathrm{S}^{12} \arrow[l, "0^9_3"'] & \mathrm{S}^{12} \arrow[l, "0^0_{12}"'] & \mathrm{S}^{12} \arrow[l, "0^0_{12}"'] & \mathrm{S}^{19}. \arrow[l, "0^7_{12}"'] \end{tikzcd}$$

Then $(*^9_3, *^0_{12}, *^0_{12}, *^7_{12}; A_1, A_2, A_3)$ is an admissible representative of it for every respective null homotopies A_i. We can write $A_i = \widehat{A}_i \circ \pi$, where $\widehat{A}_1 : \mathrm{S}^{13} \to \mathrm{S}^3$, $\widehat{A}_2 : \mathrm{S}^{13} \to \mathrm{S}^{12}$, $\widehat{A}_3 : \mathrm{S}^{20} \to \mathrm{S}^{12}$ are maps and $\pi : C\,\mathrm{S}^m \to E\,\mathrm{S}^m = \mathrm{S}^{m+1}$ is the quotient map for $m = 12, 19$. Then $\{A_1, A_2, A_3\}^{(0)}$ consists of a single element $\widehat{A}_1 \circ E\widehat{A}_3$ which generates $\{A_1, A_2, A_3\}^{(1)}$. We know $\{A_1, A_2, A_3\}^{(k)}$ $(k = 1, 2, 3)$ from Table k which will be given in the proof.

Proof. Recall from [**19**, Theorem 7.1, Theorem 7.3, Theorem 12.8, (7.7)] that $\pi_{13}(\mathrm{S}^3) = \mathbb{Z}_4\{\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_3\mu_4\} \oplus \mathbb{Z}_3$, $\pi_{20}(\mathrm{S}^{12}) = \mathbb{Z}_2^2\{\overline{\nu}_{12}, \varepsilon_{12}\}$, $\pi_{21}(\mathrm{S}^3) = \mathbb{Z}_4\{\mu'\sigma_{14}\} \oplus \mathbb{Z}_2^2\{\nu'\overline{\varepsilon}_6, \eta_3\overline{\mu}_4\}$ and $2\mu' = \eta_3^2\mu_5$. We have

$$\{[*^9_3, A_1, *^0_{12}], (*^0_{12}, A_2, *^0_{12}), -E*^7_{12}\} = \widehat{A}_1 \circ \pi_{21}(\mathrm{S}^{13}),$$

$$\{*^9_3, [*^0_{12}, A_2, *^0_{12}], (*^0_{12}, A_3, *^7_{12})\} = \pi_{13}(\mathrm{S}^3) \circ E\widehat{A}_3,$$

$$\{A_1, A_2, A_3\}^{(3)} = \widehat{A}_1 \circ \pi_{21}(\mathrm{S}^{13}) + \pi_{13}(\mathrm{S}^3) \circ E\widehat{A}_3,$$

$$\{A_1, A_2, A_3\}^{(2)} = \big(\widehat{A}_1 \circ \pi_{21}(\mathrm{S}^{13})\big) \cap \big(\pi_{13}(\mathrm{S}^3) \circ E\widehat{A}_3\big).$$

We use the following relations [**14**, (2.13)(7),(8), (2.17)(8)]:

$$\mu_3\varepsilon_{12} \equiv \eta_3\mu_4\sigma_{13} \pmod{2\overline{\varepsilon}'}, \quad \mu_3\overline{\nu}_{12} \equiv 0 \pmod{2\overline{\varepsilon}'},$$

$$\varepsilon'\varepsilon_{13} = \varepsilon'\overline{\nu}_{13} = \overline{\varepsilon}'\eta_{20} = \nu'\overline{\varepsilon}_6.$$

Note that, although there is a few of gaps in [**14**], the above relations were correctly proved. We then easily obtain Table k ($k = 1, 2, 3$):

TABLE 1. $\widehat{A}_1 \circ E\widehat{A}_3$

$\widehat{A}_3$ $\widehat{A}_1$	ε_{12}	$\overline{\nu}_{12}$	$\varepsilon_{12} + \overline{\nu}_{12}$
ε'	$\nu'\overline{\varepsilon}_6$	$\nu'\overline{\varepsilon}_6$	0
$\eta_3\mu_4$	$2\mu'\sigma_{14}$	0	$2\mu'\sigma_{14}$
$\varepsilon' + \eta_3\mu_4$	$\nu'\overline{\varepsilon}_6 + 2\mu'\sigma_{14}$	$\nu'\overline{\varepsilon}_6$	$2\mu'\sigma_{14}$

TABLE 2. $\{A_1, A_2, A_3\}^{(2)}$

$\widehat{A}_3$ $\widehat{A}_1$	ε_{12}	$\overline{\nu}_{12}$	$\varepsilon_{12} + \overline{\nu}_{12}$	0
ε'	$\mathbb{Z}_2\{\nu'\overline{\varepsilon}_6\}$	$\mathbb{Z}_2\{\nu'\overline{\varepsilon}_6\}$	0	0
$\eta_3\mu_4$	$\mathbb{Z}_2\{2\mu'\sigma_{14}\}$	0	$\mathbb{Z}_2\{2\mu'\sigma_{14}\}$	0
$\varepsilon' + \eta_3\mu_4$	$\mathbb{Z}_2^2\{\nu'\overline{\varepsilon}_6, 2\mu'\sigma_{14}\}$	$\mathbb{Z}_2\{\nu'\overline{\varepsilon}_6\}$	$\mathbb{Z}_2\{2\mu'\sigma_{14}\}$	0
0	0	0	0	0

TABLE 3. $\{A_1, A_2, A_3\}^{(3)}$

$\widehat{A}_3$ $\widehat{A}_1$	ε_{12}	$\overline{\nu}_{12}$	$\varepsilon_{12} + \overline{\nu}_{12}$	0
ε'	$\Gamma := \mathbb{Z}_2^2\{\nu'\overline{\varepsilon}_6, 2\mu'\sigma_{14}\}$	$\mathbb{Z}_2\{\nu'\overline{\varepsilon}_6\}$	Γ	$\mathbb{Z}_2\{\nu'\overline{\varepsilon}_6\}$
$\eta_3\mu_4$	Γ	Γ	$\mathbb{Z}_2\{2\mu'\sigma_{14}\}$	$\mathbb{Z}_2\{2\mu'\sigma_{14}\}$
$\varepsilon' + \eta_3\mu_4$	Γ	Γ	Γ	Γ
0	Γ	$\mathbb{Z}_2\{\nu'\overline{\varepsilon}_6\}$	$\mathbb{Z}_2\{2\mu'\sigma_{14}\}$	0

In the rest of the proof, we shall compute $\{A_1, A_2, A_3; D_2\}^{(1)}$ for all $D_2 : i_{*^0_{12}} \circ [*^0_{12}, A_2, *^0_{12}] \simeq (*^0_{12}, A_2, *^0_{12}) \circ q_{*^0_{12}}$. Given any homotopies

$$B_1 : [*^9_3, A_1, *^0_{12}] \circ (*^0_{12}, A_2, *^0_{12}) \simeq *, \quad B_2 : [*^0_{12}, A_2, *^0_{12}] \circ (*^0_{12}, A_3, *^7_{12}) \simeq *,$$

we define $f_{D_2} : EE\,\mathrm{S}^{19} \to \mathrm{S}^3$ by

$$f_{D_2} = [*^9_3, \underline{B_1 \circ Cq_{*^0_{12}}}_{(1_{*^9_3}, D_2)}, [*^0_{12}, A_2, *^0_{12}]] \circ \big([*^0_{12}, A_2, *^0_{12}], B_2, (*^0_{12}, A_3, *^7_{12})\big).$$

Then $f_{D_2}(x \wedge \overline{s} \wedge \overline{t})$ is

$$\begin{cases} [*^9_3, A_1, *^0_{12}] \circ D_2(\widehat{A}_3(x \wedge \overline{2s-1}), 2-6t) & \frac{1}{2} \leq s \leq 1,\ \frac{1}{6} \leq t \leq \frac{1}{3} \\ * & \text{otherwise.} \end{cases}$$

Hence f_{D_2} does not depend on B_1, B_2 so that $\{A_1, A_2, A_3; D_2\}^{(1)}$ consists of a single element f_{D_2} (cf. Proposition 5.6). Let $g_{D_2} : EE\,\mathrm{S}^{19} \to \mathrm{S}^3$ be defined by $g_{D_2}(x \wedge \overline{s} \wedge \overline{t}) = [*^9_3, A_1, *^0_{12}] \circ D_2(\widehat{A}_3(x \wedge \overline{s}), 1-t)$. Then, as is easily shown, $f_{D_2} \simeq g_{D_2}$.

There is a map $h : E\,\mathrm{S}^{12} \to \mathrm{S}^{12} \vee E\,\mathrm{S}^{12}$ which makes the following diagram commutative:

$$\begin{tikzcd}
E\,\mathrm{S}^{19} \times I \arrow[r, "\widehat{A}_3 \times (-1)"] \arrow[d, "\pi"'] & \mathrm{S}^{12} \times I \arrow[r, "i_{*^0_{12}} \times 1_I"] \arrow[d, "\pi"'] & (\mathrm{S}^{12} \vee E\,\mathrm{S}^{12}) \times I \arrow[d, "D_2"'] & \\
EE\,\mathrm{S}^{19} \arrow[r, "-E\widehat{A}_3"'] & E\,\mathrm{S}^{12} \arrow[r, "h"'] & \mathrm{S}^{12} \vee E\,\mathrm{S}^{12} \arrow[r, "{[*^9_3, A_1, *^0_{12}]}"] \arrow[d, "\mathrm{pr}_2"'] & \mathrm{S}^3 \\
 & & E\,\mathrm{S}^{12} \arrow[ur, "\widehat{A}_1"'] &
\end{tikzcd}$$

where $(-1)(t) = 1-t$, π's are quotient maps and pr_2 is the projection to the second factor. Let $y_{D_2} \in \mathbb{Z}$ such that $\mathrm{pr}_2 \circ h \simeq y_{D_2} 1_{\mathrm{S}^{13}}$. Then

$$\begin{aligned} g_{D_2}(x \wedge \overline{s} \wedge \overline{t}) &= (*^9_3 \underline{\vee} \widehat{A}_1) \circ D_2(\widehat{A}_3(x \wedge \overline{s}), 1-t) \\ &= \widehat{A}_1 \circ \mathrm{pr}_2 \circ h(\widehat{A}_3(x \wedge \overline{s}) \wedge \overline{1-t}) = \widehat{A}_1 \circ \mathrm{pr}_2 \circ h \circ (-E\widehat{A}_3)(x \wedge \overline{s} \wedge \overline{t}). \end{aligned}$$

Since $2\widehat{A}_1 \circ E\widehat{A}_3 \simeq *$, we then have $g_{D_2} \simeq y_{D_2} \widehat{A}_1 \circ E\widehat{A}_3$ and

$$\{A_1, A_2, A_3; D_2\}^{(1)} = \{y_{D_2} \widehat{A}_1 \circ E\widehat{A}_3\}. \tag{5.1}$$

If we take the typical homotopy $\widetilde{G}$ for $(*^0_{12}, *^0_{12}; A_2)$ as D_2, then $y_{\widetilde{G}} = 1$ so that

$$\{A_1, A_2, A_3\}^{(0)} = \{\widehat{A}_1 \circ E\widehat{A}_3\}. \tag{5.2}$$

Next we shall show that $y_{D_2} = 0$ for some D_2. Let $\omega : I \times I \to I$ and $K : E\,\mathrm{S}^{12} \times I \to \mathrm{S}^{12}$ be defined by

$$\omega(s,t) = \begin{cases} 0 & 2s \leq t \\ 2s-t & s \leq t \leq 2s \\ s & t \leq s \end{cases}, \quad K(z \wedge \overline{s}, t) = \widehat{A}_2(z \wedge \overline{\omega(s,t)}).$$

We define $D_2' : (\mathrm{S}^{12} \vee E\,\mathrm{S}^{12}) \times I \to \mathrm{S}^{12} \vee E\,\mathrm{S}^{12}$ to be the composite of

$$(\mathrm{S}^{12} \vee E\,\mathrm{S}^{12}) \times I \xrightarrow{\mathrm{pr}_2 \times 1_I} E\,\mathrm{S}^{12} \times I \xrightarrow{K} \mathrm{S}^{12} \xrightarrow{i_{*^0_{12}}} \mathrm{S}^{12} \vee E\,\mathrm{S}^{12}.$$

Then $D'_2 : i_{*^0_{12}} \circ [*^0_{12}, A_2, *^0_{12}] \simeq (*^0_{12}, A_2, *^0_{12}) \circ q_{*^0_{12}}$ and $y_{D'_2} = 0$. Hence

$$\{A_1, A_2, A_3; D'_2\}^{(1)} = \{0\}. \tag{5.3}$$

It follows from (5.1), (5.2) and (5.3) that $\{A_1, A_2, A_3\}^{(1)}$ is a group generated by $\widehat{A}_1 \circ E\widehat{A}_3$. This completes the proof. □

Proposition 5.3. *Suppose that* $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ *is an admissible representative of (4.1) and* $0 \leq m_i \leq n_i$ $(i = 1, 2)$. *Then, for any* $D_2 : i_{a_2} \circ [a_2, A_2, E^{n_2}a_3] \simeq (a_2, A_2, E^{n_2}a_3) \circ q_{E^{n_2}a_3}$, *we have*

$$\{A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2} = \{A_1, A_2, \widetilde{E}^{n_2-m_2}A_3; D_2\}^{(1)}_{n_1,m_2},$$

$$\begin{aligned}\{A_1,& A_2, A_3; D_2\}^{(1)}_{n_1,n_2} \\ &\subset (-1)^{n_1-m_1}\{A_1, \widetilde{E}^{n_1-m_1}A_2, \widetilde{E}^{n_1+n_2-m_1-m_2}A_3; D'_2\}^{(1)}_{m_1,m_2},\end{aligned} \tag{5.4}$$

where $D'_2 = (\psi^{n_1-m_1}_{a_2})^{-1} \circ \widetilde{E}^{n_1-m_1}D_2 \circ (\psi^{n_1-m_1}_{E^{n_2}a_3} \times 1_I)$. *For* $0 \leq k \leq 3$, *we have*

$$\{A_1, A_2, A_3\}^{(k)}_{n_1,n_2} = \{A_1, A_2, \widetilde{E}^{n_2-m_2}A_3\}^{(k)}_{n_1,m_2},$$

$$\{A_1, A_2, A_3\}^{(k)}_{n_1,n_2} \subset (-1)^{n_1-m_1}\{A_1, \widetilde{E}^{n_1-m_1}A_2, \widetilde{E}^{n_1+n_2-m_1-m_2}A_3\}^{(k)}_{m_1,m_2}.$$

Proof. We prove only (5.4) because others are easier. Given null homotopies

$$B_1 : [a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1} \circ E^{n_1}(a_2, A_2, E^{n_2}a_3) \simeq *,$$

$$B_2 : [a_2, A_2, E^{n_2}a_3] \circ (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4) \simeq *,$$

we define null homotopies B'_1 and B'_2 by

$$\begin{aligned} B'_1 &= B_1 \circ C\big(1_{E^{n_2}X_3} \wedge \tau(\mathrm{S}^{n_1-m_1}, \mathrm{S}^1) \wedge 1_{\mathrm{S}^{m_1}}\big) \\ &: [a_1, A_1, E^{m_1}E^{n_1-m_1}a_2] \circ \big(\psi^{m_1}_{E^{n_1-m_1}a_2}\big)^{-1} \\ &\qquad \circ E^{m_1}(E^{n_1-m_1}a_2, \widetilde{E}^{n_1-m_1}A_2, E^{m_2}E^{n_1+n_2-m_1-m_2}a_3) \simeq *, \\ B'_2 &= \widetilde{E}^{n_1-m_1}B_2 \circ \big(1_{E^{n_2}X_4} \wedge \tau(\mathrm{S}^{n_1-m_1}, \mathrm{S}^1) \wedge 1_I\big) \\ &: [E^{n_1-m_1}a_2, \widetilde{E}^{n_1-m_1}A_2, E^{m_2}E^{n_1+n_2-m_1-m_2}a_3] \\ &\quad \circ \big(E^{m_2}E^{n_1+n_2-m_1-m_2}a_3, \widetilde{E}^{m_2}\widetilde{E}^{n_1+n_2-m_1-m_2}A_3, E^{m_2}E^{n_1+n_2-m_1-m_2}a_4\big) \\ &\quad \simeq *. \end{aligned}$$

By the definition of D'_2, we have

$$\begin{aligned} D'_2 : i_{E^{n_1-m_1}a_2} &\circ [E^{n_1-m_1}a_2, \widetilde{E}^{n_1-m_1}A_2, E^{n_1+n_2-m_1}a_3] \\ &\simeq (E^{n_1-m_1}a_2, \widetilde{E}^{n_1-m_1}A_2, E^{n_1+n_2-m_1}a_3) \circ q_{E^{n_1+n_2-m_1}a_3} \end{aligned}$$

and, under the identifications $\mathrm{S}^{n_1+n_2-m_1-m_2} \wedge \mathrm{S}^{m_2} = \mathrm{S}^{n_2} \wedge \mathrm{S}^{n_1-m_1}$, $\mathrm{S}^{n_1} = \mathrm{S}^{n_1-m_1} \wedge \mathrm{S}^{m_1}$ (see the section 2 or [**19**, pp.5-6]), we obtain the following equality by (2.3) and routine calculations.

$$\begin{aligned}
&\big[a_1, \underline{B_1 \circ CE^{n_1} q_{E^{n_2}a_3}}_{(1_{a_1}, \widetilde{E}^{n_1} D_2)}, E^{n_1}[a_2, A_2, E^{n_2}a_3]\big] \\
&\qquad \circ \big(E^{n_1}[a_2, A_2, E^{n_2}a_3], \widetilde{E}^{n_1} B_2, E^{n_1}(E^{n_2}a_3, \widetilde{E}^{n_2} A_3, E^{n_2}a_4)\big) \\
&= \big[a_1, \underline{B_1' \circ CE^{m_1} q_{E^{n_1+n_2-m_1}a_3}}_{(1_{a_1}, \widetilde{E}^{m_1} D_2')}, \\
&\qquad\qquad\qquad E^{m_1}[E^{n_1-m_1}a_2, \widetilde{E}^{n_1-m_1} A_2, E^{m_2} E^{n_1+n_2-m_1-m_2} a_3]\big] \\
&\circ \big(E^{m_1}[E^{n_1-m_1}a_2, \widetilde{E}^{n_1-m_1} A_2, E^{m_2} E^{n_1+n_2-m_1-m_2} a_3], \widetilde{E}^{m_1} B_2', \\
&\qquad\qquad\qquad E^{m_1}(E^{m_2} E^{n_1+n_2-m_1-m_2} a_3, \widetilde{E}^{m_2} \widetilde{E}^{n_1+n_2-m_1-m_2} A_3, \\
&\qquad\qquad\qquad\qquad\qquad E^{m_2} E^{n_1+n_2-m_1-m_2} a_4)\big) \\
&\circ (1_{E^{n_2}X_4} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_1-m_1}) \wedge 1_{\mathrm{S}^{m_1} \wedge \mathrm{S}^1}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\{A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2} \\
&\qquad\qquad \subset (-1)^{n_1-m_1}\{A_1, \widetilde{E}^{n_1-m_1} A_2, \widetilde{E}^{n_1+n_2-m_1-m_2} A_3; D_2'\}^{(1)}_{m_1,m_2}.
\end{aligned}$$

This proves (5.4). For the case $k = 0$, we should see that if D_2 is the typical homotopy for $(a_2, E^{n_2}a_3; A_2)$, then D_2' is the typical homotopy for $(E^{n_1-m_1}a_2, E^{n_1-m_1+n_2}a_3; \widetilde{E}^{n_1-m_1} A_2)$. This is easy to prove. □

Proposition 5.4. *If $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ is an admissible representative of (4.1) and $D_2 : i_{a_2} \circ [a_2, A_2, E^{n_2}a_3] \simeq (a_2, A_2, E^{n_2}a_3) \circ q_{E^{n_2}a_3}$, then $(Ea_1, a_2, a_3, a_4; \widetilde{E}A_1, A_2, A_3)_{n_1+1,n_2}$ is admissible and*

$$\begin{aligned}
E\{A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2} &\subset -\{\widetilde{E}A_1, A_2, A_3; D_2\}^{(1)}_{n_1+1,n_2}, \\
E\{A_1, A_2, A_3\}^{(k)}_{n_1,n_2} &\subset -\{\widetilde{E}A_1, A_2, A_3\}^{(k)}_{n_1+1,n_2} \quad (k = 0, 1, 2, 3).
\end{aligned}$$

Proof. Let $B_1 : [a_1, A_1, E^{n_1}a_2] \circ \big(\psi^{n_1}_{a_2}\big)^{-1} \circ E^{n_1}(a_2, A_2, E^{n_2}a_3) \simeq *$ and $B_2 : [a_2, A_2, E^{n_2}a_3] \circ (E^{n_2}a_3, \widetilde{E}^{n_2} A_3, E^{n_2}a_4) \simeq *$. By (2.3) and Lemma 2.4, we have

$$\begin{aligned}
&E\big([a_1, A_1, E^{n_1}a_2] \circ \big(\psi^{n_1}_{a_2}\big)^{-1} \circ E^{n_1}(a_2, A_2, E^{n_2}a_3)\big) \\
&\qquad = [Ea_1, \widetilde{E}A_1, E^{n_1+1}a_2] \circ \big(\psi^{n_1+1}_{a_2}\big)^{-1} \circ E^{n_1+1}(a_2, A_2, E^{n_2}a_3).
\end{aligned}$$

Hence $\widetilde{E}B_1 : [Ea_1, \widetilde{E}A_1, E^{n_1+1}a_2] \circ \big(\psi^{n_1+1}_{a_2}\big)^{-1} \circ E^{n_1+1}(a_2, A_2, E^{n_2}a_3) \simeq *$. As is easily shown, we have

$$\widetilde{E}\big(\underline{B_1 \circ CE^{n_1} q_{E^{n_2}a_3}}_{(1_{a_1}, \widetilde{E}^{n_1} D_2)}\big) = \underline{\widetilde{E}B_1 \circ CE^{n_1+1} q_{E^{n_2}a_3}}_{(1_{Ea_1}, \widetilde{E}^{n_1+1} D_2)}.$$

It then follows from Lemma 2.4 that

$$\begin{aligned}&E\Big(\big[a_1, \underline{B_1 \circ CE^{n_1} q_{E^{n_2}a_3}}_{(1_{a_1}, \widetilde{E}^{n_1} D_2)}, E^{n_1}[a_2, A_2, E^{n_2}a_3]\big] \\ &\qquad\qquad \circ \big(E^{n_1}[a_2, A_2, E^{n_2}a_3], \widetilde{E}^{n_1} B_2, E^{n_1}(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\big)\Big) \\ &\simeq -\big[Ea_1, \underline{\widetilde{E}B_1 \circ CE^{n_1+1} q_{E^{n_2}a_3}}_{(1_{Ea_1}, \widetilde{E}^{n_1+1} D_2)}, E^{n_1+1}[a_2, A_2, E^{n_2}a_3]\big] \\ &\qquad\qquad \circ \big(E^{n_1+1}[a_2, A_2, E^{n_2}a_3], \widetilde{E}^{n_1+1} B_2, E^{n_1+1}(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\big).\end{aligned}$$

This implies the first containment. Similarly we obtain other containments. □

The following lemma can be proved by giving a homotopy. We omit details.

Lemma 5.5. *If $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ is an admissible representative of (4.1), B and B' are null homotopies of $[a_1, A_1, E^{n_1}a_2] \circ \big(\psi^{n_1}_{a_2}\big)^{-1} \circ E^{n_1}(a_2, A_2, E^{n_2}a_3)$, and $D_2 : i_{a_2} \circ [a_2, A_2, E^{n_2}a_3] \simeq (a_2, A_2, E^{n_2}a_3) \circ q_{E^{n_2}a_3}$, then*

$$\begin{aligned}&d\big(\underline{B \circ CE^{n_1} q_{E^{n_2}a_3}}_{(1_{a_1}, \widetilde{E}^{n_1} D_2)}, \underline{B' \circ CE^{n_1} q_{E^{n_2}a_3}}_{(1_{a_1}, \widetilde{E}^{n_1} D_2)}\big) \\ &\simeq d(B, B') \circ E^{n_1+1} q_{E^{n_2}a_3}.\end{aligned}$$

The essential part of the following result can be seen in [**13**, §4].

Proposition 5.6. *If $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ is an admissible representative of (4.1) and $D_2 : i_{a_2} \circ [a_2, A_2, E^{n_2}a_3] \simeq (a_2, A_2, E^{n_2}a_3) \circ q_{E^{n_2}a_3}$, then $\{A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2}$ is a coset of*

$$[E^{n_1+n_2+2}X_3, X_0] \circ E^{n_1+n_2+2}\alpha_4 + \alpha_1 \circ E^{n_1}[E^{n_2+2}X_4, X_1].$$

Proof. Take following null homotopies arbitrarily

$$\begin{aligned}B_1, B_1' &: [a_1, A_1, E^{n_1}a_2] \circ \big(\psi^{n_1}_{a_2}\big)^{-1} \circ E^{n_1}(a_2, A_2, E^{n_2}a_3) \simeq *, \\ B_2, B_2' &: [a_2, A_2, E^{n_2}a_3] \circ (E^{n_2}a_3, E^{n_2}A_3, E^{n_2}a_4) \simeq *.\end{aligned}$$

Then, by Lemma 2.8, Corollary 2.13 and Lemma 5.5, we have

$$\begin{aligned}&\big[a_1, \underline{B_1' \circ CE^{n_1} q_{E^{n_2}a_3}}_{(1_{a_1}, \widetilde{E}^{n_1} D_2)}, E^{n_1}[a_2, A_2, E^{n_2}a_3]\big] \\ &\qquad \circ \big(E^{n_1}[a_2, A_2, E^{n_2}a_3], \widetilde{E}^{n_1} B_2', E^{n_1}(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\big) \\ &\simeq d(B_1, B_1') \circ E^{n_1+n_2+1}a_4 \\ &\qquad + \big[a_1, \underline{B_1 \circ CE^{n_1} q_{E^{n_2}a_3}}_{(1_{a_1}, \widetilde{E}^{n_1} D_2)}, E^{n_1}[a_2, A_2, E^{n_2}a_3]\big] \\ &\qquad\qquad \circ \big(E^{n_1}[a_2, A_2, E^{n_2}a_3], \widetilde{E}^{n_1} B_2, E^{n_1}(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\big) \\ &\qquad + a_1 \circ (-1)^{n_1} E^{n_1} d(B_2, B_2').\end{aligned}$$

If we fix B_1, B_2 and take all possible B_1', B_2', then the assertion follows from Lemma 2.8(1)(g). □

Corollary 5.7. *Under the notations of Theorem 4.9, its proof and* Φ_1, Φ_2 *after Definition 4.12, we have the following three results.*

(1) *If* $\Phi_1 \cap \Phi_2 = \{0\}$*, then*

$$\{A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2} = \{A_1, A_2, A_3\}^{(1)}_{n_1,n_2} = \{A_1, A_2, A_3\}^{(2)}_{n_1,n_2}.$$

(2) *If* $\Phi_1 = \{0\}$*, then*

$$\{a_1, [a_2, A_2, \widetilde{E}^{n_2} a_3], (E^{n_2} a_3, \widetilde{E}^{n_2} A_3, E^{n_2} a_4)\}_{n_1}$$

is equal to the three sets in (1).

(3) *If* $\Phi_2 = \{0\}$*, then*

$$\{[a_1, A_1, E^{n_1} a_2] \circ (\psi^{n_1}_{a_2})^{-1}, (a_2, A_2, \widetilde{E}^{n_2} a_3), -E^{n_2+1} a_4\}_{n_1}$$

is equal to the three sets in (1).

Proof. Suppose $\Phi_1 \cap \Phi_2 = \{0\}$ and take $x \in \{A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2}$ arbitrarily. Then

$$\begin{aligned} \{a_1,&[a_2, A_2, E^{n_2} a_3], (E^{n_2} a_3, \widetilde{E}^{n_2} A_3, E^{n_2} a_4)\}_{n_1} \\ &\qquad\qquad = x + \Phi_1 + a_1 \circ E^{n_1}[E^2 E^{n_2} X_4, X_1], \\ \{[a_1,&A_1, E^{n_1} a_2] \circ \left(\psi^{n_1}_{a_2}\right)^{-1}, (a_2, A_2, E^{n_2} a_3), -E E^{n_2} a_4\}_{n_1} \\ &\qquad\qquad = x + [E^{n_1+2} E^{n_2} X_3, X_0] \circ E^{n_1+2} E^{n_2} a_4 + \Phi_2. \end{aligned}$$

By taking their intersection, we have

$$\begin{aligned} \{A_1,&A_2, A_3\}^{(2)}_{n_1,n_2} \\ &= x + [E^{n_1+2} E^{n_2} X_3, X_0] \circ E^{n_1+2} E^{n_2} a_4 + a_1 \circ E^{n_1}[E^2 E^{n_2} X_4, X_1] \end{aligned}$$

which is $\{A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2}$ by Proposition 5.6. This proves (1).

The set $\mathrm{Indet}\{A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2}$ is equal to

$$\begin{cases} \mathrm{Indet}\{a_1, [a_2, A_2, \widetilde{E}^{n_2} a_3], (E^{n_2} a_3, \widetilde{E}^{n_2} A_3, E^{n_2} a_4)\}_{n_1} & \Phi_1 = \{0\} \\ \mathrm{Indet}\{[a_1, A_1, E^{n_1} a_2] \circ (\psi^{n_1}_{a_2})^{-1}, (a_2, A_2, \widetilde{E}^{n_2} a_3), -E^{n_2+1} a_4\}_{n_1} & \Phi_2 = \{0\} \end{cases}.$$

Hence (2) and (3) hold. □

The next result shows that $\{A_1, A_2, A_3\}^{(k)}_{n_1,n_2}$ $(k = 1, 2, 3)$ depend on homotopy classes of A_i $(i = 1, 2, 3)$.

Proposition 5.8. *If* $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ *is an admissible representative of (4.1),* $D_2 : i_{a_2} \circ [a_2, A_2, E^{n_2} a_3] \simeq (a_2, A_2, E^{n_2} a_3) \circ q_{E^{n_2} a_3}$*,* $A_1 \simeq A_1'$ *rel* $E^{n_1+n_2} X_2$*,* $A_2 \simeq A_2'$ *rel* $E^{n_2} X_3$*, and* $A_3 \simeq A_3'$ *rel* X_4*, then*

$$(1)\ \begin{cases}\{A_1,A_2,A_3;D_2\}^{(1)}_{n_1,n_2}=\{A_1',A_2,A_3;D_2\}^{(1)}_{n_1,n_2}\\ \{A_1,A_2,A_3\}^{(k)}_{n_1,n_2}=\{A_1',A_2,A_3\}^{(k)}_{n_1,n_2}\ (k=0,1,2,3),\end{cases}$$

$$(2)\ \{A_1,A_2,A_3\}^{(k)}_{n_1,n_2}=\{A_1,A_2',A_3\}^{(k)}_{n_1,n_2}\ (k=1,2,3),$$

$$(3)\ \begin{cases}\{A_1,A_2,A_3;D_2\}^{(1)}_{n_1,n_2}=\{A_1,A_2,A_3';D_2\}^{(1)}_{n_1,n_2}\\ \{A_1,A_2,A_3\}^{(k)}_{n_1,n_2}=\{A_1,A_2,A_3'\}^{(k)}_{n_1,n_2}\ (k=0,1,2,3).\end{cases}$$

Proof. We prove assertions only for $\{\ \}^{(1)}_{n_1,n_2}$, because others are obvious from it and (2.6).

(1) Let $B_1 : [a_1,A_1,E^{n_1}a_2]\circ\left(\psi^{n_1}_{a_2}\right)^{-1}\circ E^{n_1}(a_2,A_2,E^{n_2}a_3)\simeq *$. Let $K : CE^{n_1}E^{n_2}X_2\times I\to X_0$ be a homotopy from A_1' to A_1 relative $E^{n_1}E^{n_2}X_2$. We define $B_1' : CE^{n_1+n_2+1}X_3\to X_0$ by

$$B_1'(y,t)=\begin{cases}[a_1,K_{2t},E^{n_1}a_2]\circ\left(\psi^{n_1}_{a_2}\right)^{-1}\circ E^{n_1}(a_2,A_2,E^{n_2}a_3)(y) & 0\le t\le\frac{1}{2}\\ B_1(y,2t-1) & \frac{1}{2}\le t\le 1\end{cases}$$

where $y\in E^{n_1+n_2+1}X_3$ and $K_{2t}=K|_{CE^{n_1}E^{n_2}X_2\times\{2t\}}$. Then

$$B_1' : [a_1,A_1',E^{n_1}a_2]\circ\left(\psi^{n_1}_{a_2}\right)^{-1}\circ E^{n_1}(a_2,A_2,E^{n_2}a_3)\simeq *.$$

It is not difficult to construct a homotopy from $\underline{B_1'\circ CE^{n_1}q_{E^{n_2}a_3}}_{(1_{a_1},\widetilde{E}^{n_1}D_2)}$ to $\underline{B_1\circ CE^{n_1}q_{E^{n_2}a_3}}_{(1_{a_1},\widetilde{E}^{n_1}D_2)}$ relative $E^{n_1}(E^{n_2}X_2\cup_{E^{n_2}a_3}CE^{n_2}X_3)$. We omit details. Hence $\{A_1',A_2,A_3;D_2\}^{(1)}_{n_1,n_2}=\{A_1,A_2,A_3;D_2\}^{(1)}_{n_1,n_2}$ by (2.6).

(2) Let $H : A_2\simeq A_2'\ rel\ E^{n_2}X_3$. Define $\Psi : (E^{n_2}X_2\cup_{E^{n_2}a_3}CE^{n_2}X_3)\times I\to X_1$ and $\Psi' : EE^{n_2}X_3\times I\to X_1\cup_{a_2}CE^{n_2}X_2$ by

$$\Psi(x_2,t)=a_2(x_2),\ \Psi(x_3\wedge s,t)=H(x_3\wedge s,t)\ (x_2\in E^{n_2}X_2,\ x_3\in E^{n_2}X_3),$$

$$\Psi'(x_3\wedge\overline{s},t)=\begin{cases}E^{n_2}a_3(x_3)\wedge(1-2s) & 0\le s\le\frac{1}{2}\\ H(x_3\wedge(2s-1),t) & \frac{1}{2}\le s\le 1\end{cases}\quad(x_3\in E^{n_2}X_3).$$

Then

$$\Psi : [a_2,A_2,E^{n_2}a_3]\simeq[a_2,A_2',E^{n_2}a_3]\ rel\ E^{n_2}X_2,$$
$$\Psi' : (a_2,A_2,E^{n_2}a_3)\simeq(a_2,A_2',E^{n_2}a_3).$$

Given three homotopies

$$B_1 : [a_1,A_1,E^{n_1}a_2]\circ\left(\psi^{n_1}_{a_2}\right)^{-1}\circ E^{n_1}(a_2,A_2,E^{n_2}a_3)\simeq *,$$
$$B_2 : [a_2,A_2,E^{n_2}a_3]\circ(E^{n_2}a_3,\widetilde{E}^{n_2}A_3,E^{n_2}a_4)\simeq *,$$
$$D_2 : i_{a_2}\circ[a_2,A_2,E^{n_2}a_3]\simeq(a_2,A_2,E^{n_2}a_3)\circ q_{E^{n_2}a_3},$$

we define three homotopies

$$B_1' : [a_1,A_1,E^{n_1}a_2]\circ\left(\psi^{n_1}_{a_2}\right)^{-1}\circ E^{n_1}(a_2,A_2',E^{n_2}a_3)\simeq *,$$

$$B_2' : [a_2, A_2', E^{n_2}a_3] \circ (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4) \simeq *,$$
$$D_2' : i_{a_2} \circ [a_2, A_2', E^{n_2}a_3] \simeq (a_2, A_2', E^{n_2}a_3) \circ q_{E^{n_2}a_3}$$

as follows:

$$\begin{aligned}
&B_1'(x_3 \wedge \overline{s} \wedge s_1, t) \\
&\quad = \begin{cases} [a_1, A_1, E^{n_1}a_2] \circ \big(\psi_{a_2}^{n_1}\big)^{-1}\big(\Psi'(x_3 \wedge \overline{s}, 1-2t) \wedge s_1\big) & 0 \leq t \leq \frac{1}{2} \\ B_1(x_3 \wedge \overline{s} \wedge s_1, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}, \\
&B_2'(x_4 \wedge \overline{s}, t) = \begin{cases} \Psi\big((a_3, A_3, a_4)(x_4 \wedge \overline{s}), 1-2t\big) & 0 \leq t \leq \frac{1}{2} \\ B_2(x_4 \wedge \overline{s}, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}, \\
&D_2'(x, t) = \begin{cases} i_{a_2} \circ \Psi(x, 1-3t) & 0 \leq t \leq \frac{1}{3} \\ D_2(x, 3t-1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ \Psi'(q_{E^{n_2}a_3}(x), 3t-2) & \frac{2}{3} \leq t \leq 1 \end{cases}, \\
&(x_3 \in E^{n_2}X_3, x_4 \in E^{n_2}X_4, s_1 \in \mathrm{S}^{n_1}, s, t \in I, x \in E^{n_2}X_2 \cup_{E^{n_2}a_3} CE^{n_2}X_3).
\end{aligned}$$

Consider the following diagrams, where h_i $(0 \leq i \leq 3)$ are identity maps of respective spaces:

$$\begin{tikzcd}
X_0 \arrow[d, "h_0", "="'] & E^{n_1}X_1 \arrow[l, "a_1"'] \arrow[d, "E^{n_1}h_1", "="'] \\
X_0 & E^{n_1}X_1 \arrow[l, "a_1"]
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \arrow[d, "h_1", "="'] & E^{n_2}X_2 \cup_{E^{n_2}a_3} CE^{n_2}X_3 \arrow[l, "{[a_2,A_2,E^{n_2}a_3]}"'] \arrow[d, "h_2", "="'] & EE^{n_2}X_4 \arrow[l, "{(E^{n_2}a_3,\widetilde{E}^{n_2}A_3,E^{n_2}a_4)}"'] \arrow[d, "h_3", "="'] \\
X_1 & E^{n_2}X_2 \cup_{E^{n_2}a_3} CE^{n_2}X_3 \arrow[l, "{[a_2,A_2',E^{n_2}a_3]}"] & EE^{n_2}X_4 \arrow[l, "{(E^{n_2}a_3,\widetilde{E}^{n_2}A_3,E^{n_2}a_4)}"]
\end{tikzcd}$$

Let $D_3 : EE^{n_2}X_4 \times I \to E^{n_2}X_2 \cup_{E^{n_2}a_3} CE^{n_2}X_3$ be the constant homotopy of $(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)$. Define

$$\widetilde{B}_1 = \underline{B_1 \circ CE^{n_1}q_{E^{n_2}a_3}}_{(1_{a_1}, \widetilde{E}^{n_1}D_2)}, \quad \widetilde{B}_1' = \underline{B_1' \circ CE^{n_1}q_{E^{n_2}a_3}}_{(1_{a_1}, \widetilde{E}^{n_1}D_2')}.$$

We shall prove

$$\widetilde{B}_1 \simeq \underline{\widetilde{B}_1'}_{(1_{a_1}, \widetilde{E}^{n_1}\Psi)} \ rel\ E^{n_1}(E^{n_2}X_2 \cup_{E^{n_2}a_3} CE^{n_2}X_3), \tag{5.5}$$

$$B_2 \simeq \underline{B_2'}_{(\Psi, D_3)} \ rel\ EE^{n_2}X_4. \tag{5.6}$$

Before proving these relations, we deduce the assertion (2) from them. If these relations hold, then $(h_0, h_1, h_2, h_3; 1_{a_1}, \Psi, D_3)$ is a *map* between representatives of null triples

$$\begin{aligned}&(a_1,[a_2, A_2, E^{n_2}a_3], (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4); \widetilde{B}_1, B_2)_{n_1}\\ &\qquad\longrightarrow (a_1, [a_2, A'_2, E^{n_2}a_3], (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4); \widetilde{B}'_1, B'_2)_{n_1}.\end{aligned}$$

It follows from Proposition 4.11 that

$$\begin{aligned}&\big[a_1,\underline{B_1 \circ CE^{n_1}q_{E^{n_2}a_3}}_{(1_{a_1},\widetilde{E}^{n_1}D_2)}, E^{n_1}[a_2, A_2, E^{n_2}a_3]\big]\\ &\qquad\qquad \circ \big(E^{n_1}[a_2, A_2, E^{n_2}a_3], \widetilde{E}^{n_1}B_2, E^{n_1}(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\big)\\ &\simeq\big[a_1, \underline{B'_1 \circ CE^{n_1}q_{E^{n_2}a_3}}_{(1_{a_1},\widetilde{E}^{n_1}D'_2)}, E^{n_1}[a_2, A'_2, E^{n_2}a_3]\big]\\ &\qquad \circ \big(E^{n_1}[a_2, A'_2, E^{n_2}a_3], \widetilde{E}^{n_1}B'_2, E^{n_1}(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\big)\end{aligned}$$

so that $\{A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2} \subset \{A_1, A'_2, A_3; D'_2\}^{(1)}_{n_1,n_2}$ which are the same since they have the same indeterminacies. This proves (2).

Now we prove (5.5). We decompose $I \times I = K_1 \cup \cdots \cup K_{10}$ as follows: let $(s, t) \in I \times I$ and

$$K_1 = \{(s, t)\,|\, s \le 1/3\},\ K_2 = \{(s, t)\,|\, 0 \le 3s - 1 \le t\},$$

$$K_3 = \{(s, t)\,|\, \frac{9}{4}s - \frac{3}{4} \le t \le 3s - 1\},\ K_4 = \{(s, t)\,|\, \frac{27}{13}s - \frac{9}{13} \le t \le \frac{9}{4}s - \frac{3}{4}\},$$

$$K_5 = \{(s, t)\,|\, \frac{27}{14}s - \frac{9}{14} \le t \le \frac{27}{13}s - \frac{9}{13}\},$$

$$K_6 = \{(s, t)\,|\, \frac{27}{5}s - \frac{18}{5} \le t \le \frac{27}{14}s - \frac{9}{14}\},$$

$$K_7 = \{(s, t)\,|\, \frac{9}{2}s - 3 \le t \le \frac{27}{5}s - \frac{18}{5}\},$$

$$K_8 = \{(s, t)\,|\, \frac{18}{5}s - \frac{12}{5} \le t \le \frac{9}{2}s - 3\},$$

$$K_9 = \{(s, t)\,|\, 3s - 2 \le t \le \frac{18}{5}s - \frac{12}{5}\},\ K_{10} = \{(s, t)\,|\, t \le 3s - 2\}.$$

If we define $u : K_1 \cup \cdots \cup K_6 \to I$ and $v : K_7 \cup \cdots \cup K_{10} \to I$ by moving respectively (s, t) to

$$\begin{cases} 0 & (s, t) \in K_1\\ 3s - 1 & (s, t) \in K_2\\ t & (s, t) \in K_3\\ -27s + 13t + 9 & (s, t) \in K_4\\ 27s - 13t - 9 & (s, t) \in K_5\\ 3s - \frac{5}{9}t - 1 & (s, t) \in K_6 \end{cases} \quad\text{and}\quad \begin{cases} 27s - 5t - 18 & (s, t) \in K_7\\ -18s + 5t + 12 & (s, t) \in K_8\\ 18s - 5t - 12 & (s, t) \in K_9\\ 3s - 2 & (s, t) \in K_{10} \end{cases},$$

then the map $\Theta : CE^{n_1}(E^{n_2}X_2 \cup_{E^{n_2}a_3} CE^{n_2}X_3) \times I \to X_0$ which moves $(x \wedge s_1 \wedge s, t)$ to

$$\begin{cases} a_1\big(\Psi(x, u(s,t)) \wedge s_1\big) & (s,t) \in K_{1,4} \\ [a_1, A_1, E^{n_1}a_2] \circ \big(\psi^{n_1}_{a_2}\big)^{-1}\big(D_2(x, u(s,t)) \wedge s_1\big) & (s,t) \in K_5 \cup K_6 \\ [a_1, A_1, E^{n_1}a_2] \circ \big(\psi^{n_1}_{a_2}\big)^{-1}\big(\Psi'(q_{E^{n_2}a_3}(x), v(s,t)) \wedge s_1\big) & (s,t) \in K_7 \cup K_8 \\ B_1(q_{E^{n_2}a_3}(x) \wedge s_1, v(s,t)) & (s,t) \in K_9 \cup K_{10} \end{cases}$$

$$(x \in E^{n_2}X_2 \cup_{E^{n_2}a_3} CE^{n_2}X_3, s_1 \in \mathrm{S}^{n_1}, s, t \in I, K_{1,4} = K_1 \cup \cdots \cup K_4),$$

is well defined and $\Theta : \widetilde{B}_1 \simeq \underline{\widetilde{B}'_1}_{(1_{a_1}, \widetilde{E}^{n_1}\Psi)}$ *rel* $E^{n_1}(E^{n_2}X_2 \cup_{E^{n_2}a_3} CE^{n_2}X_3)$. This proves (5.5).

If we define $w : I \times I \to I$ by

$$w(s,t) = \begin{cases} 3s & t \geq 3s \\ t & \frac{3}{2}s \leq t \leq 3s \\ -6s + 5t & \frac{6}{5}s \leq t \leq \frac{3}{2}s \\ 6s - 5t & s \leq t \leq \frac{6}{5}s \\ s & t \leq s \end{cases},$$

then the map $\Phi : CEE^{n_2}X_4 \times I \to X_1$ which moves $(x \wedge s, t)$ to

$$\begin{cases} \Psi((E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)(x), w(s,t)) & t \geq \frac{6}{5}s \\ B_2(x, w(s,t)) & t \leq \frac{6}{5}s \end{cases} \quad (x \in EE^{n_2}X_4)$$

is a homotopy from B_2 to $\underline{B'_2}_{(\Psi, D_3)}$ relative $EE^{n_2}X_4$. This proves (5.6).

We omit the proof of (3), because it is easier than (2). $\square$

The following two examples suggest that it is worth to consider $\{\ \}^{(k)}_{n_1,n_2}$ for $k = 1, 2$.

Example 5.9. *Since* $\{\eta_3, \nu', 8\iota_6\}_1 \ni 0$ *and* $\{\nu', 8\iota_5, \nu_5\}_1 = \{0\}$ *by* [**19**, pp.54-56], *it follows from Proposition 4.4 that* $(\eta_3, \nu', 8\iota_5, \nu_5)_{1,1}$ *is admissible. If* $(\eta_3, \nu', 8\iota_5, \nu_5; A_1, A_2, A_3)_{1,1}$ *is admissible, then so is* $(\eta_3, \nu', 8\iota_5, \nu_5; A_1 \dotplus \nu'\eta_6^2, A_2, A_3)_{1,1}$ *by Corollary 2.13. Hence, it follows from Corollary 4.7(2) that* $(\eta_3, \nu', 8\iota_5, \nu_5; A_1, A_2, A_3)_{1,1}$ *is admissible for any* $A_1 : \eta_3 \circ E\nu' \simeq *$, $A_2 : \nu' \circ 8\iota_6 \simeq *$ *and* $A_3 : 8\iota_5 \circ \nu_5 \simeq *$. *Take* $\mu_3 \in \{\eta_3, [\nu', A_2, 8\iota_6], (8\iota_6, \widetilde{E}A_3, \nu_6)\}_1$. *It follows from* [**19**, Chapter VII, ChapterXIII] *that* $\pi_{12}(\mathrm{S}^3) = \mathbb{Z}_2^2\{\mu_3, \eta_3\varepsilon_4\}$. *By indeterminacies, 5.1 and 5.6, we have*

$$\begin{aligned} \mu_3 + \mathbb{Z}_2\{\eta_3\varepsilon_4\} = \{A_1, A_2, A_3; D_2\}^{(1)}_{1,1} &= \{A_1, A_2, A_3\}^{(1)}_{1,1} \\ = \{A_1, A_2, A_3\}^{(2)}_{1,1} &= \{\eta_3, [\nu', A_2, 8\iota_6], (8\iota_6, \widetilde{E}A_3, \nu_6)\}_1 \end{aligned}$$

for any $D_2 : i_{\nu'} \circ [\nu', A_2, 8\iota_6] \simeq (\nu', A_2, 8\iota_6) \circ q_{8\iota_6}$.

Example 5.10. *Since* $\{\nu_7, \eta_9, 2\iota_{10}\}_1 \subset \pi_{12}(\mathrm{S}^7) = 0$ *and* $\{\eta_9, 2\iota_9, \overline{\nu}_9\}_1 \ni 0$ *by* [**19**, (10.1)], $(\nu_7, \eta_9, 2\iota_9, \overline{\nu}_9)_{1,1}$ *is admissible by Proposition 4.4. Let* $(\nu_7, \eta_9, 2\iota_9, \overline{\nu}_9; A_1, A_2, A_3)_{1,1}$ *be any admissible representative and take* $\kappa_7 \in \{\nu_7, [\eta_9, A_2, 2\iota_{10}], (2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})\}_1$ *arbitrarily. It follows from* [**19**, pp.95-101, Chapter XIII] *that* $\pi_{21}(\mathrm{S}^7) = \mathbb{Z}_4\{\kappa_7\} \oplus \mathbb{Z}_8\{\sigma'\sigma_{14}\} \oplus \mathbb{Z}_3$ *and the Hopf invariant* $H : \pi_{21}(\mathrm{S}^7) \to \pi_{21}(\mathrm{S}^{13}) = \mathbb{Z}_2^2\{\varepsilon_{13}, \overline{\nu}_{13}\}$ *is surjective.*

(1) *All of the following sets are equal to* $\kappa_7 + \mathbb{Z}_2\{4\sigma'\sigma_{14}\}$.

$$\{A_1, A_2, A_3; D_2\}^{(1)}_{1,1},\ \{A_1, A_2, A_3\}^{(k)}_{1,1},\ \{\nu_7, [\eta_9, A_2, 2\iota_{10}], (2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})\}_1,$$
$$-\{A_1, \widetilde{E}A_2, \widetilde{E}^2A_3; D'_2\}^{(1)},\ -\{A_1, \widetilde{E}A_2, \widetilde{E}^2A_3\}^{(k)},$$
$$-\{\nu_7, [\eta_{10}, \widetilde{E}A_2, 2\iota_{11}], (2\iota_{11}, \widetilde{E}^2A_3, \overline{\nu}_{11})\},$$

where $k = 1, 2$ *and* $D_2 : i_{\eta_9} \circ [\eta_9, A_2, 2\iota_{10}] \simeq (\eta_9, A_2, 2\iota_{10}) \circ q_{2\iota_{10}}$ *and* $D'_2 : i_{\eta_{10}} \circ [\eta_{10}, \widetilde{E}A_2, 2\iota_{11}] \simeq (\eta_{10}, \widetilde{E}A_2, 2\iota_{11}) \circ q_{2\iota_{11}}$.

(2) $(A_1, A_2 \dotplus \eta_9^2, A_3)_{1,1}$ *and* $(A_1, A_2, A_3 \dotplus \nu_9^3)_{1,1}$ *are admissible and*

$$\begin{aligned} -\kappa_7 + \mathbb{Z}_2\{4\sigma'\sigma_{14}\} &= \{A_1, \widetilde{E}A_2, \widetilde{E}^2A_3\}^{(1)} \\ &= \{A_1, A_2 \dotplus \eta_9^2, A_3\}^{(1)}_{1,1} = \{A_1, A_2, A_3 \dotplus \nu_9^3\}^{(1)}_{1,1}, \end{aligned} \tag{5.7}$$

$$\begin{aligned} \kappa_7 + \mathbb{Z}_2\{4\sigma'\sigma_{14}\} &= \{A_1, \widetilde{E}A_2, \widetilde{E}^2(A_3 \dotplus \nu_9^3)\}^{(1)} \\ &= \{A_1, \widetilde{E}(A_2 \dotplus \eta_9^2), \widetilde{E}^2A_3\}^{(1)} = \{A_1, A_2, A_3\}^{(1)}_{1,1}. \end{aligned} \tag{5.8}$$

(3) $H(\kappa_7) = \overline{\nu}_{13}$ *and* $H(\sigma'\sigma_{14}) = \varepsilon_{13} + \overline{\nu}_{13}$.

Proof. We shall use the following equalities:

$$\nu_7 \circ E\pi_{20}(\mathrm{S}^9) = \nu_7 \circ \pi_{21}(\mathrm{S}^{10}) = \mathbb{Z}_2\{4\sigma'\sigma_{14}\}, \tag{5.9}$$

$$\{\nu_7, \eta_9^2, \overline{\nu}_{11}\}_1 = \{\nu_7, \eta_{10}, \nu_{11}^3\} = \overline{\nu}_7\nu_{15}^2 + \mathbb{Z}_2\{4\sigma'\sigma_{14}\}, \tag{5.10}$$

$$\overline{\nu}_7\nu_{15}^2 \equiv 2\kappa_7 \pmod{4\sigma'\sigma_{14}} \tag{5.11}$$

which follow from [**19**, Lemma 5.14, Theorem 7.4, (10.7)], [**19**, Proposition 1.2, Lemma 6.2, Lemma 6.5] and [**19**, Lemma 5.4, Lemma 10.1, (10.7)], respectively.

(1) Since $[\mathrm{S}^{12} \cup_{2\iota_{12}} C\,\mathrm{S}^{12}, \mathrm{S}^7] = \mathbb{Z}_2\{\nu_7^2 \circ q_{2\iota_{12}}\}$ by a Puppe sequence and $\nu_7^2\overline{\nu}_{13} = 0$ by [**19**, (7.17),(7.18)], it follows from (2.2) that

$$\begin{aligned} &[E^2(\mathrm{S}^{10} \cup_{2\iota_{10}} C\,\mathrm{S}^{10}), \mathrm{S}^7] \circ E^2(2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10}) \\ &\qquad = [E(\mathrm{S}^{11} \cup_{2\iota_{11}} C\,\mathrm{S}^{11}), \mathrm{S}^7] \circ E(2\iota_{11}, \widetilde{E}^2A_3, \overline{\nu}_{11}) = 0. \end{aligned}$$

Hence from Corollary 5.7 we have

$$\{A_1, A_2, A_3; D_2\}^{(1)}_{1,1} = \{A_1, A_2, A_3\}^{(1)}_{1,1} = \{\nu_7, [\eta_9, A_2, 2\iota_{10}], (2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})\}_1,$$
$$\{A_1, \widetilde{E}A_2, \widetilde{E}^2A_3; D'_2\}^{(1)} = \{A_1, \widetilde{E}A_2, \widetilde{E}^2A_3\}^{(1)}$$

$$= \{\nu_7, [\eta_{10}, \widetilde{E}A_2, 2\iota_{11}], (2\iota_{11}, \widetilde{E}^2A_3, \overline{\nu}_{11})\}$$

of which indeterminacies are $\mathbb{Z}_2\{4\sigma'\sigma_{14}\}$ by (5.9). Therefore, for the six sets of (1), the first three are equal and so are the last three. By Proposition 5.3, we have $\{A_1, A_2, A_3\}^{(1)}_{1,1} \subset -\{A_1, \widetilde{E}A_2, \widetilde{E}^2A_3\}^{(1)}$. Hence they are the same because of indeterminacies. This completes the proof of (1).

(2) Since $\{A_1, \widetilde{E}A_2, \widetilde{E}^2A_3\}^{(1)} = -\{\nu_7, [\eta_9, A_2, 2\iota_{10}], (2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})\}_1 = -\kappa_7 + \mathbb{Z}_2\{4\sigma'\sigma_{14}\}$ by (1), we obtain (5.7).

By Corollary 2.13, $(A_1, A_2 \dotplus \eta_9^2, A_3)_{1,1}$ and $(A_1, A_2, A_3 \dotplus \nu_9^3)_{1,1}$ are admissible. We have

$$\begin{aligned}
&\{A_1, \widetilde{E}A_2, \widetilde{E}^2(A_3 \dotplus \nu_9^3)\}^{(1)}\\
&= \{\nu_7, [\eta_{10}, \widetilde{E}A_2, 2\iota_{11}], (2\iota_{11}, \widetilde{E}^2(A_3 \dotplus \nu_9^3), \overline{\nu}_{11})\} \quad \text{(by (1))}\\
&= \{\nu_7, [\eta_{10}, \widetilde{E}A_2, 2\iota_{11}], (2\iota_{11}, \widetilde{E}^2A_3, \overline{\nu}_{11}) + i_{2\iota_{11}} \circ \nu_{11}^3\} \quad \text{(by 2.5 and (2.5))}\\
&\subset \{\nu_7, [\eta_{10}, \widetilde{E}A_2, 2\iota_{11}], (2\iota_{11}, \widetilde{E}^2A_3, \overline{\nu}_{11})\} + \{\nu_7, [\eta_{10}, \widetilde{E}A_2, 2\iota_{11}], i_{2\iota_{11}} \circ \nu_{11}^3\}\\
&\qquad\qquad\qquad\qquad\qquad \text{(by [\textbf{19}, Proposition 1.6])}\\
&\subset \{\nu_7, [\eta_{10}, \widetilde{E}A_2, 2\iota_{11}], (2\iota_{11}, \widetilde{E}^2A_3, \overline{\nu}_{11})\} + \{\nu_7, \eta_{10}, \nu_{11}^3\}\\
&\qquad\qquad\qquad\qquad\qquad \text{(by [\textbf{19}, Proposition 1.2])}\\
&= \kappa_7 + \mathbb{Z}_2\{4\sigma'\sigma_{14}\} \quad \text{(by (1), (5.10) and (5.11))}.
\end{aligned}$$

Hence (5.8) is obtained. Also

$$\begin{aligned}
&\{A_1, A_2 \dotplus \eta_9^2, A_3\}^{(1)}_{1,1} = \big\{\nu_7, [\eta_9, A_2 \dotplus \eta_9^2, 2\iota_{10}], (2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})\big\}_1 \quad \text{(by (1))}\\
&= \big\{\nu_7, \big([\eta_9, A_2, 2\iota_{10}] \underline{\vee}\, \eta_9^2\big) \circ \theta_{2\iota_{10}}, (2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})\big\}_1 \quad \text{(by (2.4))}\\
&\supset \big\{\nu_7, [\eta_9, A_2, 2\iota_{10}] \underline{\vee}\, \eta_9^2, \theta_{2\iota_{10}} \circ (2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})\big\}_1\\
&\qquad\qquad\qquad\qquad\qquad \text{(by [\textbf{19}, Proposition 1.2])}\\
&= -\big\{\nu_7, [\eta_9, A_2, 2\iota_{10}] \underline{\vee}\, \eta_9^2, \big(-(2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10}) \vee \overline{\nu}_{11}\big) \circ \theta_{S^{19}}\big\}_1 \quad \text{(by 2.7)}\\
&= -\Big(\big\{\nu_7, [\eta_9, A_2, 2\iota_{10}], -(2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})\big\}_1 + \{\nu_7, \eta_9^2, \overline{\nu}_{11}\}_1\Big) \quad \text{(by 3.5)}\\
&= -\kappa_7 + \mathbb{Z}_2\{4\sigma'\sigma_{14}\} \quad \text{(by (5.10) and (5.11))}.
\end{aligned}$$

Hence $\{A_1, A_2 \dotplus \eta_9^2, A_3\}^{(1)}_{1,1} = -\kappa_7 + \mathbb{Z}_2\{4\sigma'\sigma_{14}\}$. Then other equalities of (2) are obtained from (1).

(3) By [**19**, Proposition 2.2, Lemma 5.14, Lemma 6.4], we have

$$H(\sigma'\sigma_{14}) = H(\sigma') \circ \sigma_{14} = \eta_{13} \circ \sigma_{14} = \varepsilon_{13} + \overline{\nu}_{13}. \tag{5.12}$$

Since $\kappa_7 \in \{\nu_7, [\eta_9, A_2, 2\iota_{10}], (2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})\}_1$, we have

$$H(\kappa_7) \in \{H(\nu_7), [\eta_9, A_2, 2\iota_{10}], (2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})\}_1 \quad \text{(by [\textbf{19}, Proposition 2.3])}$$

$$= \mathrm{Indet}\{H(\nu_7), [\eta_9, A_2, 2\iota_{10}], (2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})\}_1$$
$$\text{(since } H(\nu_7) = 0 : \mathrm{S}^{10} \to \mathrm{S}^{13}\text{)}$$
$$= [E^2(\mathrm{S}^{10} \cup_{2\iota_{10}} e^{11}), \mathrm{S}^{13}] \circ E^2(2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10})$$
$$= \mathbb{Z}_2\{E^2 q_{2\iota_{10}}\} \circ E^2(2\iota_{10}, \widetilde{E}A_3, \overline{\nu}_{10}) = \mathbb{Z}_2\{\overline{\nu}_{13}\} \quad \text{(by (2.2))}.$$

Therefore $H(\kappa_7) = \overline{\nu}_{13}$ by (5.12) and the surjectivity of H. This completes the proof. □

Proposition 5.11. *Suppose that* $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ *is an admissible representative of (4.1) and that* $a_5 \in \alpha_5 \in [X_5, X_4]$ *and* $A_4 : a_4 \circ a_5 \simeq *$ *are given.*

(1) *We have*

$$\{A_1, A_2, A_3\}^{(1)}_{n_1,n_2} \circ (-E^{n_1+n_2+2} a_5)$$
$$\subset (-1)^{n_2}\{a_1, a_2, E^{n_2}([a_3, A_3, a_4] \circ (a_4, A_4, a_5))\}_{n_1}.$$

(2) *If moreover* $(a_2, a_3, a_4, a_5; A_2, A_3, A_4)_{n_2}$ *is admissible, then*

$$\{A_1, A_2, A_3\}^{(3)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5 \cap (-1)^{n_1+n_2}\big(\alpha_1 \circ E^{n_1}\{A_2, A_3, A_4\}^{(3)}_{n_2}\big)$$
$$\supset (-1)^{n_1+n_2}\big(\alpha_1 \circ E^{n_1}\{[a_2, A_2, E^{n_2}a_3] \circ \big(\psi^{n_2}_{a_3}\big)^{-1}, (a_3, A_3, a_4), -Ea_5\}_{n_2}\big)$$
$$\supset (-1)^{n_1+n_2}\Big(\alpha_1 \circ E^{n_1}\{A_2, A_3, A_4\}^{(2)}_{n_2}\Big),$$
$$\{a_1, a_2, a_3, a_4; A_1, A_2, A_3\}^{(1)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5$$
$$= \{a_1, a_2, a_3, a_4; A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5$$
$$= (-1)^{n_1+n_2}\big(\alpha_1 \circ E^{n_1}\{a_2, a_3, a_4, a_5; A_2, A_3, A_4; D_2'\}^{(1)}_{n_2}\big)$$
$$= (-1)^{n_1+n_2}\big(\alpha_1 \circ E^{n_1}\{a_2, a_3, a_4, a_5; A_2, A_3, A_4\}^{(1)}_{n_2}\big)$$

for any $D_2 : i_{a_2} \circ [a_2, A_2, E^{n_2}a_3] \simeq (a_2, A_2, E^{n_2}a_3) \circ q_{E^{n_2}a_3}$ *and* $D_2' : i_{a_3} \circ [a_3, A_3, a_4] \simeq (a_3, A_3, a_4) \circ q_{a_4}$.

Proof. We have

$$\{a_1, a_2, a_3, a_4; A_1, A_2, A_3\}^{(1)}_{n_1,n_2} \circ (-E^{n_1+n_2+2}\alpha_5)$$
$$\subset \{a_1, [a_2, A_2, E^{n_2}a_3], (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\}_{n_1} \circ (-E^{n_1+n_2+2}a_5)$$
$$\text{(by 4.12 and 5.1)}$$
$$\subset \{a_1, [a_2, A_2, E^{n_2}a_3], (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4) \circ (-E^{n_2+1}a_5)\}_{n_1}$$
$$\text{(by [\mathbf{19}, Proposition 1.2(i)])}$$
$$= \{a_1, [a_2, A_2, E^{n_2}a_3], (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4) \circ q_{E^{n_2}a_4}$$
$$\circ (E^{n_2}a_4, \widetilde{E}^{n_2}A_4, E^{n_2}a_5)\}_{n_1} \quad \text{(by (2.2))}$$

$$
\begin{aligned}
&= \{a_1, [a_2, A_2, E^{n_2}a_3], i_{E^{n_2}a_3} \circ [E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4] \\
&\qquad\qquad \circ (E^{n_2}a_4, \widetilde{E}^{n_2}A_4, E^{n_2}a_5)\}_{n_1} \quad \text{(by 3.6)} \\
&\subset \{a_1, a_2, [E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4] \circ (E^{n_2}a_4, \widetilde{E}^{n_2}A_4, E^{n_2}a_5)\}_{n_1} \\
&\qquad\qquad \text{(by [\textbf{19}, Proposition 1.2(ii)])} \\
&= \{a_1, a_2, E^{n_2}([a_3, A_3, a_4] \circ (a_4, A_4, a_5)) \circ (1_{X_5} \wedge \tau(\mathrm{S}^{n_2}, \mathrm{S}^1)\}_{n_1} \quad \text{(by 2.4)} \\
&= (-1)^{n_2}\{a_1, a_2, E^{n_2}([a_3, A_3, a_4] \circ (a_4, A_4, a_5))\}_{n_1}.
\end{aligned}
$$

This proves (1).

We have

$$
\begin{aligned}
&\{A_1, A_2, A_3\}^{(3)}_{n_1, n_2} \circ E^{n_1+n_2+2}\alpha_5 \\
&\quad = (-1)^{n_1+1}\Big([a_1, A_1, E^{n_1}a_2] \circ \big(\psi^{n_1}_{a_2}\big)^{-1} \circ E^{n_1}\big\{i_{a_2} \circ [a_2, A_2, E^{n_2}a_3], \\
&\qquad\qquad (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4), E^{n_2+1}a_5\big\}\Big) \\
&\qquad\qquad\qquad \text{(by [\textbf{19}, Proposition 1.4])} \\
&\quad \supset (-1)^{n_1+1}\Big(\alpha_1 \circ E^{n_1}\big\{[a_2, A_2, E^{n_2}a_3], (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4), \\
&\qquad\qquad E^{n_2+1}a_5\big\}\Big) \quad \text{(by [\textbf{19}, Proposition 1.2(iv)])} \\
&\quad = (-1)^{n_1+n_2}\Big(\alpha_1 \circ E^{n_1}\big\{[a_2, A_2, E^{n_2}a_3] \circ \big(\psi^{n_2}_{a_3}\big)^{-1}, E^{n_2}(a_3, A_3, a_4), \\
&\qquad\qquad E^{n_2}(-Ea_5)\big\}\Big) \quad \text{(by 2.4)} \\
&\quad \supset (-1)^{n_1+n_2}\Big(\alpha_1 \circ E^{n_1}\big\{[a_2, A_2, E^{n_2}a_3] \circ \big(\psi^{n_2}_{a_3}\big)^{-1}, (a_3, A_3, a_4), -Ea_5\big\}_{n_2}\Big) \\
&\quad \subset (-1)^{n_1+n_2}\Big(\alpha_1 \circ E^{n_1}\big\{[a_2, A_2, E^{n_2}a_3] \circ \big(\psi^{n_2}_{a_3}\big)^{-1}, (a_3, A_3, a_4) \circ q_{a_4}, \\
&\qquad\qquad (a_4, A_4, a_5)\big\}_{n_2}\Big) \text{ (by [\textbf{19}, Proposition 1.2(ii)] and (2.2))} \\
&\quad = (-1)^{n_1+n_2}\Big(\alpha_1 \circ E^{n_1}\big\{A_2, A_3, A_4\big\}^{(3)}_{n_2}\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\{A_1, A_2, A_3\}^{(3)}_{n_1, n_2} \circ E^{n_1+n_2+2}\alpha_5 \cap (-1)^{n_1+n_2}\Big(\alpha_1 \circ E^{n_1}\big\{A_2, A_3, A_4\big\}^{(3)}_{n_2}\Big) \\
&\supset (-1)^{n_1+n_2}\Big(\alpha_1 \circ E^{n_1}\big\{[a_2, A_2, E^{n_2}a_3] \circ \big(\psi^{n_2}_{a_3}\big)^{-1}, (a_3, A_3, a_4), -Ea_5\big\}_{n_2}\Big) \\
&\supset (-1)^{n_1+n_2}\big(\alpha_1 \circ E^{n_1}\{A_2, A_3, A_4\}^{(2)}_{n_2}\big).
\end{aligned}
$$

Therefore we have the first part of (2).

It suffices for the rest of (2) to show

$$\begin{aligned}(5.13)\qquad &\alpha_1 \circ E^{n_1}\{a_2, a_3, a_4, a_5; A_2, A_3, A_4; D_2'\}_{n_2}^{(1)} \\ &= (-1)^{n_1+n_2}\{a_1, a_2, a_3, a_4; A_1, A_2, A_3; D_2\}_{n_1,n_2}^{(1)} \circ E^{n_1+n_2+2}\alpha_5\end{aligned}$$

for every D_2 and D_2'. By Lemma 2.4, we have

$$\begin{aligned}&[a_2, A_2, E^{n_2}a_3] \circ (\psi_{a_3}^{n_2})^{-1} \circ E^{n_2}(a_3, A_3, a_4) \\ &= [a_2, A_2, E^{n_2}a_3] \circ (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4) \circ (1_{X_4} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2})).\end{aligned}$$

Hence null homotopies

$$\begin{aligned}B_1' &: [a_2, A_2, E^{n_2}a_3] \circ (\psi_{a_3}^{n_2})^{-1} \circ E^{n_2}(a_3, A_3, a_4) \simeq *, \\ B_2 &: [a_2, A_2, E^{n_2}a_3] \circ (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4) \simeq *\end{aligned}$$

correspond bijectively each other by the equality

$$(5.14)\quad B_2 = B_1' \circ C\big(1_{X_4} \wedge \tau(\mathrm{S}^{n_2}, \mathrm{S}^1)\big) \quad \textit{i.e.} \quad B_2 \circ C\big(1_{X_4} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2})\big) = B_1'.$$

Any element of $\alpha_1 \circ E^{n_1}\{a_2, a_3, a_4, a_5; A_2, A_3, A_4; D_2'\}_{n_2}^{(1)}$ has a form

$$\begin{aligned}f := a_1 \circ E^{n_1}\Big(&\big[[a_2, A_2, E^{n_2}a_3] \circ \big(\psi_{a_3}^{n_2}\big)^{-1}, B_1', E^{n_2}(a_3, A_3, a_4)\big] \\ &\circ \big(E^{n_2}(a_3, A_3, a_4), \widetilde{E}^{n_2}\overline{i_{a_3} \circ B_2'}^{(D_2', D_3')}, -E^{n_2+1}a_5\big)\Big),\end{aligned}$$

where $B_2' : [a_3, A_3, a_4] \circ (a_4, A_4, a_5) \simeq *$ and $D_3' : q_{a_4} \circ (a_4, A_4, a_5) \simeq -Ea_5$. Let H be any null homotopy of $a_1 \circ E^{n_1}[a_2, A_2, E^{n_2}a_3]$. We have

$$\begin{aligned}f &= a_1 \circ E^{n_1}\Big(\big[[a_2, A_2, E^{n_2}a_3], B_1', \big(\psi_{a_3}^{n_2}\big)^{-1} \circ E^{n_2}(a_3, A_3, a_4)\big] \\ &\quad \circ \big((\psi_{a_3}^{n_2})^{-1} \cup 1_{CE^{n_2}EX_4}\big) \circ \big(E^{n_2}(a_3, A_3, a_4), \widetilde{E}^{n_2}\overline{i_{a_3} \circ B_2'}^{(D_2', D_3')}, \\ &\qquad - E^{n_2+1}a_5\big)\Big) \qquad \text{(by 2.2(3))} \\ &\simeq \big[a_1, H, E^{n_1}[a_2, A_2, E^{n_2}a_3]\big] \\ &\quad \circ \Big(E^{n_1}[a_2, A_2, E^{n_2}a_3], \widetilde{E}^{n_1}B_1', E^{n_1}\big((\psi_{a_3}^{n_2})^{-1} \circ E^{n_2}(a_3, A_3, a_4)\big)\Big) \\ &\quad \circ \big(1_{E^{n_2+1}X_4} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_1})\big) \circ E^{n_1}q_{\left(\psi_{a_3}^{n_2}\right)^{-1} \circ E^{n_2}(a_3, A_3, a_4)} \\ &\quad \circ E^{n_1}\big((\psi_{a_3}^{n_2})^{-1} \cup 1_{CE^{n_2}EX_4}\big) \\ &\quad \circ E^{n_1}\big(E^{n_2}(a_3, A_3, a_4), \widetilde{E}^{n_2}\overline{i_{a_3} \circ B_2'}^{(D_2', D_3')}, -E^{n_2+1}a_5\big) \quad \text{(by 3.8)} \\ &\simeq \big[a_1, H, E^{n_1}[a_2, A_2, E^{n_2}a_3]\big] \\ &\quad \circ \Big(E^{n_1}[a_2, A_2, E^{n_2}a_3], \widetilde{E}^{n_1}B_1', E^{n_1}\big((\psi_{a_3}^{n_2})^{-1} \circ E^{n_2}(a_3, A_3, a_4)\big)\Big) \\ &\quad \circ \big(1_{E^{n_2+1}X_4} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_1})\big) \circ E^{n_1+n_2+2}a_5 \qquad \text{(by (2.2))}\end{aligned}$$

$$
\begin{aligned}
&= \bigl[a_1, H, E^{n_1}[a_2, A_2, E^{n_2}a_3]\bigr] \circ \bigl(\psi^{n_1}_{[a_2,A_2,E^{n_2}a_3]}\bigr)^{-1}\\
&\quad \circ E^{n_1}\bigl([a_2, A_2, E^{n_2}a_3], B'_1, (\psi^{n_2}_{a_3})^{-1} \circ E^{n_2}(a_3, A_3, a_4)\bigr)\\
&\quad \circ \bigl(1_{E^{n_2+1}X_4} \wedge \tau(\mathrm{S}^{n_1}, \mathrm{S}^1)\bigr) \circ \bigl(1_{E^{n_2+1}X_4} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_1})\bigr) \circ E^{n_1+n_2+2}a_5 \ \text{(by 2.4)}\\
&= \bigl[a_1, H, E^{n_1}[a_2, A_2, E^{n_2}a_3]\bigr] \circ \bigl(\psi^{n_1}_{[a_2,A_2,E^{n_2}a_3]}\bigr)^{-1}\\
&\qquad \circ E^{n_1}\bigl([a_2, A_2, E^{n_2}a_3], B'_1, (\psi^{n_2}_{a_3})^{-1} \circ E^{n_2}(a_3, A_3, a_4)\bigr) \circ E^{n_1+n_2+2}a_5\\
&= \bigl[a_1, H, E^{n_1}[a_2, A_2, E^{n_2}a_3]\bigr] \circ \bigl(\psi^{n_1}_{[a_2,A_2,E^{n_2}a_3]}\bigr)^{-1}\\
&\qquad \circ E^{n_1}\Bigl([a_2, A_2, E^{n_2}a_3], B_2 \circ C(1_{X_4} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2})),\\
&\qquad (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4) \circ (1_{X_4} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2}))\Bigr) \circ E^{n_1+n_2+2}a_5\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (5.14) and 2.4)}\\
&= \bigl[a_1, H, E^{n_1}[a_2, A_2, E^{n_2}a_3]\bigr] \circ \bigl(\psi^{n_1}_{[a_2,A_2,E^{n_2}a_3]}\bigr)^{-1}\\
&\qquad \circ E^{n_1}\Bigl(\bigl([a_2, A_2, E^{n_2}a_3], B_2, (E^{n_2}a_3, \widetilde{E}^{n_2}B_2, E^{n_2}a_4)\bigr)\\
&\qquad \circ E(1_{X_4} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2}))\Bigr) \circ E^{n_1+n_2+2}a_5 \quad \text{(by 2.2(1))}\\
&= \bigl[a_1, H, E^{n_1}[a_2, A_2, E^{n_2}a_3]\bigr]\\
&\qquad \circ \bigl(E^{n_1}[a_2, A_2, E^{n_2}a_3], \widetilde{E}^{n_1}B_2, E^{n_1}(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\bigr)\\
&\qquad \circ \bigl(1_{E^{n_2+1}X_4} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_1})\bigr) \circ E^{n_1+1}\bigl(1_{X_4} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2})\bigr) \circ E^{n_1+n_2+2}a_5\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by 2.4)}\\
&\simeq (-1)^{n_1+n_2}\bigl[a_1, H, E^{n_1}[a_2, A_2, E^{n_2}a_3]\bigr] \circ \bigl(E^{n_1}[a_2, A_2, E^{n_2}a_3], \widetilde{E}^{n_1}B_2,\\
&\qquad\quad E^{n_1}(E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4)\bigr) \circ E^{n_1+n_2+2}a_5.
\end{aligned}
$$

If we take $H = \underline{B_1 \circ CE^{n_1}q_{E^{n_2}a_3}}_{(1_{a_1},\widetilde{E}^{n_1}D_2)}$, where

$$B_1 : [a_1, A_1, E^{n_1}a_2] \circ \bigl(\psi^{n_2}_{a_2}\bigr) \circ E^{n_1}(a_2, A_2, E^{n_2}a_3) \simeq *,$$

then we know that

$$f \in (-1)^{n_1+n_2}\{a_1, a_2, a_3, a_4; A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5$$

so that

$$
\begin{aligned}
&\alpha_1{\circ}E^{n_1}\{a_2, a_3, a_4, a_5; A_2, A_3, A_4; D'_2\}^{(1)}_{n_2}\\
&\qquad \subset (-1)^{n_1+n_2}\{a_1, a_2, a_3, a_4; A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5.
\end{aligned}
$$

If (5.14) holds, then, by tracing the above discussion reversely, we obtain

$$\alpha_1{\circ}E^{n_1}\{a_2, a_3, a_4, a_5; A_2, A_3, A_4; D'_2\}^{(1)}_{n_2}$$

$$\supset (-1)^{n_1+n_2}\{a_1, a_2, a_3, a_4; A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5.$$

Hence we obtain (5.13). □

We owe the next remark to Ôguchi [**13**].

Remark 5.12. *The hypotheses of Proposition 5.11 are satisfied if one of the following five conditions holds.*

(1) $\{\alpha_1, \alpha_2, E^{n_2}\alpha_3\}_{n_1} = \{0\},\ \{\alpha_2, \alpha_3, \alpha_4\}_{n_2} \ni 0,\ \{\alpha_3, \alpha_4, \alpha_5\} = \{0\}$.

(2) $\{\alpha_1, \alpha_2, E^{n_2}\alpha_3\}_{n_1} \ni 0,\ \{\alpha_2, \alpha_3, \alpha_4\}_{n_2} = \{0\},\ \{\alpha_3, \alpha_4, \alpha_5\} \ni 0$.

(3) $\{\alpha_1, \alpha_2, E^{n_2}\alpha_3\}_{n_1} \ni 0,\ \{\alpha_2, \alpha_3, \alpha_4\}_{n_2} \ni 0,\ \{\alpha_3, \alpha_4, \alpha_5\} = \{0\}$,

$$G_1 + G_2 = [E^{n_2+1}X_3, X_1].$$

(4) $\{\alpha_1, \alpha_2, E^{n_2}\alpha_3\}_{n_1} \ni 0,\ \{\alpha_2, \alpha_3, \alpha_4\}_{n_2} \ni 0,\ \{\alpha_3, \alpha_4, \alpha_5\} \ni 0$,

$$G_1 + G_2 = [E^{n_2+1}X_3, X_1], \quad \overline{G}_3 + G_4 = [EX_4, X_2].$$

(5) $\{\alpha_1, \alpha_2, E^{n_2}\alpha_3\}_{n_1} \ni 0,\ \{\alpha_2, \alpha_3, \alpha_4\}_{n_2} \ni 0,\ \{\alpha_3, \alpha_4, \alpha_5\} \ni 0$,

$$G_1 + \overline{G}_2 = [E^{n_2+1}X_3, X_1], \quad G_3 + G_4 = [EX_4, X_2].$$

Here G_1, G_2 *are subgroups of* $[E^{n_2+1}X_3, X_1]$ *defined for* $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_{n_1,n_2}$ *in Proposition 4.1, and* G_3, G_4 *are similarly defined subgroups of* $[EX_4, X_2]$ *for* $(\alpha_2, \alpha_3, \alpha_4, \alpha_5)_{n_2}$*. Also* $\overline{G}_2$ *and* $\overline{G}_3$ *are respectively the kernels of*

$$E^{n_2+1}{\alpha_4}^* : [E^{n_2+1}X_3, X_1] \to [E^{n_2+1}X_4, X_1],$$
$$\alpha_{2*} \circ E^{n_2} : [EX_4, X_2] \to [E^{n_2+1}X_4, X_1].$$

Proof. The assertions for the cases (1), (2) and (3) follow from Proposition 4.6 and Proposition 4.1, respectively. Take $a_i \in \alpha_i$ $(1 \le i \le 5)$. Assume (4). Then there exist

$$A_1 : a_1 \circ E^{n_1}a_2 \simeq *,\ A_2 : a_2 \circ E^{n_2}a_3 \simeq *,\ A_3, A_3' : a_3 \circ a_4 \simeq *,\ A_4' : a_4 \circ a_5 \simeq *$$

such that $[a_1, A_1, E^{n_1}a_2] \circ (E^{n_1}a_2, \widetilde{E}^{n_1}A_2, E^{n_1+n_2}a_3) \simeq *$ and

$$[a_2, A_2, E^{n_2}a_3] \circ (E^{n_2}a_3, \widetilde{E}^{n_2}A_3, E^{n_2}a_4) \simeq *, \quad [a_3, A_3', a_4] \circ (a_4, A_4', a_5) \simeq *.$$

By the assumption, $\delta(A_3, A_3') = \gamma_3 + \gamma_4$ with $\gamma_3 \in \overline{G}_3$ and $\gamma_4 \in G_4$. Hence $\alpha_2 \circ E^{n_2}\gamma_3 = 0$ and there exists $\gamma \in [EX_5, X_3]$ such that $\alpha_3 \circ \gamma = \gamma_4 \circ E\alpha_5$. Take $c_i \in \gamma_i$ $(i = 3, 4)$ and $c \in \gamma$. Then

$$a_2 \circ E^{n_2}c_3 \simeq *, \quad a_3 \circ c \simeq c_4 \circ Ea_5, \quad d(A_3, A_3') \simeq c_3 + c_4 \tag{5.15}$$

and so $[a_2, A_2, E^{n_2}a_3] \circ (E^{n_2}a_3, \widetilde{E}^{n_2}(A_3 \dotplus c_3), E^{n_2}a_4) \simeq *$ by Corollary 2.13. Hence $(A_1, A_2, A_3 \dotplus c_3)_{n_1,n_2}$ is admissible. It follows from Proposition 2.6, Lemma 2.8 and (5.15) that

$$\begin{aligned} A_3 \dotplus c_3 &\simeq (A_3' \dotplus d(A_3', A_3)) \dotplus c_3 \\ &\simeq A_3' \dotplus (d(A_3', A_3) + c_3) \simeq A_3' \dotplus (-c_4)\ rel\ X_4 \end{aligned}$$

so that

$$\begin{aligned}[a_3,&A_3 \dotplus c_3, a_4] \circ (a_4, A_4' \dotplus (-c), a_5)\\ &\simeq [a_3, A_3' \dotplus (-c_4), a_4] \circ (a_4, A_4' \dotplus (-c), a_5)\\ &\simeq (-c_4) \circ (-Ea_5) + [a_3, A_3', a_4] \circ (a_4, A_4', a_5) + a_3 \circ (-c) \simeq *.\end{aligned}$$

Hence $(A_2, A_3 \dotplus c_3, A_4' \dotplus (-c))_{n_2}$ is admissible. This proves the assertion when (4) holds. The same argument holds for (5). $\square$

Proposition 5.13. (1) *Suppose that* $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ *is an admissible representative of (4.1) and that* $a_0 \in \alpha_0 \in [X_0, X_{-1}]$ *and* $a_5 \in \alpha_5 \in [X_5, X_4]$ *are given. Then* $(a_0 \circ a_1, a_2, a_3, a_4; a_0 \circ A_1, A_2, A_3)_{n_1,n_2}$ *and* $(a_1, a_2, a_3, a_4 \circ a_5; A_1, A_2, A_3 \circ Ca_5)_{n_1,n_2}$ *are admissible, and*

$$\begin{aligned}&\alpha_0 \circ \{a_1, a_2, a_3, a_4; A_1, A_2, A_3\}^{(k)}_{n_1,n_2}\\ &\qquad\qquad\subset \{a_0 \circ a_1, a_2, a_3, a_4; a_0 \circ A_1, A_2, A_3\}^{(k)}_{n_1,n_2},\\ &\{a_1, a_2, a_3, a_4; A_1, A_2, A_3\}^{(k)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5\\ &\qquad\qquad\subset \{a_1, a_2, a_3, a_4 \circ a_5; A_1, A_2, A_3 \circ Ca_5\}^{(k)}_{n_1,n_2},\\ &\alpha_0 \circ \{a_1, a_2, a_3, a_4; A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2}\\ &\qquad\qquad\subset \{a_0 \circ a_1, a_2, a_3, a_4; a_0 \circ A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2},\\ &\{a_1, a_2, a_3, a_4; A_1, A_2, A_3; D_2\}^{(1)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5\\ &\qquad\qquad\subset \{a_1, a_2, a_3, a_4 \circ a_5; A_1, A_2, A_3 \circ Ca_5; D_2\}^{(1)}_{n_1,n_2}\end{aligned}$$

for $k = 0, 1, 2, 3$ *and every* $D_2 : i_{a_2} \circ [a_2, A_2, E^{n_2}a_3] \simeq (a_2, A_2, E^{n_2}a_3) \circ q_{E^{n_2}a_3}$.

(2) *Suppose the following data are given:*

$$\beta_k \in [E^{n_k}Y_k, Y_{k-1}]\,(k = 1, 2, 3), \quad \beta_\ell \in [Y_\ell, Y_{\ell-1}](\ell = 4, 5),$$
$$\beta_1 \circ E^{n_1}\beta_2 \circ E^{n_1+n_2}\beta_3 = 0, \quad \beta_3 \circ E^{n_3}\beta_4 = 0, \quad \beta_4 \circ \beta_5 = 0.$$

If $b_k \in \beta_k\,(1 \le k \le 5)$ *and* $(b_1 \circ E^{n_1}b_2, b_3, b_4, b_5; B_1, B_2, B_3)_{n_1+n_2,n_3}$ *is admissible, then* $(B_1, b_2 \circ \widetilde{E}^{n_2}B_2, B_3)_{n_1,n_2+n_3}$ *is admissible and*

$$\begin{aligned}&\{b_1 \circ E^{n_1}b_2, b_3, b_4, b_5; B_1, B_2, B_3\}^{(k)}_{n_1+n_2,n_3}\\ &\quad\subset (-1)^{n_2}\{b_1, b_2 \circ E^{n_2}b_3, b_4, b_5; B_1, b_2 \circ \widetilde{E}^{n_2}B_2, B_3\}^{(k)}_{n_1,n_2+n_3}\ (k = 0, 1, 2, 3),\\ &\{b_1 \circ E^{n_1}b_2, b_3, b_4, b_5; B_1, B_2, B_3; D_2\}^{(1)}_{n_1+n_2,n_3}\\ &\quad\subset (-1)^{n_2}\{b_1, b_2 \circ E^{n_2}b_3, b_4, b_5; B_1, b_2 \circ \widetilde{E}^{n_2}B_2, B_3; D_2'\}^{(1)}_{n_1,n_2+n_3},\end{aligned}$$

where $D_2 : i_{b_3} \circ [b_3, B_2, E^{n_3}b_4] \simeq (b_3, B_2, E^{n_3}b_4) \circ q_{E^{n_3}b_4}$ *and*

$$D_2' = (b_2 \cup 1_{CE^{n_2+n_3}Y_3}) \circ \left(\psi^{n_2}_{b_3}\right)^{-1} \circ \widetilde{E}^{n_2}D_2 \circ \left(\psi^{n_2}_{E^{n_3}b_4} \times 1_I\right).$$

(3) *Suppose the following data are given:*

$$\beta_k \in [E^{n_k}Y_k, Y_{k-1}] \ (k=1,2), \quad \beta_\ell \in [Y_\ell, Y_{\ell-1}] \ (\ell = 3,4,5),$$
$$\beta_1 \circ E^{n_1}\beta_2 = 0, \quad \beta_2 \circ E^{n_2}\beta_3 = 0, \quad \beta_3 \circ \beta_4 \circ \beta_5 = 0.$$

If $b_k \in \beta_k$ $(1 \le k \le 5)$ *and* $(b_1, b_2, b_3, b_4 \circ b_5; B_1, B_2, B_3)_{n_1,n_2}$ *is admissible, then* $(B_1, B_2 \circ CE^{n_2}b_4, B_3)_{n_1,n_2}$ *is admissible and*

$$\{b_1, b_2, b_3, b_4 \circ b_5; B_1, B_2, B_3\}^{(k)}_{n_1,n_2}$$
$$\subset \{b_1, b_2, b_3 \circ b_4, b_5; B_1, B_2 \circ CE^{n_2}b_4, B_3\}^{(k)}_{n_1,n_2} \quad (k = 0,1,2,3),$$
$$\{b_1, b_2, b_3, b_4 \circ b_5; B_1, B_2, B_3; D_2\}^{(1)}_{n_1,n_2}$$
$$\subset \{b_1, b_2, b_3 \circ b_4, b_5; B_1, B_2 \circ CE^{n_2}b_4, B_3; D''_2\}^{(1)}_{n_1,n_2},$$

where $D_2 : i_{b_2} \circ [b_2, B_2, E^{n_2}b_3] \simeq (b_2, B_2, E^{n_2}b_3) \circ q_{E^{n_2}b_3}$ *and* $D''_2 = D_2 \circ ((1_{E^{n_2}Y_2} \cup CE^{n_2}b_4) \times 1_I)$.

Proof. We give a proof of the second containment of (2). Proofs of others are similar or easy. Let

$$\big[b_1 \circ E^{n_1}b_2, \underline{K_1 \circ CE^{n_1+n_2}q_{E^{n_3}b_4}}_{(1_{b_1 \circ E^{n_1}b_2}, \widetilde{E}^{n_1+n_2}D_2)}, E^{n_1+n_2}[b_3, B_2, E^{n_3}b_4]\big]$$
$$\circ \big(E^{n_1+n_2}[b_3, B_2, E^{n_3}b_4], \widetilde{E}^{n_1+n_2}K_2, E^{n_1+n_2}(E^{n_3}b_4, \widetilde{E}^{n_3}B_3, E^{n_3}b_5)\big) =: f$$
$$: EE^{n_1}E^{n_2}EE^{n_3}Y_5 \to Y_0$$

be any element of $\{b_1 \circ E^{n_1}b_2, b_3, b_4, b_5; B_1, B_2, B_3; D_2\}^{(1)}_{n_1+n_2,n_3}$, where

$$K_1 : CE^{n_1}E^{n_2}EE^{n_3}Y_4 \to Y_0, \quad K_2 : CEE^{n_3}Y_5 \to Y_2$$

are respectively null homotopies of

$$[b_1 \circ E^{n_1}b_2, B_1, E^{n_2+n_3}b_3] \circ (\psi^{n_1+n_2}_{b_3})^{-1} \circ E^{n_1+n_2}(b_3, B_2, E^{n_3}b_4),$$
$$[b_3, B_2, E^{n_3}b_4] \circ (E^{n_3}b_4, \widetilde{E}^{n_3}B_3, E^{n_3}b_5).$$

We define

$$K'_1 = K_1 \circ CE^{n_1}(1_{E^{n_3}Y_4} \wedge \tau(\mathrm{S}^{n_2}, \mathrm{S}^1))$$
$$: [b_1, B_1, E^{n_1}(b_2E^{n_2}b_3)] \circ (\psi^{n_1}_{b_2E^{n_2}b_3})^{-1} \circ E^{n_1}(b_2E^{n_2}b_3, b_2\widetilde{E}^{n_2}B_2, E^{n_2+n_3}b_4)$$
$$\simeq *,$$
$$K'_2 = b_2 \circ \widetilde{E}^{n_2}K_2 \circ C(1_{E^{n_3}Y_5} \wedge \tau(\mathrm{S}^{n_2}, \mathrm{S}^1))$$
$$: [b_2E^{n_2}b_3, b_2\widetilde{E}^{n_2}B_2, E^{n_2+n_3}b_4] \circ (E^{n_2+n_3}b_4, \widetilde{E}^{n_2+n_3}B_3, E^{n_2+n_3}b_5) \simeq *.$$

We define an element of $\{b_1, b_2 \circ E^{n_2}b_3, b_4, b_5; B_1, b_2 \circ \widetilde{E}^{n_2}B_2, B_3; D'_2\}^{(1)}_{n_1,n_2+n_3}$ as follows:

$$\big[b_1, \underline{K'_1 \circ CE^{n_1}q_{E^{n_2+n_3}b_4}}_{(1_{b_1}, \widetilde{E}^{n_1}D'_2)}, E^{n_1}[b_2 \circ E^{n_2}b_3, b_2 \circ \widetilde{E}^{n_2}B_2, E^{n_2+n_3}b_4]\big]$$

$$\circ \big(E^{n_1}[b_2 \circ E^{n_2}b_3, b_2 \circ \widetilde{E}^{n_2}B_2, E^{n_2+n_3}b_4], \widetilde{E}^{n_1}K'_2,$$
$$E^{n_1}(E^{n_2+n_3}b_4, \widetilde{E}^{n_2+n_3}B_3, E^{n_2+n_3}b_5)\big) =: g \ : \ EE^{n_1}EE^{n_2}E^{n_3}Y_5 \to Y_0.$$

By routine calculations, we can see $f = g \circ (1_{E^{n_3}Y_5} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2}) \wedge 1_{\mathrm{S}^{n_1} \wedge \mathrm{S}^1}) \simeq (-1)^{n_2}g$. Hence

$$\{b_1 \circ E^{n_1}b_2, b_3, b_4, b_5; B_1, B_2, B_3; D_2\}^{(1)}_{n_1+n_2,n_3}$$
$$\subset (-1)^{n_2}\{b_1, b_2 \circ E^{n_2}b_3, b_4, b_5; B_1, b_2 \circ \widetilde{E}^{n_2}B_2, B_3; D'_2\}^{(1)}_{n_1,n_2+n_3}$$

as desired. □

6. Tertiary compositions

Suppose that (4.1) is admissible and $a_i \in \alpha_i$ $(1 \leq i \leq 4)$. Then we define

$$\{a_1,\, a_2,\, a_3,\, a_4\}^{(k)}_{n_1,n_2} = \bigcup \{A_1, A_2, A_3\}^{(k)}_{n_1,n_2} \quad (k = 0, 1, 2, 3) \tag{6.1}$$

where the union is taken over A_1, A_2, A_3 with $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)_{n_1,n_2}$ admissible. From (2.7), Lemma 2.9 and Lemma 2.10, it follows that (6.1) for $k = 2, 3$ depends only on α_i so that we denote it by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(k)}_{n_1,n_2}$. We define

$$\{\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4\}^{(\ell)}_{n_1,n_2} = \bigcup \{a_1,\, a_2,\, a_3,\, a_4\}^{(\ell)}_{n_1,n_2} \quad (\ell = 0, 1),$$

where the union is taken over $a_i \in \alpha_i$ $(1 \leq i \leq 4)$. The one for $\ell = 1$ was called the second derived composition in [**13**]. Now we obtain the following four subsets of $[E^{n_1+n_2+2}X_4, X_0]$:

$$\begin{aligned}\{\alpha_1,\, \alpha_2,\, \alpha_3, \alpha_4\}^{(0)}_{n_1,n_2} &\subset \{\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4\}^{(1)}_{n_1,n_2}\\ &\subset \{\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4\}^{(2)}_{n_1,n_2} \subset \{\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4\}^{(3)}_{n_1,n_2}.\end{aligned} \tag{6.2}$$

These are called *tertiary compositions.* If $0 \leq m_i \leq n_i$ $(i = 1, 2)$ and $k = 0, 1, 2, 3$, then, by Proposition 5.3, we have

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(k)}_{n_1,n_2}$$
$$\subset (-1)^{n_1-m_1}\{\alpha_1, E^{n_1-m_1}\alpha_2, E^{n_1+n_2-m_1-m_2}\alpha_3, E^{n_1+n_2-m_1-m_2}\alpha_4\}^{(k)}_{m_1,m_2}.$$

We omit the subscripts n_1, n_2 when $n_1 = n_2 = 0$. For example, we abbreviate $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(k)}_{0,0}$ to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(k)}$. Note that $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(2)}$ is contained in the Cohen's 4-fold Toda bracket $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{C}$ (see B.6).

It seems that, in some places of [**13**, §6], Ôguchi did not distinguish between the following three sets:

$$\{a_1, a_2, a_3, a_4\}^{(1)},\ \{a_1, [a_2, A_2, a_3], (a_3, A_3, a_4)\},$$
$$\{[a_1, A_1, a_2], (a_2, A_2, a_3), -Ea_4\}.$$

As a consequence, his proofs of Proposition (6.12) and some other assertions in [**13**] are incomplete.

We can not abbreviate $\{A_1, A_2, A_3\}^{(k)}_{n_1,n_2}$ to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(k)}_{n_1,n_2}$, while Mimura [**10**] took this incorrect manner for $k = 3$ (see Proposition 7.4 below).

By Example 5.9 and Example 5.10, we have

Example 6.1. $\{\eta_3, \nu', 8\iota_5, \nu_5\}^{(k)}_{1,1} = \mu_3 + \mathbb{Z}_2\{\eta_3\varepsilon_4\}$ $(k = 0, 1, 2)$ *and*

$$\begin{aligned}\{\pm\kappa_7\} + \mathbb{Z}_2\{4\sigma'\sigma_{14}\} \subset \{\nu_7, \eta_9, 2\iota_9, \overline{\nu}_9\}^{(0)}_{1,1} &= \{\nu_7, \eta_9, 2\iota_9, \overline{\nu}_9\}^{(1)}_{1,1} \\ &= \{\nu_7, \eta_9, 2\iota_9, \overline{\nu}_9\}^{(2)}_{1,1} \subset \{\pm\kappa_7,\ \pm(\kappa_7 + 2\sigma'\sigma_{14})\} + \mathbb{Z}_2\{4\sigma'\sigma_{14}\}.\end{aligned}$$

We give a revision of [**10**, Proposition 2.9 (0)]. We omit details.

Proposition 6.2. *If one of the three conditions*

$$\alpha_1 = 0 \text{ and } \{\alpha_2, \alpha_3, \alpha_4\}_{n_2} \ni 0,$$
$$\{\alpha_1, \alpha_2, \alpha_3\}_{n_1} \ni 0 \text{ and } \alpha_4 = 0,$$
$$\alpha_2 = 0 \text{ or } \alpha_3 = 0$$

is satisfied, then (4.1) is admissible and $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(1)}_{n_1,n_2} \ni 0$.

By Proposition 5.4, we have the following generalization of [**13**, (6.11)].

Proposition 6.3. *If (4.1) is admissible, then*

$$E\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(k)}_{n_1,n_2} \subset -\{E\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(k)}_{n_1+1,n_2} \quad (k = 0, 1, 2, 3).$$

The following result is a revision of [**13**, Proposition (6.12)] and [**10**, Proposition 2.12].

Proposition 6.4. *With the hypotheses of Proposition 5.11, the set*

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(3)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5 \cap (-1)^{n_1+n_2}\alpha_1 \circ E^{n_1}\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}^{(3)}_{n_2}$$

contains

$$\begin{aligned}\{\alpha_1,\alpha_2, \alpha_3, \alpha_4\}^{(0)}_{n_1,n_2} &\circ E^{n_1+n_2+2}\alpha_5 \cap (-1)^{n_1+n_2}\alpha_1 \circ E^{n_1}\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}^{(0)}_{n_2} \\ &\supset \{a_1, a_2, a_3, a_4; A_1, A_2, A_3\}^{(1)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5 \\ &= (-1)^{n_1+n_2}\alpha_1 \circ E^{n_1}\{a_2, a_3, a_4, a_5; A_2, A_3, A_4\}^{(1)}_{n_2},\end{aligned}$$

and the following two sets are not the same in general:

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(3)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5,\ (-1)^{n_1+n_2}\alpha_1 \circ E^{n_1}\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}^{(3)}_{n_2}.$$

Proof. Containments follow from (6.2) and Proposition 5.11(2). For the last part of the assertion, consider the following sequence:

$$\begin{tikzcd}
\mathrm{S}^6 & \mathrm{S}^6 \arrow[l, "2\iota_6"'] & \mathrm{S}^7 \arrow[l, "0_6"'] & \mathrm{S}^8 \arrow[l, "\eta_7"'] & \mathrm{S}^{13} \arrow[l, "0^5_8"'] & \mathrm{S}^{14} \arrow[l, "0_{13}"']
\end{tikzcd}$$

Null quadruples $(2\iota_6, 0_6, \eta_7, 0^5_8)$ and $(0_6, \eta_7, 0^5_8, 0_{13})$ are admissible. Note that $\{2\iota_6, 0_6, \eta_7, 0^5_8\}^{(k)} \circ E^2 0_{13} = \{0\}$ $(k = 0, 1, 2, 3)$. We shall show

$$\{0_6, \eta_7, 0^5_8, 0_{13}\}^{(0)} \ni -2\nu_6\sigma_9.$$

If this holds, then $2\iota_6 \circ \{0_6, \eta_7, 0^5_8, 0_{13}\}^{(3)} \ni 2\iota_6 \circ (-2\nu_6\sigma_9) = 4\nu_6\sigma_9 \neq 0$ by [**19**, Theorem 7.3] and so the last part of the assertion follows.

In the rest of the proof, we prove the above containment.

Put $A_1 = \nu_6 \circ \pi : C\,\mathrm{S}^8 \overset{\pi}{\to} E\,\mathrm{S}^8 \overset{\nu_6}{\to} \mathrm{S}^6$, $A_2 = * : C\,\mathrm{S}^{13} \to \mathrm{S}^7$, and $A_3 = E\sigma' \circ \pi : C\,\mathrm{S}^{14} \overset{\pi}{\to} E\,\mathrm{S}^{14} \overset{E\sigma'}{\to} \mathrm{S}^8$, where π are the quotient maps. Then $(*_6, \eta_7, *^5_8, *_{13}; A_1, A_2, A_3)$ is admissible.

By definitions, we have $(\eta_7, A_2, *^5_8) = * : E\,\mathrm{S}^{13} \to \mathrm{S}^7 \cup_{\eta_7} C\,\mathrm{S}^8$. Put $B_1 = * : [*_6, A_1, \eta_7] \circ (\eta_7, A_2, *^5_8) \simeq *$ and take $B_2 : [\eta_7, A_2, *^5_8] \circ (*^5_8, A_3, *_{13}) \simeq *$ arbitrarily. See the following homotopy commutative diagram:

$$\begin{tikzcd}
\mathrm{S}^6 \arrow[d, equal] & \mathrm{S}^7 \arrow[l, "*_6"'] \arrow[d, "i_{\eta_7}"] & \mathrm{S}^8 \vee \mathrm{S}^{14} \arrow[l, "{[\eta_7, A_2, *^5_8]}"'] \arrow[d, "q_{*^5_8}"] & E\,\mathrm{S}^{14} \arrow[l, "{(*^5_8, A_3, *_{13})}"'] \arrow[d, equal] \\
\mathrm{S}^6 & \mathrm{S}^7 \cup_{\eta_7} C\,\mathrm{S}^8 \arrow[l, "{[*_6, A_1, \eta_7]}"] & E\,\mathrm{S}^{13} \arrow[l, "{(\eta_7, A_2, *^5_8)}"] & E\,\mathrm{S}^{14} \arrow[l, "*_{14}"]
\end{tikzcd}$$

Let $\widetilde{G} : (\mathrm{S}^8 \vee \mathrm{S}^{14}) \times I \to \mathrm{S}^7 \cup_{\eta_7} C\,\mathrm{S}^8$ be the typical homotopy for $(\eta_7, *^5_8; A_2)$. Then $\widetilde{G}(x, t) = x \wedge t$ for $x \in \mathrm{S}^8$, $\widetilde{G}(\mathrm{S}^{14} \times I) = *$, and the map

$$\underline{B_1 \circ Cq_{*^5_8}}_{(1_{*_6}, \widetilde{G})} : C(\mathrm{S}^8 \vee \mathrm{S}^{14}) = C\,\mathrm{S}^8 \vee C\,\mathrm{S}^{14} \to \mathrm{S}^6$$

satisfies $\underline{B_1 \circ Cq_{*^5_8}}_{(1_{*_6}, \widetilde{G})}(C\,\mathrm{S}^{14}) = *$ and

$$\underline{B_1 \circ Cq_{*^5_8}}_{(1_{*_6}, \widetilde{G})}(x \wedge t) = \begin{cases} \nu_6(x \wedge \overline{3t - 1}) & 1/3 \leq t \leq 2/3 \\ * & \text{otherwise} \end{cases} \quad (x \in \mathrm{S}^8).$$

Hence $f := [*_6, \underline{B_1 \circ Cq_{*^5_8}}_{(1_{*_6}, \widetilde{G})}, [\eta_7, A_2, *^5_8]] \circ ([\eta_7, A_2, *^5_8], B_2, (*^5_8, A_3, *_{13}))$ which is a map from $E^2\,\mathrm{S}^{14}$ to S^6 is given by

$$f(x \wedge \overline{s} \wedge \overline{t}) = \begin{cases} \nu_6\big(E\sigma'(x \wedge \overline{2s - 1}) \wedge \overline{2 - 6t}\big) & 1/2 \leq s \leq 1,\ 1/6 \leq t \leq 1/3 \\ * & \text{otherwise} \end{cases}.$$

Thus $f \simeq -\nu_6 \circ E^2\sigma' \simeq -2\nu_6\sigma_9$ so that $\{A_1, A_2, A_3\}^{(0)} = \{-2\nu_6\sigma_9\}$ by 5.6. This proves the desired containment. □

By Proposition 5.11(1) and Proposition 5.13, we have the following result which contains revisions of [**13**, (6.9) except (iii)] and [**10**, Proposition 2.9 except (0)].

Proposition 6.5. *With the hypotheses of Proposition 5.13, we have*

$$\alpha_0 \circ \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(k)}_{n_1,n_2} \subset \{\alpha_0 \circ \alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(k)}_{n_1,n_2},$$
$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(k)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5 \subset \{\alpha_1, \alpha_2, \alpha_3, \alpha_4 \circ \alpha_5\}^{(k)}_{n_1,n_2},$$
$$\{\beta_1 \circ E^{n_1}\beta_2, \beta_3, \beta_4, \beta_5\}^{(k)}_{n_1+n_2,n_3} \subset (-1)^{n_2}\{\beta_1, \beta_2 \circ E^{n_2}\beta_3, \beta_4, \beta_5\}^{(k)}_{n_1,n_2+n_3},$$
$$\{\beta_1, \beta_2, \beta_3, \beta_4 \circ \beta_5\}^{(k)}_{n_1,n_2} \subset \{\beta_1, \beta_2, \beta_3 \circ \beta_4, \beta_5\}^{(k)}_{n_1,n_2}$$

for $k = 0, 1, 2, 3$. *If moreover* $\alpha_4 \circ \alpha_5 = 0$, *then*

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}^{(1)}_{n_1,n_2} \circ E^{n_1+n_2+2}\alpha_5 \subset (-1)^{n_2+1} \bigcup \{\alpha_1, \alpha_2, E^{n_2}\lambda\}_{n_1}$$

where the union is taken over $\lambda \in \{\alpha_3, \alpha_4, \alpha_5\}$ *such that* $\alpha_2 \circ E^{n_2}\lambda = 0$.

In the rest of this section, we revise [**13**, Proposition (6.9)(iii), Proposition (6.13)(i)] (cf. the last equality in [**17**]). We suppose that the following data are given.

$$\begin{aligned} &\alpha_i \in [E^{n_i}X_i, X_{i-1}]\ (i = 1, 2), \quad \alpha_i \in [X_i, X_{i-1}]\ (i = 3, 4, 5), \\ &\alpha_1 \circ E^{n_1}\alpha_2 = 0, \quad \alpha_4 \circ \alpha_5 = 0, \quad a_i \in \alpha_i\ (1 \leq i \leq 5). \end{aligned} \tag{6.3}$$

To revise Proposition (6.9)(iii) of [**13**], we suppose (6.3) and

$$\alpha_2 \circ E^{n_2}(\alpha_3 \circ \alpha_4) = 0. \tag{6.4}$$

Let

$$A_1 : a_1 \circ E^{n_1}a_2 \simeq *, \quad A_2 : a_2 \circ E^{n_2}(a_3 \circ a_4) \simeq *, \quad A_3 : a_4 \circ a_5 \simeq *. \tag{6.5}$$

Lemma 6.6. $(a_1, a_2 \circ E^{n_2}a_3, a_4, a_5; A_1 \circ CE^{n_1+n_2}a_3, A_2, A_3)_{n_1,n_2}$ *is admissible if and only if* $(a_1, a_2, a_3 \circ a_4, a_5; A_1, A_2, a_3 \circ A_3)_{n_1,n_2}$ *is admissible.*

Proof. This follows from Lemma 2.2(4),(5). □

Lemma 6.7. *If* $(a_1, a_2 \circ E^{n_2}a_3, a_4, a_5; A_1 \circ CE^{n_1+n_2}a_3, A_2, A_3)_{n_1,n_2}$ *is admissible, then*

$$\begin{aligned} \{a_1, a_2 \circ E^{n_2}a_3, a_4, a_5;& A_1 \circ CE^{n_1+n_2}a_3, A_2, A_3; \widetilde{G}\}^{(1)}_{n_1,n_2} \\ &= \{a_1, a_2, a_3 \circ a_4, a_5; A_1, A_2, a_3 \circ A_3; \widetilde{G}'\}^{(1)}_{n_1,n_2} \end{aligned}$$

where $\widetilde{G}$ *and* $\widetilde{G}'$ *are the typical homotopies for* $(a_2 \circ E^{n_2}a_3, E^{n_2}a_4; A_2)$ *and* $(a_2, E^{n_2}(a_3 \circ a_4); A_2)$, *respectively.*

Proof. From the definitions of typical homotopies $\widetilde{G}$ and $\widetilde{G}'$, the following square is strictly commutative:

$$\begin{tikzcd}
\big(E^{n_2}X_3 \cup_{E^{n_2}a_4} CE^{n_2}X_4\big) \times I \arrow[r, "\widetilde{G}"] \arrow[d, "(E^{n_2}a_3 \cup 1_{CE^{n_2}X_4}) \times 1_I"'] & X_1 \cup_{a_2 \circ E^{n_2}a_3} CE^{n_2}X_3 \arrow[d, "1_{X_1} \cup CE^{n_2}a_3"] \\
\big(E^{n_2}X_2 \cup_{E^{n_2}(a_3 \circ a_4)} CE^{n_2}X_4\big) \times I \arrow[r, "\widetilde{G'}"] & X_1 \cup_{a_2} CE^{n_2}X_2
\end{tikzcd}$$

Hence $E^{n_1}(1_{X_1} \cup CE^{n_2}a_3) \circ \widetilde{E}^{n_1}\widetilde{G} = \widetilde{E}^{n_1}\widetilde{G}' \circ \big(E^{n_1}(E^{n_2}a_3 \cup 1_{CE^{n_2}X_4}) \times 1_I\big)$. On the other hand, by Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}
&[a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1} \circ E^{n_1}\big(a_2, A_2, E^{n_2}(a_3 \circ a_4)\big) \\
&\quad = \big[a_1, A_1 \circ CE^{n_1+n_2}a_3, E^{n_1}(a_2 \circ E^{n_2}a_3)\big] \circ (\psi^{n_1}_{a_2 \circ E^{n_2}a_3})^{-1} \\
&\qquad\qquad\qquad\qquad \circ E^{n_1}(a_2 \circ E^{n_2}a_3, A_2, E^{n_2}a_4), \\
&[a_2, A_2, E^{n_2}(a_3 \circ a_4)] \circ \big(E^{n_2}(a_3 \circ a_4), \widetilde{E}^{n_2}(a_3 \circ A_3), E^{n_2}a_5\big) \\
&\quad = [a_2 \circ E^{n_2}a_3, A_2, E^{n_2}a_4] \circ (E^{n_2}a_4, \widetilde{E}^{n_2}A_3, E^{n_2}a_5).
\end{aligned}$$

Let B_1 and B_2 be any null homotopies of the above maps, respectively. Then routine calculations show

$$\begin{aligned}
&\big[a_1, \underline{B_1 \circ CE^{n_1}q_{E^{n_2}a_4}}_{(1_{a_1}, \widetilde{E}^{n_1}\widetilde{G})}, E^{n_1}[a_2 \circ E^{n_2}a_3, A_2, E^{n_2}a_4]\big] \\
&\quad \circ \big(E^{n_1}[a_2 \circ E^{n_2}a_3, A_2, E^{n_2}a_4], \widetilde{E}^{n_1}B_2, E^{n_1}(E^{n_2}a_4, \widetilde{E}^{n_2}A_3, E^{n_2}a_5)\big) \\
&= \big[a_1, \underline{B_1 \circ CE^{n_1}q_{E^{n_2}(a_3 \circ a_4)}}_{(1_{a_1}, \widetilde{E}^{n_1}\widetilde{G}')}, E^{n_1}[a_2, A_2, E^{n_2}(a_3 \circ a_4)]\big] \\
&\quad \circ \big(E^{n_1}[a_2, A_2, E^{n_2}(a_3 \circ a_4)], \widetilde{E}^{n_1}B_2, \\
&\qquad\qquad\qquad E^{n_1}(E^{n_2}(a_3 \circ a_4), \widetilde{E}^{n_2}(a_3 \circ A_3), E^{n_2}a_5)\big).
\end{aligned}$$

Therefore we obtain the assertion. □

Proposition 6.8. *If* $(a_1, a_2, a_3 \circ a_4, a_5; A_1, A_2, a_3 \circ A_3)_{n_1,n_2}$ *is admissible, then*

$$\begin{aligned}
&\{\alpha_1, \alpha_2 \circ E^{n_2}\alpha_3, \alpha_4, \alpha_5\}^{(0)}_{n_1,n_2} \\
&\qquad \text{and } \{\alpha_1, \alpha_2, \alpha_3 \circ \alpha_4, \alpha_5\}^{(0)}_{n_1,n_2} \text{ have a common element.}
\end{aligned} \tag{6.6}$$

Proof. This follows from Lemma 6.6 and Lemma 6.7. □

A revision of [**13**, (6.9)(iii)] is

Corollary 6.9. *We have (6.6) if (6.3) and (6.4) satisfy one of the following three conditions.*

$$\{\alpha_1, \alpha_2, E^{n_2}(\alpha_3 \circ \alpha_4)\}_{n_1} \ni 0, \quad \{\alpha_2, \alpha_3 \circ \alpha_4, \alpha_5\}_{n_2} = \{0\}, \tag{6.7}$$

$$(6.8)\qquad \begin{cases}\{\alpha_1,\alpha_2,E^{n_2}(\alpha_3\circ\alpha_4)\}_{n_1}\ni 0, \quad \{\alpha_2,\alpha_3\circ\alpha_4,\alpha_5\}_{n_2}\ni 0,\\ G_1+G_2=[E^{n_2+1}X_4,X_1], \quad \alpha_2\circ E^{n_2}[EX_5,X_2]=\{0\},\end{cases}$$

$$(6.9)\qquad \begin{cases}\{\alpha_1,\alpha_2\circ E^{n_2}\alpha_3,E^{n_2}\alpha_4\}_{n_1}\ni 0, \quad \{\alpha_2\circ E^{n_2}\alpha_3,\alpha_4,\alpha_5\}_{n_2}\ni 0,\\ G''_1+G''_2=[E^{n_2+1}X_4,X_1],\\ [E^{n_1+n_2+1}X_3,X_0]\circ E^{n_1+n_2+1}\alpha_4=\{0\},\end{cases}$$

where G_1,G_2 *and* G''_1,G''_2 *are ones defined in Proposition 4.1 respectively for* $(\alpha_1,\alpha_2,\alpha_3\circ\alpha_4,\alpha_5)_{n_1,n_2}$ *and* $(\alpha_1,\alpha_2\circ E^{n_2}\alpha_3,\alpha_4,\alpha_5)_{n_1,n_2}$.

Proof. Under (6.7), there is (6.5) with $(a_1,a_2,a_3\circ a_4,a_5;A_1,A_2,a_3\circ A_3)_{n_1,n_2}$ admissible.

Suppose (6.8). Then there exist A_1,A_2 and $A'_3:a_3\circ a_4\circ a_5\simeq *$ such that $(a_1,a_2,a_3\circ a_4,a_5;A_1,A_2,A'_3)_{n_1,n_2}$ is admissible. We have $a_3\circ A_3\simeq A'_3\dotplus d(A'_3,a_3\circ A_3)$ *rel* X_5. From (2.6) and Corollary 2.13, we have

$$\begin{aligned}&[a_2,A_2,E^{n_2}(a_3\circ a_4)]\circ\big(E^{n_2}(a_3\circ a_4),\widetilde{E}^{n_2}(a_3\circ A_3),E^{n_2}a_5\big)\\ &\quad\simeq[a_2,A_2,E^{n_2}(a_3\circ a_4)]\circ\big(E^{n_2}(a_3\circ a_4),\widetilde{E}^{n_2}A'_3,E^{n_2}a_5\big)\\ &\qquad\qquad+a_2\circ(-1)^{n_2}E^{n_2}d(A'_3,a_3\circ A_3)\\ &\quad\simeq * \qquad \text{(by the assumption).}\end{aligned}$$

Hence $(a_1,a_2,a_3\circ a_4,a_5;A_1,A_2,a_3\circ A_3)_{n_1,n_2}$ is admissible.

By the similar argument, if (6.9) holds, then there exists (6.5) such that $(a_1,a_2\circ E^{n_2}a_3,a_4,a_5;A_1\circ CE^{n_1+n_2}a_3,A_2,A_3)_{n_1,n_2}$ is admissible. This completes the proof. □

In order to revise Proposition (6.13)(i) of [**13**], suppose (6.3) and

$$(6.10)\qquad \alpha_3\circ\alpha_4=0.$$

Let

$$(6.11)\qquad A_1:a_1\circ E^{n_1}a_2\simeq *,\quad A'_2:a_3\circ a_4\simeq *,\quad A_3:a_4\circ a_5\simeq *.$$

By Lemma 6.6, $(a_1,a_2\circ E^{n_2}a_3,a_4,a_5;A_1\circ CE^{n_1+n_2}a_3,a_2\circ\widetilde{E}^{n_2}A'_2,A_3)_{n_1,n_2}$ is admissible if and only if $(a_1,a_2,a_3\circ a_4,a_5;A_1,a_2\circ\widetilde{E}^{n_2}A'_2,a_3\circ A_3)_{n_1,n_2}$ is admissible.

Proposition 6.10. *If* $(a_1,a_2,a_3\circ a_4,a_5;A_1,a_2\circ\widetilde{E}^{n_2}A'_2,a_3\circ A_3)_{n_1,n_2}$ *is admissible, then*

$$\begin{aligned}&[a_1,A_1,E^{n_1}a_2]\circ\Big(E^{n_1}a_2,\widetilde{E}^{n_1}B_2,\\ &\qquad E^{n_1}E^{n_2}\big([a_3,A'_2,a_4]\circ(a_4,A_3,a_5)\big)\circ E^{n_1}\big(1_{X_5}\wedge\tau(\mathrm{S}^{n_2},\mathrm{S}^1)\big)\Big)\\ &\simeq\big[a_1,\underline{B_1\circ CE^{n_1}q_{E^{n_2}(a_3\circ a_4)}}_{(1_{a_1},\widetilde{E}^{n_1}\widetilde{G}')},E^{n_1}[a_2,a_2\circ\widetilde{E}^{n_2}A'_2,E^{n_2}(a_3\circ a_4)]\big]\end{aligned}$$

$$\circ \big(E^{n_1}[a_2, a_2 \circ \widetilde{E}^{n_2}A_2', E^{n_2}(a_3 \circ a_4)], \widetilde{E}^{n_1}B_2,$$
$$E^{n_1}(E^{n_2}(a_3 \circ a_4), \widetilde{E}^{n_2}(a_3 \circ A_3), E^{n_2}a_5)\big),$$

where $\widetilde{G}' : (E^{n_2}X_2 \cup_{E^{n_2}(a_3 \circ a_4)} CE^{n_2}X_4) \times I \to X_1 \cup_{a_2} CE^{n_2}X_2$ *is the typical homotopy for* $(a_2, E^{n_2}(a_3 \circ a_4); a_2 \circ \widetilde{E}^{n_2}A_2')$, *and*

$$B_1 : [a_1, A_1, E^{n_1}a_2] \circ (\psi^{n_1}_{a_2})^{-1} \circ E^{n_1}(a_2, a_2 \circ \widetilde{E}^{n_2}A_2', E^{n_2}(a_3 \circ a_4)) \simeq *,$$
$$B_2 : [a_2, a_2 \circ \widetilde{E}^{n_2}A_2', E^{n_2}(a_3 \circ a_4)] \circ (E^{n_2}(a_3 \circ a_4), \widetilde{E}^{n_2}(a_3 \circ A_3), E^{n_2}a_5) \simeq *.$$

Before proving this proposition, we give two corollaries which are revisions of Proposition (6.13)(i) of [**13**] (cf. the last equality in [**17**]).

Corollary 6.11. *If* $(a_1, a_2, a_3 \circ a_4, a_5; A_1, a_2 \circ \widetilde{E}^{n_2}A_2', a_3 \circ A_3)_{n_1,n_2}$ *is admissible, then the following two sets have a common element:*

$$(-1)^{n_2}\{a_1, a_2, E^{n_2}([a_3, A_2', a_4] \circ (a_4, A_3, a_5))\}_{n_1},$$
$$\{a_1, a_2, a_3 \circ a_4, a_5; A_1, a_2 \circ \widetilde{E}^{n_2}A_2', a_3 \circ A_3\}^{(0)}_{n_1,n_2}.$$

Proof. This follows from Proposition 6.10. □

Corollary 6.12. *If (6.3) and (6.10) satisfy* $\alpha_2 \circ E^{n_2}\{a_3, a_4, a_5\} \ni 0$ *and* $\alpha_1 \circ E^{n_1}[E^{n_2+1}X_4, X_1] = \{0\}$, *then there exists* $\lambda \in \{\alpha_3, \alpha_4, \alpha_5\}$ *such that* $\alpha_2 \circ E^{n_2}\lambda = 0$ *and the following three sets have a common element*

$$(-1)^{n_2}\{\alpha_1, \alpha_2, E^{n_2}\lambda\}_{n_1}, \quad \{\alpha_1, \alpha_2, \alpha_3 \circ \alpha_4, \alpha_5\}^{(0)}_{n_1,n_2},$$
$$\{\alpha_1, \alpha_2 \circ E^{n_2}\alpha_3, \alpha_4, \alpha_5\}^{(0)}_{n_1,n_2}.$$

Proof. First we show that under the assumptions there exists (6.11) such that $(a_1, a_2 \circ E^{n_2}a_3, a_4, a_5; A_1 \circ CE^{n_1+n_2}a_3, a_2 \circ \widetilde{E}^{n_2}A_2', A_3)_{n_1,n_2}$ is admissible.

By the assumption $a_2 \circ E^{n_2}\{a_3, a_4, a_5\} \ni 0$, there exist $A_2' : a_3 \circ a_4 \simeq *$ and $A_3 : a_4 \circ a_5 \simeq *$ such that $a_2 \circ E^{n_2}\big([a_3, A_2', a_4] \circ (a_4, A_3, a_5)\big) \simeq *$. Since

$$a_2 \circ E^{n_2}\big([a_3, A_2', a_4] \circ (a_4, A_3, a_5)\big) = a_2 \circ E^{n_2}[a_3, A_2', a_4] \circ E^{n_2}(a_4, A_3, a_5)$$
$$= a_2 \circ [E^{n_2}a_3, \widetilde{E}^{n_2}A_2', E^{n_2}a_4] \circ (E^{n_2}a_4, \widetilde{E}^{n_2}A_3, E^{n_2}a_5) \circ (1_{X_5} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2}))$$
$$\text{(by 2.4)}$$
$$= [a_2 \circ E^{n_2}a_3, a_2 \circ \widetilde{E}^{n_2}A_2', E^{n_2}a_4] \circ (E^{n_2}a_4, \widetilde{E}^{n_2}A_3, E^{n_2}a_5)$$
$$\circ (1_{X_5} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2})),$$

we have

$$(6.12) \qquad [a_2 \circ E^{n_2}a_3, a_2 \circ \widetilde{E}^{n_2}A_2', E^{n_2}a_4] \circ (E^{n_2}a_4, \widetilde{E}^{n_2}A_3, E^{n_2}a_5) \simeq *.$$

Since $a_3 \circ a_4 \simeq *$, we have $\{a_1, a_2, E^{n_2}(a_3 \circ a_4)\}_{n_1} \ni 0$. Hence there exist $A_1 : a_1 \circ E^{n_1}a_2 \simeq *$ and $A_2'' : a_2 \circ E^{n_2}(a_3 \circ a_4) \simeq *$ such that

$$[a_1, A_1, E^{n_1}a_2] \circ (E^{n_1}a_2, \widetilde{E}^{n_1}A_2'', E^{n_1}E^{n_2}(a_3 \circ a_4)) \simeq *.$$

By the lemmas 2.2, 2.4, 2.8 and Corollary 2.13, we have

$$\begin{aligned}
* &\simeq [a_1, A_1, E^{n_1}a_2] \circ (E^{n_1}a_2, \widetilde{E}^{n_1}A_2'', E^{n_1}E^{n_2}(a_3 \circ a_4)) \\
&= [a_1, A_1, E^{n_1}a_2] \circ (1_{E^{n_1}X_1} \cup CE^{n_1+n_2}a_3) \\
&\qquad\qquad \circ (E^{n_1}a_2 \circ E^{n_1+n_2}a_3, \widetilde{E}^{n_1}A_2'', E^{n_1+n_2}a_4) \\
&= [a_1, A_1 \circ CE^{n_1+n_2}a_3, E^{n_1}(a_2 \circ E^{n_2}a_3)] \\
&\qquad\qquad \circ (E^{n_1}a_2 \circ E^{n_1+n_2}a_3, \widetilde{E}^{n_1}A_2'', E^{n_1+n_2}a_4) \\
&\simeq \big[a_1, A_1 \circ CE^{n_1+n_2}a_3, E^{n_1}(a_2 \circ E^{n_2}a_3)\big] \\
&\quad \circ \big(E^{n_1}(a_2 \circ E^{n_2}a_3), \widetilde{E}^{n_1}(a_2 \circ \widetilde{E}^{n_2}A_2' \dotplus d(a_2 \circ \widetilde{E}^{n_2}A_2', A_2'')), E^{n_1+n_2}a_4\big) \\
&\simeq \big[a_1, A_1 \circ CE^{n_1+n_2}a_3, E^{n_1}(a_2 \circ E^{n_2}a_3)\big] \\
&\qquad\qquad \circ \big(E^{n_1}(a_2 \circ E^{n_2}a_3), \widetilde{E}^{n_1}(a_2 \circ \widetilde{E}^{n_2}A_2'), E^{n_1+n_2}a_4\big) \\
&\qquad\qquad + a_1 \circ (-1)^{n_1}E^{n_1}d(a_2 \circ \widetilde{E}^{n_2}A_2', A_2'') \\
&\simeq \big[a_1, A_1 \circ CE^{n_1+n_2}a_3, E^{n_1}(a_2 \circ E^{n_2}a_3)\big] \\
&\qquad\qquad \circ \big(E^{n_1}(a_2 \circ E^{n_2}a_3), \widetilde{E}^{n_1}(a_2 \circ \widetilde{E}^{n_2}A_2'), E^{n_1+n_2}a_4\big) \\
&\qquad\qquad\qquad\qquad \text{(by the assumption } a_1 \circ E^{n_1}[E^{n_2+1}X_4, X_1] = \{0\}\text{)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\big[a_1, A_1 \circ CE^{n_1+n_2}a_3, E^{n_1}(a_2 \circ E^{n_2}a_3)\big] \\
&\qquad\qquad \circ \big(E^{n_1}(a_2 \circ E^{n_2}a_3), \widetilde{E}^{n_1}(a_2 \circ \widetilde{E}^{n_2}A_2'), E^{n_1+n_2}a_4\big) \simeq *.
\end{aligned}$$

Therefore we have obtained the desired (6.11) from (6.12).

Now the assertion follows from Lemma 6.7 and Corollary 6.11. □

Proof of Proposition 6.10. Consider the following diagrams:

$$\begin{tikzcd}
X_0 \arrow[d, "1_{X_0}"'] \arrow[dr, phantom, "1_{a_1}"] & E^{n_1}X_1 \arrow[l, "a_1"'] \arrow[d, "E^{n_1}1_{X_1}"] \\
X_0 & E^{n_1}X_1 \arrow[l, "a_1"']
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \arrow[d, "1_{X_1}"'] \arrow[dr, phantom, "1_{a_2}"] & E^{n_2}X_2 \arrow[l, "a_2"'] \arrow[d, "i_{E^{n_2}(a_3a_4)}"] \arrow[dr, phantom, "D_3"] & E^{n_2}EX_5 \arrow[l, "E^{n_2}(\overline{a_3}\circ\widetilde{a_5})"'] \arrow[d, "1_{X_5}\wedge\tau(\mathrm{S}^1{,}\mathrm{S}^{n_2})"] \\
X_1 & E^{n_2}X_2 \cup_{E^{n_2}(a_3\circ a_4)} CE^{n_2}X_4 \arrow[l, "\overline{a_2}"] & EE^{n_2}X_5 \arrow[l, "\widetilde{E^{n_2}a_5}"]
\end{tikzcd}$$

where we have used abbreviations

$$\overline{a_2} = [a_2, a_2 \circ \widetilde{E}^{n_2}A_2', E^{n_2}(a_3 \circ a_4)], \quad \overline{a_3} = [a_3, A_2', a_4], \quad \widetilde{a_5} = (a_4, A_3, a_5),$$

$$\widetilde{E^{n_2}a_5} = (E^{n_2}(a_3 \circ a_4), \widetilde{E}^{n_2}(a_3 \circ A_3), E^{n_2}a_5)$$

and the homotopy

$$\begin{aligned} D_3 : i_{E^{n_2}(a_3\circ a_4)} &\circ E^{n_2}\big([a_3, A_2', a_4] \circ (a_4, A_3, a_5)\big) \\ &\simeq \big(E^{n_2}(a_3 \circ a_4), \widetilde{E}^{n_2}(a_3 \circ A_3), E^{n_2}a_5\big) \circ \big(1_{X_5} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2})\big) \end{aligned}$$

which is defined by

$$\begin{aligned} &D_3(x_5 \wedge \overline{s} \wedge s_2, t) \\ &= \begin{cases} A_2'\big(a_5(x_5) \wedge (1-2s)(1-2t)\big) \wedge s_2 & 0 \le s \le \frac{1}{2},\ 0 \le t \le \frac{1}{2} \\ a_5(x_5) \wedge s_2 \wedge (1-2s)(2t-1) & 0 \le s \le \frac{1}{2},\ \frac{1}{2} \le t \le 1 \\ a_3\big(A_3(x_5 \wedge (2s-1))\big) \wedge s_2 & \frac{1}{2} \le s \le 1,\ 0 \le t \le 1 \end{cases} \\ &\qquad (x_5 \in X_5, s_2 \in \mathrm{S}^{n_2}, s, t \in I). \end{aligned}$$

We shall prove the following three relations.

$$\begin{aligned} &[a_2, a_2 \circ \widetilde{E}^{n_2}A_2', E^{n_2}(a_3 \circ a_4)] \circ \big(E^{n_2}(a_3 \circ a_4), \widetilde{E}^{n_2}(a_3 \circ A_3), E^{n_2}a_5\big) \\ &= a_2 \circ E^{n_2}\big([a_3, A_2', a_4] \circ (a_4, A_3, a_5)\big) \circ (1_{X_5} \wedge \tau(\mathrm{S}^{n_2}, \mathrm{S}^1)), \end{aligned} \tag{6.13}$$

$$\begin{aligned} A_1 \simeq &\underline{\underline{B_1 \circ CE^{n_1}q_{E^{n_2}(a_3\circ a_4)}}_{(1_{a_1}, \widetilde{E}^{n_1}\widetilde{G}')} \circ CE^{n_1}i_{E^{n_2}(a_3\circ a_4)}}_{(1_{a_1}, \widetilde{E}^{n_1}1_{a_2})} \\ &\qquad rel\ E^{n_1}E^{n_2}X_2, \end{aligned} \tag{6.14}$$

$$\begin{aligned} &B_2 \circ C(1_{X_5} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2})) \\ &\quad \simeq \underline{B_2 \circ C(1_{X_5} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2}))}_{(1_{a_2}, D_3)}\ rel\ E^{n_2}EX_5. \end{aligned} \tag{6.15}$$

If they hold, then the above diagram consists of null triples and the following quasi-map is a map.

$$\begin{aligned} &\big(1_{X_0}, 1_{X_1}, i_{E^{n_2}(a_3\circ a_4)}, 1_{X_5} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2}); 1_{a_1}, 1_{a_2}, D_3\big) : \\ &\quad \big(a_1, a_2, E^{n_2}([a_3, A_2', a_4] \circ (a_4, A_3, a_5)); A_1, B_2 \circ C(1_{X_5} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2}))\big)_{n_1} \\ &\quad \longrightarrow \big(a_1, [a_2, a_2 \circ \widetilde{E}^{n_2}A_2', E^{n_2}(a_3 \circ a_4)], \\ &\qquad\qquad (E^{n_2}(a_3 \circ a_4), \widetilde{E}^{n_2}(a_3 \circ A_3), E^{n_2}a_5); \\ &\qquad\qquad\qquad \underline{B_1 \circ CE^{n_1}q_{E^{n_2}(a_3\circ a_4)}}_{(1_{a_1}, \widetilde{E}^{n_1}\widetilde{G}')}, B_2\big)_{n_1} \end{aligned}$$

Hence, by Proposition 4.11, we have

$$\begin{aligned} &[a_1, A_1, E^{n_1}a_2] \circ \big(E^{n_1}a_2, \widetilde{E}^{n_1}(B_2 \circ C(1_{X_5} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2}))), \\ &\qquad\qquad E^{n_1}E^{n_2}([a_3, A_2', a_4] \circ (a_4, A_3, a_5))\big) \\ &\simeq \big[a_1, \underline{B_1 \circ CE^{n_1}q_{E^{n_2}(a_3\circ a_4)}}_{(1_{a_1}, \widetilde{E}^{n_1}\widetilde{G}')}, E^{n_1}[a_2, a_2 \circ \widetilde{E}^{n_2}A_2', E^{n_2}(a_3 \circ a_4)]\big] \\ &\quad \circ \big(E^{n_1}[a_2, a_2 \circ \widetilde{E}^{n_2}A_2', E^{n_2}(a_3 \circ a_4)], \widetilde{E}^{n_1}B_2, \end{aligned}$$

$$\begin{aligned} &E^{n_1}\big(E^{n_2}(a_3\circ a_4), \widetilde{E}^{n_2}(a_3\circ A_3), E^{n_2}a_5)\big) \\ &\circ EE^{n_1}(1_{X_5}\wedge \tau(\mathrm{S}^1,\mathrm{S}^{n_2})). \end{aligned}$$

Pre-composing $EE^{n_1}(1_{X_5}\wedge\tau(\mathrm{S}^{n_2},\mathrm{S}^1))$ to both sides, we have the assertion of Proposition 6.10 by Lemma 2.2(1), Lemma 2.4 and (6.13).

To complete the proof, we should prove (6.13), (6.14) and (6.15).

(6.13). This is easily obtained from Lemma 2.2 and Lemma 2.4.

(6.14). Since the first two squares of the above diagrams are strictly commutative, it can be easily seen that

$$\underline{\underline{B_1\circ CE^{n_1}q_{E^{n_2}(a_3\circ a_4)}}_{(1_{a_1},\widetilde{E}^{n_1}\widetilde{G}')}\circ CE^{n_1}i_{E^{n_2}(a_3\circ a_4)}}_{(1_{a_1},\widetilde{E}^{n_1}1_{a_2})}$$
$$\simeq f := \underline{B_1\circ CE^{n_1}q_{E^{n_2}(a_3\circ a_4)}}_{(1_{a_1},\widetilde{E}^{n_1}\widetilde{G}')}\circ CE^{n_1}i_{E^{n_2}(a_3\circ a_4)}\ rel\ E^{n_1}E^{n_2}X_2.$$

From definitions, we have

$$f(x_2\wedge s_2\wedge s_1\wedge t) = \begin{cases} A_1(x_2\wedge s_2\wedge s_1, 0) & 0\le t\le \frac{1}{3} \\ A_1(x_2\wedge s_2\wedge s_1, 3t-1) & \frac{1}{3}\le t\le\frac{2}{3} \\ A_1(x_2\wedge s_2\wedge s_1, 1) & \frac{2}{3}\le t\le 1 \end{cases},$$

where $x_2\in X_2,\ s_i\in\mathrm{S}^{n_i}\ (i=1,2),\ t\in I$. Hence $f\simeq A_1\ rel\ E^{n_1}E^{n_2}X_2$.

(6.15). From definitions, the map

$$\underline{B_2\circ C(1_{X_5}\wedge\tau(\mathrm{S}^1,\mathrm{S}^{n_2}))}_{(1_{a_2},D_3)} : CE^{n_2}EX_5\to X_1$$

moves $(x_5\wedge\overline{s}\wedge s_2\wedge t)$ $(x_5\in X_5, s_2\in\mathrm{S}^{n_2}, s,t\in I)$ to

$$\begin{cases} a_2(A_2'(a_5(x_5)\wedge(1-2s))\wedge s_2) & 0\le s\le\frac{1}{2},\ 0\le t\le\frac{1}{3} \\ a_2(A_2'(a_5(x_5)\wedge(1-2s)(3-6t))\wedge s_2) & 0\le s\le\frac{1}{2},\ \frac{1}{3}\le t\le\frac{1}{2} \\ a_2(A_2'(a_5(x_5)\wedge(1-2s)(6t-3))\wedge s_2) & 0\le s\le\frac{1}{2},\ \frac{1}{2}\le t\le\frac{2}{3} \\ a_2(a_3(A_3(x_5\wedge(2s-1)))\wedge s_2) & \frac{1}{2}\le s\le 1,\ 0\le t\le\frac{2}{3} \\ B_2(x_5\wedge s_2\wedge\overline{s}\wedge(3t-2)) & 0\le s\le 1,\ \frac{2}{3}\le t\le 1 \end{cases}.$$

Let $\widetilde{B_2} : CE^{n_2}EX_5\to X_1$ be defined by

$$\begin{aligned} &\widetilde{B_2}(x_5\wedge\overline{s}\wedge s_2, t) \\ &= \begin{cases} B_2\circ C(1_{X_5}\wedge\tau(\mathrm{S}^1,\mathrm{S}^{n_2}))(x_5\wedge\overline{s}\wedge s_2, 0) & 0\le t\le\frac{2}{3} \\ B_2\circ C(1_{X_5}\wedge\tau(\mathrm{S}^1,\mathrm{S}^{n_2}))(x_5\wedge\overline{s}\wedge s_2, 3t-2) & \frac{2}{3}\le t\le 1 \end{cases} \\ &= \begin{cases} a_2(A_2'(a_5(x_5)\wedge(1-2s))\wedge s_2) & 0\le s\le\frac{1}{2},\ 0\le t\le\frac{2}{3} \\ a_2(a_3(A_3(x_5\wedge(2s-1)))\wedge s_2) & \frac{1}{2}\le s\le 1,\ 0\le t\le\frac{2}{3} \\ B_2(x_5\wedge s_2\wedge\overline{s}\wedge(3t-2)) & 0\le s\le 1,\ \frac{2}{3}\le t\le 1 \end{cases}. \end{aligned}$$

Then, as is easily seen, $\widetilde{B_2} \simeq B_2 \circ C(1_{X_5} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2}))$ *rel* $E^{n_2}EX_5$. Let $H : CE^{n_2}EX_5 \times I \to X_1$ be the map which moves $(x_5 \wedge \overline{s} \wedge s_2 \wedge t, u)$ to

$$\begin{cases} a_2(A_2'(a_5(x_5) \wedge (1-2s)) \wedge s_2) & (s,t) \in K_1 \\ a_2(A_2'(a_5(x_5) \wedge (u(1-2s) + (1-u)(1-2s)(3-6t))) \wedge s_2) & (s,t) \in K_2 \\ a_2(A_2'(a_5(x_5) \wedge (u(1-2s) + (1-u)(1-2s)(6t-3))) \wedge s_2) & (s,t) \in K_3\ , \\ a_2(a_3(A_3(x_5 \wedge (2s-1))) \wedge s_2) & (s,t) \in K_4 \\ B_2(x_5 \wedge s_2 \wedge \overline{s} \wedge (3t-2)) & (s,t) \in K_5 \end{cases}$$

where $x_5 \in X_5,\ s_2 \in \mathrm{S}^{n_2},\ s,t,u \in I$ and $K_1 = [0, \frac{1}{2}] \times [0, \frac{1}{3}],\ K_2 = [0, \frac{1}{2}] \times [\frac{1}{3}, \frac{1}{2}],\ K_3 = [0, \frac{1}{2}] \times [\frac{1}{2}, \frac{2}{3}],\ K_4 = [\frac{1}{2}, 1] \times [0, \frac{2}{3}],\ K_5 = [0,1] \times [\frac{2}{3}, 1]$. Then $H : \underline{B_2 \circ C(1_{X_5} \wedge \tau(\mathrm{S}^1, \mathrm{S}^{n_2}))}_{(1_{a_2}, D_3)} \simeq \widetilde{B_2}$ *rel* $E^{n_2}EX_5$. Hence we have (6.15). This completes the proof of Proposition 6.10. □

7. Secondary and tertiary compositions in $SU(3)$

We use results and notations of Mimura-Toda [**12**] for homotopy groups of SU(3). For example, $\pi_3(\mathrm{SU}(3)) = \mathbb{Z}\{i\}$, where $i : \mathrm{S}^3 = \mathrm{SU}(2) \to \mathrm{SU}(3)$ is the inclusion map, $\pi_4(\mathrm{SU}(3)) = 0$, $\pi_5(\mathrm{SU}(3)) = \mathbb{Z}\{[2\iota_5]\}$, $\pi_6(\mathrm{SU}(3)) = \mathbb{Z}_2\{i_*\nu'\} \oplus \mathbb{Z}_3$, $\pi_7(\mathrm{SU}(3)) = 0$, $\pi_8(\mathrm{SU}(3)) = \mathbb{Z}_4\{[2\iota_5]\nu_5\} \oplus \mathbb{Z}_3$, $\pi_9(\mathrm{SU}(3)) = \mathbb{Z}_3$, and $\pi_{10}(\mathrm{SU}(3)) = \mathbb{Z}_2\{[\nu_5\eta_8^2]\} \oplus \mathbb{Z}_{15}$.

We denote the cofibre sequence $\mathrm{S}^{n+1} \overset{q_{2\iota_n}}{\leftarrow} \mathrm{S}^n \cup_{2\iota_n} e^{n+1} \overset{i_{2\iota_n}}{\leftarrow} \mathrm{S}^n \overset{2\iota_n}{\leftarrow} \mathrm{S}^n$ by $(\mathrm{Cofib})_n$. Let

$$\overline{\eta_{n-1}}' \in [\mathrm{S}^n \cup_{2\iota_n} e^{n+1}, \mathrm{S}^{n-1}]\ (n \geq 4), \quad \widetilde{\eta_n}' \in \pi_{n+2}(\mathrm{S}^n \cup_{2\iota_n} e^{n+1})\ (n \geq 3)$$

be an extension of η_{n-1} and a coextension of η_n, respectively. It follows from a Puppe sequence and a stable homotopy exact sequence of $(\mathrm{Cofib})_n$ that the orders of groups $[\mathrm{S}^n \cup_{2\iota_n} e^{n+1}, \mathrm{S}^{n-1}]$ and $\pi_{n+2}(\mathrm{S}^n \cup_{2\iota_n} e^{n+1})$ for $n \geq 4$ are 4. On the other hand, if $n \geq 4$, then $\overline{\eta_{n-1}}' \circ \widetilde{\eta_n}' = \pm E^{n-4}\nu'$ of which the order is 4 by [**19**, (5.4), (5.5), Lemma 5.4, Proposition 5.6]. Hence

$$[\mathrm{S}^n \cup_{2\iota_n} e^{n+1}, \mathrm{S}^{n-1}] = \mathbb{Z}_4\{\overline{\eta_{n-1}}'\}, \quad 2\overline{\eta_{n-1}}' = \eta_{n-1}^2 \circ q_{2\iota_n} \quad (n \geq 4),$$
$$\pi_{n+2}(\mathrm{S}^n \cup_{2\iota_n} e^{n+1}) = \mathbb{Z}_4\{\widetilde{\eta_n}'\}, \quad 2\widetilde{\eta_n}' = i_{2\iota_n} \circ \eta_n^2 \quad (n \geq 4)^*.$$

Lemma 7.1. *We have*

$$\{[2\iota_5], 4\nu_5, \eta_8\} = \{[2\iota_5]\overline{\eta_5}', 2\widetilde{\eta_6}', \eta_8\} = \{[2\iota_5]\overline{\eta_5}', i_{2\iota_6} \circ \eta_6^2, \eta_8\}$$

which consist of a single element, and $[2\iota_5]\overline{\eta_5}' \in \mathrm{Ext}_{2\iota_6}([2\iota_5]\eta_5)$ *and*

$$2\widetilde{\eta_6}' \in \mathrm{Coext}_{2\iota_6}(*_6). \tag{7.1}$$

---

*This holds also for $n = 3$.

Proof. Since $E^2\nu' = 2\nu_5$ by [**19**, Lemma 5.4], we have $4\nu_5 = 2(\overline{\eta_5}' \circ \widetilde{\eta_6}') = \overline{\eta_5}' \circ 2\widetilde{\eta_6}'$ and

$$\begin{aligned}\{[2\iota_5], 4\nu_5, \eta_8\} &= \{[2\iota_5], \overline{\eta_5}' \circ 2\widetilde{\eta_6}', \eta_8\}\\ &\supset \{[2\iota_5]\overline{\eta_5}', 2\widetilde{\eta_6}', \eta_8\} = \{[2\iota_5]\overline{\eta_5}', i_{2\iota_6} \circ \eta_6^2, \eta_8\},\end{aligned}$$

where $[2\iota_5]\overline{\eta_5}' \in \mathrm{Ext}_{2\iota_6}([2\iota_5]\eta_5)$ and $2\widetilde{\eta_6}' \in \mathrm{Coext}_{2\iota_6}(0_6)$ by Lemma 2.3. The indeterminacy of $\{[2\iota_5], 4\nu_5, \eta_8\}$ is $\pi_9(\mathrm{SU}(3)) \circ \eta_9 + [2\iota_5] \circ \pi_{10}(\mathrm{S}^5) = 0$ by [**12**]. This completes the proof. □

To determine $\{[2\iota_5]\overline{\eta_5}', i_{2\iota_6} \circ \eta_6^2, \eta_8\}$, we use the following null quadruple:

(7.2)
$$\begin{tikzcd}\mathrm{SU}(3) & \mathrm{S}^6 \arrow[l, "{[2\iota_5]\eta_5}"'] & \mathrm{S}^6 \arrow[l, "2\iota_6"'] & \mathrm{S}^7 \arrow[l, "0_6"'] & \mathrm{S}^8 \arrow[l, "\eta_7"']\end{tikzcd}$$

Lemma 7.2. (1) $\{[2\iota_5]\eta_5, 2\iota_6, 0_6\} = \{0\}$.

(2) $\{2\iota_6, 0_6, \eta_7\} \ni 0$.

(3) $\pi_8(\mathrm{S}^6) \circ \eta_8 \subset 2\iota_6 \circ \pi_9(\mathrm{S}^6)$.

(4) $[2\iota_5]\eta_5 = i \circ \nu'$ *in* $\pi_6(\mathrm{SU}(3))$.

(5) $([2\iota_5]\eta_5, 2\iota_6, 0_6, \eta_7)$ *is admissible.*

Proof. We have (1) and (2), since

$$\begin{aligned}\{[2\iota_5]\eta_5, 2\iota_6, 0_6\} &= \mathrm{Indet}\{[2\iota_5]\eta_5, 2\iota_6, 0_6\} = [2\iota_5]\eta_5 \circ \pi_8(\mathrm{S}^6) = \{0\} \text{ (by [\textbf{10}])},\\ \{2\iota_6, 0_6, \eta_7\} &= \mathrm{Indet}\{2\iota_6, 0_6, \eta_7\} \ni 0.\end{aligned}$$

Since $\eta_6^2 \circ \eta_8 = 4\nu_6 = 2\iota_6 \circ 2\nu_6$, we obtain (3). If we apply [**12**, Theorem 2.1] for $\alpha = \iota_5$, $\beta = 2\iota_4$, $\gamma = \eta_4$, then we have $[2\iota_5]\eta_5 = i \circ \nu'$. Hence we obtain (4) by [**12**, Theorem 4.1]. By Proposition 4.4, (1) and (2), we have (5). □

Let $A_1 : [2\iota_5]\eta_5 \circ 2\iota_6 \simeq *$, $A_2 : 2\iota_6 \circ *_6 \simeq *$, and $A_3 : *_6 \circ \eta_7 \simeq *$. Then there exists uniquely a map $\widehat{A}_m : \mathrm{S}^{6+m} \to \mathrm{S}^6$ for $m = 2, 3$ such that $A_m = \widehat{A}_m \circ \pi$, where $\pi : C\,\mathrm{S}^{6+m-1} \to E\,\mathrm{S}^{6+m-1} = \mathrm{S}^{6+m}$ is the quotient map.

Lemma 7.3. $([2\iota_5]\eta_5, 2\iota_6, *_6, \eta_7; A_1, A_2, A_3)$ *is an admissible representative of* (7.2) *if and only if the pair of homotopy classes of* $\widehat{A}_2$ *and* $\widehat{A}_3$ *is one of* $(0_6^2, 0_6^3), (0_6^2, 4\nu_6)$, $(\eta_6^2, 2\nu_6)$ *and* $(\eta_6^2, -2\nu_6)$. *In that case, we have*

$$\begin{aligned}&\{[2\iota_5]\overline{\eta_5}', i_{2\iota_6} \circ \widehat{A}_2, \eta_8\} = \{A_1, A_2, A_3\}^{(3)}\\ &= \{i \circ \nu', 2\iota_6 \underline{\vee} \widehat{A}_2, (\widehat{A}_3 \vee \eta_8) \circ \theta_{\mathrm{S}^9}\} = \begin{cases}[\nu_5\eta_8^2] & \widehat{A}_2 = \eta_6^2\\ 0 & \widehat{A}_2 = 0_6^2\end{cases}.\end{aligned}$$

Proof. Since $\pi_7(\mathrm{SU}(3)) = 0$ by [**12**], it follows from Lemma 7.2(4) that $[\mathrm{S}^6 \cup_{2\iota_6} e^7, \mathrm{SU}(3)] = \mathbb{Z}_2\{[2\iota_5]\overline{\eta_5}'\}$ and

$$[[2\iota_5]\eta_5, A_1, 2\iota_6] \simeq [2\iota_5]\overline{\eta_5}'. \tag{7.3}$$

We easily have

$$[2\iota_6, A_2, *_6] = 2\iota_6 \underline{\vee} \widehat{A}_2 : \mathrm{S}^6 \vee \mathrm{S}^8 \to \mathrm{S}^6, \tag{7.4}$$

$$(2\iota_6, A_2, *_6) \simeq i_{2\iota_6} \circ \widehat{A}_2. \tag{7.5}$$

Also

$$\begin{aligned}(*_6, A_3, \eta_7) &= i_2 \circ (-E\eta_7) + i_1 \circ \widehat{A}_3 \simeq i_1 \circ \widehat{A}_3 + i_2 \circ (-E\eta_7) \\ &\qquad\qquad\qquad (\text{since } \pi_9(\mathrm{S}^6 \vee \mathrm{S}^8) \text{ is abelian}) \\ &= \big(\widehat{A}_3 \vee (-E\eta_7)\big) \circ \theta_{\mathrm{S}^9} \simeq (\widehat{A}_3 \vee \eta_8) \circ \theta_{\mathrm{S}^9} \quad (\text{since } -E\eta_7 \simeq \eta_8),\end{aligned}$$

where $i_1 : \mathrm{S}^6 \to \mathrm{S}^6 \vee \mathrm{S}^8$ and $i_2 : \mathrm{S}^8 \to \mathrm{S}^6 \vee \mathrm{S}^8$ are the inclusion maps. Hence

$$(*_6, A_3, \eta_7) \simeq (\widehat{A}_3 \vee \eta_8) \circ \theta_{\mathrm{S}^9}. \tag{7.6}$$

If $([2\iota_5]\eta_5, 2\iota_6, *_6, \eta_7; A_1, A_2, A_3)$ is admissible, then

$$0 \simeq [2\iota_6, A_2, *_6] \circ (*_6, A_3, \eta_7) \simeq 2\widehat{A}_3 + \widehat{A}_2 \circ \eta_8$$

by (7.4), (7.6) and Lemma 2.1 so that the pair of homotopy classes of $\widehat{A}_2$ and $\widehat{A}_3$ is one of the four pairs $(0^2_6, 0^3_6)$, $(0^2_6, 4\nu_6)$, $(\eta^2_6, 2\nu_6)$ and $(\eta^2_6, -2\nu_6)$. Conversely if the pair is one of the four pairs, then, as is easily seen, $([2\iota_5]\eta_5, 2\iota_6, *_6, \eta_7; A_1, A_2, A_3)$ is admissible.

Suppose $([2\iota_5]\eta_5, 2\iota_6, *_6, \eta_7; A_1, A_2, A_3)$ is admissible. We have

$$\begin{aligned}&\{A_1, A_2, A_3\}^{(3)} = \big\{\big[[2\iota_5]\eta_5, A_1, 2\iota_6\big], (2\iota_6, A_2, *_6) \circ q_{*_6}, (*_6, A_3, \eta_7)\big\} \\ &\quad = \{[2\iota_5]\overline{\eta_5}', i_{2\iota_6} \circ \widehat{A}_2 \circ q_{*_6}, (\widehat{A}_3 \vee \eta_8) \circ \theta_{\mathrm{S}^9}\} \quad (\text{by (7.3), (7.5), (7.6)}) \\ &\quad \supset \{[2\iota_5]\overline{\eta_5}', i_{2\iota_6} \circ \widehat{A}_2, \eta_8\} \quad (\text{by Proposition 1.2(ii) of } [\mathbf{19}]), \\ &\mathrm{Indet}\{A_1, A_2, A_3\}^{(3)} \\ &\quad = [\mathrm{S}^7 \vee \mathrm{S}^9, \mathrm{SU}(3)] \circ (E\widehat{A}_3 \vee \eta_9) \circ \theta_{\mathrm{S}^{10}} + [2\iota_5]\overline{\eta_5}' \circ \pi_{10}(\mathrm{S}^6 \cup_{2\iota_6} e^7) \\ &\quad = [2\iota_5]\overline{\eta_5}' \circ \pi_{10}(\mathrm{S}^6 \cup_{2\iota_6} e^7) \\ &\qquad\qquad (\text{since } \pi_7(\mathrm{SU}(3)) = 0 \text{ and } \pi_9(\mathrm{SU}(3)) = \mathbb{Z}_3 \text{ by } [\mathbf{12}]) \\ &\quad = [2\iota_5]\overline{\eta_5}' \circ \mathbb{Z}_2\{\widetilde{\eta_6}' \circ \eta^2_8\} = 0.\end{aligned}$$

Hence $\{A_1, A_2, A_3\}^{(3)} = \{[2\iota_5]\overline{\eta_5}', i_{2\iota_6} \circ \widehat{A}_2, \eta_8\}$ and they consist of a single element. We also have

$$\begin{aligned}\{A_1, A_2, A_3\}^{(3)} &= \{[[2\iota_5]\eta_5, A_1, 2\iota_6], i_{2\iota_6} \circ [2\iota_6, A_2, *_6], (*_6, A_3, \eta_7)\} \\ &= \{[2\iota_5]\overline{\eta_5}', i_{2\iota_6} \circ (2\iota_6 \underline{\vee} \widehat{A}_2), (\widehat{A}_3 \vee \eta_8) \circ \theta_{\mathrm{S}^9}\} \quad (\text{by (7.3), (7.4) and (7.6)}) \\ &= \{[2\iota_5]\eta_5, 2\iota_6 \underline{\vee} \widehat{A}_2, (\widehat{A}_3 \vee \eta_8) \circ \theta_{\mathrm{S}^9}\} \quad (\text{by } [\mathbf{19}, \text{Proposition 1.2}]) \\ &= \{i \circ \nu', 2\iota_6 \underline{\vee} \widehat{A}_2, (\widehat{A}_3 \vee \eta_8) \circ \theta_{\mathrm{S}^9}\} \quad (\text{by 7.2(4)})\end{aligned}$$

$$\begin{aligned}&\in \{i, \nu' \circ (2\iota_6 \underline{\vee} \widehat{A}_2), (\widehat{A}_3 \vee \eta_8) \circ \theta_{S^9}\} = \{i, 2\nu' \underline{\vee} (\nu' \circ \widehat{A}_2), (\widehat{A}_3 \vee \eta_8) \circ \theta_{S^9}\}\\ &\subset \pi_{10}(\mathrm{SU}(3)).\end{aligned}$$

Let $p : \mathrm{SU}(3) \to \mathrm{S}^5$ be the canonical bundle projection. Then it follows from Proposition 1.4 of [**19**] that

$$p\circ\{i,\ 2\nu' \underline{\vee} (\nu'\circ\widehat{A}_2),\ (\widehat{A}_3\vee\eta_8)\circ\theta_{S^9}\} = -\{p,\ i,\ 2\nu' \underline{\vee} (\nu'\circ\widehat{A}_2)\}\circ(E\widehat{A}_3\vee\eta_9)\circ\theta_{S^{10}}.$$

Let $x \in \{0, 1\}$ satisfy $\widehat{A}_2 = x\eta_6^2$. Since $2\nu' = \eta_3^3$ and $\nu'\eta_6 = \eta_3\nu_4$ by [**19**, (5.3),(5.9)], we have

$$\{p,\ i,\ 2\nu' \underline{\vee} x\nu'\eta_6^2\} = \{p,\ i,\ \eta_3 \circ (\eta_4^2 \underline{\vee} x\nu_4\eta_7)\} \supset \{p,\ i,\ \eta_3\} \circ (\eta_5^2 \underline{\vee} x\nu_5\eta_8).$$

Let $j : \mathrm{S}^3 \cup_{\eta_3} C\,\mathrm{S}^4 \to \mathrm{SU}(3)$ be the inclusion map. Since $i = j \circ i_{\eta_3}$, it follows from Lemma 3.2 that

$$\{p,\ i,\ \eta_3\} \supset \{p \circ j,\ i_{\eta_3},\ \eta_3\} = \{q_{\eta_3},\ i_{\eta_3},\ \eta_3\} \ni \iota_5$$

so that $\{p,\ i,\ \eta_3\} = \iota_5 + \mathbb{Z}\{2\iota_5\}$ and $\{p,\ i,\ \eta_3\} \circ (\eta_5^2 \underline{\vee} x\nu_5\eta_8) = \eta_5^2 \underline{\vee} x\nu_5\eta_8$. Hence

$$p\circ\{i,\ 2\nu' \underline{\vee} (\nu'\circ\widehat{A}_2),\ (\widehat{A}_3\vee\eta_8)\circ\theta_{S^9}\} \ni -(\eta_5^2 \underline{\vee} x\nu_5\eta_8)\circ(E\widehat{A}_3\vee\eta_9)\circ\theta_{S^{10}} = x\nu_5\eta_8^2.$$

On the other hand, since $\pi_{10}(\mathrm{SU}(3)) = \mathbb{Z}_2\{[\nu_5\eta_8^2]\}\oplus\mathbb{Z}_{15}\{i_*\alpha\}$ by [**12**], where α is a generator of $\pi_{10}(\mathrm{S}^3) \cong \mathbb{Z}_{15}$, and since $\pi_{10}(\mathrm{S}^5) = \mathbb{Z}_2\{\nu_5\eta_8^2\}$ by [**19**], we easily see that the indeterminacy of $\{i,\ 2\nu' \underline{\vee} (\nu' \circ \widehat{A}_2),\ (\widehat{A}_3 \vee \eta_8) \circ \theta_{S^9}\}$ is $\mathbb{Z}_{15}$. Therefore

$$\{i,\ 2\nu' \underline{\vee} (\nu' \circ \widehat{A}_2),\ (\widehat{A}_3 \vee \eta_8) \circ \theta_{S^9}\} = x[\nu_5\eta_8^2] + \mathbb{Z}_{15}.$$

Hence $\{i \circ \nu',\ 2\iota_6 \underline{\vee} \widehat{A}_2,\ (\widehat{A}_3 \vee \eta_8) \circ \theta_{S^9}\} = x[\nu_5\eta_8^2]$, since

$$\begin{aligned}&4\{i \circ \nu',\ 2\iota_6 \underline{\vee} \widehat{A}_2,\ (\widehat{A}_3 \vee \eta_8) \circ \theta_{S^9}\} = \psi^4 \circ \{i \circ \nu',\ 2\iota_6 \underline{\vee} \widehat{A}_2,\ (\widehat{A}_3 \vee \eta_8) \circ \theta_{S^9}\}\\ &\in \{\psi^4 \circ i \circ \nu',\ 2\iota_6 \underline{\vee} \widehat{A}_2,\ (\widehat{A}_3 \vee \eta_8) \circ \theta_{S^9}\} = \{0, 2\iota_6 \underline{\vee} \widehat{A}_2, (\widehat{A}_3 \vee \eta_8) \circ \theta_{S^9}\}\\ &= \{0\},\end{aligned}$$

where $\psi^4 : \mathrm{SU}(3) \to \mathrm{SU}(3)$ is the map defined by $\psi^4(z) = z^4$. □

Proposition 7.4. *If* $([2\iota_5]\eta_5, 2\iota_6, *_6, \eta_7; A_1, A_2, A_3)$ *is admissible, then*

$$\{A_1, A_2, A_3\}^{(k)} = \begin{cases}[\nu_5\eta_8^2] & \widehat{A}_2 \simeq \eta_6^2\\ 0 & \widehat{A}_2 \simeq *_6^2\end{cases},\ \{[2\iota_5]\eta_5, 2\iota_6, 0_6, \eta_7\}^{(k)} = \mathbb{Z}_2\{[\nu_5\eta_8^2]\}$$

for $0 \le k \le 3$.

Proof. By Proposition 5.1 and Lemma 7.3, we have the first equality. By Corollary 4.7(1) and Lemma 7.2, every element of $\pi_8(\mathrm{S}^6) = \{0, \eta_6^2\}$ can be the homotopy class of $\widehat{A}_2$. Hence we obtain the second equality. □

Proposition 7.5. *The Toda bracket* $\{[2\iota_5]\eta_5, 4\nu_5, \eta_8\}$ *consists of a single element* $[\nu_5\eta_8^2]$.

Proof. Take $A_1 : [2\iota_5] \circ \eta_5 \circ 2\iota_6 \simeq *$ arbitrarily. By (7.1), there exist $A_2 : 2\iota_6 \circ *_6 \simeq *$ such that $(2\iota_6, A_2, *_6) \simeq 2\widetilde{\eta_6}'$. Since $2\widetilde{\eta_6}' \simeq i_{2\iota_6} \circ \eta_6^2$, we have $\widehat{A}_2 \simeq \eta_6^2$ by (7.5). By Lemma 7.2(1),(2),(3) and Corollary 4.7(1), there exists $A_3 : *_6 \circ \eta_7 \simeq *$ such that $([2\iota_5]\eta_5, 2\iota_6, *_6, \eta_7; A_1, A_2, A_3)$ is an admissible representative of (7.2). It follows from Lemma 7.3 that $\{[2\iota_5]\overline{\eta_5}', 2\widetilde{\eta_6}', \eta_8\} = [\nu_5\eta_8^2]$. Hence we obtain the assertion from Lemma 7.1. □

8. HAMANAKA-KONO'S RESULTS

We use results and notations of [**12**] for homotopy groups of SU(4). For example, $\pi_{10}(\mathrm{SU}(4)) = \mathbb{Z}_8\{[\nu_7]\} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{15}$. Recall that $C_{\eta_n} = \mathrm{S}^n \cup_{\eta_n} e^{n+2}$ is the mapping cone of η_n for $n \geq 2$, and that $i : \mathrm{S}^3 = \mathrm{SU}(2) \to \mathrm{SU}(3)$ is the inclusion map. Let $j : C_{\eta_3} \to \mathrm{SU}(3)$ and $i_{3,4} : \mathrm{SU}(3) \to \mathrm{SU}(4)$ be the inclusion maps, $q_3 : C_{\eta_3} \to \mathrm{S}^5$ and $q_8 : C_{\eta_8} \to \mathrm{S}^{10}$ the quotient maps. Let $\langle\ ,\ \rangle : [C_{\eta_3}, \mathrm{SU}(3)] \times \pi_5(\mathrm{SU}(3)) \to [C_{\eta_3} \wedge \mathrm{S}^5, \mathrm{SU}(3)] = [C_{\eta_8}, \mathrm{SU}(3)]$ be the Samelson product [**1**].

Lemma 8.1. *We have* $\pi_6(\mathrm{SU}(3)) = \mathbb{Z}_6\{\langle i, i\rangle\}$, $\pi_8(\mathrm{SU}(3)) = \mathbb{Z}_{12}\{\langle i, [2\iota_5]\rangle\}$ *and* $\pi_{10}(\mathrm{SU}(3)) = \mathbb{Z}_{30}\{\langle [2\iota_5], [2\iota_5]\rangle\}$.

Proof. These are easily obtained from [**2**, Theorem 1] and [**12**]. □

We shall prove the following Hamanaka-Kono's results [**4**, Theorem 2.5, Theorem 2.3] as a corollary to Proposition 7.5.

Proposition 8.2. (1) $[C_{\eta_8}, \mathrm{SU}(3)] = \mathbb{Z}_8\{15\langle j, [2\iota_5]\rangle\} \oplus \mathbb{Z}_3\{40\langle j, [2\iota_5]\rangle\} \oplus \mathbb{Z}_{15}\{2\langle q_3^*[2\iota_5], [2\iota_5]\rangle\}$.

(2) $[C_{\eta_8}, \mathrm{SU}(4)] = \mathbb{Z}_8\{q_8^*[\nu_7]\} \oplus \mathbb{Z}_4\{i_{3,4*}\overline{[2\iota_5]\nu_5} - q_8^*[\nu_7]\} \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_5$ *and* $i_{3,4*} : [C_{\eta_8}, \mathrm{SU}(3)] \to [C_{\eta_8}, \mathrm{SU}(4)]$ *is an isomorphism onto a direct summand, where* $\overline{[2\iota_5]\nu_5} \in [C_{\eta_8}, \mathrm{SU}(3)]$ *is an extension of* $[2\iota_5]\nu_5$ *with order* 8.

Proof. Let $\mathrm{S}^n \xrightarrow{i_{\eta_n}} \mathrm{S}^n \cup_{\eta_n} C\,\mathrm{S}^{n+1} \xrightarrow{q_{\eta_n}} \mathrm{S}^{n+2}$ be the cofibre sequence for $n \geq 2$. For simplicity, we put $i_n = i_{\eta_n}$ and $q_n = q_{\eta_n}$. By [**12**], we have the following exact sequence.

$$0 \to \mathbb{Z}_2\{[\nu_5\eta_8^2]\} \oplus \mathbb{Z}_{15} \xrightarrow{q_8^*} [C_{\eta_8}, \mathrm{SU}(3)] \xrightarrow{i_8^*} \mathbb{Z}_4\{[2\iota_5]\nu_5\} \oplus \mathbb{Z}_3 \to 0 \tag{8.1}$$

By [**9**, (3.1),Table 1,Lemma 3.2] and [**19**], (8.1) splits about odd components. So it suffices to show that (8.1) does not split about 2-primary components. Let $\overline{[2\iota_5]\nu_5} \in [C_{\eta_8}, \mathrm{SU}(3)]$ be an extension of $[2\iota_5]\nu_5$. Then $45\overline{[2\iota_5]\nu_5}$ is also an extension of $[2\iota_5]\nu_5$ with $8 \cdot 45\overline{[2\iota_5]\nu_5} = 0$. From now on, we take $\overline{[2\iota_5]\nu_5}$ with $8\overline{[2\iota_5]\nu_5} = 0$. We shall prove

$$4\,\overline{[2\iota_5]\nu_5} = q_8^*[\nu_5\eta_8^2]. \tag{8.2}$$

If this is established, then the 2-primary part of $[C_{\eta_8}, \mathrm{SU}(3)]$ is $\mathbb{Z}_8\{\overline{[2\iota_5]\nu_5}\}$ and hence (1) of Proposition 8.2 is obtained from Lemma 8.1 and (8.1).

Let $\psi^n : \mathrm{SU}(3) \to \mathrm{SU}(3)$ be defined by $\psi^n(x) = x^n$. We have $4\,\overline{[2\iota_5]\nu_5} = \psi^4 \circ \overline{[2\iota_5]\nu_5} \in \{\psi^4, [2\iota_5]\nu_5, \eta_8\} \circ q_8$ by [**19**, Proposition 1.9] and

$$\mathrm{Indet}\{\psi^4, [2\iota_5]\nu_5, \eta_8\} = \pi_9(\mathrm{SU}(3)) \circ \eta_9 + \psi^4 \circ \pi_{10}(\mathrm{SU}(3)) = \mathbb{Z}_{15}. \tag{8.3}$$

Hence it suffices for (8.2) to prove $\{\psi^4, [2\iota_5]\nu_5, \eta_8\} = [\nu_5\eta_8^2] + \mathbb{Z}_{15}$. We have

$$\begin{aligned}
\{\psi^4, [2\iota_5]\nu_5, \eta_8\} &\subset \{\psi^2, \psi^2 \circ [2\iota_5]\nu_5, \eta_8\} = \{\psi^2, [2\iota_5] \circ 2\nu_5, \eta_8\} \\
&\qquad\qquad\qquad\qquad (\text{since } \psi^4 = \psi^2 \circ \psi^2) \\
&\supset \{\psi^2 \circ [2\iota_5], 2\nu_5, \eta_8\} = \{[2\iota_5] \circ 2\iota_5, 2\nu_5, \eta_8\} \\
&\subset \{[2\iota_5], 2\iota_5 \circ 2\nu_5, \eta_8\} = \{[2\iota_5], 4\nu_5, \eta_8\} = [\nu_5\eta_8^2] \quad (\text{by } 7.5),
\end{aligned}$$

$$\mathrm{Indet}\{\psi^2, [2\iota_5] \circ 2\nu_5, \eta_8\} = \pi_9(\mathrm{SU}(3)) \circ \eta_9 + \psi^2 \circ \pi_{10}(\mathrm{SU}(3)) = \mathbb{Z}_{15}. \tag{8.4}$$

Hence

$$\{\psi^4, [2\iota_5]\nu_5, \eta_8\} = \{\psi^2, [2\iota_5] \circ 2\nu_5, \eta_8\} = [\nu_5\eta_8^2] + \mathbb{Z}_{15}$$

by (8.3) and (8.4). This proves (8.2) and completes the proof of (1) of Proposition 8.2.

Next we shall prove (2) of Proposition 8.2. Let $\mathrm{SU}(3) \xrightarrow{i_{3,4}} \mathrm{SU}(4) \xrightarrow{\pi} S^7$ be the canonical SU(3)-bundle. By [**12**], we have the following commutative diagram whose rows and columns are exact.

$$\begin{tikzcd}
 & \mathbb{Z}_3 \arrow[r] \arrow[d, "\eta_9^*"] & \mathbb{Z}_2\{[\nu_5 \oplus \eta_7]\eta_8\} \arrow[d, "\eta_9^*"] \\
0 \arrow[r] \arrow[d] & \mathbb{Z}_2\{[\nu_5\eta_8^2]\} \oplus \mathbb{Z}_{15} \arrow[r, "i_{3,4*}"] \arrow[d, "q_8^*"] & \mathbb{Z}_2\{[\nu_5\eta_8^2]\} \oplus \mathbb{Z}_8\{[\nu_7]\} \oplus \mathbb{Z}_{15} \arrow[d, "q_8^*"] \\
0 \arrow[r] \arrow[d, "i_9^*"] & {[C_{\eta_8}, \mathrm{SU}(3)]} \arrow[r, "i_{3,4*}"] \arrow[d, "i_8^*"] & {[C_{\eta_8}, \mathrm{SU}(4)]} \arrow[d, "i_8^*"] \\
\mathbb{Z}_2\{\eta_7^2\} \arrow[r] \arrow[d, "\eta_9^*"] & \mathbb{Z}_4\{[2\iota_5]\nu_5\} \oplus \mathbb{Z}_3 \arrow[r, "i_{3,4*}"] \arrow[d, "\eta_8^*"] & \mathbb{Z}_8\{[\nu_5 \oplus \eta_7]\} \oplus \mathbb{Z}_3 \arrow[d, "\eta_8^*"] \\
\mathbb{Z}_8\{\nu_7\} \oplus \mathbb{Z}_3 \arrow[r] & \mathbb{Z}_3 \arrow[r, "i_{3,4*}"] & \mathbb{Z}_2\{[\nu_5 \oplus \eta_7]\eta_8\}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathbb{Z}_2\{\eta_7^2\} \arrow[d, "\eta_9^*"] & \\
{} \arrow[r] & \mathbb{Z}_8\{\nu_7\} \oplus \mathbb{Z}_3 \arrow[r] \arrow[d, "q_8^*"] & \mathbb{Z}_3 \arrow[d, "q_7^*", "\cong"'] \\
{} \arrow[r, "\pi_*"] & {[C_{\eta_8}, \mathrm{S}^7]} \arrow[r] \arrow[d, "i_8^*"] & \mathbb{Z}_3 \arrow[d, "i_7^*"] \\
{} \arrow[r] & \mathbb{Z}_2\{\eta_7\} \arrow[r] \arrow[d, "\eta_8^*", "\cong"'] & 0 \\
{} \arrow[r, "\cong"] & \mathbb{Z}_2\{\eta_7^2\} &
\end{tikzcd}$$

Since $\eta_9^*([\nu_5 \oplus \eta_7]\eta_8) = [\nu_5\eta_8^2] \oplus 4[\nu_7]$, we have the following commutative diagram whose rows and columns are short exact, where $4q_8^*[\nu_7] = q_8^*[\nu_5\eta_8^2]$.

$$\begin{tikzcd}
\mathbb{Z}_2\{q_8^*[\nu_5\eta_8^2]\} \oplus \mathbb{Z}_{15} \arrow[r] \arrow[d] & \mathbb{Z}_8\{q_8^*[\nu_7]\} \oplus \mathbb{Z}_{15} \arrow[r] \arrow[d] & \mathbb{Z}_4\{q_8^*\nu_7\} \arrow[d, equal] \\
{[C_{\eta_8}, \mathrm{SU}(3)]} \arrow[r, "i_{3,4*}"] \arrow[d, "i_8^*"] & {[C_{\eta_8}, \mathrm{SU}(4)]} \arrow[r, "\pi_*"] \arrow[d, "i_8^*"] & \mathbb{Z}_4\{q_8^*\nu_7\} \\
\mathbb{Z}_4\{[2\iota_5]\nu_5\} \oplus \mathbb{Z}_3 \arrow[r, "\cong"] & \mathbb{Z}_4\{2[\nu_5 \oplus \eta_7]\} \oplus \mathbb{Z}_3 &
\end{tikzcd}$$

As seen in [**9**], the odd component of $[C_{\eta_8}, \mathrm{SU}(3)]$ is $\mathbb{Z}_{15} \oplus \mathbb{Z}_3$. Hence so is about $[C_{\eta_8}, \mathrm{SU}(4)]$. Thus it suffices to see 2-primary components. Let $\overline{[2\iota_5]\nu_5} \in [C_{\eta_8}, \mathrm{SU}(3)]$ be an extension of $[2\iota_5]\nu_5$ whose order is 8. Then the above discussion implies that the 2-primary part of $[C_{\eta_8}, \mathrm{SU}(4)]$ is equal to $\mathbb{Z}_8\{q_8^*[\nu_7]\} \oplus \mathbb{Z}_4\{i_{3,4*}\overline{[2\iota_5]\nu_5} - q_8^*[\nu_7]\}$. This completes the proof of (2) of Proposition 8.2. □

Remark 8.3. *Let* $\mathrm{map}_*(\mathrm{SU}(3), \mathrm{SU}(3))$ *be the space of based self maps of* $\mathrm{SU}(3)$. *By (1) of Proposition 8.1, we can solve an ambiguity in* [**9**, Theorem 7.1]*:* $\pi_5(\mathrm{map}_*(\mathrm{SU}(3), \mathrm{SU}(3))) \cong \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_5$. *Hence we have determined* $\pi_n(\mathrm{map}_*(\mathrm{SU}(3), \mathrm{SU}(3)))$ *for* $n \leq 11$ [**11**, **9**, **15**].

APPENDIX A. A COUNTEREXAMPLE TO A PROPOSITION OF ÔGUCHI

Consider the following null quadruple.

$$\mathrm{S}^3 \xleftarrow{\eta_3\varepsilon_4} \mathrm{S}^{12} \xleftarrow{0_{12}^0} \mathrm{S}^{12} \xleftarrow{0_{12}^0} \mathrm{S}^{12} \xleftarrow{0_{12}^7} \mathrm{S}^{19} \tag{A.1}$$

Calculations show that $G_1 = \{0\}$ and $G_2 = \pi_{13}(\mathrm{S}^{12})$ so that (A.1) satisfies the hypotheses of Proposition (6.5)(i) of [**13**]. Also $\pi_{14}(\mathrm{S}^3) \circ E^2 0^7_{12} + \eta_3\varepsilon_4 \circ \pi_{21}(\mathrm{S}^{12}) = \mathbb{Z}_2\{2\mu'\sigma_{14}\}$ by [**19**] and [**14**, (2.13)(7)]. Hence the following result implies that $\{\eta_3\varepsilon_4, 0^0_{12}, 0^0_{12}, 0^7_{12}\}^{(0)}$ and $\{\eta_3\varepsilon_4, 0^0_{12}, 0^0_{12}, 0^7_{12}\}^{(1)}$ are not cosets of $\mathbb{Z}_2\{2\mu'\sigma_{14}\}$. Therefore (A.1) is a counterexample to Proposition (6.5)(i) of [**13**].

Example A.1. $\{\eta_3\varepsilon_4, *^0_{12}, *^0_{12}, *^7_{12}\}^{(0)} = \mathbb{Z}_2^2\{2\mu'\sigma_{14}, \nu'\bar{\varepsilon}_6\}$.

Proof. Let $A_1 : C\,\mathrm{S}^{12} \to \mathrm{S}^3$, $A_2 : C\,\mathrm{S}^{12} \to \mathrm{S}^{12}$ and $A_3 : C\,\mathrm{S}^{19} \to \mathrm{S}^{12}$ be null homotopies of trivial maps. Then we can write $A_1 = \widehat{A}_1 \circ \pi$, $A_2 = \widehat{A}_2 \circ \pi$ and $A_3 = \widehat{A}_3 \circ \pi'$, where $\widehat{A}_1 : E\,\mathrm{S}^{12} \to \mathrm{S}^3$, $\widehat{A}_2 : E\,\mathrm{S}^{12} \to \mathrm{S}^{12}$ and $\widehat{A}_3 : E\,\mathrm{S}^{19} \to \mathrm{S}^{12}$ are maps and $\pi : C\,\mathrm{S}^{12} \to E\,\mathrm{S}^{12}$ and $\pi' : C\,\mathrm{S}^{19} \to E\,\mathrm{S}^{19}$ are the quotient maps.

First we show that (A_1, A_2, A_3) is admissible if and only if $\widehat{A}_2 \simeq *$. We have $[*^0_{12}, A_2, *^0_{12}] = *^0_{12} \underline{\vee}\, \widehat{A}_2$ and $(*^0_{12}, A_3, *^7_{12}) \simeq i_{*^0_{12}} \circ \widehat{A}_3$. Hence $[*^0_{12}, A_2, *^0_{12}] \circ (*^0_{12}, A_3, *^7_{12}) \simeq *$. We have $[\eta_3\varepsilon_4, A_1, *^0_{12}] = \eta_3\varepsilon_4 \underline{\vee}\, \widehat{A}_1$ and $(*^0_{12}, A_2, *^0_{12}) \simeq i_{*^0_{12}} \circ \widehat{A}_2$, Hence $[\eta_3\varepsilon_4, A_1, *^0_{12}] \circ (*^0_{12}, A_2, *^0_{12}) \simeq \eta_3\varepsilon_4 \circ \widehat{A}_2$. Since $\eta_3\varepsilon_{4*} : \pi_{13}(\mathrm{S}^{12}) \to \pi_{13}(\mathrm{S}^3)$ is injective, we have the desired assertion.

Let (A_1, A_2, A_3) be admissible. Take $B_1 : [\eta_3\varepsilon_4, A_1, *^0_{12}] \circ (*^0_{12}, A_2, *^0_{12}) \simeq *$ arbitrarily. Since $[*^0_{12}, A_2, *^0_{12}] \circ (*^0_{12}, A_3, *^7_{12}) = *$, any null homotopy $B_2 : [*^0_{12}, A_2, *^0_{12}] \circ (*^0_{12}, A_3, *^7_{12}) \simeq *$ has a form $B_2 = \widehat{B}_2 \circ \pi$ for any map $\widehat{B}_2 : E^2\,\mathrm{S}^{19} = \mathrm{S}^{21} \to \mathrm{S}^{12}$. Let $\widetilde{G} : (\mathrm{S}^{12} \vee \mathrm{S}^{13}) \times I \to \mathrm{S}^{12} \vee \mathrm{S}^{13}$ be the typical homotopy for $(*^0_{12}, *^0_{12}; A_2)$. Then

$$f = [\eta_3\varepsilon_4, \underline{B_1 \circ Cq_{*^0_{12}}}_{(1,\widetilde{G})}, [*^0_{12}, A_2, *^0_{12}]] \circ ([*^0_{12}, A_2, *^0_{12}], B_2, (*^0_{12}, A_3, *^7_{12}))$$

is a map from $E^2\,\mathrm{S}^{19}$ to S^3 such that

$$f(x \wedge \overline{s} \wedge \overline{t}) = \begin{cases} (-\widehat{A}_1)(\widehat{A}_3(x \wedge \overline{2s-1}) \wedge \overline{6t-1}) & \frac{1}{2} \le s \le 1,\ \frac{1}{6} \le t \le \frac{1}{3} \\ \eta_3\varepsilon_4\widehat{B}_2(x \wedge \overline{s} \wedge \overline{2t-1}) & \frac{1}{2} \le t \le 1 \\ * & \text{otherwise} \end{cases}.$$

Then $f \simeq (-\widehat{A}_1) \circ E\widehat{A}_3 + \eta_3\varepsilon_4\widehat{B}_2$. This can be proved by giving a homotopy. We omit details. Then $\{\eta_3\varepsilon_4, *^0_{12}, *^0_{12}, *^7_{12}; A_1, A_2, A_3; \widetilde{G}\}^{(1)} = (-\widehat{A}_1) \circ E\widehat{A}_3 + \eta_3\varepsilon_4 \circ \pi_{21}(\mathrm{S}^{12})$. Therefore

$$\begin{aligned} \{\eta_3\varepsilon_4, *^0_{12}, *^0_{12}, *^7_{12}\}^{(0)} &= \pi_{13}(\mathrm{S}^3) \circ E\pi_{20}(\mathrm{S}^{12}) + \eta_3\varepsilon_4 \circ \pi_{21}(\mathrm{S}^{12}) \\ &= \mathbb{Z}_2^2\{2\mu'\sigma_{14}, \nu'\bar{\varepsilon}_6\}. \end{aligned}$$

This completes the proof. □

Appendix B. Cohen's higher Toda brackets

Definition B.1. *A finitely filtered space is a space X together with subspaces $F_0X = \{*\} \subset F_1X \subset F_2X \subset \cdots$ of X such that $F_nX = X$ for some n and pairs $(F_{k+1}X, F_kX)$ $(k \geq 1)$ have homotopy extension property, that is, cofibred pairs.*

Definition B.2. *Given an integer $n \geq 2$ and a sequence of maps*

$$X_1 \xleftarrow{a_2} X_2 \xleftarrow{a_3} \cdots \xleftarrow{a_n} X_n,$$

we say that the finitely filtered space X is of type $(a_2, \cdots, a_n)$ if and only if $F_nX = X$ and there are homotopy equivalences $g_k : E^kX_{k+1} \underset{\simeq}{\to} F_{k+1}X/F_kX$ for $0 \leq k \leq n-1$ such that following diagrams are homotopy commutative for $1 \leq k \leq n-1$.

$$\begin{tikzcd}
EE^{k-1}X_k \arrow[r, equal] \arrow[d, "Eg_{k-1}"'] & E^kX_k & E^kX_{k+1} \arrow[l, "E^ka_{k+1}"'] \arrow[d, "\simeq"', "g_k"] \\
E(F_kX/F_{k-1}X) & EF_kX \arrow[l, "Eq"] & F_{k+1}X/F_kX \arrow[l, "\delta"]
\end{tikzcd}$$

Here $q : F_kX \to F_kX/F_{k-1}X$ are the quotient maps and δ are connecting maps of the cofibre sequences $F_{k+1}X/F_kX \longleftarrow F_{k+1}X \overset{\supset}{\longleftarrow} F_kX$ $(1 \leq k \leq n-1)$. We put

$$j_X : X_1 = E^0X_1 \xrightarrow{g_0} F_1X \subset X,$$

$$\sigma_X : X = F_nX \overset{q}{\twoheadrightarrow} F_nX/F_{n-1}X \overset{(g_{n-1})^{-1}}{\longrightarrow} E^{n-1}X_n,$$

where $(g_{n-1})^{-1}$ is a homotopy inverse of g_{n-1}.

Definition B.3. *Given an integer $n \geq 2$ and a sequence of maps*

$$X_0 \xleftarrow{a_1} X_1 \xleftarrow{a_2} X_2 \xleftarrow{a_3} \cdots \xleftarrow{a_n} X_n \xleftarrow{a_{n+1}} X_{n+1},$$

the $(n+1)$-fold Cohen's Toda bracket $\{a_1, a_2, \cdots, a_{n+1}\}^C$ is the set of all $\theta \in [E^{n-1}X_{n+1}, X_0]$ such that there are some space X of type $(a_2, \ldots, a_n)$ and maps g, h which make the following diagram homotopy commutative and θ is the homotopy class of $h \circ g$.

$$\begin{tikzcd}
 & E^{n-1}X_{n+1} \arrow[dl, "E^{n-1}a_{n+1}"'] \arrow[d, "g"] & \\
E^{n-1}X_n & X \arrow[l, "\sigma_X"] \arrow[d, "h"'] & X_1 \arrow[l, "j_X"'] \arrow[dl, "a_1"] \\
 & X_0 &
\end{tikzcd}$$

Proposition B.4. *If* $X_0 \xleftarrow{a_1} X_1 \xleftarrow{a_2} X_2 \xleftarrow{a_3} X_3$ *is a null triple, then*

$$\{a_1, a_2, a_3\} \subset \{a_1, a_2, a_3\}^C.$$

Proof. There is a cofibre sequence

$$EX_1 \xleftarrow{-Ea_2} EX_2 \xleftarrow{\quad q \quad} X_1 \cup_{a_2} CX_2 \xleftarrow{\quad i_{a_2} \quad} X_1 \xleftarrow{\quad a_2 \quad} X_2$$

which gives the following homotopy commutative diagram:

$$\begin{tikzcd}
EX_1 \arrow[r, equal] \arrow[d, equal] & EX_1 & EX_2 \arrow[l, "Ea_2"'] \arrow[d, "\simeq"', "-1_{EX_2}"] \\
EX_1 \arrow[r, equal] & EX_1 & (X_1 \cup_{a_2} CX_2)/X_1 \arrow[l, "\delta"]
\end{tikzcd}$$

We construct a space X of type (a_2) as follows:

$$X = X_1 \cup_{a_2} CX_2 = F_2 \supset F_1 = X_1,$$

$$g_0 = 1_{X_1} : X_1 \to F_1, \quad g_1 = -1_{EX_2} : EX_2 \to EX_2 = F_2/F_1.$$

Then

$$j_X : X_1 = F_1 \subset F_2 = X, \quad \sigma_X : X = F_2 \overset{q}{\twoheadrightarrow} F_2/F_1 = EX_2 \xrightarrow{-1} EX_2.$$

Take any element $\overline{a_1} \circ \widetilde{a}_3 \in \{a_1, a_2, a_3\}$. Then we obtain the following homotopy commutative diagram.

$$\begin{tikzcd}
 & & EX_3 \arrow[dll, bend right, "Ea_3"'] \arrow[dl, "-Ea_3"] \arrow[d, "\widetilde{a_3}"] & & & X_3 \arrow[dl, "a_3"'] \\
EX_2 & EX_2 \arrow[l, "-1"] & X \arrow[l, "q"] \arrow[d, "\overline{a_1}"'] & X_1 \arrow[l, "j_X"'] \arrow[dl, "a_1"] & X_2 \arrow[l, "a_2"'] & \\
 & & X_0 & & &
\end{tikzcd}$$

Hence $\overline{a_1} \circ \widetilde{a}_3 \in \{a_1, a_2, a_3\}^C$. Therefore $\{a_1, a_2, a_3\} \subset \{a_1, a_2, a_3\}^C$. □

Remark B.5. *In some cases,* $\{a_1, a_2, a_3\} \subsetneqq \{a_1, a_2, a_3\}^C$. *For example,*

(1) $\{2\iota_5, \nu_5\eta_8, 2\iota_9\} = \{\nu_5\eta_8^2\} \subsetneqq \mathbb{Z}_2\{\nu_5\eta_8^2\} = \pi_{10}(\mathrm{S}^5) = \{2\iota_5, \nu_5\eta_8, 2\iota_9\}^C$.

(2) *Let* $\mathbb{H}P^n$ *be the quaternionic projective* n*-space and* $p^n : \mathrm{S}^{4n+3} \to \mathbb{H}P^n$ *the canonical projection. Then* $\{*^1_3, p^1, 2E^3p^1\}^C$ *contains* 0*, while* $\{*^1_3, p^1, 2E^3p^1\}$ *is empty, that is, the triple* $(0^1_3, p^1, 2E^3p^1)$ *is not a null triple.*

Proof. (1) We define a space X of type $(\nu_5\eta_8)$ as follows.

$$X = \mathrm{S}^5 \vee \mathrm{S}^{10} = F_2, \quad F_1 = \mathrm{S}^5, \quad g_0 = \iota_5 : \mathrm{S}^5 \to \mathrm{S}^5, \quad g_1 = -\iota_9 : \mathrm{S}^9 \to \mathrm{S}^9,$$

$$j_X : F_1 \subset F_2, \quad \sigma_X : F_2 \xrightarrow{q} F_2/F_1 = \mathrm{S}^{10} \xrightarrow{-\iota_{10}} \mathrm{S}^{10}.$$

The following diagram is homotopy commutative.

$$\begin{tikzcd}
 & \mathrm{S}^{10} \arrow[dl, "2\iota_{10}"'] \arrow[d, "(*\vee-2\iota_{10})\circ\theta_{E\,\mathrm{S}^9}"] & \\
\mathrm{S}^{10} & \mathrm{S}^5\vee\mathrm{S}^{10} \arrow[l, "\sigma_X"] \arrow[d, "2\iota_5\,\underline{\vee}\,*"'] & \mathrm{S}^5 \arrow[l, "j_X"'] \arrow[dl, "2\iota_5"] \\
 & \mathrm{S}^5 &
\end{tikzcd}$$

Hence we have $\{2\iota_5,\nu_5\eta_8,2\iota_9\}^C \ni (2\iota_5\,\underline{\vee}\,0)\circ(0\vee-2\iota_{10})\circ\theta_{E\,\mathrm{S}^9}=0$ by Lemma 2.1. On the other hand, it follows from [**19**] that $\pi_{10}(\mathrm{S}^5)=\mathbb{Z}_2\{\nu_5\eta_8^2\}$, $\mathrm{Indet}\{2\iota_5,\nu_5\eta_8,2\iota_9\}=0$ and $\{2\iota_5,\nu_4\eta_7,2\iota_8\}_1\ni\nu_5\eta_8^2$ by [**19**, Corollary 3.7]. Therefore $\{2\iota_5,\nu_5\eta_8,2\iota_9\}=\{\nu_5\eta_8^2\}$ which does not contain 0. Hence (1) is proved by B.4.

(2) We can write

$$p^1=a\nu_4+bE\nu'+c\alpha_1(4)\in\pi_7(\mathrm{S}^4)=\mathbb{Z}\{\nu_4\}\oplus\mathbb{Z}_4\{E\nu'\}\oplus\mathbb{Z}_3\{\alpha_1(4)\},$$

where $a,b,c\in\mathbb{Z}$ with $|a|=|c|=1$. The space $\mathbb{H}P^2$ is of type (p^1):

$$\mathbb{H}P^2=\mathrm{S}^4\cup_{p^1}C\,\mathrm{S}^7=F_2,\quad F_1=\mathrm{S}^4,\quad g_0=\iota_4:\mathrm{S}^4\to\mathrm{S}^4,$$
$$g_1=-\iota_8:\mathrm{S}^8\to\mathrm{S}^8,\quad j=j_{\mathbb{H}P^2}:F_1=\mathrm{S}^4\subset F_2=\mathbb{H}P^2,$$
$$\sigma_{\mathbb{H}P^2}=(-\iota_8)\circ q:\mathbb{H}P^2\xrightarrow{q}\mathbb{H}P^2/\mathbb{H}P^1=\mathrm{S}^8\xrightarrow{-\iota_8}\mathrm{S}^8\,.$$

Recall from [**7**, (2.10a)] that $q\circ p^n=\pm nE^{4n-4}p^1$ so that $q\circ(\mp p^n)=-nE^{4n-4}p^1$, where $q:\mathbb{H}P^n\to\mathbb{H}P^n/\mathbb{H}P^{n-1}=\mathrm{S}^{4n}$ is the quotient map. Then we have the following homotopy commutative diagram.

$$\begin{tikzcd}
 & \mathrm{S}^{11} \arrow[dl, "2E^4p^1"'] \arrow[d, "-2E^4p^1"] & \mathrm{S}^{11} \arrow[l, "="'] \arrow[d, "\mp p^2"] & & & \mathrm{S}^{10} \arrow[dl, "2E^3p^1"'] \\
\mathrm{S}^8 & \mathrm{S}^8 \arrow[l, "-1"] & \mathbb{H}P^2 \arrow[l, "q"] \arrow[d, "*"'] & \mathrm{S}^4 \arrow[l, "j"'] \arrow[dl, "*_3^1"] & \mathrm{S}^7 \arrow[l, "p^1"'] & \\
 & & \mathrm{S}^3 & & &
\end{tikzcd}$$

Hence we have $\{0_3^1,p^1,2E^3p^1\}^C\ni 0\circ(\mp p^2)=0$. On the other hand, since $p^1\circ(2E^3p^1)=(2+4ab)\nu_4^2-\alpha_1(4)\alpha_1(7)$ is not null homotopic, the triple $(0_3^1,p^1,2E^3p^1)$ is not a null triple so that $\{0_3^1,p^1,2E^3p^1\}$ is not defined. □

Proposition B.6. *If a null quadruple (4.1) with* $n_1=n_2=0$ *has an admissible representative* $(a_1,a_2,a_3,a_4;A_1,A_2,A_3)$*, then*

$$\{[a_1,A_1,a_2],(a_2,A_2,a_3),-Ea_4\}=\{[a_1,A_1,a_2],-(a_2,A_2,a_3),Ea_4\}$$

$$\subset \{a_1, a_2, a_3, a_4\}^C,$$

$$\{a_1, a_2, a_3, a_4\}^{(2)} \subset \bigcup \{[a_1, A_1, a_2], (a_2, A_2, a_3), -Ea_4\} \subset \{a_1, a_2, a_3, a_4\}^C,$$

where the union is taken over A_1, A_2, A_3 *with* $(a_1, a_2, a_3, a_4; A_1, A_2, A_3)$ *admissible.*

Proof. We define a space X of type (a_2, a_3) as follows.

$$X = (X_1 \cup_{a_2} CX_2) \cup_{-\widetilde{a_3}} CEX_3 = F_3 \overset{i'}{\supset} F_2 = X_1 \cup_{a_2} CX_2 \overset{i}{\supset} F_1 = X_1,$$
$$g_0 = 1_{X_1} : X_1 \to F_1, \quad g_1 = -1_{EX_2} : EX_2 \to EX_2 = F_2/F_1,$$
$$g_2 = -1_{E^2X_3} : E^2X_3 \to E^2X_3 = F_3/F_2,$$

where $\widetilde{a_3} = (a_2, A_2, a_3)$. Then

$$j_X : X_1 = F_1 \subset F_3 = X, \quad \sigma_X : F_3 \twoheadrightarrow F_3/F_2 = E^2X_3 \xrightarrow{-1} E^2X_3.$$

We obtain the assertion from the following homotopy commutative diagram, where $\overline{a_1} = [a_1, A_1, a_2]$, $\overline{\overline{a_1}}$ is an extension of $\overline{a_1}$ with respect to $-\widetilde{a_3}$, and $\widetilde{Ea_4}$ is a coextension of Ea_4 with respect to $-\widetilde{a_3}$.

$$\begin{tikzcd}
 & & E^2X_4 \arrow[dll, bend right, "E^2a_4"'] \arrow[dl, "-E^2a_4"] \arrow[d, "\widetilde{Ea_4}"] & & & EX_4 \arrow[dl, "Ea_4"'] \\
E^2X_3 & E^2X_3 \arrow[l, "-1"] & X \arrow[l, "q"] \arrow[d, "\overline{\overline{a_1}}"'] & F_2 \arrow[l, "i'"'] \arrow[dl, "\overline{a_1}"] & EX_3 \arrow[l, "-\widetilde{a_3}"'] & \\
 & & X_0 & X_1 \arrow[l, "a_1"] \arrow[u, "i"'] & &
\end{tikzcd}$$

□

ACKNOWLEDGEMENT

The first author would like to thank Ken-ichi Maruyama for a meaningful discussion on $[C_{\eta_8}, \mathrm{SU}(3)]$.

REFERENCES

- **[1]** M. Arkowitz, Homotopy products for H-spaces, Michigan Math. J. **10** (1963), 1–9.
- **[2]** R. Bott, A note on the Samelson products in the classical groups, Comment. Math. Helv. **34** (1960), 249–256.
- **[3]** J. Cohen, The decomposition of stable homotopy, Ann. of Math. **87** (1968), 305–320.
- **[4]** H. Hamanaka and A. Kono, Homotopy type of gauge groups of SU(3)-bundles over S^6, Topology and its Appl. **154** (2007), 1377-1380.
- **[5]** P. Hilton, Homotopy theory and duality, Gordon and Breach, New York, 1965.
- **[6]** I. James, On spaces with a multiplication, Pacific J. Math. **7** (1957), 1083–1100.

[7] I. James, Spaces associated with Stiefel manifolds, Proc. London Math. Soc. **9** (1959), 115–140.

[8] S. Mac Lane, Categories for the working mathematician, second edition, GTM **5**, Springer, New York, 1998

[9] K. Maruyama and H. Ōshima, Homotopy groups of the spaces of self-maps of Lie groups, J. Math. Soc. Japan **60** (2008), 767–792.

[10] M. Mimura, On the generalized Hopf homomorphism and the higher composition, Part I, J. Math. Kyoto Univ. **4-1** (1964), 171–190.

[11] M. Mimura and H. Ōshima, Self homotopy groups of Hopf spaces with at most three cells, J. Math. Soc. Japan **51** (1999), 71–92.

[12] M. Mimura and H. Toda, Homotopy groups of SU(3), SU(4) and Sp(2), J. Math. Kyoto Univ. **3** (1964), 217–250.

[13] K. Ôguchi, A generalization of secondary composition and its application, J. Fac. Sci. Univ. Tokyo **X** (1963), 29–79.

[14] K. Ôguchi, Generators of 2-primary components of homotopy groups of spheres, unitary groups and symplectic groups, J. Fac. Sci. Univ. Tokyo **XI** (1964), 65–111.

[15] K. Ōshima and H. Ōshima, Homotopy groups of the spaces of self-maps of Lie groups II, Kodai Math. J. **32** (2009), 530–546.

[16] H. Toda, Generalized Whitehead products and homotopy groups of spheres, J. Inst. Poly. Osaka City Univ. Series A Math. **3** (1952), 43–82.

[17] H. Toda, Sur les groupes d'homotopie des sphères, C. R. Acad. Sci. Paris **240** (1955), 42–44.

[18] H. Toda, p-primary components of homotopy groups IV. Composition and toric constructions, Mem. Coll. Sci. Univ. Kyoto Ser. A **XXXII** (1959), 297–332.

[19] H. Toda, Composition methods in the homotopy groups of spheres, Ann. of Math. Studies **49**, Princeton Univ., Princeton, 1962.

H. ŌSHIMA
EMERITUS PROFESSOR
IBARAKI UNIVERSITY, MITO, IBARAKI 310-8512, JAPAN
e-mail address: ooshima@mx.ibaraki.ac.jp

K. ŌSHIMA
1-4001-5 ISHIKAWA, MITO, IBARAKI 310-0905, JAPAN
e-mail address: k-oshima@mbr.nifty.com

(*Received October 29, 2012*)
(*Revised February 22, 2013*)