Math. J. Okayama Univ. **56** (2014), 179–198

ON THE SOLVABILITY OF CERTAIN (SSIE) WITH OPERATORS OF THE FORM $B(r,s)$

Bruno de Malafosse and Eberhard Malkowsky

Abstract. Given any sequence $z=(z_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_z for the set of all sequences $y=(y_n)_{n\geq 1}$ such that $y/z=(y_n/z_n)_{n\geq 1}\in E$; in particular, $\mathbf{s}_z^{(c)}$ denotes the set of all sequences y such that y/z converges. In this paper we deal with *sequence spaces inclusion equations (SSIE)*, which are determined by an inclusion each term of which is a *sum or a sum of products of sets of sequences of the form $\chi_a(T)$ and $\chi_x(T)$* where a is a given sequence, the sequence x is the unknown, T is a given triangle, and $\chi_a(T)$ and $\chi_x(T)$ are the matrix domains of T in the set χ. Here we determine the set of all positive sequences x for which the *(SSIE)* $\mathbf{s}_x^{(c)}(B(r,s))\subset \mathbf{s}_x^{(c)}(B(r',s'))$ holds, where r,r',s' and s are real numbers, and $B(r,s)$ is the generalized operator of the first difference defined by $(B(r,s)y)_n=ry_n+sy_{n-1}$ for all $n\geq 2$ and $(B(r,s)y)_1=ry_1$. We also determine the set of all positive sequences x for which

$$\frac{ry_n+sy_{n-1}}{x_n}\to l \text{ implies } \frac{r'y_n+s'y_{n-1}}{x_n}\to l \ \ (n\to\infty) \text{ for all } y$$

and for some scalar l. Finally, for a given sequence a, we consider the a–Tauberian problem which consists of determining the set of all x such that $\mathbf{s}_x^{(c)}(B(r,s))\subset \mathbf{s}_a^{(c)}$.

1. Introduction

As usual we denote by ω the set of all complex sequences $x=(x_n)_{n\geq 1}$, and by c_0, c and ℓ_∞ the subsets of all null, convergent and bounded sequences, respectively; we write cs for the set of all convergent complex series. Also let U^+ denote the set of all sequences $u=(u_n)_{n\geq 1}$ with $u_n>0$ for all n. Given a sequence $a\in\omega$ and a subset E of ω, Wilansky [15] introduced the notation $a^{-1}*E=\{y\in\omega : ay=(a_ny_n)_{n\geq 1}\in E\}$. The sets $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$ were introduced in [3] by $((1/a_n)_{n\geq 1})^{-1}*E$ for any sequence $a\in U^+$ and $E\in\{\ell_\infty,c_0,c\}$. In [4, 5] the sum $\chi_a+\chi'_b$ and the product $\chi_a*\chi'_b$ were defined, where χ and χ' are any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$; also matrix transformations in the sets $\mathbf{s}_a+\mathbf{s}_b^0(\Delta^q)$ and $\mathbf{s}_a+\mathbf{s}_b^{(c)}(\Delta^q)$ were characterized, where Δ is the *operator of the first difference.* In [9] de Malafosse and

Mathematics Subject Classification. 40H05, 46A45.

Key words and phrases. Matrix transformations; BK space; the spaces $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$; (SSIE); (SSE) with operator; band matrix $B(r,s)$; Tauberian result.

Research of the second author supported by the research project #174007 of the Serbian Ministry of Science, Technology and Environmental Development.

Malkowsky gave the properties of the spectrum of the matrix of weighted means $\overline{N}_q$ considered as an operator in the set $\mathbf{s}_a$. In [10] characterizations can be found of the classes of matrix transformations from $\mathbf{s}_a(\Delta^q)$ into χ_b, where χ is any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$. Using the spectral properties of the operator of the first difference in the sets $\mathbf{s}^0_\alpha$ and $\mathbf{s}^{(c)}_\beta$, in [5] we were able to simply the set $\mathbf{s}^0_\alpha((\Delta-\lambda I)^h)+\mathbf{s}^{(c)}_\beta((\Delta-\mu I)^l)$, where h and l are complex numbers, and α and β are given sequences; also matrix transformations in this set were characterized in [5]. In [11] de Malafosse and Rakočević gave applications of the measure of noncompactness to operators on the spaces $\mathbf{s}_\alpha$, $\mathbf{s}^0_\alpha$, $\mathbf{s}^{(c)}_\alpha$ and ℓ^p_α to determine compact operators between some of these spaces. *Sequence spaces inclusion equations (SSIE)* and *sequence spaces equations (SSE)* were introduced and studied in [2, 8, 7]. They are determined by an inclusion or identity each term of which is a *sum* or a *sum of products of sets of the form* $\chi_a(T)$ *and* $\chi_{f(x)}(T)$ where χ is any of the symbols $\mathbf{s}$, $\mathbf{s}^0$, or $\mathbf{s}^{(c)}$, a is a given sequence in U^+, x is the unknown, f maps U^+ to itself, and T is a triangle. In this paper we use the operator represented by the triangle $B(r,s)$, called the generalized operator of the first difference and defined by $(B(r,s)y)_n = ry_n + sy_{n-1}$ for all $n \geq 2$ and $(B(r,s)y)_1 = ry_1$. Then we deal with the (SSIE) $\mathbf{s}^{(c)}_x(B(r,s)) \subset \mathbf{s}^{(c)}_x(B(r',s'))$, which is equivalent to

$$\frac{ry_n + sy_{n-1}}{x_n} \to l \text{ implies } \frac{r'y_n + s'y_{n-1}}{x_n} \to l' \;\; (n \to \infty) \text{ for all } y.$$

We then obtain extensions of results stated in [3, 2, 8, 7, 6]. The notion of an *a–Tauberian theorem* was introduced in [6] as follows. For a given sequence a, an *a–Tauberian theorem* is one in which the convergence of a sequence $y/a = (y_n/a_n)_{n\geq 1}$ is deduced from the convergence of some transform of the sequence together with some side conditions, the so–called *a–Tauberian conditions*. In [6], for given sequences λ and μ, we determined the set of all sequences a such that

$$\frac{1}{\lambda_n}\sum_{k=1}^{n}\mu_k\left(\sum_{i=k}^{\infty} y_i\right) \to l \text{ implies } \frac{y_n}{a_n} \to l' \;\; (n \to \infty)$$

for all $y \in cs$. In [6] *a–Tauberian theorem* is an extension of *Hardy's Tauberian theorem*. In *Hardy's Tauberian theorem* it is shown that under some condition for $y = (y_n)_{n\geq 1}$, we have $n^{-1}\sum_{k=1}^{n} y_k \to l$ implies $y_n \to l$ as n tends to infinity. In a similar way, for a given sequence a, we will determine the set of all positive sequences x for which

$$\frac{ry_n + sy_{n-1}}{x_n} \to l \text{ implies } \frac{y_n}{a_n} \to l \;\; (n \to \infty) \text{ for all } y.$$

If $a_n = 1$ for all n we obtain the classical *Tauberian problems.* In [14] we considered the (C, λ, μ) summability that generalizes the $(C, 1)$ summability and established conditions for the equivalence between the convergence of x_n/μ_n and the convergence of the sequence

$$\mu'_n = 1/\lambda_n \sum_{m=1}^{n} \widehat{\mu}_m(x),$$

where $\widehat{\mu}_n(x) = (x_1 + ... + x_n)/\mu_n$, and also for the equivalence between the convergence of $\widehat{\mu}_n(x)$ and the convergence of μ'_n.

This paper is organized as follows. In Section 2 we recall some results on AK and BK spaces and on the set $S_{a,b}$. In Section 3 we consider the operator $C(\xi)$ and its inverse $\Delta(\xi)$, and recall the definitions and properties of the sets $\widehat{\Gamma}$, $\widehat{C}$, Γ and $\widehat{C_1}$. In Section 4 we solve the (SSIE) $s_x^{(c)}(B(r,s)) \subset \mathbf{s}_x^{(c)}(B(r',s'))$ where $B(r,s)$ is the generalized operator of the first difference defined above. In Section 5 we determine the set of all sequences x of positive real numbers such that $(ry_n + sy_{n-1})/x_n \to l$ implies $(r'y_n + s'y_{n-1})/x_n \to l$ as n tends to infinity, for some scalar l and for given reals r, s, r' and s'. Finally in Section 6 we consider some *a–Tauberian theorems*; this is achieved by determining the set of all x such that $s_x^{(c)}(B(r,s)) \subset \mathbf{s}_a^{(c)}$.

2. Notations and preliminary results

Let $A = (a_{nk})_{n,k\geq 1}$ be an infinite matrix and $y = (y_k)_{k\geq 1}$ be a sequence. Then we write

$$A_n y = \sum_{k=1}^{\infty} a_{nk} y_k \text{ for any integer } n \geq 1 \tag{2.1}$$

and $Ay = (A_n y)_{n\geq 1}$ provided all the series in (2.1) converge.

Let E and F be any subsets of ω. Then we write (E, F) for the class of all infinite matrices A for which the series in (2.1) converge for all $y \in E$ and all n, and $Ay \in F$ for all $y \in E$. So if $A \in (E, F)$ then we are led to the study of the operator $\Lambda = \Lambda_A : E \to F$ defined by $\Lambda y = Ay$ and we identify the operator Λ with the matrix A.

A Banach space E of complex sequences is said to be a *BK space* if each projection $P_n : E \to \mathbb{C}$ defined by $P_n(y) = y_n$ for all $y = (y_n)_{n\geq 1} \in E$ is continuous. A BK space E is said to have *AK* if every sequence $y = (y_k)_{k\geq 1} \in E$ has a unique representation $y = \sum_{k=1}^{\infty} y_k e^{(k)}$ where $e^{(k)}$ is the sequence with 1 in the k-th position and 0 otherwise.

If u and v are sequences and E and F are two subsets of ω, then we write $uv = (u_n v_n)_{n\geq 1}$ and

$$M(E,F) = \{u = (u_n)_{n\geq 1} : \ uv \in F \text{ for all } v \in E\},$$

for the *multiplier space of* E *and* F.

To simplify notations, we use the diagonal matrix D_a defined by $[D_a]_{nn} = a_n$ for all n, write

$$D_a * E = (1/a)^{-1} * E = \{(y_n)_{n\geq 1} \in \omega \ : (y_n/a_n)_n \in E\}$$

for any $a \in U^+$ and any $E \subset \omega$, and define $\mathbf{s}_a = D_a * \ell_\infty$, $\mathbf{s}_a^0 = D_a * c_0$ and $\mathbf{s}_a^{(c)} = D_a * c$, (see, for instance, [4, 3, 11]). Each of the spaces $D_\alpha * \chi$, where $\chi \in \{\ell_\infty, c_0, c\}$, is a BK space normed by $\|\xi\|_{\mathbf{s}_a} = \sup_{n\geq 1}(|\xi_n|/a_n)$ and $\mathbf{s}_a^0$ has AK (see [15, Theorem 4.3.6]).

Now let $a = (a_n)_{n\geq 1}, b = (b_n)_{n\geq 1} \in U^+$. By $S_{a,b}$ we denote the set of all infinite matrices $\Lambda = (\lambda_{nk})_{n,k\geq 1}$ such that

$$\|\Lambda\|_{S_{a,b}} = \sup_{n\geq 1}\left(\frac{1}{b_n}\sum_{k=1}^{\infty}|\lambda_{nk}|a_k\right) < \infty.$$

It is well known that $\Lambda \in (\mathbf{s}_a, \mathbf{s}_b)$ if and only if $\Lambda \in S_{a,b}$. So we can write $(\mathbf{s}_a, \mathbf{s}_b) = S_{a,b}$.

When $\mathbf{s}_a = \mathbf{s}_b$ we obtain the *Banach algebra with identity* $S_{a,b} = S_a$ (see [3]), normed by $\|\Lambda\|_{S_a} = \|\Lambda\|_{S_{a,a}}$. We also have $\Lambda \in (\mathbf{s}_a, \mathbf{s}_a)$ if and only if $\Lambda \in S_a$.

If $a = (r^n)_{n\geq 1}$, the sets S_a, $\mathbf{s}_a$, $\mathbf{s}_a^0$ and $\mathbf{s}_a^{(c)}$ are denoted by S_r, $\mathbf{s}_r$, $\mathbf{s}_r^0$ and $\mathbf{s}_r^{(c)}$, respectively (see [4]). When $r = 1$, we obtain $\mathbf{s}_1 = \ell_\infty$, $\mathbf{s}_1^0 = c_0$ and $\mathbf{s}_1^{(c)} = c$, and witing $e = (1, 1, ...)$ we have $S_1 = S_e$. It is well known that $(\mathbf{s}_1, \mathbf{s}_1) = (c_0, \mathbf{s}_1) = (c, \mathbf{s}_1) = S_1$ (see, for instance, [15, Example 8.4.5A]).

In the sequel we will frequently use the obvious fact that $\Lambda \in (\chi_a, \chi'_b)$ if and only if $D_{1/b}\Lambda D_a \in (\chi_e, \chi'_e)$ where χ, χ' are any of the symbols $\mathbf{s}^0$, $\mathbf{s}^{(c)}$, or $\mathbf{s}$.

For any subset E of ω, we put $\Lambda E = \{\eta \in \omega \ : \eta = \Lambda y \text{ for some } y \in E\}$. If F is a subset of ω, we write $F(\Lambda) = F_\Lambda = \{y \in \omega : \ \Lambda y \in F\}$ *for the matrix domain of* Λ *in* F.

3. The operators $C(\xi)$, $\Delta(\xi)$ and the sets $\widehat{\Gamma}$, $\widehat{C}$, Γ and $\widehat{C_1}$

An infinite matrix $T = (t_{nk})_{n,k\geq 1}$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all n. Now let U be the set of all sequences $(u_n)_{n\geq 1} \in \omega$ with $u_n \neq 0$ for all n. If $\xi = (\xi_n)_{n\geq 1} \in U$, we write $C(\xi)$ for the triangle with

$$[C(\xi)]_{nk} = \begin{cases} \dfrac{1}{\xi_n} & \text{if } k \le n, \\ 0 & \text{otherwise,} \end{cases}$$

(see, for instance, [12]-[14]). It is easy to see that the triangle $\Delta(\xi)$ defined by

$$[\Delta(\xi)]_{nk} = \begin{cases} \xi_n & \text{if } k = n, \\ -\xi_{n-1} & \text{if } k = n-1 \text{ and } n \ge 2, \\ 0 & \text{otherwise,} \end{cases}$$

is the inverse of $C(\xi)$, that is, $C(\xi)(\Delta(\xi)y) = \Delta(\xi)(C(\xi)y) = y$ for all $y \in \omega$. If $\xi = e$ we get $\Delta(e) = \Delta$, where Δ is the well–known operator of the first difference defined by $\Delta_n y = y_n - y_{n-1}$ for all $y \in \omega$ and all $n \ge 1$, with the convention $y_0 = 0$. It is usual to write $\Sigma = C(e)$. We note that Δ and Σ are inverse to one another, and $\Delta, \Sigma \in S_R$ for any $R > 1$.

To simplify notation, for $t > 0$ and $\xi \in U^+$, we write $\xi'_n = t^{-n}\xi_n$ and

$$c_n(t,\xi) = \left[C\left(\xi'\right)\xi'\right]_n = \frac{t^n}{\xi_n}\sum_{k=1}^{n}\frac{\xi_k}{t^k} \text{ for all } n,$$

and

$$c_n(\xi) = c_n(1,\xi) = \frac{1}{\xi_n}\sum_{k=1}^{n}\xi_k \text{ for all } n.$$

We also consider the sets

$$\widehat{C} = \left\{\xi \in U^+ : c_n(\xi) \to l \ (n \to \infty) \text{ for some scalar } l\right\},$$

$$\widehat{C_1} = \left\{\xi \in U^+ : \quad \sup_n c_n(\xi) < \infty\right\},$$

$$\widehat{\Gamma} = \left\{\xi \in U^+ : \lim_{n\to\infty}\left(\frac{\xi_{n-1}}{\xi_n}\right) < 1\right\},$$

$$\Gamma = \left\{\xi \in U^+ : \limsup_{n\to\infty}\left(\frac{\xi_{n-1}}{\xi_n}\right) < 1\right\}$$

and

$$G_1 = \left\{\xi \in U^+ : \text{there are } C > 0 \text{ and } \gamma > 1 \text{ such that } \xi_n \ge C\gamma^n \text{ for all } n\right\}.$$

We obtain the next lemma by [3, Proposition 2.1, p. 1786] and [9, Proposition 2.2, p. 88].

Lemma 3.1. *We have* $\widehat{C} = \widehat{\Gamma} \subset \Gamma \subset \widehat{C_1} \subset G_1$.

4. On the (SSIE) $\mathbf{s}_x^{(c)}(B(r,s)) \subset \mathbf{s}_x^{(c)}(B(r',s'))$ for real numbers r, s, r' and s'

In this subsection we determine, for given real numbers r, s, r' and s', the set of all $x \in U^+$ such that

$$\frac{ry_n + sy_{n-1}}{x_n} \to l \text{ implies } \frac{r'y_n + s'y_{n-1}}{x_n} \to l' \ (n \to \infty) \text{ for all } y$$

and for some scalars l and l'. We will see that this is equivalent to determining the set of all $x \in U^+$ that satisfy the (SSIE)

$$\mathbf{s}_x^{(c)}(B(r,s)) \subset \mathbf{s}_x^{(c)}(B(r',s')), \tag{4.1}$$

where $B(r,s)$ and $B(r',s')$ are the generalized operators of the first difference.

We recall the next result which is a direct consequence of the famous Silverman-Toeplitz theorem.

Lemma 4.1. *We have:*

i) $\Lambda \in (c,c)$ *if and only if*

$$\Lambda \in S_1,\ \lim_{n\to\infty} \sum_{k=1}^{\infty} \lambda_{nk} = l \text{ and } \lim_{k\to\infty} \lambda_{nk} = l_k \text{ for all } k \geq 1$$

for some scalars l and l_k (see, for instance, [15, Theorem 1.3.6]).

ii) *Let $\Lambda \in (c,c)$ and $y \in c$. If $\lim_{k\to\infty} \lambda_{nk} = 0$ for all $k \geq 1$, then*

$$\lim_{n\to\infty} y_n = L \text{ implies } \lim_{n\to\infty} \Lambda_n y = lL$$

(see, for instance, [15, Theorem 1.3.8]).

To state the next theorem we need the following result.

Proposition 4.2. *Let $x \in U^+$. Then*

$$c_n(x) = \frac{1}{x_n} \sum_{k=1}^{n} x_k \to l \text{ if and only if } \frac{x_{n-1}}{x_n} \to 1 - \frac{1}{l} \ (n \to \infty)$$

for some scalar l.

Proof. We put $L = 1 - 1/l$ and $\Sigma_n = \sum_{k=1}^{n} x_k$ and note that $l \geq 1$, since $\Sigma_n / x_n = 1 + \Sigma_{n-1}/x_n \geq 1$ for all n.
It was shown in [3, Proposition 2.1, p. 1786] that $c_n(x) \to l$ $(n \to \infty)$ implies $x_{n-1}/x_n \to 1 - 1/l$ $(n \to \infty)$.
To show the converse implication, we assume $x_{n-1}/x_n \to 1 - 1/l$ $(n \to \infty)$.

Since we have $\widehat{C} = \widehat{\Gamma}$ by Lemma 3.1, we can write $\Sigma_n/x_n \to l_1$ $(n \to \infty)$ for some scalar l_1, and must show $l_1 = l$. We have for every $n > 2$

$$\frac{x_{n-1}}{x_n} = \frac{\Sigma_{n-1} - \Sigma_{n-2}}{x_n} = \frac{\Sigma_{n-1}}{x_{n-1}}\frac{x_{n-1}}{x_n} - \frac{\Sigma_{n-2}}{x_{n-2}}\frac{x_{n-2}}{x_{n-1}}\frac{x_{n-1}}{x_n}$$

and

$$\frac{\Sigma_{n-1} - \Sigma_{n-2}}{x_n} \to l_1 L - l_1 L^2 = L \ (n \to \infty).$$

If $L \neq 0$ then we have $l_1 = 1/(1-L)$ and since $L = 1 - 1/l$, we conclude

$$l_1 = \frac{1}{1 - \left(1 - \dfrac{1}{l}\right)} = l.$$

If $L = 0$ then we have $l = 1$ and

$$\frac{\Sigma_n}{x_n} = \frac{\Sigma_{n-1}}{x_{n-1}}\frac{x_{n-1}}{x_n} + 1 \to 1 \ (n \to \infty).$$

□

We recall that $B(r,s)$, where r and s are real numbers, is the lower triangular matrix

$$B(r,s) = \begin{pmatrix} r & & & \\ s & r & & 0 \\ & s & r & \\ 0 & & \cdot & \cdot \\ & & & \cdot & \cdot \end{pmatrix}.$$

For $r, s \neq 0$, the matrix $B(r,s)$ was introduced by Altay and Basar [1] and was called the *generalized operator of the first difference.*

In the next theorem we confine our studies to the case when $\alpha = -s/r > 0$ if $\delta = rs' - r's \neq 0$.

Theorem 4.3. *Let r, s, r' and s' be real numbers with $r, s \neq 0$, and $\delta = rs' - r's$.*

i) *If $\delta = 0$, then (SSIE) (4.1) holds for all x.*
ii) *If $\delta \neq 0$ and $\alpha = -s/r > 0$, then (4.1) holds if and only if*

$$\lim_{n\to\infty} \frac{x_{n-1}}{x_n} < \frac{1}{\alpha}.$$

Proof. Inclusion (4.1) is equivalent to $I \in (\mathbf{s}_x^{(c)}(B(r,s)), \mathbf{s}_x^{(c)}(B(r',s')))$, that is, to

$$\widetilde{B} = B(r',s')B^{-1}(r,s) \in \left(\mathbf{s}_x^{(c)}, \mathbf{s}_x^{(c)}\right).$$

This means

$$D_{1/x}\widetilde{B}D_x \in (c,c). \tag{4.2}$$

Since $r \neq 0$, the matrix $B(r,s)$ is invertible, its inverse is a triangle and elementary calculations give

$$\left[B^{-1}(r,s)\right]_{nk} = \frac{1}{r}\alpha^{n-k} \text{ for } 1 \leq k \leq n.$$

Then we obtain $\widetilde{B}_{nn} = r'/r$, and have for $k \leq n-1$

$$\begin{aligned}\widetilde{B}_{nk} &= s'\left[B^{-1}(r,s)\right]_{n-1,k} + r'\left[B^{-1}(r,s)\right]_{nk} \\ &= s'\frac{1}{r}\alpha^{n-k-1} + \frac{r'}{r}\alpha^{n-k} \\ &= \alpha^{n-k-1}\left(\frac{s'}{r} + \frac{r'}{r}\alpha\right) = \alpha^{n-k-1}\frac{\delta}{r^2}.\end{aligned}$$

It follows that

$$\left[D_{1/x}\widetilde{B}D_x\right]_{nk} = \begin{cases} \dfrac{1}{x_n}\alpha^{n-k-1}\dfrac{\delta}{r^2}x_k & \text{for } k \leq n-1, \\ \dfrac{r'}{r} & \text{for } k = n. \end{cases}$$

We deduce from the characterization of (c,c) in Lemma 4.1 (i) that (4.2) holds if and only if

$$\sum_{k=1}^{n}\left[D_{1/x}\widetilde{B}D_x\right]_{nk} = \frac{r'}{r} - \frac{\delta}{rs}\widetilde{c}_n(\alpha,x) \to l \ (n \to \infty) \tag{4.3}$$

for some scalar l, where

$$\widetilde{c}_n(\alpha,x) = c_n(\alpha,x) - 1 = \frac{1}{\dfrac{x_n}{\alpha^n}}\sum_{k=1}^{n-1}\frac{x_k}{\alpha^k}.$$

Indeed this condition implies $D_{1/x}\widetilde{B}D_x \in S_1$ and $(x_n/\alpha^n)_n \in \widehat{C}$. Since we have $\widehat{C} \subset G_1$ by Lemma 3.1, we deduce $x_n/\alpha^n \to \infty \ (n \to \infty)$ and have for each k and for $n > k$

$$\left[D_{1/x}\widetilde{B}D_x\right]_{nk} = \frac{1}{x_n}\alpha^{n-k-1}\frac{\delta}{r^2}x_k = \frac{\alpha^n}{x_n}\left(\alpha^{-k-1}\frac{\delta}{r^2}x_k\right) = o(1) \ (n \to \infty).$$

i) If $\delta = 0$ then the sum in (4.3) reduces to r'/r and inclusion (4.1) holds for all x.

ii) If $\delta \neq 0$ then inclusion (4.1) means that (4.3) is convergent and

$$\widetilde{c}_n(\alpha,x) \to -\frac{l - \dfrac{r'}{r}}{\dfrac{1}{rs}\delta} \ (n \to \infty),$$

so we have $(x_n/\alpha^n)_n \in \widehat{C}$. By Lemma 3.1 we have $\widehat{C} = \widehat{\Gamma}$, and so (4.2) is equivalent to

$$\lim_{n\to\infty} \frac{x_{n-1}}{\alpha^{n-1}} \frac{\alpha^n}{x_n} = \alpha \lim_{n\to\infty} \frac{x_{n-1}}{x_n} < 1.$$

This shows ii).

□

The following result can easily be shown when $r = 0$ or $s = 0$.

Theorem 4.4. *Let r, s, r' and s' be real numbers.*

i) *Let $r \neq 0$ and $s = 0$.*

a) *If $s' \neq 0$, then (4.1) holds if and only if*

$$\frac{x_{n-1}}{x_n} \to l \ (n \to \infty) \text{ for some scalar } l.$$

b) *If $s' = 0$, then (4.1) holds for all x.*

ii) *Let $r = 0$ and $s \neq 0$.*

a) *If $r' \neq 0$, then (4.1) holds if and only if*

$$\frac{x_n}{x_{n-1}} \to l' \ (n \to \infty) \text{ for some scalar } l'.$$

b) *If $r' = 0$, then (4.1) holds for all x.*

iii) *Let $r = s = 0$.*

a) *If $r' \neq 0$, or $s' \neq 0$, then (4.1) has no solution.*

b) *If $r' = s' = 0$, then (4.1) holds for all x.*

Proof. We only prove Part i), the proofs of the other parts are left to the reader.

i) Let $r \neq 0$ and $s = 0$.
Since $B(r,s) = rI$ we have $\mathbf{s}_x^{(c)}(B(r,s)) = \mathbf{s}_x^{(c)}$. So inclusion (4.1) is equivalent to $D_{1/x}B(r',s')D_x \in (c,c)$. This means that there are $K \geq 0$ and L such that

$$(*) \qquad \begin{cases} |r'| + |s'|\dfrac{x_{n-1}}{x_n} \leq K \text{ for all } n, \\ r' + s'\dfrac{x_{n-1}}{x_n} \to L \ (n \to \infty). \end{cases}$$

a) If $s' \neq 0$ then we have

$$\frac{x_{n-1}}{x_n} \to \frac{L - r'}{s'} \ (n \to \infty).$$

b) If $s' = 0$ then the system (*) is satisfied for all x.

□

In the general case when r, s, δ, $\alpha \neq 0$ we can state the following remark.

Remark. Condition (4.1) holds if and only if

$$\text{(i)} \qquad \frac{\alpha^n}{x_n}\sum_{k=1}^{n-1}\frac{x_k}{\alpha^k} \to l \ (n\to\infty),$$

$$\text{(ii)} \qquad \frac{|\alpha|^n}{x_n}\sum_{k=1}^{n-1}\frac{x_k}{|\alpha|^k} \le K \text{ for all } n$$

and

$$\text{(iii)} \qquad \frac{\alpha^n}{x_n} \to l' \ (n\to\infty)$$

for some scalars l and l', and a constant $K > 0$. This result is a direct consequence of condition (4.2) in the proof of Theorem 4.3.

5. The case of regularity

5.1. The set of all $x \in U^+$ such that $x_n^{-1}B(r,s)y_n \to l$ implies $x_n^{-1}B(r',s')y_n \to l$ $(n\to\infty)$ for all y and for some l. A matrix $A \in (c,c)$ and the corresponding operator Λ are said to be *regular* if $y_n \to l$ implies $A_n y \to l$ $(n\to\infty)$ for all $y \in \omega$ and for some scalar l. We then write $A \in (c,c)_{reg}$. As a direct consequence of Lemma 4.1, we have the known result (see, for instance, [15, Theorem 1.3.9])

Lemma 5.1. *We have $\Lambda \in (c,c)_{reg}$ if and only if the next statements hold,*
a) $\Lambda \in S_1$,
b) $\sum_{k=1}^{\infty}\lambda_{nk} \to 1$ $(n\to\infty)$,
c) $\lambda_{nk} \to 0$ $(n\to\infty)$ *for* $k = 1, 2, \dots$.

Now we consider the next question, where r, s, r' and s' are real numbers. What is the set of all $x \in U^+$ such that

$$\frac{ry_n + sy_{n-1}}{x_n} \to l \text{ implies } \frac{r'y_n + s'y_{n-1}}{x_n} \to l \ (n\to\infty) \text{ for all } y \tag{5.1}$$

and for some scalar l? The answer to this question is given by the following theorem where we confine our studies to the case $-s/r > 0$ when $\delta \neq 0$.

Theorem 5.2. *Let r, s, r' and s' be real numbers.*

i) *Let $\delta \neq 0$ and $\alpha = -s/r > 0$.*
   a) *If $\tau = (r-r')/(s-s') \le 0$, then (5.1) holds if and only if*
   $$\lim_{n\to\infty}\frac{x_{n-1}}{x_n} = -\tau.$$
   b) *If $\tau > 0$, then (5.1) has no solutions.*

ii) *Let $\delta = 0$ and $r \neq 0$.*
   a) *If $r = r'$, then (5.1) holds for all x.*

b) *If $r \neq r'$, then (5.1) has no solution.*

Proof. First we note that statement (5.1) obviously means that

(5.2)
$$z_n = \left[D_{1/x}B(r,s)y\right]_n \to l \text{ implies } t_n = \left[D_{1/x}B(r',s')y\right]_n \to l \ (n \to \infty)$$

for all y and for some scalar l. Since $y = B^{-1}(r,s)D_x z$, for $r \neq 0$ statement (5.2) is equivalent to

$$z_n \to l \text{ implies } \left[D_{1/x}\widetilde{B}D_x z\right]_n \to l \ (n \to \infty)$$

where $\widetilde{B} = B(r',s')B^{-1}(r,s)$. Then (5.1) is equivalent to

$$D_{1/x}\widetilde{B}D_x \in (c,c)_{reg}, \tag{5.3}$$

which, by Lemma 5.1, is equivalent to

$$\begin{aligned} D_{1/x}\widetilde{B}D_x &\in S_1, \\ \sum_{k=1}^{n}\left[D_{1/x}\widetilde{B}D_x\right]_{nk} &\to 1 \ (n \to \infty), \end{aligned}$$

and

$$\left[D_{1/x}\widetilde{B}D_x\right]_{nk} \to 0 \ (n \to \infty) \text{ for all } k.$$

Using this characterization of $(c,c)_{reg}$ and reasoning as in Theorem 4.3, we deduce that (5.3) holds if and only if

$$\sum_{k=1}^{n}\left[D_{1/x}\widetilde{B}D_x\right]_{nk} = \frac{r'}{r} - \frac{\delta}{rs}\widetilde{c}_n(\alpha,x) \to 1 \ (n \to \infty). \tag{5.4}$$

i) Now we can show a) and b).
Putting $z_n = x_n\alpha^{-n}$, we have

$$\widetilde{c}_n(z) = \frac{1}{z_n}\sum_{k=1}^{n-1} z_k \to L \ (n \to \infty),$$

where

$$L = \frac{1 - \dfrac{r'}{r}}{-\dfrac{\delta}{rs}} = -\frac{r - r'}{\delta}s \geq 0. \tag{5.5}$$

Then we obtain $c_n(z) = \widetilde{c}_n(z) + 1 \to L + 1$ $(n \to \infty)$, and deduce by Proposition 4.2 that (5.1) is equivalent to

$$\frac{z_{n-1}}{z_n} \to 1 - \frac{1}{L+1} = \frac{L}{L+1} \ (n \to \infty).$$

Using (5.5) we immediately obtain $L/(L+1) = -\alpha\tau$. We conclude

$$\frac{x_{n-1}}{x_n} = \frac{z_{n-1}}{z_n}\frac{1}{\alpha} \to -\tau \geq 0 \ (n \to \infty).$$

ii) If $\delta = 0$ the sum defined in (5.4) reduces to $r'/r = 1$, that is, $r = r'$. We then have $s = s'$ and (5.1) holds for all x.

□

Now give a remark in which we consider a Tauberian problem using the operator of the generalized difference sequence.

Remark. If $r > 1$ or $r < 0$, then $ry_n + (1-r)y_{n-1} \to l$ implies $y_n \to l$ $(n \to \infty)$ for all y and for some scalar l. Indeed, it is enough to take $r' = 1$, $s' = 0$ and $x = e$ in Theorem 4.3. Then we have $1 = -(r-1)/s$ with $-s/r > 0$.

Now we consider the equivalence

$$(5.6) \quad \frac{ry_n + sy_{n-1}}{x_n} \to l \text{ if and only if } \frac{r'y_n + s'y_{n-1}}{x_n} \to l \ (n \to \infty) \text{ for all } y$$

and for some scalar l. Note that in [3] we determined the set of all $x \in U^+$ such that $\mathbf{s}_x^{(c)}(\Delta) = \mathbf{s}_x^{(c)}$. In [7] we gave a necessary and sufficient condition under which a, $b \in U^+$ satisfy $\mathbf{s}_a^{(c)}(\Delta) = \mathbf{s}_b^{(c)}$. Since we have $B(-1,1) = \Delta$ and $B(1,0) = I$, then $\mathbf{s}_x^{(c)}(B(-1,1)) = \mathbf{s}_x^{(c)}(\Delta)$ and $\mathbf{s}_x^{(c)}(B(1,0)) = \mathbf{s}_x^{(c)}$. Thus we see that condition (5.6) is an extension of [3, 7].

We obtain the next result as a direct consequence of Theorem 5.2.

Theorem 5.3. *Let r, s, r' and s' be real numbers, all different from zero.*

i) *Let $\delta \neq 0$ and r/s, $r'/s' < 0$.*
   a) *If $\tau = (r-r')/(s-s') \leq 0$, then the solutions of (5.6) are defined by*
   $$\lim_{n\to\infty} \frac{x_{n-1}}{x_n} = -\tau.$$
   b) *If $\tau > 0$, then (5.6) has no solutions.*

ii) *Let $\delta = 0$.*
   a) *If $r = r'$, then (5.6) holds for all x.*
   b) *If $r \neq r'$, then (5.6) has no solution.*

Now we deal with the case when $r = 0$ or $s = 0$.

Theorem 5.4. i) *We assume $r \neq 0$ and $s = 0$.*
   a) *Let $s' \neq 0$.*
      α) *If $\tau_1 = (r-r')/s' \geq 0$, then (5.1) holds if and only if*

$$(5.7) \quad \lim_{n\to\infty} \frac{x_{n-1}}{x_n} = \tau_1.$$

β) *If* $\tau_1 < 0$, *then (5.1) has no solution.*

b) *Let* $s' = 0$.

α) *If* $r = r'$, *then (5.1) holds for all* x.

β) *If* $r \neq r'$, *then (5.1) has no solution.*

(ii) *We assume* $r = 0$ *and* $s \neq 0$.

a) *Let* $r' \neq 0$.

α) *If* $l = 0$, *then (5.1) is equivalent to* $(x_n/x_{n-1})_n \in \ell_\infty$.

β) *If* $l \neq 0$, *then condition (5.1) holds if and only if*

$$\lim_{n\to\infty} \frac{x_n}{x_{n-1}} = \frac{s - s'}{r'} \geq 0.$$

b) *Let* $r' = 0$.

α) *If* $s' = s$, *then (5.1) holds for all* x.

β) *If* $s' \neq s$, *then (5.1) has no solution.*

(iii) *Let* $r = s = 0$.

a) *If* $r' \neq 0$, *or* $s' \neq 0$, *then (5.1) has no solution.*

b) *If* $r' = s' = 0$, *then (5.1) holds for all* x.

Proof. i) We assume $r \neq 0$ and $s = 0$. Since $B(r, s) = rI$, statement (5.1) is equivalent to $D_{1/x}B(r'/r, s'/r)D_x \in (c, c)_{reg}$, that is,

$$\left|\frac{r'}{r}\right| + \left|\frac{s'}{r}\right| \frac{x_{n-1}}{x_n} \leq K \text{ for all } n, \tag{5.8}$$

$$\frac{r'}{r} + \frac{s'}{r}\frac{x_{n-1}}{x_n} \to 1 \ (n \to \infty). \tag{5.9}$$

a) Let $s' \neq 0$. Since condition (5.9) implies (5.8), statement (5.1) is equivalent to (5.7).

b) Let $s' = 0$.

α) If $r = r'$, then the previous system holds for all x.

β) If $r \neq r'$, then the system has no solution.

ii) We assume $r = 0$ and $s \neq 0$.

a) Let $r' \neq 0$. Then statement (5.1) reduces to

$$s\frac{y_{n-1}}{x_n} \to l \text{ implies } t_n = \frac{r'y_n + s'y_{n-1}}{x_n} \to l \ (n \to \infty). \tag{5.10}$$

α) If $l = 0$, then we have

$$\mathbf{s}_x^0\left(B(0, s)\right) = \left\{ y \in \omega : \frac{y_n}{x_{n+1}} = o(1) \ (n \to \infty) \right\} = \mathbf{s}_{x^+},$$

where $x^+ = (x_{n+1})_n$. Then statement (5.1) with $l = 0$ is equivalent to $\mathbf{s}^0_{x^+} \subset \mathbf{s}^0_x(B(r', s'))$, $B(r', s') \in (\mathbf{s}^0_{x^+}, \mathbf{s}^0_x)$, that is, to

$$|r'|\frac{x_{n+1}}{x_n} + |s'| \leq K \text{ for all } n. \tag{5.11}$$

Obviously the condition in (5.11) is equivalent to

$$(x_n/x_{n-1}) \in \ell_\infty.$$

β) If $l \neq 0$, we put $z_n = sy_{n-1}/x_n$. Then (5.1) is equivalent to

$$z_n \to l \text{ implies } t_n = \frac{r'}{s} z_{n+1} \frac{x_{n+1}}{x_n} + \frac{s'}{s} z_n \to l \ (n \to \infty),$$

that is, to

$$\frac{x_{n+1}}{x_n} = \frac{t_n - \frac{s'}{s} z_n}{\frac{r'}{s} z_{n+1}} \to \frac{s - s'}{r'} \ (n \to \infty).$$

b) Let $r' = 0$. Then $z_n = sy_{n-1}/x_n \to l$ implies $s'y_{n-1}/x_n \to l = ls'/s$ $(n \to \infty)$.
   α) If $s' = s$, then statement (5.1) holds for all $x \in U^+$.
   β) If $s' \neq s$, then (5.1) has no solution.

iii) We assume $r = s = 0$. Then we must have $B(r', s') \in (\omega, \mathbf{s}_x^0)$ which implies $r' = s' = 0$. Indeed we assume either $r' \neq 0$ or $s' \neq 0$.
Let $r' \neq 0$. We consider the cases $s'/r' \geq 0$ and $s'/r' < 0$.
If $s'/r' \geq 0$, then we take $y = (R^n x_n)_n \in \omega$ with $R > 1$, and obtain

$$\left|\frac{B(r', s')y_n}{x_n}\right| = \frac{|r'|}{x_n}\left|y_n + \frac{s'}{r'} y_{n-1}\right| \geq |r'|R^n \text{ for all } n.$$

Then we have $|B(r', s')y_n/x_n| \to \infty$ $(n \to \infty)$ and $\omega \subset s_x(B(r', s'))$ is impossible.
If $s'/r' < 0$, then we take $y_n = (-R)^n x_n$ with $R > 1$, and obtain

$$\begin{aligned}\left|\frac{B(r', s')y_n}{x_n}\right| &= \left|\frac{r'}{x_n}\left(y_n + \frac{s'}{r'} y_{n-1}\right)\right| = |r'|R^n\left(1 - \frac{s'}{r'}\frac{x_{n-1}}{Rx_n}\right) \\ &\geq |r'|R^n \text{ for all } n,\end{aligned}$$

and we conclude as above.
The case $s' \neq 0$ can be treated similarly.

$\square$

5.2. **Applications.** Let $r<0$ and $s>-1$, and different from 0 and consider the sets

$$S_1(r)=\Big\{x\in U^+ : \frac{ry_n+y_{n-1}}{x_n}\to l \text{ implies } \frac{\Delta y_n}{x_n}\to l\ (n\to\infty)$$
$$\text{for all } y\in\omega \text{ and for some scalar } l\}$$

and

$$S_2(s)=\Big\{x\in U^+ : \frac{\Delta y_n}{x_n}\to l \text{ implies } \frac{sy_n}{x_n}\to l\ \ (n\to\infty)$$
$$\text{for all } y\in\omega \text{ and for some scalar } l\}\,.$$

We can determine the set $S_1(r)\cap S_2(s)$. Since $\delta=-r+1\neq 0$, we have by Theorem 5.2

$$S_1(r)=\left\{x\in U^+ : \frac{x_{n-1}}{x_n}\to\frac{1-r}{2}\ (n\to\infty)\right\},$$

and similarly

$$S_2(s)=\left\{x\in U^+ : \frac{x_{n-1}}{x_n}\to\frac{1}{1+s}\ (n\to\infty)\right\}.$$

We conclude

$$S_1(r)\cap S_2(s)=\begin{cases} S_2(s) & \text{if } s=(1+r)/(1-r),\\ \emptyset & \text{otherwise.}\end{cases}$$

Note that if $r<0$, then $S_1(r)\cap S_2(s)\neq\emptyset$ implies $|s|<1$ and $s\neq 0$.

6. The a-Tauberian (SSIE) $\mathbf{s}_x^{(c)}(B(r,s))\subset\mathbf{s}_a^{(c)}$

6.1. ***a*-Tauberian (SSIE) with operators of the form $B(r,s)$.** Here we consider the a-Tauberian (SSIE) problem for given $a\in U^+$, (see [6]), stated as follows. Let r, s, r' and s' be real numbers, and let a be a given sequence; what is the set $\mathbb{S}_a$ of all $x\in U^+$ such that

$$\frac{ry_n+sy_{n-1}}{x_n}\to l \text{ implies } \frac{y_n}{a_n}\to l'\ (n\to\infty) \text{ for all } y,$$

and for some scalars l and l'? This statement is equivalent to the solvability of the (SSIE)

$$\mathbf{s}_x^{(c)}\,(B(r,s))\subset\mathbf{s}_a^{(c)}. \tag{6.1}$$

As we will see in Proposition 6.1, since the condition on the sequence a is less restrictive for (6.1) than for the (SSIE) $\mathbf{s}_a^{(c)}\,(B(r,s))\subset\mathbf{s}_x^{(c)}$ it is natural to begin with the study of the set $\mathbb{S}_a$. To state the next result, we use the set cs_b of all $x \in U^+$ such that $\sum_{k=1}^{\infty} x_k/b_k < \infty$, where $b \in U^+$. For $b = e$ we obtain $cs_b = cs \cap U^+$. Throughout this section we assume $\alpha = -s/r > 0$.

Proposition 6.1. *We assume $(\alpha^n/a_n)_n \in c$. Then $x \in \mathbb{S}_a$ if and only if*

$$\left(\frac{\alpha^n}{a_n}\sum_{k=1}^{n}\frac{x_k}{\alpha^k}\right)_n \in c. \tag{6.2}$$

Moreover if $a_n \sim \lambda\alpha^n$ $(n \to \infty)$ for $\lambda > 0$, that is, $a_n/\lambda\alpha^n \to 1$ $(n \to \infty)$, then we have

$$S_a = cs_{(\alpha^n)_n}.$$

Proof. We have $x \in \mathbb{S}_a$ if and only if (6.1) holds, which is equivalent to

$$B^{-1}(r,s) \in \left(\mathbf{s}_x^{(c)}, \mathbf{s}_a^{(c)}\right), \tag{6.3}$$

that is, to $D_{1/a}B^{-1}(r,s)D_x \in (c,c)$. From the expression of $B^{-1}(r,s)$ in the proof of Theorem 5.2, and the characterization of (c,c), condition (6.3) is equivalent to (6.2) and $(\alpha^n/a_n)_n \in c$. Now we assume $a_n/\alpha^n \to \lambda > 0$ $(n \to \infty)$. Then we have $x \in \mathbb{S}_a$ if and only if

$$u_n = \frac{\alpha^n}{a_n}\sum_{k=1}^{n}\frac{x_k}{\alpha^k} \to L \ (n \to \infty)$$

for some scalar L, that is,

$$\sum_{k=1}^{n}\frac{x_k}{\alpha^k} = \frac{u_n}{\frac{\alpha^n}{a_n}} \to \frac{L}{\lambda} \ (n \to \infty),$$

and $x \in cs_{(\alpha^n)_n}$. □

When $a = e$, we obtain the next Tauberian result.

Corollary 6.2. i) *If $0 < \alpha \le 1$, then $x \in \mathbb{S}_e$ if and only if*

$$\left(\alpha^n\sum_{k=1}^{n}\frac{x_k}{\alpha^k}\right)_n \in c.$$

ii) *If $\alpha = 1$, then $\mathbb{S}_e = cs \cap U^+$.*

As a direct application we also have the next result,

Corollary 6.3. *We assume $0 < \alpha < 1$. Then $(x^n)_n \in \mathbb{S}_e$ if and only if $0 < x \le 1$.*

Proof. First we assume $x \neq \alpha$. Since $x_k = x^k$ for all k, we have $(x^n)_n \in \mathbb{S}_e$ if and only if

$$\begin{aligned}\alpha^n \sum_{k=1}^{n} \frac{x_k}{\alpha^k} &= \alpha^n \frac{x}{\alpha}\frac{1}{1-\dfrac{x}{\alpha}} - \alpha^n \left(\frac{x}{\alpha}\right)^{n+1} \frac{1}{1-\dfrac{x}{\alpha}} \\ &= \alpha^{n-1} x \frac{1}{1-\dfrac{x}{\alpha}} - \frac{x^{n+1}}{\alpha}\frac{1}{1-\dfrac{x}{\alpha}},\end{aligned}$$

is convergent as n tends to infinity, that is, for $0 < x \leq 1$ and $x \neq \alpha$. If $x = \alpha < 1$, we have $\alpha^n \sum_{k=1}^{n} (x/\alpha)^k = n\alpha^n = o(1)$ $(n \to \infty)$. □

We immediately deduce the next examples.

Example. Let u, $v > 0$. Then $x \in U^+$ satisfies the condition

$$\frac{uy_n - vy_{n-1}}{x_n} \to l \text{ implies } \left(\frac{u}{v}\right)^n y_n \to l' \ (n \to \infty)$$

for all y and for some scalars l and l',

if and only if $\sum_{k=1}^{\infty} (u/v)^k x_k < \infty$. This result can be obtained writing $\alpha = v/u$ and $a_n = \alpha^n$ in Proposition 6.1. In particular, if $u = v = 1$, then the set of all $x \in U^+$ such that

$$\frac{\Delta y_n}{x_n} \to l \text{ implies } y_n \to l' \ (n \to \infty) \text{ for all } y \text{ and for some scalars } l \text{ and } l'$$

is equal to $cs \cap U^+$.

Remark. We obtain a similar result when a and x are interchanged in (SSIE) (6.1). Indeed, let $a \in cs_{(\alpha^n)_n}$ and let $\overline{\mathbb{S}}_a$ be the set of all $x \in U^+$ such that the (SSIE) $\mathbf{s}_a^{(c)}(B(r,s)) \subset \mathbf{s}_x^{(c)}$ holds. Then $x \in \overline{\mathbb{S}}_a$ if and only if

$$\left(\frac{\alpha^n}{x_n}\right)_n \in c. \tag{6.4}$$

This result follows from the fact that here the condition $D_{1/x} B^{-1}(r,s) D_a \in (c,c)$ is equivalent to (6.4) and

$$\left(\frac{\alpha^n}{x_n} \sum_{k=1}^{n} \frac{a_k}{\alpha^k}\right)_n \in c, \tag{6.5}$$

and we conclude since (6.4) implies (6.5).

We immediately deduce the following Tauberian result.

Remark. If $a \in cs_{(\alpha^n)_n}$, then

$$\frac{B(r,s)y_n}{a_n} \to l \text{ implies } y_n \to l' \quad (n \to \infty)$$

for all y and for some scalars l and l', if and only if

$$0 < -s/r \leq 1. \tag{6.6}$$

This result comes from the fact that $e \in \overline{\mathbb{S}}_a$ if and only if (6.6) holds.

6.2. **The case of the operator of the first difference.**

6.2.1. *The general case.* If $r = -s = 1$, then we obtain $B(r,s) = \Delta$. We confine our studies to the case when $a_n \to \infty$ $(n \to \infty)$. We denote by $\widetilde{\mathbb{S}}_a$ the set of all $x \in U^+$ such that

$$\frac{\Delta y_n}{x_n} \to l \text{ implies } \frac{y_n}{a_n} \to l' \quad (n \to \infty) \tag{6.7}$$

for all y and for some scalars l and l'.

We state the next elementary result.

Proposition 6.4. *We assume $a_n \to \infty$ $(n \to \infty)$. Then the set $\widetilde{\mathbb{S}}_a$ is equal to the set of all $x \in U^+$ such that*

$$\frac{1}{a_n}\sum_{k=1}^{n} x_k \to L \ (n \to \infty) \text{ for some scalar } L; \tag{6.8}$$

moreover we have $l' = lL$ in (6.7).

Proof. It is enough to apply Proposition 6.1 with $\alpha = 1$, and $\alpha^n/a_n = 1/a_n \to 0$ $(n \to \infty)$. By Lemma 4.1, we have $l' = lL$. □

6.2.2. *Applications to the case when $a_n = n^{\beta+1}$ with $\beta > -1$, or $a_n = \ln n$.* It is well known that if $\xi > -1$, then

$$\sum_{k=1}^{n} k^{\xi} \sim \frac{n^{\xi+1}}{\xi+1} \ (n \to \infty). \tag{6.9}$$

The next result is a direct consequence of Proposition 6.4 and (6.9).

Corollary 6.5. *Let β be a real number.*

i) *If $\beta > -1$, then*

$$\frac{\Delta y_n}{n^{\beta}} \to l \text{ implies } \frac{y_n}{n^{\beta+1}} \to \frac{l}{\beta+1}(n \to \infty)$$

for all y and for some scalar l.

ii) *If $\beta = -1$, then*

$$\frac{\Delta y_n}{n^\beta} = n\Delta y_n \to l \textit{ implies } \frac{y_n}{\ln n} \to l \ (n \to \infty)$$

for all y and for some scalar l.

Proof. i) Part i) is a direct consequence of Proposition 6.4 and (6.9), since

$$v_n = \frac{1}{n^{\beta+1}} \sum_{k=1}^{n} k^\beta \to \frac{1}{\beta+1} \ (n \to \infty).$$

ii) Trivially we have

$$\begin{aligned} 1 + \ln\left(\frac{n+1}{2}\right) = 1 + \int_2^{n+1} \frac{dx}{x} \le s_n = \sum_{k=1}^{n} \frac{1}{k} \le 1 + \int_1^n \frac{dx}{x} \\ = 1 + \ln n \text{ for all } n. \end{aligned}$$

We immediately deduce that $s_n/\ln n \to 1$ $(n \to \infty)$ and $n\Delta y_n \to l$ imply

$$\frac{y_n}{\ln n} \to l \lim_{n\to\infty} \frac{s_n}{\ln n} = l \qquad (n \to \infty)$$

for all y.

□

As a direct consequence of the preceding result we obtain,

Corollary 6.6. i) *If $\beta > -1$, then*

$$y_n - \left(1 - \frac{1}{n}\right)^\beta y_{n-1} \to L \textit{ implies } \frac{y_n}{n} \to \frac{L}{\beta+1} \ (n \to \infty)$$

for all y.

ii) *If $\beta = -1$, then*

$$y_n - \left(1 - \frac{1}{n}\right)^\beta y_{n-1} = y_n - \frac{n}{n-1} y_{n-1} \to L$$

implies

$$\frac{y_n}{n \ln n} \to L \qquad (n \to \infty)$$

for all y.

References

[1] Altay, B., Başar, F., *On the space of sequences of p–bounded variation and related matrix mappings*, Ukrainian Math. J. **55** (2003), 1, 136–147
[2] Farés A., de Malafosse, B., *Sequence spaces equations and application to matrix transformations* Int. Forum **3**, (2008) 19 911-927.
[3] de Malafosse, B., *On some BK space*, Int. J. Math. Math. Sc. **28** (2003) 1783-1801.
[4] de Malafosse, B., *Sum of sequence spaces and matrix transformations,* Acta Math. Hung. **113 (3)** (2006) 289-313.
[5] de Malafosse, B., *Sum of sequence spaces and matrix transformations mapping in* $\mathbf{s}_{\alpha}^{0}\left((\Delta-\lambda I)^{h}\right)+\mathbf{s}_{\beta}^{(c)}\left((\Delta-\mu I)^{l}\right)$, Acta Math. Hung. **122** (2008) 217-230
[6] de Malafosse, B., *α–Tauberian results*, Stud. Univ. Babeş-Bolyai Math. **56 (4)** (2011) 95-111.
[7] de Malafosse, B., *Application of the infinite matrix theory to the solvability of certain sequence spaces equations with operators.* Mat. Vesnik **64, 1** (2012) 39-52.
[8] de Malafosse, B., *Solvability of certain sequence spaces inclusion equations with operators,* Demonstratio Math. **46, 2** (2013).
[9] de Malafosse, B., Malkowsky, E., *Matrix transformations in the sets* $\chi\left(\overline{N}_p\overline{N}_q\right)$ *where* χ *is in the form* $\mathbf{s}_{\xi}$*, or* $\mathbf{s}_{\xi}^{0}$*, or* $\mathbf{s}_{\xi}^{(c)}$, Filomat **17** (2003) 85-106.
[10] de Malafosse, B., Malkowsky, E., *Sets of difference sequences of order m*, Acta Sci. Math. (Szeged) **70** (2004) 659-682.
[11] de Malafosse, B., Rakočević, V., *Applications of measure of noncompactness in operators on the spaces* $\mathbf{s}_{\alpha}$*,* $\mathbf{s}_{\alpha}^{0}$*,* $\mathbf{s}_{\alpha}^{(c)}$ *and* l_{α}^{p}, J. Math. Anal. Appl. **323** (2006) 131-145.
[12] de Malafosse, B., Rakočević V., *A generalization of a Hardy theorem*, Linear Algebra Appl. **421** (2007) 306-314.
[13] de Malafosse, B., Rakočević V., *Matrix Transformations and Statistical convergence,* Linear Algebra Appl. **420** (2007) 377-387.
[14] de Malafosse, B., Rakočević V., *Series summable* (C,λ,μ) and applications, Linear Algebra Appl. (2011), doi: 10,1016/j.aa.2011.11.006.
[15] Wilansky, A., *Summability through Functional Analysis,* North-Holland Mathematics Studies **85**, 1984.

Bruno de Malafosse
LMAH Université du Havre
25 rue Philippe Lebon, BP 540
76058 Le Havre cedex, France
e-mail address: bdemalaf@wanadoo.fr

Eberhard Malkowsky
Fatih University
34500 Büyükçekmece
Istanbul Turkey
e-mail address: eberhard.malkowsky@math.uni-giessen.de

(*Received March 28, 2012*)
(*Revised November 13, 2012*)