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WEIL ALGEBRAS ASSOCIATED TO FUNCTORS OF THIRD ORDER SEMIHOLONOMIC VELOCITIES

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Abstract. The structure of Weil algebras associated to functors of third order semiholonomic velocities is completely described including the explicit expression of widths.

Introduction

Weil algebras play a very important role in the modern differential geometry; the story starts in 1950's, when — motivated by algebraic geometry — André Weil suggested the treatment of infinitesimal objects as homomorphisms from algebras of smooth functions into some real finite-dimensional commutative algebra with unit. Now, new concepts, called Weil algebras, Weil functors, Weil bundles at present, are widely studied, because of their considerable generality. Especially, we can refer to the monographical work Natural Operations in Differential Geometry [5] (Ivan Kolář, Peter W. Michor and Jan Slovák, 1993) and, moreover, also to number of papers of Włodzimierz M. Mikulski, where methods for finding natural operators use Weil algebra techniques. We recall that the Weil algebra is a local commutative $\mathbb{R}$-algebra A with identity, the nilradical $\mathfrak{n}_A$ of which has finite dimension as a vector space and $A/\mathfrak{n}_A = \mathbb{R}$; some further definitions are given below and geometric constructions one can find e.g. in [5], [9], [10].

Roughly speaking, Weil bundles generalize higher order velocities bundles including also higher order semiholonomic velocities bundles. Semiholonomic jets and velocities play an important role in the modern physics. However, the description of Weil algebras associated to functors of higher order semiholonomic velocities is not known up to now because of their complicated structure mainly in view of some technical problems of a combinatorial character. The case of the order 1 is trivial, the second order case is known in the community of specialists (and described e.g. in [7], where one reference is also to unpublished notes of Ivan Kolář) and the further step — the third order case — is the object of concern of this paper.

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1. Weil algebras and Weil bundles, algebras $\mathbb{D}^r_k$ and $\tilde{\mathbb{D}}^r_k$

Let $r, k \in \mathbb{N}$ and let us denote by $\mathbb{D}^r_k$ the $\mathbb{R}$-algebra

$$\mathbb{R}[x_1, \dots, x_k]/\langle x_1, \dots, x_k\rangle^{r+1},$$

where $\mathbb{R}[x_1, \dots, x_k]$ is the $\mathbb{R}$-algebra of real polynomials in k indeterminates and $\langle x^1, \dots, x^k\rangle^{r+1}$ the $(r+1)$-th power of its maximal ideal, i.e. of the ideal of polynomials without a constant term, which is generated by $x_1, \dots, x_k$.

The *Weil algebra* is defined as an arbitrary non-trivial quotient $\mathbb{R}$-algebra of $\mathbb{D}^r_k$. We shall denote by π_A a (non-unique) epimorphism from $\mathbb{D}^r_k$ to a Weil algebra A. Of course, the first fundamental example of a Weil algebra is $\mathbb{D}^r_k$ itself. We remark that $\mathbb{D}^r_k$ is isomorphic to the $\mathbb{R}$-algebra $J^r_0(\mathbb{R}^k, \mathbb{R})$ of r-th order jets of maps from $\mathbb{R}^k$ into $\mathbb{R}$ having a source in 0.

Every Weil algebra A is a commutative local artinian $\mathbb{R}$-algebra with identity. It follows the maximal ideal of A identifies with the Jacobson radical of A as well as with the nilradical of A: we will use the notation $\mathfrak{n}_A$ for this ideal. Obviously, $\mathfrak{n}_A$ is a finite dimensional real vector space, $A/\mathfrak{n}_A = \mathbb{R}$ and there is $s \in \mathbb{N} \cup \{0\}$ such that $\mathfrak{n}_A^{s+1} = 0$; the least of such s is called the *order* of A and denoted by $\operatorname{ord}(A)$. (Some authors write the *height* of A instead the order of A; furthermore, the name *Loewy length* in meaning $\operatorname{ord}(A) + 1$ is also sometimes used.)

Further, for every $i \in \mathbb{N} \cup \{0\}$, $\mathfrak{n}^i_A/\mathfrak{n}^{i+1}_A$ is a finite dimensional real vector space. (We take $\mathfrak{n}^0_A = A$.) We denote by

$$\mathrm{w}_i(A) = \dim \mathfrak{n}^i_A/\mathfrak{n}^{i+1}_A;$$

$\mathrm{w}_i(A)$ will be called the *i-width* of A. Trivially, $\mathrm{w}_0(A) = 1$. The 1-width of A will be called only *width* of A and written $\mathrm{w}(A)$ instead $\mathrm{w}_1(A)$. Alternatively, the *embedding dimension* of A is defined as the minimal number of elements of A generating the ideal $\mathfrak{n}_A$ and denoted by $\operatorname{embdim}(A)$. The known consequence of Nakayama's Lemma is the fact that a $\mathbb{R}$-basis for $\mathfrak{n}_A/\mathfrak{n}^2_A$ generates $\mathfrak{n}_A$ as an ideal (see e.g. [2], Chapter 5), in particular $\operatorname{embdim}(A) = \mathrm{w}(A)$ holds.

A *gradation* on A is a family $(A_i)_{i\in\mathbb{Z}}$ of vector subspaces of A such that $A = \bigoplus_i A_i$ and $A_iA_j \subseteq A_{i+j}$, for all $i, j \in \mathbb{Z}$. The gradation is called *by the radical* when A is isomorphic to $G(A)$, where $G(A)$ is so-called the *associated graded algebra* of A defined by $G(A)_i = \mathfrak{n}^i_A/\mathfrak{n}^{i+1}_A$ (or equivalently, $A_i = 0$ for $i < 0$ and $\mathfrak{n}^j_A = \bigoplus_{i\geq j} A_i$ for each $j \geq 0$), see [12].

In differential geometry, the following construction has the fundamental importance. Let M be a smooth manifold and let A be a Weil algebra. Two smooth maps $g, h\colon \mathbb{R}^k \to M$ are said to *determine the same A-jet* $j^A g = j^A h$, if for every smooth function $\phi\colon M \to \mathbb{R}$

$$\pi_A\left(j_0^r(\phi \circ g)\right) = \pi_A\left(j_0^r(\phi \circ h)\right)$$

is satisfied. The space $T^A M$ of all A-jets on M is fibered over M and is called the *Weil bundle*. The functor T^A from $\mathcal{M}f$ of manifolds into the category $\mathcal{FM}$ of fibered manifolds is called the *Weil functor*. It is well-known that Weil bundles $T^A M$ represent a powerful generalization of the velocities bundles $T_k^r M$. For further explication of the importance of Weil bundles see e.g. [5]. Analogously, we can define the *Weil evolution bundle* by

$$\mathbb{R}^{\mathrm{w}(A)} \times T^A M$$

and it is clear that the concept of the Weil evolution bundle generalizes the evolution velocities bundles $\mathbb{R}^k \times T_k^r M$, the role of which is crucial in the variational calculus.

Remark. Indeed, the simple Lagrangian known from the classical calculus of variations has a form

$$L = L\big(t, x(t), x'(t)\big)$$

but it is nothing but a function $L\colon \mathbb{R} \times TM \to \mathbb{R}$ (where $M = \mathbb{R}$ in the most fundamental case). The notion of the *evolution space* is used by Manuel de León and Paolo R. Rodrigues in [11] for the jet space $J^1(\mathbb{R}, M)$.

As the Weil evolution bundle, many classification theorems from the theory of natural operators need also the explicit enumerating of the width of a Weil algebra A. We refer e.g. to [9] and [10], where one can find details. Mentioned facts can be readed as direct applications of our results presented in Section 2. (We remark that the formalism is productive also for gauge bundles where modules over Weil algebra occur, see [6].)

The well-known result, which we can obtain by the direct evaluation, is the following. We take $m = \dim M$.

Proposition 1.1. *For every $i \in \{0, \dots, r\}$,*

$$\mathrm{w}_i(\mathbb{D}_k^r) = \binom{k+i-1}{i}.$$

It follows

$$\dim T_k^r M = m \sum_{i=0}^{r} \binom{k+i-1}{i} = \frac{m(r+1)}{k} \binom{r+k}{r+1}.$$

The bundle of nonholonomic r-jets of maps from N into M is defined by the following induction. For $r = 1$, the bundle of nonholonomic 1-jets $\tilde{J}^1(N, M) := J^1(N, M)$. By induction, let $\alpha\colon \tilde{J}^{r-1}(N, M) \to N$ denote the source projection and $\beta\colon \tilde{J}^{r-1}(N,M) \to M$ the target projection of $(r-1)$-th nonholonomic jets. Then Z is said to be a *nonholonomic r-jet* with the source $x \in N$ and the target $\bar{x} \in M$, if there is a local section $\sigma\colon N \to \tilde{J}^{r-1}(N,M)$ such that $Z = j^1_x\sigma$ and $\beta(\sigma(x)) = \bar{x}$. For $N = \mathbb{R}^k$, we have the *bundle of nonoholonomic velocities* $\tilde{T}^r_k M = \tilde{J}^r_0(\mathbb{R}^k, M)$. This is a Weil bundle; its Weil algebra is denoted by $\tilde{\mathbb{D}}^r_k$. The basis of $\tilde{\mathbb{D}}^r_k$ is constituted by x_α, where $\alpha = (\alpha_1, \dots, \alpha_r)$ is a multiindex with $\alpha_i \in \{0, 1, \dots, k\}, i = 1, \dots, r$, where 1 is used instead $x_{0\dots0}$. The multiplication in $\tilde{\mathbb{D}}^r_k$ acts by the following way.

$$x_{\alpha_1\dots\alpha_r} x_{\beta_1\dots\beta_r} = \begin{cases} x_{\alpha_1+\beta_1\dots\alpha_r+\beta_r} & \text{if either } \alpha_i = 0 \text{ or } \beta_i = 0 \text{ is satisfied for} \\ & \text{all } i = 1, \dots, r \\ 0 & \text{if } \alpha_i \neq 0 \text{ and } \beta_i \neq 0 \text{ for any } i = 1, \dots, r \end{cases}$$

It follows that all elements of the basis except 1, i.e. with at least one non-zero index, belongs also to the basis of $\mathfrak{n}_{\tilde{\mathbb{D}}^r_k}$. As well as this, it is clear that elements belonging also to the basis of $\mathfrak{n}^2_{\tilde{\mathbb{D}}^r_k}$ have at least two non-zero indexes, etc. Hence we have obtain with ease this result:

Proposition 1.2. *For every $i \in \{0, \dots, r\}$,*

$$\mathrm{w}_i(\tilde{\mathbb{D}}^r_k) = \binom{r}{i} k^i.$$

It follows

$$\dim \tilde{T}^r_k M = m \sum_{i=0}^{r} \binom{r}{i} k^i = m(1+k)^r.$$

2. Weil algebra $\bar{\mathbb{D}}^r_k$ of semiholonomic velocities

The requirement for the equalization of certain projections of the bundle $\tilde{T}^r_k M$ gives rise to the bundle of the semiholonomic velocities $\bar{T}^r_k$ having a meaningful importance in applications, but we do not go to details about applications here. We introduce the related Weil algebra $\bar{\mathbb{D}}^r_k$ by the following way. For the elements of the basis of $\tilde{\mathbb{D}}^r_k$ (except 1), let $\bar{\alpha}$ denotes the multiindex obtained from α by the omission of all zeros, nevertheless, by the full respecting of the original order in non-zero indexes. So, we define the algebra $\bar{\mathbb{D}}^r_k$ as the algebra $\tilde{\mathbb{D}}^r_k$ with added relations

$$x_\alpha = x_\beta \text{ for all } \alpha, \beta \text{ satisfying } \bar{\alpha} = \bar{\beta}.$$

The multiplication realizes by the nonholonomic multiplication defined above.

2.1. **The second order.** First, we present it for $r = 2$. Basis elements in the nonholonomic case are $x_{00} = 1$, $x_{\iota 0}$, $x_{0\iota}$, $x_{\iota\kappa}$, where $\iota, \kappa \in \{1, \dots, k\}$. In the semiholonomic case we identify $x_{\iota 0}$ with $x_{0\iota}$. Further, if we have in mind the semiholonomic case, we denote it by bar over x's; so, we have 1, $\bar{x}_\iota$, $\bar{x}_{\iota\kappa}$. We write generating equations for $\bar{\mathbb{D}}^2_k$ as

$$\begin{aligned}
2\bar{x}_\iota \bar{x}_\kappa &= x_{\iota 0}x_{0\kappa} + x_{0\iota}x_{\kappa 0} = x_{\iota\kappa} + x_{\kappa\iota} = \bar{x}_{\iota\kappa} + \bar{x}_{\kappa\iota} \\
2\bar{x}_{\iota\kappa}\bar{x}_\lambda &= x_{\iota\kappa}x_{\lambda 0} + x_{\iota\kappa}x_{0\lambda} = 0 + 0 = 0 \\
\bar{x}_{\iota\kappa}\bar{x}_{\lambda\mu} &= x_{\iota\kappa}x_{\lambda\mu} = 0, \\
&\ \text{where } \iota, \kappa, \lambda, \mu \in \{1, \dots, k\}.
\end{aligned}$$

We observe that

$$\begin{aligned}
\bar{x}_{\iota\iota} &= \bar{x}_\iota \bar{x}_\iota \\
\bar{x}_{\kappa\iota} &= 2x_\iota x_\kappa - x_{\iota\kappa} \quad \text{for } \iota < \kappa.
\end{aligned}$$

Thus, we are able express a general element of $\bar{\mathbb{D}}^2_k$ as

$$a + b\bar{x}_\iota + c\bar{x}_{\iota\kappa} + d\bar{x}_\iota\bar{x}_\iota + e\bar{x}_\iota\bar{x}_\kappa \qquad (\text{only with } \iota < \kappa),$$

where $a, b, c, d, e \in \mathbb{R}$. An element of $\mathfrak{n}_{\bar{\mathbb{D}}^2_k}$ has a form $b\bar{x}_\iota + c\bar{x}_{\iota\kappa} + d\bar{x}_\iota\bar{x}_\iota + e\bar{x}_\iota\bar{x}_\kappa$ and an element of $\mathfrak{n}^2_{\bar{\mathbb{D}}^2_k}$ has a form $d\bar{x}_\iota\bar{x}_\iota + e\bar{x}_\iota\bar{x}_\kappa$. Thus, we have derived the following assertion (cf. [7]).

Proposition 2.1. *The Weil algebra associated to the functors of second order semiholonomic velocities has a form*

$$\bar{\mathbb{D}}^2_k = \mathbb{R}[\bar{x}_{\iota\kappa}, \bar{x}_\lambda] / \left\langle \langle \bar{x}_{\iota\kappa} \rangle^2, \bar{x}_{\iota\kappa}\bar{x}_\lambda, \langle \bar{x}_\lambda \rangle^3 \right\rangle \qquad (\textit{where } \iota < \kappa)$$

and

$$\mathrm{w}_1(\bar{\mathbb{D}}^2_k) = \frac{k^2 + k}{2}, \quad \mathrm{w}_2(\bar{\mathbb{D}}^2_k) = \frac{k^2 + k}{2}.$$

2.2. **The third order.** The third (and any higher) order case was not investigated from the presented point of view up to now. We have proved:

Proposition 2.2. *The Weil algebra associated to the functors of third order semiholonomic velocities has a form*

$$\begin{aligned}
\bar{\mathbb{D}}^3_k &= \mathbb{R}[\bar{x}_{\iota\kappa\lambda}, \bar{x}_{\mu\nu}, \bar{x}_\pi] / \left\langle \langle \bar{x}_{\iota\kappa\lambda} \rangle^2, \bar{x}_{\iota\kappa\lambda}\bar{x}_{\mu\nu}, \bar{x}_{\iota\kappa\lambda}\bar{x}_\pi, \langle \bar{x}_{\mu\nu} \rangle^2, \bar{x}_{\mu\nu}\langle \bar{x}_\pi \rangle^2, \langle \bar{x}_\pi \rangle^4 \right\rangle \\
&\quad (\textit{where } (\iota < \kappa < \lambda \textit{ or } \kappa \le \iota < \lambda \textit{ or } \iota < \lambda \le \kappa) \textit{ and } \mu < \nu)
\end{aligned}$$

and

$$\mathrm{w}_1(\bar{\mathbb{D}}^3_k) = \frac{k^3 + k}{2}, \quad \mathrm{w}_2(\bar{\mathbb{D}}^3_k) = \frac{2k^3 + 3k^2 + k}{6}, \quad \mathrm{w}_3(\bar{\mathbb{D}}^3_k) = \frac{k^3 + 3k^2 + 2k}{6}.$$

Proof. We use an analogous technique as in the second order case and write newly emerging generating equations for $\bar{\mathbb{D}}_k^3$ as

$$\begin{aligned}
6\bar{x}_\iota\bar{x}_\kappa\bar{x}_\lambda &= x_{\iota00}x_{0\kappa0}x_{00\lambda} + x_{\iota00}x_{00\kappa}x_{0\lambda0} + x_{0\iota0}x_{\kappa00}x_{00\lambda} + \\
&\quad x_{0\iota0}x_{00\kappa}x_{\lambda00} + x_{00\iota}x_{\kappa00}x_{0\lambda0} + x_{00\iota}x_{0\kappa0}x_{\lambda00} = \\
&\quad x_{\iota\kappa\lambda} + x_{\iota\kappa\lambda} + x_{\kappa\iota\lambda} + x_{\kappa\lambda\iota} + x_{\lambda\iota\kappa} + x_{\lambda\kappa\iota} = \\
&\quad \bar{x}_{\iota\kappa\lambda} + \bar{x}_{\iota\kappa\lambda} + \bar{x}_{\kappa\iota\lambda} + \bar{x}_{\kappa\lambda\iota} + \bar{x}_{\lambda\iota\kappa} + \bar{x}_{\lambda\kappa\iota} \\
3\bar{x}_{\iota\kappa}\bar{x}_\lambda &= x_{\iota\kappa0}x_{00\lambda} + x_{\iota0\kappa}x_{0\lambda0} + x_{0\iota\kappa}x_{\lambda00} = \\
&\quad x_{\iota\kappa\lambda} + x_{\iota\lambda\kappa} + x_{\lambda\iota\kappa} = \bar{x}_{\iota\kappa\lambda} + \bar{x}_{\iota\lambda\kappa} + \bar{x}_{\lambda\iota\kappa}, \\
&\quad \text{where } \iota,\kappa,\lambda \in \{1,\dots,k\},
\end{aligned}$$

but, apart from the second order case, we do not write zero products here. In particular, we have

$$\begin{aligned}
\bar{x}_{\iota\iota\iota} &= \bar{x}_\iota\bar{x}_\iota\bar{x}_\iota \\
\bar{x}_{\iota\kappa\iota} &= 3\bar{x}_{\iota\kappa}\bar{x}_\iota - 2\bar{x}_{\iota\iota\kappa} \\
\bar{x}_{\kappa\iota\kappa} &= 3\bar{x}_{\iota\kappa}\bar{x}_\kappa - 2\bar{x}_{\iota\kappa\kappa} \\
\bar{x}_{\kappa\iota\iota} &= 3\bar{x}_\iota\bar{x}_\iota\bar{x}_\kappa - \bar{x}_{\iota\iota\kappa} - \bar{x}_{\iota\kappa\iota} \\
\bar{x}_{\kappa\kappa\iota} &= 3\bar{x}_\iota\bar{x}_\kappa\bar{x}_\kappa - \bar{x}_{\iota\kappa\kappa} - \bar{x}_{\kappa\iota\kappa} \\
\bar{x}_{\lambda\iota\kappa} &= 3\bar{x}_{\iota\kappa}\bar{x}_\lambda - \bar{x}_{\iota\kappa\lambda} - \bar{x}_{\iota\lambda\kappa} \\
\bar{x}_{\kappa\lambda\iota} &= 3\bar{x}_{\kappa\lambda}\bar{x}_\iota - \bar{x}_{\iota\kappa\lambda} - \bar{x}_{\kappa\iota\lambda} \\
\bar{x}_{\iota\lambda}\bar{x}_\kappa &= \frac{1}{3}\left(\bar{x}_{\iota\kappa\lambda} + \bar{x}_{\iota\lambda\kappa} + \bar{x}_{\kappa\iota\lambda}\right) \\
\bar{x}_{\lambda\kappa\iota} &= 6\bar{x}_\iota\bar{x}_\kappa\bar{x}_\lambda - \bar{x}_{\iota\kappa\lambda} - \bar{x}_{\iota\lambda\kappa} - \bar{x}_{\kappa\iota\lambda} - \bar{x}_{\kappa\lambda\iota} - \bar{x}_{\lambda\iota\kappa} \\
&\quad \text{thinking constantly } \iota < \kappa < \lambda.
\end{aligned}$$

Thus, considering the form of elements of $\bar{\mathbb{D}}_k^2$ and a feasibility to express left hand sides of equations above by the described way, we are able express a general element of $\bar{\mathbb{D}}_k^3$ as

$$\begin{aligned}
&a + b\bar{x}_\iota + c\bar{x}_{\iota\kappa} + d\bar{x}_{\iota\iota\kappa} + e\bar{x}_{\iota\kappa\kappa} + f\bar{x}_{\iota\kappa\lambda} + g\bar{x}_{\iota\lambda\kappa} + h\bar{x}_{\kappa\iota\lambda} + \\
&i\bar{x}_\iota\bar{x}_\iota + j\bar{x}_\iota\bar{x}_\kappa + k\bar{x}_{\iota\kappa}\bar{x}_\iota + l\bar{x}_{\iota\kappa}\bar{x}_\kappa + m\bar{x}_{\iota\kappa}\bar{x}_\lambda + n\bar{x}_{\kappa\lambda}\bar{x}_\iota + \\
&o\bar{x}_\iota\bar{x}_\iota\bar{x}_\iota + p\bar{x}_\iota\bar{x}_\iota\bar{x}_\kappa + q\bar{x}_\iota\bar{x}_\kappa\bar{x}_\kappa + r\bar{x}_\iota\bar{x}_\kappa\bar{x}_\lambda \quad (\text{only with } \iota < \kappa < \lambda),
\end{aligned}$$

where $a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r \in \mathbb{R}$. An element of $\mathfrak{n}_{\bar{\mathbb{D}}_k^3}$ has a form

$$\begin{aligned}
&b\bar{x}_\iota + c\bar{x}_{\iota\kappa} + d\bar{x}_{\iota\iota\kappa} + e\bar{x}_{\iota\kappa\kappa} + f\bar{x}_{\iota\kappa\lambda} + g\bar{x}_{\iota\lambda\kappa} + h\bar{x}_{\kappa\iota\lambda} + \\
&i\bar{x}_\iota\bar{x}_\iota + j\bar{x}_\iota\bar{x}_\kappa + k\bar{x}_{\iota\kappa}\bar{x}_\iota + l\bar{x}_{\iota\kappa}\bar{x}_\kappa + m\bar{x}_{\iota\kappa}\bar{x}_\lambda + n\bar{x}_{\kappa\lambda}\bar{x}_\iota + \\
&o\bar{x}_\iota\bar{x}_\iota\bar{x}_\iota + p\bar{x}_\iota\bar{x}_\iota\bar{x}_\kappa + q\bar{x}_\iota\bar{x}_\kappa\bar{x}_\kappa + r\bar{x}_\iota\bar{x}_\kappa\bar{x}_\lambda,
\end{aligned}$$

an element of $\mathfrak{n}^2_{\bar{\mathbb{D}}^3_k}$ has a form

$$\begin{aligned}&i\bar{x}_\iota\bar{x}_\iota + j\bar{x}_\iota\bar{x}_\kappa + k\bar{x}_{\iota\kappa}\bar{x}_\iota + l\bar{x}_{\iota\kappa}\bar{x}_\kappa + m\bar{x}_{\iota\kappa}\bar{x}_\lambda + n\bar{x}_{\kappa\lambda}\bar{x}_\iota +\\ &o\bar{x}_\iota\bar{x}_\iota\bar{x}_\iota + p\bar{x}_\iota\bar{x}_\iota\bar{x}_\kappa + q\bar{x}_\iota\bar{x}_\kappa\bar{x}_\kappa + r\bar{x}_\iota\bar{x}_\kappa\bar{x}_\lambda,\end{aligned}$$

and an element of $\mathfrak{n}^3_{\bar{\mathbb{D}}^3_k}$ has a form

$$o\bar{x}_\iota\bar{x}_\iota\bar{x}_\iota + p\bar{x}_\iota\bar{x}_\iota\bar{x}_\kappa + q\bar{x}_\iota\bar{x}_\kappa\bar{x}_\kappa + r\bar{x}_\iota\bar{x}_\kappa\bar{x}_\lambda.$$

A little of combinatorics provides the computed widths. □

Example. The first non-trivial example is the algebra $\bar{\mathbb{D}}^3_2$. It has the form

$$\begin{aligned}\bar{\mathbb{D}}^3_2 = {}&\mathbb{R}[U,V,W,X,Y]/\\ &\big\langle\langle U,V\rangle^2, UW, VW, UX, UY, VX, VY, W^2, W\langle X,Y\rangle^2, \langle X,Y\rangle^4\big\rangle\,.\end{aligned}$$

Example. We describe the algebra $\bar{\mathbb{D}}^3_3$ in more detail. We have

$$\begin{aligned}\bar{\mathbb{D}}^3_3 = {}&\mathbb{R}[K,L,M,N,P,Q,R,S,T,U,V,W,X,Y,Z]/\\ &\langle\langle K,L,M,N,P,Q,R,S,T\rangle^2, KU, LU, MU, NU, PU, QU, RU, SU, TU,\\ &KV, LV, MV, NV, PV, QV, RV, SV, TV, KW, LW, MW, NW, PW,\\ &QW, RW, SW, TW, KX, LX, MX, NX, PX, QX, RX, SX, TX,\\ &KY, LY, MY, NY, PY, QY, RY, SY, TY, KZ, LZ, MZ, NZ, PZ,\\ &QZ, RZ, SZ, TZ, \langle U,V,W\rangle^2, U\langle X,Y,Z\rangle^2, V\langle X,Y,Z\rangle^2, W\langle X,Y,Z\rangle^2,\\ &\langle X,Y,Z\rangle^4\rangle\end{aligned}$$

where $K=\bar{x}_{123}$, $L=\bar{x}_{112}$, $M=\bar{x}_{122}$, $N=\bar{x}_{113}$, $P=\bar{x}_{133}$, $Q=\bar{x}_{223}$, $R=\bar{x}_{233}$, $S=\bar{x}_{213}$, $T=\bar{x}_{132}$, $U=\bar{x}_{12}$, $V=\bar{x}_{13}$, $W=\bar{x}_{23}$, $X=\bar{x}_1$, $Y=\bar{x}_2$, $Z=\bar{x}_3$.

This example has its individual importance: we recall that, for a manifold M, $\dim M = m$, the r-th order semiholonomic frame bundle $\bar{P}^rM$ is defined as the space of invertible jets

$$\bar{P}^rM = \operatorname{inv}\bar{J}^r_0(\mathbb{R}^m, M)$$

and hence, if $\dim M = 3$,

$$\bar{P}^3M = \operatorname{inv}\bar{J}^3_0(\mathbb{R}^3, M).$$

Indeed, $\bar{P}^3M$ formed by 3-jets of local diffeomorphisms is an open dense subset of $\bar{T}^3_3M$ having the Weil algebra $\bar{\mathbb{D}}^3_3$.

Remark. Let μ, ν are multindexes $\mu = (\mu_1\dots\mu_p)$, $\nu = (\nu_1\dots\nu_q)$. Let $\rho = (\rho_1\dots\rho_{p+q})$ is a multiindex constructed by the following way. We choose p indexes $\lambda_1 < \dots < \lambda_p$ from $\{1,\dots,p+q\}$ and denote by $\kappa_1 < \dots < \kappa_q$ the remained indexes. Now, we put $\rho_{\lambda_1} = \mu_1, \dots, \rho_{\lambda_p} = \mu_p$, $\rho_{\kappa_1} = \nu_1, \dots, \rho_{\kappa_q} =$

ν_q. Every such ρ will be called the *admissible product* of μ and ν. It is not difficult realize that there is $\frac{(p+q)!}{p!q!}$ eventual admissible products of μ and ν.

$\boxed{\mathrm{w}(\mathbb{D}^r_k)}$			
$k \setminus r$	1	2	3
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
…	…	…	…

$\boxed{\mathrm{w}(\tilde{\mathbb{D}}^r_k)}$			
$k \setminus r$	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9
4	4	8	12
…	…	…	…

$\boxed{\mathrm{w}(\bar{\mathbb{D}}^r_k)}$			
$k \setminus r$	1	2	3
1	1	1	1
2	2	3	5
3	3	6	15
4	4	10	34
…	…	…	…

3. Semiholonomic jets in applications

When studied the prolongations of a differential system of higher order connections, Claude Ehresmann was led to use the terminology of mechanics: so, he called jets in question nonholonomic and semiholonomic jets. However, applications of semiholonomic jets of the *third* were taken into account very rarely up to now and we only present some possible directions and inspirations for them.

3.1. **Special types of semiholonomic jets.** Ivan Kolář starts in his recent investigation ([4]) with some general ideas concerning the concept of special type of nonholonomic r-jets. Then he classifies the special types of semiholonomic 3-jets. Namely, he used the general concept of nonholonomic r-jet category and then he described semiholonomic 3-jet categories. Kolář in [4] also has proved

$$\mathrm{Hom}(\bar{\mathbb{D}}^r_k, \bar{\mathbb{D}}^r_n) \cong \bar{L}^r_{n,k},$$

where $\bar{L}^r_{n,k} = \bar{J}^r_0(\mathbb{R}^n, \mathbb{R}^k)$; So, the structure of Weil algebras $\bar{\mathbb{D}}^3_k$ associated to functors of third order semiholonomic velocities play an important role in the algebraic description of semiholonomic third order jet spaces in general.

On the other hand, in [8] is considered more formal concept of so-called symmetrisation (arising from permutations of lower indexes of their local coordinates) it offers several fully geometrically interpreted operations. The construction of the generalised involution, the classification of all symmetrised nonholonomic jets of the third order and the geometrical interpretation of them are main results of [6]. Third order symmetrised nonholonomic jets can be also restricted to the case of velocities and the question is if they can be found from the opposite side: is it possible to obtain (all?)

symmetrised velocities by a similar way as we had derived the structure of $\bar{\mathbb{D}}^3_k$?

In [8] are also some references to the use of third order jets (e.g. relations of semiholonomic jets with Stieffel and Grassmann bundles, the application of semiholonomic jets in the theory of Verma modules, etc.) which are not recalled here.

3.2. **Cosserat bodies.** We recall that the *Cosserat body* is defined as a frame bundle P^1M of a material body M and its *bundle configuration* is defined as the principal fiber bundle morphism $K\colon P^1M \to P^1\mathbb{R}^3$ over the base map $\kappa\colon M \to \mathbb{R}^3$. Marcelo Epstein in [3] investigates the use of the semiholonomic functor $\bar{P}^2$ instead P^1 and comments the interpration in continuum mechanics. So, $\bar{P}^3$ can present a generalization in this direction for higher order grade materials. Nevertheless, such an idea needs a clear interpretation in continuum mechanics. We will present a progress in this direction in the next work.

3.3. **Derivative and differential strings.** Let $t, m \in \mathbb{N}$. The *multiindex of length t with entries up to m* is a finite sequence

$$I = (i_1, \dots, i_t) \in N(m)^t = \underbrace{N(m) \times \dots \times N(m)}_{t\text{-times}},$$

where $N(m) = \{1, \dots, m\}$. The *standard ordered partition* of a multiindex I to u multiindexes $I_1, \dots, I_u$, $u \le t$ is a map

$$\pi|_u^{t(m)}\colon N(m)^t \to N(m)^{t_1} \times \dots \times N(m)^{t_u},$$

$$\pi|_u^{t(m)}(I) = (I_1, \dots, I_u) = \Big((i_{s_1^1}, \dots, i_{s_{t_1}^1}), \dots, (i_{s_1^u}, \dots, i_{s_{t_u}^u})\Big)$$

(where $t_1 + \dots + t_u = t$, $s_1^1, \dots, s_{t_u}^u \in N(m)$) satisfying

(i) $s_1^1 < \dots < s_{t_1}^1$, …, $s_u^1 < \dots < s_{t_u}^u$,
(ii) $s_1^1 < \dots < s_1^u$.

The set of all standard ordered partitions of a multiindex I of the length t with entries up to m to u multiindexes $I_1, \dots, I_u$ will be denoted by $\Pi|_u^{t(m)}$. Further, we put $\Pi|^{t(m)} = \Pi|_1^{t(m)} \cup \dots \cup \Pi|_t^{t(m)}$.

Let M be an m-dimensional manifold. Given two systems $(x^1, \dots, x^m)$ and $(y^1, \dots, y^m)$ of local coordinates on M and a multiindex $I \in N(m)^t$, we define the functions $y^j_{/I}$ ($j \in N(m)$) by

$$y^j_{/I} = \frac{\partial^t y^j}{\partial x^{i_1} \dots \partial x^{i_t}}.$$

For $J = (j_1, \dots, j_u) \in N(m)^u$, we put

$$y^J_{/I} = \sum_{\Pi|_u^{t(m)}} y^{j_1}_{/I_1} \dots y^{j_u}_{/I_u},$$

where the sum runs over $\Pi|_u^{t(m)}$, i.e. over all standard ordered partitions of I to u multiindexes.

Example. Suppose $J = (j_1, j_2)$ and $I = (i_1, i_2, i_3)$. Then

$$y^J_{/I} = \sum_{\Pi|_2^{3(m)}} y^{j_1}_{/I_1} y^{j_2}_{/I_2} = y^{j_1}_{/(i_1)} y^{j_2}_{/(i_2,i_3)} + y^{j_1}_{/(i_1,i_2)} y^{j_2}_{/(i_3)} + y^{j_1}_{/(i_1,i_3)} y^{j_2}_{/(i_2)}.$$

In classical tensor calculus, tensors are introduced as multidimensional arrays satisfying certain transformation laws. Given a point $p \in M$, a *derivative string* at p assigns to each local coordinate system $(y^1 \dots, y^m)$ round p a multidimensional array H^{IJ}_{KL} which are required to transform under coordinate change from $(y^1 \dots, y^m)$ to $(x^1 \dots, x^m)$ by

$$H^{AD}_{BC} = x^A_{/I} y^J_{/B} H^{IJ}_{KL} y^K_{/C} x^D_{/L}$$

Furthermore, *differential strings* which form a generalization of derivative strings are also studied. Peter E. Jupp (see [1] for a detailed explanation and also for relations to stochastic calculus) has described spaces of strings of some lengths of multiindexes by so called zero-truncated semiholonomic jets. In the third order, it means the bundle $\bar{J}^3(M, \mathbb{R})_0 = \bar{T}^{*3}M$ of third order covelocities.

3.4. **Cartan connections on geometries modeled on homogeneous spaces.** Jan Slovák in [13] studied principal fiber bundles obtaining as reductions of certain universal bundles equipped with canonical forms. (Classically, higher order holonomic frame bundle are such universal bundles.) In particular, if H is a closed subgroup of a structure group G (corresponding Lie algebras are denoted by $\mathfrak{h}$ and $\mathfrak{g}$) a Cartan connection is defined as certain $\mathfrak{g}$-valued 1-form on the principal right H-bundle P. Slovák in his introductory note of [13] has remarked that in typical examples, the conformal Riemannian and almost Grassmannian geometries, already in the conformal case we cannot obtain the canonical Cartan connections via reductions of higher order holonomic bundles. On the other hand, the canonical Cartan connections for both structures are available via reductions of third order semiholonomic frame bundles.

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