Math. J. Okayama Univ. **56** (2014), 1–16

MUTATING BRAUER TREES

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Abstract. In this paper we introduce mutation of Brauer trees. We show that our mutation of Brauer trees explicitly describes the tilting mutation of Brauer tree algebras introduced by Okuyama and Rickard.

Mutation plays an important role in representation theory, especially in tilting theory and cluster tilting theory. Cluster tilting theory deals with the combinatorial structure of 2-Calabi-Yau triangulated categories and is applied to categorification of Fomin-Zelevinsky cluster algebras. This is closely related with tilting theory of hereditary algebras, and the class of tilting complexes is one of the most important classes from Morita theoretic viewpoint [R2]. Now it is an important problem to study the combinatorial structure of tilting complexes for finite dimensional algebras. In this paper we consider this problem for Brauer tree algebras, which form one of the most basic classes of symmetric algebras. The main result of this paper is to describe explicitly the combinatorics of tilting mutation for Brauer tree algebras. This is given by mutation of Brauer trees, which is a new operation introduced in this paper. We point out that our mutation of Brauer trees can be regarded as a generalization of flip of triangulations of surfaces [FST].

In Section 1 we recall a construction of a tilting complex T of a symmetric algebra A which we call an *Okuyama-Rickard complex*. It was introduced by Okuyama and Rickard, and has been playing an important role in the study of Broué's abelian defect group conjecture. Since it is a special case of tilting mutation as we pointed out in [AI], we call the endomorphism algebra $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\text{-}A)}(T)$ *tilting mutation* of A.

In Section 2 we introduce a combinatorial operation of Brauer trees which we call *mutation of Brauer trees*. Our main result shows that tilting mutation of Brauer tree algebras which we explained above is compatible with our mutation of Brauer trees. A special case was given in [KZ] and [Z] (see Remark 2.3). As an application of our main result, we give braid-type relations for tilting mutation of Brauer tree algebras (Theorem 2.5). In Section 3 we prove our main result. Our proof is very simple and seems to be interesting by itself.

Mathematics Subject Classification. 16G10, 16G20, 20C05.

Key words and phrases. Brauer tree, Brauer tree algebra, tilting mutation, mutation of Brauer tree.

1. Preliminary

Throughout this paper, let A be a finite dimensional k-algebra over an algebraically closed field k and we assume that A is basic and indecomposable as an A-A-bimodule. Let $\{e_1, e_2, \cdots, e_n\}$ be a basic set of orthogonal local idempotents in A and put $E = \{1, 2, \cdots, n\}$. For each $i \in E$, we set $P_i = e_iA$ and $S_i = P_i/\mathrm{rad}P_i$.

We denote by mod-A the category of finitely generated right A-modules, by proj-A the full subcategory of mod-A consisting of finitely generated projective right A-modules, and by $\mathsf{K}^{\mathrm{b}}(\text{proj-}A)$ the homotopy category of bounded complexes over proj-A.

Let us start with recalling the complex introduced by Okuyama and Rickard [O, R], which is a special case of silting mutation defined in [AI].

Definition 1.1. Let E_0 be a subset of E and put $e = \sum_{i \in E_0} e_i$. For any $i \in E$, we define a complex by

$$T_i = \begin{cases} \quad (0\text{th}) \qquad\qquad (1\text{st}) & \\ \quad P_i \longrightarrow 0 & (i \in E_0) \\ \quad Q_i \xrightarrow[\pi_i]{} P_i & (i \notin E_0) \end{cases}$$

where $Q_i \xrightarrow{\pi_i} P_i$ is a minimal projective presentation of e_iA/e_iAeA. Now we define $T := T(E_0) := \oplus_{i \in E} T_i$ and call it the *Okuyama-Rickard complex with respect to* E_0.

The following observation shows the importance of Okuyama-Rickard complexes.

Proposition 1.2. [O, Proposition 1.1] *If A is a symmetric algebra, then any Okuyama-Rickard complex T is tilting. In particular* $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T)$ *is derived equivalent to A.*

If we drop the assumption that A is symmetric, an Okuyama-Rickard complex is not necessarily a tilting complex but still a silting complex.

Definition 1.3. Let B be a finite dimensional k-algebra. For any $i \in E$, we say that B is the *tilting mutation of A with respect to i* if $B \simeq \mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T(E\backslash\{i\}))$ and write $A \xrightarrow{i} B$ or $B = \mu_i(A)$.

The aim of this paper is to introduce mutation of Brauer trees and study the relationship with tilting mutation of Brauer tree algebras.

Let us recall the definitions of Brauer trees and Brauer tree algebras

Definition 1.4. [Alp, GR] A *Brauer graph G* is a finite connected graph, together with the following data:

(i) There exists a cyclic ordering of the edges adjacent to each vertex, usually described by the clockwise ordering given by a fixed planar representation of G;
(ii) For each vertex v, there exists a positive integer m_v assigned to v, called the *multiplicity*. We call a vertex v *exceptional* if $m_v > 1$.

A *Brauer tree* G is a Brauer graph which is a tree and having at most one exceptional vertex.

A *Brauer tree algebra* $A = A_G$ is a basic algebra given by a Brauer tree G as follows:

(i) There exists a one-to-one correspondence between simple A-modules S_i and edges i of G;
(ii) For any edge i of G, the projective indecomposable A-module P_i has $\mathrm{soc}(P_i) \simeq P_i/\mathrm{rad}(P_i)$ and $\mathrm{rad}(P_i)/\mathrm{soc}(P_i)$ is the direct sum of two uniserial modules whose composition factors are, for the cyclic ordering $(i, i_1, \cdots, i_a, i)$ of the edges adjacent to a vertex v,

$$S_{i_1}, \cdots, S_{i_a}, S_i, S_{i_1}, \cdots, S_{i_a} \text{ (from the top to the socle)}$$

where S_i appears $m_v - 1$ times.

Note that A_G is uniquely determined by G up to isomorphism. Moreover it is a symmetric algebra.

We say that a Brauer tree G is a *star* if there exists a vertex v of G such that all edges of G appear in the cyclic ordering adjacent to v.

The following well-known result shows that derived equivalence classes of Brauer tree algebras are determined by certain numerical invariants.

Theorem 1.5. [R, Theorem 4.2]

(1) *For every Brauer tree algebra A, there exists a tilting complex P in* $\mathsf{K}^{\mathrm{b}}(\text{proj-}A)$ *such that the endomorphism algebra* $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(P)$ *of P is a Brauer tree algebra for a star with exceptional vertex in the center if it exists.*
(2) *In particular, derived equivalence classes of Brauer tree algebras are determined by the number of the edges and the multiplicities of the vertices.*

2. Mutating Brauer trees

Let us start with the definition of mutation of Brauer trees.

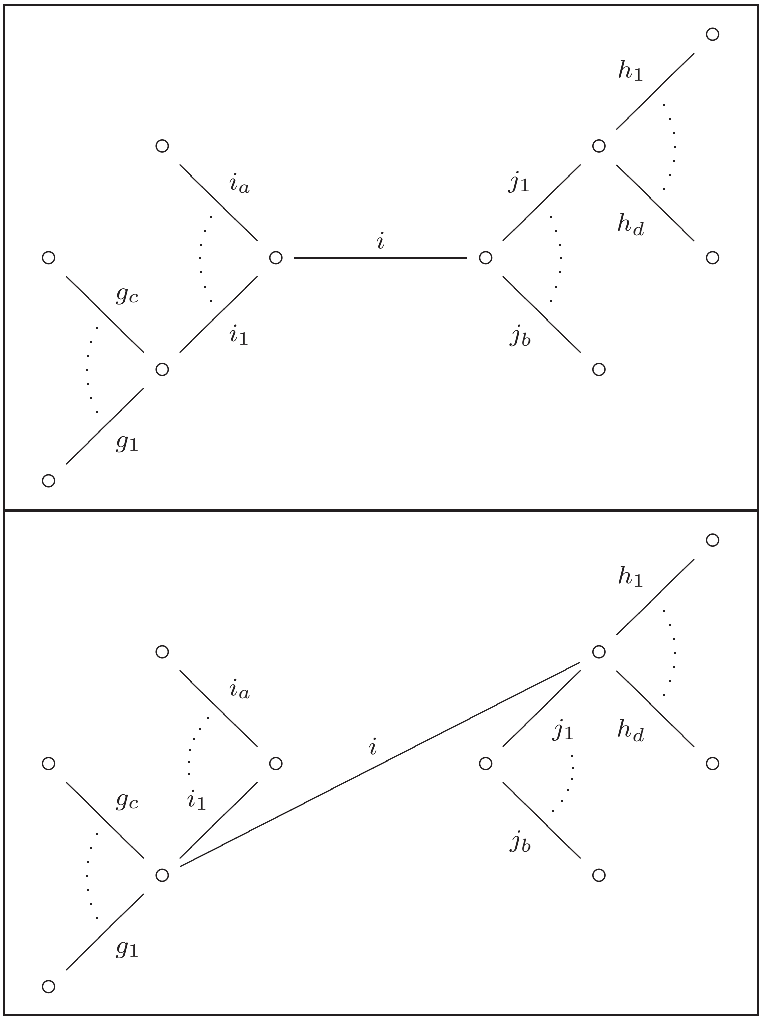
Definition 2.1. Let G be a Brauer tree and i be an edge of G. We define a Brauer tree $\mu_i(G)$ which is called *mutation* of G with respect to i as follows:

Case (1) The edge i is an internal edge: Let $(i, i_1, \cdots, i_a, i)$ and $(i, j_1, \cdots, j_b, i)$ be the cyclic orderings containing i adjacent to vertices v and u, respectively. Let $(i_1, g_1, \cdots, g_c, i_1)$ and $(j_1, h_1, \cdots, h_d, j_1)$ be the cyclic orderings containing i_1 and j_1, which do not contain i and is adjacent to vertices v' and u'.

Then detach i from v and u, and attach it to v' and u' as having cyclic orderings $(i_1, i, g_1, \cdots, g_c, i_1)$ and $(j_1, i, h_1, \cdots, h_d, j_1)$, respectively.

$G \ :=$

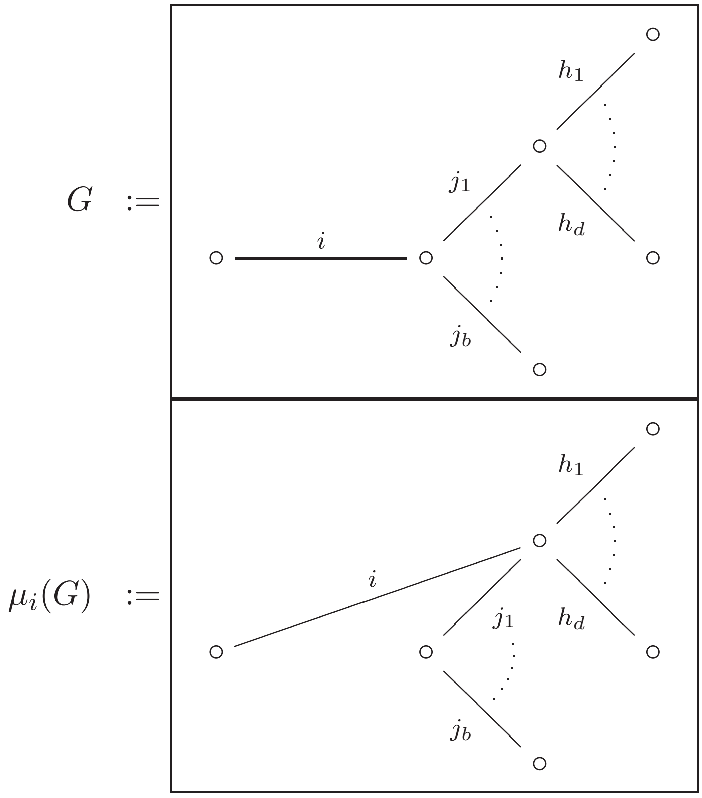
$\mu_i(G) \ :=$



The multiplicities of vertices do not change.

Case (2) The edge i is an external edge: Let $(i, j_1, \cdots, j_b, i)$ be the cyclic ordering containing i adjacent to a vertex u. Let $(j_1, h_1, \cdots, h_d, j_1)$ be the cyclic ordering containing j_1, which does not contain i and is adjacent to a vertex u'.

Then detach i from u and attach it to u' as having a cyclic ordering $(j_1, i, h_1, \cdots, h_d, j_1)$.



The multiplicities of vertices do not change.

Now we state the main theorem in this paper.

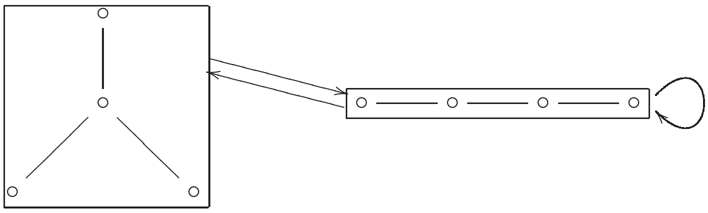
Theorem 2.2. *For any Brauer tree G and any edge i of G, we have an isomorphism $\mu_i(A_G) \simeq A_{\mu_i(G)}$ of k-algebras, sending each idempotent e_j to e_j.*

We prove the theorem above in the next section. This can be regarded as an analogue of derived equivalences associated with Bernstein-Gelfand-Ponomarev reflection of quivers [BGP] and Derksen-Weyman-Zelevinsky mutation of quivers with potentials [DWZ, BIRS].

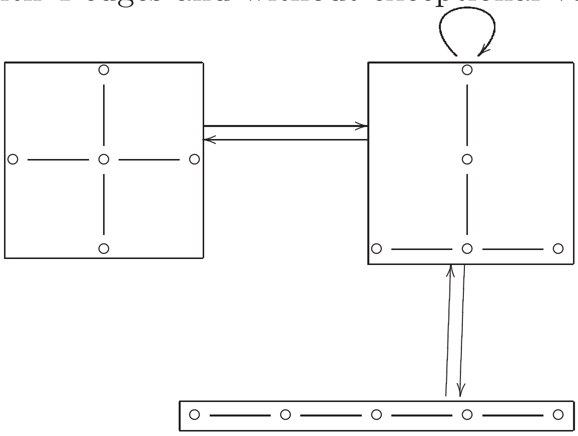
Remark 2.3. Note that a special case of Theorem 2.2 was given in [KZ] and [Z], where they only considered the case (2) in Definition 2.1 such that there are no exceptional vertices.

Example 2.4. We give graphs of mutation of Brauer trees.

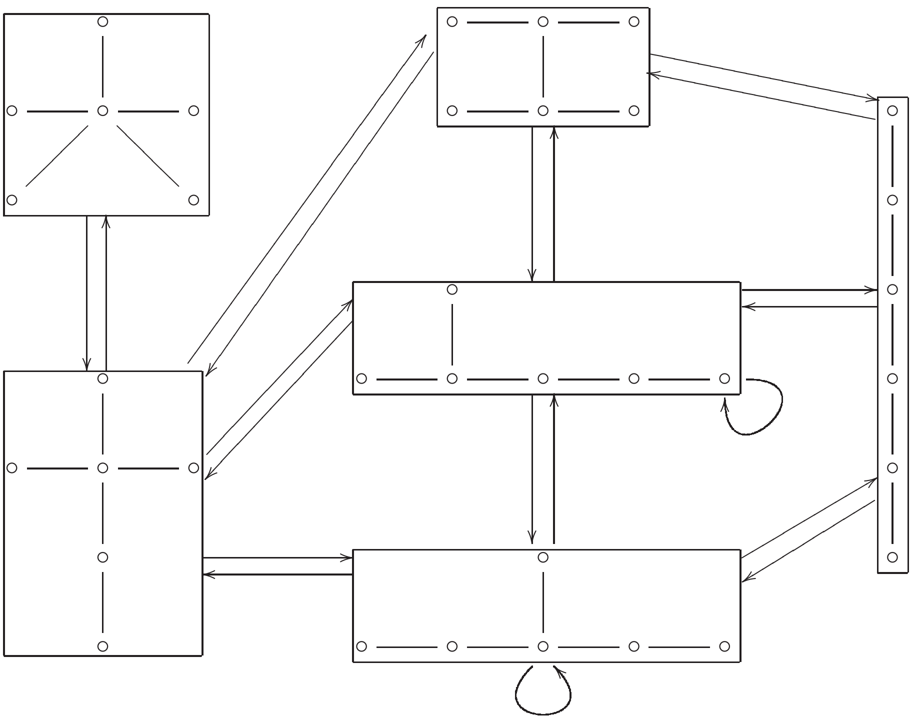
(1) Brauer trees with 3 edges and without exceptional vertices:



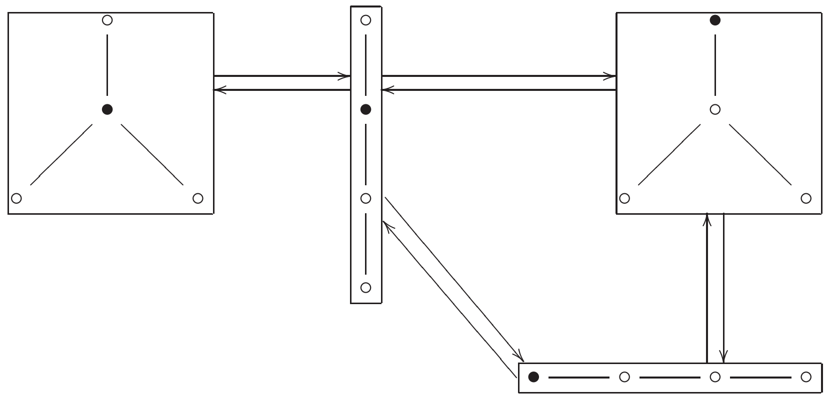
(2) Brauer trees with 4 edges and without exceptional vertices:



(3) Brauer trees with 5 edges and without exceptional vertices:



(4) Brauer trees with 3 edges and exceptional vertex $\bullet$:



We give some applications of Theorem 2.2.

For edges i and j in a Brauer tree we say that j *follows* i if there exists a cyclic ordering of the form $(\cdots, i, j, \cdots)$.

Theorem 2.5. *Let A be a Brauer tree algebra. Then for any edges i, j of the Brauer tree of A, we have the following relations:*

(1) $(\mu_i)^s(A) \simeq A$ for some positive integer s;

(2) $\mu_j\mu_i(A) \simeq \mu_i\mu_j(A)$ if i does not follow j and j does not follow i;

(3) $\mu_i\mu_j\mu_i(A) \simeq \mu_i\mu_j(A)$ if i does not follow j and j follows i;

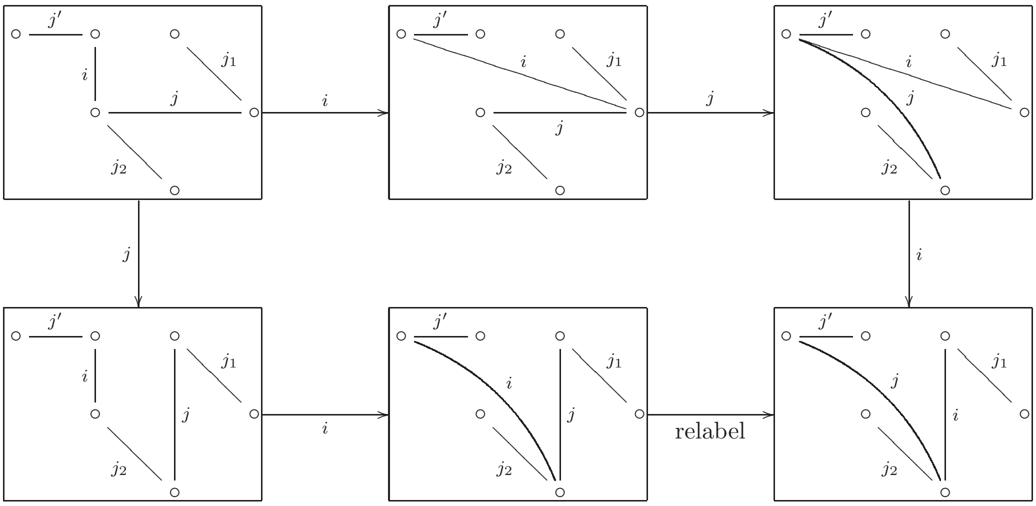
(4) $\mu_i\mu_j\mu_i(A) \simeq \mu_j\mu_i\mu_j(A)$ if i follows j and j follows i.

Proof. Let G be the Brauer tree of A. By Theorem 2.2, we have only to show the corresponding isomorphisms of Brauer trees for G.

(1) We denote by $\mathrm{Br}(n, m)$ the set of labeled Brauer trees which have n edges and multiplicity m of the exceptional vertex. Since $\mathrm{Br}(n, m)$ is a finite set, the symmetric group $\mathcal{G}$ of $\mathrm{Br}(n, m)$ is also finite. We have only to prove that each μ_i belongs to $\mathcal{G}$. For a given $G \in \mathrm{Br}(n, m)$, we can obtain $G' \in \mathrm{Br}(n, m)$ such that $\mu_i(G') = G$, by applying to G an operation similar to μ_i using the anti-clockwise ordering. This shows that each μ_i is a surjection from $\mathrm{Br}(n, m)$ to itself, and thus belongs to $\mathcal{G}$. Hence it follows that the order of mutation is finite.

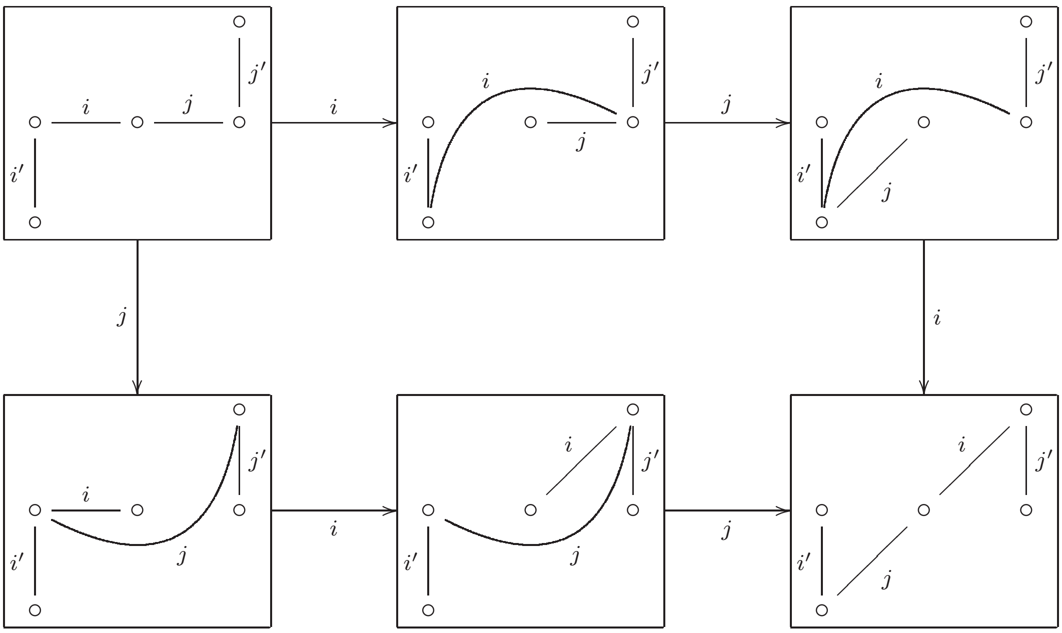
(2) This assertion can be checked easily.

(3) Let j' follow i being not j and j_1, j_2 follow j. We have the following mutation:



Hence we obtain $\mu_i\mu_j\mu_i(G) \simeq \mu_i\mu_j(G)$.

(4) Let i' follow i being not j and j' follow j being not i. We have the following mutation:



Hence we get $\mu_i\mu_j\mu_i(G) \simeq \mu_j\mu_i\mu_j(G)$. $\square$

As another application of Theorem 2.2, we give a simple proof of the following stronger statement than Theorem 1.5.

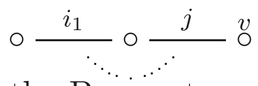
Corollary 2.6. *Let G be a Brauer tree and $\ell \geq 1$. For every vertex v of G there exists a tilting complex $T \in \mathsf{K}^{\mathrm{b}}(\text{proj-}A_G)$ of the form* $(\cdots \to 0 \to$

$P^0 \to P^1 \to 0 \to \cdots)$ *with* $P^0, P^1 \in \text{proj-}A_G$ *such that the Brauer tree of* $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T)$ *is a star with the vertex* v *in the center.*

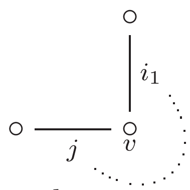
To prove the corollary above, we need the following preliminary results.

Lemma 2.7. *For any Brauer tree* G *and any vertex* v *of* G*, there exists a sequence* $i_1, ..., i_\ell$ *of distinct edges such that* $\mu_{i_\ell} \cdots \mu_{i_1}(G)$ *is a star with the vertex* v *in the center.*

Proof. Take an edge i_1 which is followed by an edge j with the vertex v:



By Theorem 2.2, we see that the Brauer tree $\mu_{i_1}(G)$ is of the form



which says that the edge i_1 has the vertex v. Continuing the argument, we obtain a star with the vertex v in the center. $\square$

Lemma 2.8. *Let* G *be a Brauer tree and* $\ell \geq 1$*. If* $i_1, \cdots, i_\ell$ *are distinct edges of* G*, then there exists a tilting complex* $T \in \mathsf{K}^{\mathrm{b}}(\text{proj-}A_G)$ *of the form* $(\cdots \to 0 \to P^0 \to P^1 \to 0 \to \cdots)$ *with* $P^0, P^1 \in \text{proj-}A_G$ *such that* $\mu_{i_\ell} \cdots \mu_{i_1}(A_G) \simeq \mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A_G)}(T)$.

Proof. For any edge i of G we have a derived equivalence $F_i{:}\mathsf{K}^{\mathrm{b}}(\text{proj-}\mu_i(A_G)) \xrightarrow{\sim} \mathsf{K}^{\mathrm{b}}(\text{proj-}A_G)$, which sends $\mu_i(A_G)$ to the Okuyama-Rickard complex $T(E\backslash\{i\})$ of A_G. Put $T_\ell := F_{i_1} \cdots F_{i_\ell}(\mu_{i_\ell} \cdots \mu_{i_1}(A_G))$, which is a tilting complex in $\mathsf{K}^{\mathrm{b}}(\text{proj-}A_G)$. We show that T_ℓ has a form

$$T_\ell = \left\{ \begin{array}{ccc} \text{(0th)} & & \text{(1st)} \\ P & \longrightarrow & 0 \\ \oplus & & \\ Q^0 & \longrightarrow & Q^1 \end{array} \right. \tag{2.8.1}$$

where $P = \bigoplus_{j \in E\backslash\{i_1, \cdots, i_\ell\}} P_j$ and $Q^0, Q^1 \in \text{proj-}A_G$. We use induction on $\ell \geq 1$. If $\ell = 1$, then we observe $T_1 = T(E\backslash\{i_1\})$. This says that T_1 is of the form (2.8.1). Assume $\ell \geq 2$. It follows from the induction hypothesis that P_{i_ℓ} is a direct summand of $T_{\ell-1}$ and the complement $T_{\ell-1}\backslash P_{i_\ell}$ is of the form $R := (\cdots \to 0 \to R^0 \to R^1 \to 0 \to \cdots)$. We see that T_ℓ is given by the direct sum of R and a complex P' which admits a triangle

$P' \to R' \to P_{i_\ell} \to P'[1]$ with $R' \in \mathsf{add}R$: see [AI]. Since P_{i_ℓ} and R concentrate on degree 0 and $(0,1)$ respectively, it is observed that P' is of the form $(\cdots \to 0 \to (P')^0 \to (P')^1 \to 0 \to \cdots)$. This implies that T_ℓ is of the form (2.8.1). □

Now Corollary 2.6 is an immediate consequence of Lemma 2.7 and Lemma 2.8. □

The following result is a direct consequence of Corollary 2.6.

Corollary 2.9. *Let A be a Brauer tree algebra. Any basic algebra which is derived equivalent to A is obtained from A by iterated tilting mutation.*

More generally, the statement above is shown for representation-finite symmetric algebras in [A].

3. Proof of main theorem

In this section we prove the main theorem of this paper.

We define an $(n \times n)$-matrix C^A as $C^A_{ij} = \dim_k \mathrm{Hom}_A(P_i, P_j)$ for any $i, j \in E$, called the *Cartan matrix* of A. Note that if A is a symmetric algebra, then we have $C^A_{ij} = C^A_{ji}$ for any $i, j \in E$.

We have the following property of the Cartan matrix of a Brauer tree algebra.

Lemma 3.1. *Let G be a Brauer tree. Then the Cartan matrix C^{A_G} of A_G is determined as follows:*

$$C^{A_G}_{ij} = \begin{cases} m_v + m_u & \text{if } i = j \text{ and the ends of } i \text{ are } u, v; \\ m_v & \text{if } i \neq j \text{ and } i, j \text{ have a common end } v; \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, the Cartan matrix C^A of a Brauer tree algebra A and the data of extensions among simple A-modules determine the Brauer tree of A by the following:

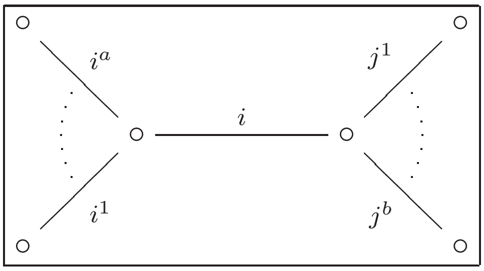
Method 3.2. Let A be a Brauer tree algebra. Assume that we know the Cartan matrix C^A of A and which of $\dim_k \mathrm{Ext}^1_A(S_i, S_j)$ are not zero. We explicitly determine the Brauer tree G of A.

(1) We give the cyclic ordering containing each edge. Fix any $i \in E$. We define a subset I of E by $I = \{j \in E \mid C^A_{ij} \neq 0\}$. Since G is a Brauer tree, we have a disjoint union $I = \{i\} \cup I_0 \cup I_1$ satisfying $C^A_{i_0 i_1} = 0$ for any $i_0 \in I_0$ and $i_1 \in I_1$. Moreover, for any $\ell \in \{0, 1\}$ and any $j \in \{i\} \cup I_\ell$ there exists a unique $j' \in \{i\} \cup I_\ell$ such that $\mathrm{Ext}^1_A(S_j, S_{j'}) \neq 0$. Thus we can take sequences

$$i = i^0, i^1, \cdots, i^a, i^{a+1} = i \text{ in } \{i\} \cup I_0$$

$$i = j^0, j^1, \cdots, j^b, j^{b+1} = i \text{ in } \{i\} \cup I_1$$

such that $\mathrm{Ext}^1_A(S_{i^x}, S_{i^{x+1}}) \neq 0$ for any $0 \leq x \leq a$ and $\mathrm{Ext}^1_A(S_{j^y}, S_{j^{y+1}}) \neq 0$ for any $0 \leq y \leq b$. Hence we can explicitly determine the cyclic ordering containing i by $(i, i^1, \cdots, i^a, i)$ and $(i, j^1, \cdots, j^b, i)$:



(2) We give the position and the multiplicity of the exceptional vertex if it exists: The multiplicities of non-exceptional vertices are 1. Note that the exceptional vertex exists if and only if there is $i \in E$ satisfying $C^A_{ii} > 2$. Put $\mathcal{E} := \{i \in E \mid C^A_{ii} > 2\}$ and assume that $\mathcal{E}$ is not an empty set. Since the Brauer tree G has only one exceptional vertex, we observe that all edges in $\mathcal{E}$ have a common vertex v and any edge having the vertex v belongs to $\mathcal{E}$. Thus the vertex v is exceptional with multiplicity $C^A_{ii} - 1$ for $i \in \mathcal{E}$. $\square$

We show the following easy observation.

Proposition 3.3. *Let A and B be derived equivalent symmetric k-algebras. If A is a Brauer tree algebra, then so is B.*

Proof. By Theorem 1.5, A is derived equivalent to a Brauer tree algebra C for a star with the exceptional vertex in the center if it exists. Since A and B are derived equivalent, it follows that B and C are also derived equivalent. This implies that B is stable equivalent to C. Note that C is a symmetric Nakayama algebra. Hence the assertion follows from [ARS, X, Theorem 3.14]. $\square$

We denote by $\mathsf{D}^{\mathrm{b}}(\text{mod-}A)$ and $\underline{\text{mod}}\text{-}A$ the bounded derived category and the stable module category of mod-A, respectively.

We also need the result below.

Lemma 3.4. [O, Lemma 2.1] *Let E_0 be a subset of E and put $e := \sum_{i \in E_0} e_i$. Let $T := T(E_0)$ be the Okuyama-Rickard complex with respect to E_0. Assume that A is a symmetric algebra. Now the endomorphism algebra $B = \mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T)$ of T is stable equivalent to A and we denote the stable equivalence by $F : \underline{\text{mod}}\text{-}A \xrightarrow{\sim} \underline{\text{mod}}\text{-}B$. Then the following hold:*

(1) *If $i \notin E_0$, then $F(\Omega(S_i))$ is a simple B-module;*
(2) *If $i \in E_0$, then $F(Y_i)$ is a simple B-module, where Y_i is maximal amongst submodules of P_i such that any S_j $(j \in E_0)$ is not a composition factor of Y_i/S_i.*

Proof. For the convenience of the reader, we give full details here.

We have to calculate $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\text{mod-}A)}(T, -[n])$ for any $n \in \mathbb{Z}$.

(i) Assume $i \notin E_0$. It is easy to see that $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\text{mod-}A)}(T, S_i[n])$ is isomorphic to k if $n = -1$ and is otherwise zero. Since $\Omega(S_i)$ is isomorphic to $S_i[-1]$ in $\underline{\text{mod}}$-A, it is sent by F to a simple B-module.

(ii) Assume $i \in E_0$. We can easily check $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\text{mod-}A)}(T, Y_i[n]) = 0$ for any $n \neq 0$. For $j \in E \backslash E_0$, the complex T_j is the (-1)-shift of a minimal projective presentation $\pi_j : Q_j \to P_j$ of e_jA/e_jAeA where $e = \sum_{\ell \in E_0} e_\ell$. Let $f \in \mathrm{Hom}_A(Q_j, Y_i)$. The A-module Y_i is a submodule of P_i with $\mathrm{Hom}_A(eA, Y_i) \simeq k$ and Q_j belongs to $\mathsf{add}(eA)$. Therefore we observe that f factors through the canonical homomorphism $Q_j \to X_j := e_jAeA$. Since Y_i is maximal amongst submodules of P_i with $\mathrm{Hom}_A(eA, Y_i) \simeq k$, we see that $\mathrm{Hom}_A(P_j/X_j, P_i/Y_i) = 0$. This implies that $\mathrm{Ext}^1_A(P_j/X_j, Y_i) = 0$, and so any homomorphism from X_j to Y_i factors through the inclusion $X_j \to P_j$. Thus there exists $\alpha : P_j \to Y_i$ satisfying $f = \alpha\pi_j$, which says that $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\text{mod-}A)}(T_j, Y_i) = 0$ for any $j \in E\backslash E_0$. Hence we have $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\text{mod-}A)}(T, Y_i) \simeq k$, and it follows that $F(Y_i)$ is a simple B-module. $\square$

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let G be the Brauer tree of A and we use the notation of Definition 2.1. Since the Okuyama-Rickard complex $T := T(E\backslash\{i\})$ is tilting by Proposition 1.2, $B := \mu_i(A)$ is a Brauer tree algebra by Proposition 3.3. Our goal is to show that the Brauer tree of B coincides with $\mu_i(G)$. To do this, we have only to calculate the Cartan matrix of B and which of the dimensions of extensions among simple B-modules are not zero.

Recall that T is defined as the direct sum of the following complexes:

$$\begin{array}{lccc} & \text{(0th)} & & \text{(1st)} & \\ T_i: & P_{i_1} \oplus P_{j_1} & \longrightarrow & P_i & \\ T_\ell: & P_\ell & \longrightarrow & 0 & \quad (\ell \neq i) \end{array}$$

(If the edge i is external, then replace the above first complex with $P_{i_1} \to P_i$ or $P_{j_1} \to P_i$.)

(1) Let C^A and C^B be Cartan matrices of A and B, respectively. We calculate $C^B_{\ell m}$. For each $\ell \in E$, we denote by P^B_ℓ a projective indecomposable B-module corresponding to T_ℓ.

(i) We can easily check $C^B_{\ell m} = C^A_{\ell m}$ for any $\ell \neq i$ and $m \neq i$.

(ii) We calculate $C^B_{i\ell}$ for $\ell \neq i$. If $\ell \neq i$, then we have equalities

$$\begin{aligned} C^B_{i\ell} &= \dim_k \mathrm{Hom}_B(P^B_i, P^B_\ell) \\ &= \dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T_i, T_\ell) \\ &= \dim_k \mathrm{Hom}_A(P_{i_1}, P_\ell) + \dim_k \mathrm{Hom}_A(P_{j_1}, P_\ell) - \dim_k \mathrm{Hom}_A(P_i, P_\ell) \\ &= C^A_{i_1\ell} + C^A_{j_1\ell} - C^A_{i\ell}. \end{aligned}$$

Therefore we see the following:

$$C^B_{i\ell} = \begin{cases} 0 & (\ell \in \{i_2, \cdots, i_a\} \text{ or } \{j_2, \cdots, j_b\}); \\ C^A_{i_1\ell} \neq 0 & (\ell \in \{g_1, \cdots, g_c\}); \\ C^A_{j_1\ell} \neq 0 & (\ell \in \{h_1, \cdots, h_d\}); \\ m_{v(\ell)} & (\ell = i_1 \text{ or } j_1); \\ 0 & (\text{otherwise}) \end{cases}$$

where $v(\ell)$ is the vertex of ℓ that i does not have.

(iii) We show $C^B_{ii} = m_{v(i_1)} + m_{v(j_1)}$. We observe equalities

$$\begin{aligned} C^B_{ii} &= \dim_k \mathrm{Hom}_B(P^B_i, P^B_i) \\ &= \dim_k \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\text{proj-}A)}(T_i, T_i) \\ &= C^A_{ii} + C^A_{i_1i_1} + C^A_{j_1j_1} + 2C^A_{i_1j_1} - 2(C^A_{ii_1} + C^A_{ij_1}) \\ &= m_{v(i_1)} + m_{v(j_1)}. \end{aligned}$$

(2) For each $\ell \in E$, we put $S^B_\ell = P^B_\ell/\mathrm{rad}P^B_\ell$. We calculate $\mathrm{Ext}^1_B(S^B_\ell, S^B_m)$. We denote by $F : \underline{\mathrm{mod}}\text{-}A \to \underline{\mathrm{mod}}\text{-}B$ the stable equivalence between A and B given by T. By Lemma 3.4, it is observed that F sends

$$X_\ell := \begin{cases} \Omega(S_i) & (\ell = i) \\ Y_\ell & (\ell = i_1 \text{ or } j_1) \\ S_\ell & (\text{otherwise}) \end{cases}$$

to S^B_ℓ, where Y_ℓ is a unique submodule of P_ℓ whose Loewy series is $\begin{pmatrix} S_i \\ S_\ell \end{pmatrix}$.

(a) We can easily check $\mathrm{Ext}^1_B(S^B_\ell, S^B_m) \simeq \mathrm{Ext}^1_A(S_\ell, S_m)$ for $\ell, m \notin \{i, i_1, j_1\}$ or $\ell = m = i$.

(b) We calculate $\mathrm{Ext}^1_B(S^B_\ell, S^B_m)$ for $\ell \in \{i, i_1, j_1\}$ and $m \notin \{i, i_1, j_1\}$. We have isomorphisms

$$\begin{aligned}
\mathrm{Ext}^1_B(S^B_\ell, S^B_m) &\simeq \mathrm{Ext}^1_A(X_\ell, S_m) \\
&\simeq \underline{\mathrm{Hom}}_A(\Omega(X_\ell), S_m) \\
&\simeq \mathrm{Hom}_A(\Omega(X_\ell), S_m) \\
&\simeq \begin{cases} \mathrm{Hom}_A(\Omega^2(S_i), S_m) & (\ell = i) \\ \mathrm{Hom}_A(\Omega(Y_\ell), S_m) & (\ell = i_1 \text{ or } j_1) \end{cases} \\
&\begin{cases} \neq 0 & (\ell = i \text{ and } m \in \{g_1, h_1\}) \\ \neq 0 & ((\ell, m) = (i_1, i_2) \text{ or } (j_1, j_2)) \\ = 0 & (\text{otherwise}). \end{cases}
\end{aligned}$$

Similarly, for $\ell \notin \{i, i_1, j_1\}$ and $m \in \{i, i_1, j_1\}$ we obtain isomorphisms

$$\begin{aligned}
\mathrm{Ext}^1_B(S^B_\ell, S^B_m) &\simeq \mathrm{Hom}_A(S_\ell, \Omega^{-1}(X_m)) \\
&\simeq \begin{cases} \mathrm{Hom}_A(S_\ell, S_i) & (m = i) \\ \mathrm{Hom}_A(S_\ell, \Omega^{-1}(Y_m)) & (m = i_1 \text{ or } j_1) \end{cases} \\
&\begin{cases} \neq 0 & ((\ell, m) = (i_a, i_1), (g_c, i_1), (j_b, j_1) \text{ or } (h_d, j_1)) \\ = 0 & (\text{otherwise}). \end{cases}
\end{aligned}$$

(c) We show $\mathrm{Ext}^1_B(S^B_\ell, S^B_i) \neq 0$ for $\ell \in \{i_1, j_1\}$. We see isomorphisms

$$\begin{aligned}
\mathrm{Ext}^1_B(S^B_\ell, S^B_i) &\simeq \mathrm{Ext}^1_A(Y_\ell, \Omega(S_i)) \\
&\simeq \underline{\mathrm{Hom}}_A(Y_\ell, S_i) \\
&\simeq \mathrm{Hom}_A(Y_\ell, S_i) \\
&\neq 0.
\end{aligned}$$

Similarly, for $\ell \in \{i_1, j_1\}$ we have an isomorphism

$$\begin{aligned}
\mathrm{Ext}^1_B(S^B_i, S^B_\ell) &\simeq \mathrm{Hom}_A(S_i, \Omega^{-2}(Y_\ell)) \\
&\begin{cases} = 0 & \text{if } \ell \text{ is an internal edge} \\ \neq 0 & \text{otherwise} \end{cases}
\end{aligned}$$

.

(d) We calculate $\mathrm{Ext}^1_B(S^B_\ell, S^B_m)$ for $\ell, m \in \{i_1, j_1\}$. If $\ell \neq m$, it follows from (i) that $C^B_{\ell m} = 0$, which implies $\mathrm{Ext}^1_B(S^B_\ell, S^B_m) = 0$. Let $\ell = m$. Since

$\mathrm{Hom}_A(Y_\ell, Y_\ell) \simeq \mathrm{Hom}_A(P_i, Y_\ell)$, we obtain isomorphisms

$$\begin{aligned}\mathrm{Ext}^1_B(S^B_\ell, S^B_\ell) &\simeq \mathrm{Ext}^1_A(Y_\ell, Y_\ell)\\ &\simeq \mathrm{Hom}_A(\Omega(Y_\ell), Y_\ell)\\ &\begin{cases} \neq 0 & \text{if } \mathrm{Ext}^1_A(S_\ell, S_i) \neq 0 \text{ and } m_v > 1;\\ = 0 & \text{otherwise}\end{cases}\end{aligned}$$

where v is the common vertex of ℓ and i.

Applying Method 3.2, we conclude that the Brauer tree of B is given by $\mu_i(G)$. □

Acknowledgement

The author would like to give his deep gratitude to Shigeo Koshitani, Osamu Iyama, Sefi Ladkani and Joseph Grant, who read the paper carefully and gave an important information on earlier literature and a lot of helpful comments and suggestions. The author thanks the refree for his/her careful reading.

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(*Received April 6, 2012*)
(*Revised April 22, 2013*)