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AN ALGEBRAIC APPROACH TO THE CAMERON-MARTIN-MARUYAMA-GIRSANOV FORMULA

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Abstract. In this paper, we will give a new perspective to the Cameron-Martin-Maruyama-Girsanov formula by giving a totally algebraic proof to it. It is based on the exponentiation of the Malliavin-type differentiation and its adjointness.

1. Introduction.

Let $(\mathscr{W}, \mathscr{B}(\mathscr{W}), \gamma)$ be the Wiener space on the interval $[0,1]$, that is, $\mathscr{W}$ is the set of all continuous paths in $\mathbb{R}$ defined on $[0,1]$ which starts from zero, $\mathscr{B}(\mathscr{W})$ is the σ-field generated by the topology of uniform convergence. and γ is the Wiener measure on the measurable space $(\mathscr{W}, \mathscr{B}(\mathscr{W}))$. Then the canonical Wiener process $(W(t))_{t\geq 0}$ is defined by $W(t,w) = w(t)$ for $0 \leq t \leq 1$ and $w \in \mathscr{W}$.

Let $\mathscr{H}$ denote the Cameron-Martin subspace of $\mathscr{W}$, i.e., $h(t) \in \mathscr{W}$ belongs to $\mathscr{H}$ if and only if $h(t)$ is absolutely continuous in t and the derivative $\dot{h}(t)$ is square-integrable. Note that $\mathscr{H}$ is a Hilbert space under the inner product

$$\langle h_1, h_2 \rangle_{\mathscr{H}} = \int_0^1 \dot{h_1}(t)\dot{h_2}(t)dt, \qquad h_1, h_2 \in \mathscr{H}.$$

It is a fundamental fact in stochastic calculus that the Cameron-Martin (henceforth CM) formula (see, e.g. [5], pp 25) in the following form holds:

$$\begin{aligned} &\int_{\mathscr{W}} F(w+\theta)\gamma(dw) \\ &\qquad = \int_{\mathscr{W}} F(w) \exp\Big\{ \int_0^1 \dot{\theta}(t)dw(t) - \frac{1}{2}\int_0^1 \dot{\theta}(t)^2 dt \Big\} \gamma(dw) \end{aligned} \tag{1.1}$$

where F is a bounded measurable function on $\mathscr{W}$ and $\theta \in \mathscr{H}$.

The motivation of the present study comes from the following observation(s). In the above CM formula (1.1), the integrand of the left-hand-side can be seen as an action of a translation operator, which is an exponentiation of a differentiation D_θ:

$$\int_{\mathscr{W}} F(w+\theta)\gamma(dw) \text{ ``$=$'' } E\Big[e^{D_\theta}F\Big]. \tag{1.2}$$

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On the other hand, the right-hand-side can be seen as a "coupling" of the exponential martingale and F:

$$\begin{aligned}\int_{\mathscr{W}} F(w) \exp\Big\{\int_0^1 \dot\theta(t)dw(t) - \frac{1}{2}\int_0^1 \dot\theta(t)^2 dt\Big\}\gamma(dw) \\ = \Big\langle F, \exp\Big\{\int_0^1 \dot\theta(t)dW(t) - \frac{1}{2}\int_0^1 \dot\theta(t)^2 dt\Big\}\Big\rangle .\end{aligned}$$

Since we can read the right-hand-side of (1.2) as

$$E\Big[e^{D_\theta} F\Big] \text{ “ = ” } \langle 1, e^{D_\theta} F\rangle ,$$

the Cameron Martin formula

$$\langle 1, e^{D_\theta} F\rangle \text{ “ = ” } \Big\langle F, \exp\Big\{\int_0^1 \dot\theta(t)dW(t) - \frac{1}{2}\int_0^1 \dot\theta(t)^2 dt\Big\}\Big\rangle$$

leads to the following interpretation:

$$\exp\Big\{\int_0^1 \dot\theta(t)dW(t) - \frac{1}{2}\int_0^1 \dot\theta(t)^2 dt\Big\} \text{ “ = ” } e^{D_\theta^*}(1),$$

where D_θ^* is an "adjoint operator" of D_θ.

The observation, conversely, suggests that the CM formula could be proved directly by the duality relation between e^{D_θ} and $e^{D_\theta^*}$, without resorting to the stochastic calculus. The program is successfully carried out in section 2. We may say this program runs by the calculus of functionals of Wiener integrals.

Along the line, we also give an algebraic proof of the Maruyama-Girsanov (henceforth MG) formula (see e.g. [10, IV.38, Theorem (38.5)]), an extension of the CM formula. Note that MG formula cannot be written in the quasi-invariant form as (1.1), but in the following way:

$$\begin{aligned}&\int_{\mathscr{W}} F(w)\gamma(dw) \\ (1.3)\quad &= \int_{\mathscr{W}} F(w - Z(w)) \exp\Big\{\int_0^1 \dot Z(t,w)dw(t) - \frac{1}{2}\int_0^1 \dot Z(t,w)^2 dt\Big\}\gamma(dw).\end{aligned}$$

Here $Z : \mathscr{W} \to \mathscr{H}$ is a "predictable" map such that

$$\int_{\mathscr{W}} \exp\Big\{\int_0^1 \dot Z(t,w)dw(t) - \frac{1}{2}\int_0^1 \dot Z(t,w)^2 dt\Big\}\gamma(dw) = 1.$$

In this non-linear situation, infinite dimensional vector fields like $X_Z \equiv Z^i D_{e_i}$[1], where $\{e_i\}$ is a basis of $\mathscr{H}$ and $Z^i = \langle Z, e_i\rangle_{\mathscr{H}}$, may play a role of D_θ

[1]Here we use Einstein's convention.

in the linear case, but its exponentiation e^{X_Z} does not make sense anymore. Instead, we need to consider "tensor fields"

$$D_Z^{\otimes n} = Z^{i_1} \cdots Z^{i_n} D_{e_{i_1}} \cdots D_{e_{i_n}}$$

and its formal series

$$\sum_{n=0}^{\infty} \frac{1}{n!} D_Z^{\otimes n} =: \widetilde{e}^{D_Z}.$$

We will show in Proposition 3.2 that the operator $\widetilde{e}^{D_Z}$ is the translation by Z; $\widetilde{e}^{D_Z}(f(W)) = f(W+Z)$. To understand MG formula (1.3) in terms of the translation operator $\widetilde{e}^{D_Z}$, we additionally introduce another sequence $\{L_n\}$ of tensor fields (see subsection 3.2 for the definition), which has the property (Lemma 3.4) of

$$\sum_{n=1}^{\infty} \frac{1}{n!} L_n = \exp\Big\{ \int_0^1 \dot{Z}(t) dw(t) - \frac{1}{2} \int_0^1 \dot{Z}^2(t) dt \Big\} (\widetilde{e}^{D_Z} - 1).$$

Then, as a corollary to the adjoint formula for L_n (Theorem 3.3), MG formula can be obtained (Corollary 3.5).

The proof of key theorem (Theorem 3.3), however, is not "algebraic" since it involves the use of Itô's formula. This means, we feel, a considerable part of the "algebraic structure" of MG formula is still unrevealed. We then try to give a purely algebraic proof (=without resorting the results from the stochastic calculus) to MG formula in section 4 at the cost that we only consider the case where $\dot{Z}$ is a simple predictable process such as

$$\dot{Z} = \sum_{i=1}^{N} z_i 1_{(t_i, t_{i+1}]}(t).$$

We will consider a family of vector fields like $z_i D_i$, where D_i is the differentiation in the direction of $\int 1_{(t_i,t_{i+1}]}(t)\, dt$. A key ingredient in our (second) algebraic proof of MG formula is the following semi-commutativity:

$$z_i D_j = D_j z_i \quad \text{if } j \geq i, \tag{1.4}$$

which may be understood as "causality".

Actually, the relation (1.4) implies that the Jacobian matrix $DZ = (D_{e_i} Z_j)_{ij}$, if it is defined, is upper triangular. In a coordinate-free language, it is nilpotent. Equivalently, $\mathrm{Tr}(DZ)^n = 0$ for every n, or $\mathrm{Tr} \wedge^n DZ = 0$ for every n. Since the statements are coordinate-free(=independent of the choice of $\{e_i\}$), they can be a characterization of the causality (=predictability) in the infinite dimensional setting as well. This observation retrieves the result in [12] that Ramer-Kusuoka formula ([9],[4]) is reduced to MG formula when DZ is nilpotent in this sense. The observation also implies that Ramer-Kusuoka formula itself can be approached in our algebraic way. This program has been successfully carried out in [1].

In the present paper, the domains of the operators are basically restricted to "polynomials" (precise definition of which will be given soon) in order to concentrate on algebraic structures. We leave in Appendix a lemma and its proof to ensure the continuity of the operators and hence to have a standard version of CM-MG formula.

To the best of our knowledge, an algebraic proof like ours for CMMG formula have never been proposed. Though we only treat a simplest one-dimensional Brownian case, our method can be applied to more general cases if only they have a proper action of the infinite dimensional Heisenberg algebra. The present study is largely motivated by P. Malliavin's way to look at stochastic calculus, which for example appears in [5] and [6], and also by some operator calculus often found in the quantum fields theory (see e.g. [7]).

2. An Algebraic Proof of the Cameron-Martin Formula.

2.1. Preliminaries. For any $h \in \mathscr{H}$, we set

$$[h](w) := \int_0^1 \dot{h}(t)dw(t), \quad w \in \mathscr{W}.$$

A function $F : \mathscr{W} \to \mathbb{R}$ is called a *polynomial functional* if there exist an $n \in \mathbb{N}$, $h_1, h_2, \cdots, h_n \in \mathscr{H}$ and a polynomial $p(x_1, x_2, \cdots, x_n)$ of n-variables such that

$$F(w) = p\Big([h_1](w), [h_2](w), \cdots, [h_n](w)\Big), \quad w \in \mathscr{W}.$$

The set of all polynomial functionals is denoted by $\mathbf{P}$. This is an algebra over $\mathbb{R}$ included densely in $L^p(\mathscr{W})$ for any $1 \leq p < \infty$ (see e.g. [3], pp 353, Remark 8.2).

Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$. If we set

$$\xi_i(w) := [e_i](w) = \int_0^1 \dot{e_i}(t)dw(t), \qquad i = 1, 2, \cdots$$

then $\xi_1, \xi_2, \cdots$ are mutually independent standard Gaussian random variables. Let $H_n[\xi]$, $n = 1, 2, \cdots$ be the n-th Hermite polynomial in ξ defined by the generating function identity

$$\exp\Big(\lambda\xi - \frac{\lambda^2}{2}\Big) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n[\xi], \quad \lambda \in \mathbb{R},$$

and put

$$\mathbf{\Lambda} := \left\{ \mathbf{a} = (a_i)_{i=1}^{\infty} : \begin{array}{l} a_i \in \mathbb{Z}^+, \\ a_i = 0 \text{ except for a finite number of } i\text{'s} \end{array} \right\}.$$

We write $\mathbf{a}! := \prod_{i=1}^{\infty} a_i!$ for $\mathbf{a} = (a_i)_{i=1}^{\infty} \in \mathbf{\Lambda}$. We define $H_{\mathbf{a}}(w) \in \mathbf{P}$, $\mathbf{a} \in \mathbf{\Lambda}$ by

$$H_{\mathbf{a}}(w) := \prod_{i=1}^{\infty} H_{a_i}[\xi_i(w)], \qquad w \in \mathscr{W}$$

and then $\{\frac{1}{\sqrt{\mathbf{a}!}} H_{\mathbf{a}} : \mathbf{a} \in \mathbf{\Lambda}\}$ forms an orthonormal basis of $L^2(\mathscr{W})$ (see e.g. [3]).

For a differentiable function f on $\mathbb{R}$ measured by $N_1(d\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} d\xi$, if we define ∂ and ∂^* as

$$\partial f(\xi) = f'(\xi) \text{ and } \partial^* f(\xi) = -\partial f(\xi) + \xi f(\xi), \quad \xi \in \mathbb{R}$$

then ∂^* is adjoint to ∂ on the differentiable class in $L^2(\mathbb{R}, N_1)$. We note that the n-th Hermite polynomial H_n can be given by $H_n[\xi] = (\partial^{*n} 1)(\xi)$.

2.2. **Directional differentiations and its exponentials.** For a function F on $\mathscr{W}$ and $\theta \in \mathscr{H}$, the differentiation of F in the direction θ $D_\theta F$ is defined by

$$D_\theta F(w) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big\{ F(w + \varepsilon\theta) - F(w) \Big\}, \quad w \in \mathscr{W}$$

if it exists(see e.g. [3]). Note that $D_\theta F(w)$ is linear in θ and F and satisfies the Leibniz' formula $D_\theta(FG)(w) = D_\theta F(w) \cdot G(w) + F(w) D_\theta G(w)$ for functions F and G on $\mathscr{W}$ such that $D_\theta F(w)$ and $D_\theta G(w)$ exist. If $F(w)$ is of the form $F(w) = f([h](w))$ where f is a differentiable function on $\mathbb{R}$ and $h \in \mathscr{H}$, then we have

$$D_\theta F(w) = \langle \theta, h \rangle_{\mathscr{H}} f'([h](w)). \tag{2.1}$$

For $\theta \in \mathscr{H}$, we define the *exponential* of D_θ by

$$e^{D_\theta} F(w) := \sum_{n=0}^{\infty} \frac{1}{n!} D_\theta^n F(w), \quad F \in \mathbf{P} \text{ and } w \in \mathscr{W}$$

which is actually a finite sum by (2.1).

Lemma 2.1. *For $F, G \in \mathbf{P}$, we have*

$$e^{D_\theta}(FG) = e^{D_\theta}(F) \cdot e^{D_\theta}(G). \tag{2.2}$$

Proof. is a straightforward computation:

$$
\begin{aligned}
e^{D_\theta}(F) \cdot e^{D_\theta}(G) &= \Big(\sum_{n=0}^{\infty} \frac{1}{n!} D_\theta^n F\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{1}{n!} D_\theta^n G\Big) \\
&= \Big(F + D_\theta F + \frac{1}{2!} D_\theta^2 F + \frac{1}{3!} D_\theta^3 F + \cdots\Big) \\
&\qquad\qquad \cdot \Big(G + D_\theta G + \frac{1}{2!} D_\theta^2 G + \frac{1}{3!} D_\theta^3 G + \cdots\Big) \\
&= FG + \Big\{D_\theta F \cdot G + F D_\theta G\Big\} \\
&\qquad + \Big\{\frac{1}{2!} D_\theta^2 F \cdot G + D_\theta F \cdot D_\theta G + F \cdot \frac{1}{2!} D_\theta^2 G\Big\} \\
&\quad + \Big\{\frac{1}{3!} D_\theta^3 F \cdot G + \frac{1}{2!} D_\theta^2 F \cdot D_\theta G + D_\theta F \cdot \frac{1}{2!} D_\theta^2 G + F \cdot \frac{1}{3!} D_\theta^3 G\Big\} \\
&\qquad\qquad + \cdots \\
&= FG + D_\theta(FG) + \frac{1}{2!} D_\theta^2 (FG) + \frac{1}{3!} D_\theta^3 (FG) + \cdots = e^{D_\theta}(FG).
\end{aligned}
$$

□

Proposition 2.2. *For every $F \in \mathbf{P}$, we have*

$$
e^{D_\theta} F(w) = F(w + \theta), \quad w \in \mathscr{W}. \tag{2.3}
$$

Proof. By Lemma 2.1, it suffices to show (2.3) for the functional $F \in \mathbf{P}$ of the form $F(w) = f([h](w))$ where $f(x)$ is a polynomial in one-variable and $h \in \mathscr{H}$. Then using (2.1), we obtain

$$
\begin{aligned}
e^{D_\theta} F(w) &= \sum_{n=0}^{\infty} \frac{1}{n!} D_\theta^n f([h](w)) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \theta, h \rangle_{\mathscr{H}}^n f^{(n)}([h](w)) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}([h](w)) \Big\{ \Big([h](w) + \langle \theta, h \rangle_{\mathscr{H}} \Big) - [h](w) \Big\}^n \\
&= f\Big([h](w) + \langle \theta, h \rangle_{\mathscr{H}} \Big) = F(w + \theta),
\end{aligned}
$$

where $f^{(n)}(x)$ denotes the n-th derivative of $f(x)$. □

2.3. **Formal adjoint operator and its exponential.** In the analogy of ∂ and ∂^* in the previous section, we define D_θ^*, $\theta \in \mathscr{H}$ by

$$
D_\theta^* F(w) := -D_\theta F(w) + \int_0^1 \dot{\theta}(t) dw(t) \cdot F(w), \quad F \in \mathbf{P}, w \in \mathscr{W}.
$$

Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$ and put $\xi_i(w) := [e_i](w)$ for $i = 1, 2, \cdots$. Then we have

Lemma 2.3. *It holds that*

$$E\Big[D_\theta H_n[\xi_k]\cdot H_m[\xi_l]\Big] = E\Big[H_n[\xi_k]D_\theta^* H_m[\xi_l]\Big]$$

for any $k, l, m, n = 1, 2, \cdots$.

Proof. Since $t \mapsto H_n[\int_0^t e_k(s)dw(s)]$ $(n \geq 1)$ is a martingale with initial value zero, if $k \neq l$ the independence of ξ_k and ξ_l and the formula (2.1) imply that both sides become zero when $n, m \geq 1$. If $n = m = 0$, it is clear that the left-hand side is zero. Then the right-hand side equals to

$$E[D_\theta^* 1] = E[-D_\theta 1 + \int_0^1 \dot\theta(t)dw(t)] = E[\int_0^1 \dot\theta(t)dw(t)] = 0.$$

Hence the case $k = l$ suffices. Noting that ξ_k is a normal Gaussian random variable, we have

$$\begin{aligned} E\Big[D_\theta H_n[\xi_k]\cdot H_m[\xi_k]\Big] &= \langle\theta, e_k\rangle_{\mathscr{H}} E\Big[H_n'[\xi_k]H_m[\xi_k]\Big] \\ &= \langle\theta, e_k\rangle_{\mathscr{H}} \int_{-\infty}^{\infty} \partial H_n[\xi]\cdot H_m[\xi]\gamma_1(d\xi) \\ &= \langle\theta, e_k\rangle_{\mathscr{H}} \int_{-\infty}^{\infty} H_n[\xi]\partial^* H_m[\xi]\gamma_1(d\xi) \\ &= \langle\theta, e_k\rangle_{\mathscr{H}} \int_{-\infty}^{\infty} H_n[\xi]\Big\{-H_m'[\xi] + \xi H_m[\xi]\Big\}\gamma_1(d\xi) \\ &= \langle\theta, e_k\rangle_{\mathscr{H}} E\Big[H_n[\xi_k]\Big\{-H_m'[\xi_k] + \xi_k H_m[\xi_k]\Big\}\Big] \\ &= E\Big[H_n[\xi_k]\Big\{-D_\theta H_m[\xi_k] + \langle\theta, e_k\rangle_{\mathscr{H}}\xi_k H_m[\xi_k]\Big\}\Big]. \end{aligned}$$

Since θ can be written as $\theta = \sum_{k=1}^\infty \langle\theta, e_k\rangle_{\mathscr{H}} e_k$, $\int_0^1 \dot\theta(t)dw(t)$ admits the L^2-expansion

$$\int_0^1 \dot\theta(t)dw(t) = \sum_{k=1}^\infty \langle\theta, e_k\rangle_{\mathscr{H}}\xi_k.$$

Now the independence of $\{\xi_i\}_{i=1}^\infty$ shows that

$$E\Big[H_n[\xi_k]\int_0^1 \dot\theta(t)dw(t)H_m[\xi_k]\Big] = E\Big[H_n[\xi_k]\langle\theta, e_k\rangle_{\mathscr{H}}\xi_k H_m[\xi_k]\Big].$$

□

Proposition 2.4. *For every* $F, G \in \mathbf{P}$, *it holds that*

$$E[D_\theta F\cdot G] = E[F D_\theta^* G]. \tag{2.4}$$

Proof. For fixed $F, G \in \mathbf{P}$, there exist a positive integer $n \in \mathbb{N}$ and an orthonormal system $\{e_1, e_2, \cdots, e_n\}$ in $\mathscr{H}$ and polynomials $f(x_1, x_2, \cdots, x_n)$ and $g(x_1, x_2, \cdots, x_n)$ of n-variables such that

$$\begin{aligned}
F(w) &= f\Big([e_1](w), [e_2](w), \cdots, [e_n](w)\Big) \quad \text{and} \\
G(w) &= g\Big([e_1](w), [e_2](w), \cdots, [e_n](w)\Big).
\end{aligned}$$

Extend $\{e_1, e_2, \cdots, e_n\}$ to an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of $\mathscr{H}$. Since the degree of the n-th Hermite polynomial is exactly n, f and g can be written as linear combinations of finite products of Hermite polynomials. From this fact and by the linearity of D_θ and D_θ^* and the independence, F and G may be assumed without loss of generality to be of the form

$$F(w) = \prod_{i=0}^{p} H_{n_i}[\xi_{k_i}(w)] \quad \text{and} \quad G(w) = \prod_{i=0}^{p} H_{m_i}[\xi_{k_i}(w)].$$

where $\xi_k(w) = [e_k](w)$ and $k_1, k_2, \cdots, k_p$ are mutually distinct. Then, using the Leibniz' rule, Lemma 2.3 and the independence of $\xi_1, \xi_2, \cdots$, we have

$$\begin{aligned}
E[D_\theta F \cdot G] &= E\Big[D_\theta \prod_{i=1}^{p} H_{n_i}[\xi_{k_i}] \cdot \prod_{i=1}^{p} H_{m_i}[\xi_{k_i}]\Big] \\
&= \sum_{i=1}^{p} E\Big[D_\theta H_{n_i}[\xi_{k_i}] \cdot \prod_{j \neq i} H_{n_j}[\xi_{k_j}] \cdot \prod_{i=1}^{p} H_{m_i}[\xi_{k_i}]\Big] \\
&= \sum_{i=1}^{p} E\Big[D_\theta H_{n_i}[\xi_{k_i}] \cdot H_{m_i}[\xi_{k_i}]\Big] E\Big[\prod_{j \neq i} H_{n_j}[\xi_{k_j}] H_{m_j}[\xi_{k_j}]\Big] \\
&= \sum_{i=1}^{p} E\Big[H_{n_i}[\xi_{k_i}]\Big\{ - D_\theta H_{m_i}[\xi_{k_i}] + \langle e_{k_i}, \theta \rangle_{\mathscr{H}} \xi_{k_i} H_{m_i}[\xi_{k_i}]\Big\}\Big] \\
&\qquad \times E\Big[\prod_{j \neq i} H_{n_j}[\xi_{k_j}] H_{m_j}[\xi_{k_j}]\Big] \\
&= \sum_{i=1}^{p} E\Big[\prod_{j=1}^{p} H_{n_j}[\xi_{k_j}]\Big\{ - D_\theta H_{m_i}[\xi_{k_i}] + \langle e_{k_i}, \theta \rangle_{\mathscr{H}} \xi_{k_i} H_{m_i}[\xi_{k_i}]\Big\} \prod_{j \neq i} H_{m_j}[\xi_{k_j}]\Big] \\
&= \sum_{i=1}^{p} E\Big[\prod_{j=1}^{p} H_{n_j}[\xi_{k_j}]\Big(- D_\theta H_{m_i}[\xi_{k_i}]\Big)\Big] \\
&\qquad + E\Big[\prod_{j=1}^{p} H_{n_j}[\xi_{k_j}]\Big\{ \sum_{i=1}^{p} \langle e_{k_i}, \theta \rangle_{\mathscr{H}} \xi_{k_i}\Big\} \prod_{j=1}^{p} H_{m_j}[\xi_{k_j}]\Big].
\end{aligned}$$

By the orthogonality of $\xi_1, \xi_2, \cdots$, the last term is equal to

$$E\Big[\prod_{j=1}^{p} H_{n_j}[\xi_{k_j}] \cdot \int_0^1 \dot{\theta}(t)dw(t) \prod_{j=1}^{p} H_{m_j}[\xi_{k_j}]\Big],$$

which completes the proof. □

Remark 2.5. Note that $\{D_\theta : \theta \in \mathscr{H}\}$ determines a linear operator $D : \mathbf{P} \to \mathbf{P} \otimes \mathscr{H}$ such that $\langle DF, \theta\rangle_{\mathscr{H}} = D_\theta F$ for each $F \in \mathbf{P}$ and $\theta \in \mathscr{H}$. The operator can be extended to an operator $D : \mathbf{P} \otimes \mathscr{H} \to \mathbf{P} \otimes \mathscr{H} \otimes \mathscr{H}$ by $D(F \otimes \theta) = DF \otimes \theta$. This operator is commonly used in Malliavin calculus (see e.g. [3]). Its "adjoint" $D^* : \mathbf{P} \otimes \mathscr{H} \to \mathbf{P}$ is defined by $D^*F(w) = -\mathrm{tr}\, DF(w) + [F](w),\ F \in \mathbf{P}\otimes\mathscr{H}$. Then the "integration by parts formula";

$$\int_{\mathscr{W}} \langle DF(w), G(w)\rangle_{\mathscr{H}} \gamma(dw) = \int_{\mathscr{W}} F(w) D^* G(w) \gamma(dw)$$

holds for all $F \in \mathbf{P}$ and $G \in \mathbf{P} \otimes \mathscr{H}$ (see e.g. [3], pp 361). Under these notations, $D^*_\theta F = D^*(F \otimes \theta)$ for each $F \in \mathbf{P}$ and hence the above adjointness follows immediately from our result and vice versa.

Next we define the *exponential* $e^{D^*_\theta}$ of D^*_θ by the formal series

$$e^{D^*_\theta} := \sum_{n=0}^{\infty} \frac{1}{n!} D^{*n}_\theta.$$

Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis of $\mathscr{H}$ as above.

Theorem 2.6. *For every $\theta \in \mathscr{H}$ such that $|\theta|_{\mathscr{H}} = 1$, it holds that*

$$D^{*n}_\theta 1 = H_n[\int_0^1 \dot{\theta}(t)dw(t)] \in \mathbf{P}, \quad n = 0, 1, 2, \cdots \tag{2.5}$$

and hence $e^{D^*_\theta}1$ can be defined. In fact, it is the exponential martingale

$$e^{D^*_\theta}1(w) = \exp\Big\{ \int_0^1 \dot{\theta} dw(t) - \frac{1}{2} \Big\}, \quad w \in \mathscr{W}. \tag{2.6}$$

Furthermore, it holds that

$$E\Big[e^{D_\theta} F\Big] = E\Big[F \cdot e^{D^*_\theta} 1\Big], \quad F \in \mathbf{P}. \tag{2.7}$$

Proof. We use the induction on n to prove (2.5). It is clear that

$$D^*_\theta 1(w) = \int_0^1 \dot{\theta}(t)dw(t) = H_1[\int_0^1 \dot{\theta}(t)dw(t)].$$

Suppose that (2.5) holds for n. We recall that the Hermite polynomials satisfy the identity

$$H_{n+1}[x] = xH_n[x] - nH_{n-1}[x]. \tag{2.8}$$

Put $\Theta(w) := \int_0^1 \dot\theta(t)dw(t)$. Then, noting that $\langle\theta,\theta\rangle_{\mathscr{H}} = 1$ and using (2.1),

$$\begin{aligned} D_\theta^{*(n+1)}1 &= D_\theta^* H_n[\Theta] = -D_\theta H_n[\Theta] + \Theta H_n[\Theta] \\ &= \Theta H_n[\Theta] - nH_{n-1}[\Theta] = H_{n+1}[\Theta]. \end{aligned}$$

Hence (2.5) holds for every $n = 0, 1, 2, \cdots$. Then (2.6) follows immediately from (2.5).

Finally we shall prove (2.7). By using Proposition 2.4, for $F \in \mathbf{P}$ we have

$$E\Big[e^{D_\theta}F\Big] = \sum_{n=0}^{\infty}\frac{1}{n!}E\Big[D_\theta^n F\Big] = \sum_{n=0}^{\infty}\frac{1}{n!}E\Big[F\cdot D_\theta^{*n}1\Big] = E\Big[F\cdot e^{D_\theta^*}1\Big].$$

□

Corollary 2.7. *For every $\theta \in \mathscr{H}$, it holds that*

$$e^{D_\theta^*}1(w) = \exp\Big\{\int_0^1 \dot\theta(t)dw(t) - \frac{1}{2}\int_0^1 \dot\theta(t)^2 dt\Big\}, \quad w \in \mathscr{W}. \tag{2.9}$$

Furthermore, it holds that

$$E\Big[e^{D_\theta}F\Big] = E\Big[F\cdot e^{D_\theta^*}1\Big], \quad F \in \mathbf{P}. \tag{2.10}$$

Proof. Let $\eta = \theta/|\theta|_{\mathscr{H}}$ and then it follows that

$$D_\theta^{*n}1(w) = |\theta|_{\mathscr{H}}^n D_\eta^{*n}1(w) = |\theta|_{\mathscr{H}}^n H_n[\int_0^1 \dot\eta(t)dw(t)]$$

for $n = 0, 1, 2, \cdots$ and $w \in \mathscr{W}$ by Theorem 2.6. Hence we have

$$e^{D_\theta^*}1(w) = \sum_{n=0}^{\infty}\frac{|\theta|_{\mathscr{H}}^n}{n!}H_n[\int_0^1 \dot\eta(t)dw(t)] = \exp\Big\{|\theta|_{\mathscr{H}}\int_0^1 \dot\eta(t)dw(t) - \frac{|\theta|_{\mathscr{H}}^2}{2}\Big\}.$$

The identity (2.10) can be shown by the same argument as Theorem 2.6. □

Now, we have the Cameron-Martin formula in this polynomial framework.

Corollary 2.8. *For every $\theta \in \mathscr{H}$ and $F \in \mathbf{P}$, it holds that*

$$\begin{aligned} &\int_{\mathscr{W}} F(w+\theta)\gamma(dw) = E\Big[e^{D_\theta}F\Big] = E\Big[F\cdot e^{D_\theta^*}1\Big] \\ &= \int_{\mathscr{W}} F(w)\exp\Big\{\int_0^1 \dot\theta dw(t) - \frac{1}{2}\int_0^1 \dot\theta(t)^2 dt\Big\}\gamma(dw). \end{aligned} \tag{2.11}$$

3. An Algebraic Proof of MG Formula.

In this section, we will give an algebraic proof of the MG formula using an adjoint relation similar to (2.7). As we have announced in the introduction, for the proof of the adjoint relation we will rely on the standard stochastic calculus.

Let $Z : \mathscr{W} \to \mathscr{H}$ be a predictable map; i.e. $\dot{Z}(t)$, $0 \le t \le 1$ is a predictable process such that

$$\|Z\|_{\mathscr{H}}^2 = \int_0^1 \dot{Z}(s)^2 ds < \infty \quad \text{a.s.}$$

Suppose $\mathcal{E}(\int \dot{Z} dW)$ is a true martingale where for a martingale $M = (M(t))_{0 \le t \le 1}$ the process $\mathcal{E}(M)$ is defined by

$$\mathcal{E}(M)_t = \exp\Big\{M(t) - \frac{1}{2}\langle M\rangle(t)\Big\}.$$

3.1. **Infinite dimensional tensor fields.** We fix a c.o.n.s. $\{e_i : i \in \mathbf{N}\}$ of $\mathscr{H}$ and will write simply D_i for D_{e_i} for each $i \in \mathbf{N}$. We define a differentiation along Z. For $\phi \in \mathbf{P}$, we define D_Z in the following way:

$$D_Z \phi(W) := \sum_{i=1}^{\infty} \langle Z, e_i\rangle(W) D_i \phi(W),$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathscr{H}$. Moreover, we define the n-th D_Z, which we write as $D_Z^{\otimes n}$ by the following:

$$\begin{aligned} D_Z^{\otimes n} &:= \underbrace{D_Z \otimes D_Z \otimes \cdots \otimes D_Z}_{n} \\ &:= \sum_{i,j,k,\cdots} \underbrace{\langle Z, e_i\rangle\langle Z, e_j\rangle\langle Z, e_k\rangle \cdots}_{n} \underbrace{D_i D_j D_k \cdots}_{n}. \end{aligned}$$

Next we define the exponential of D_Z by the formal series of

$$\begin{aligned} \widetilde{e}^{D_Z} &:= 1 + D_Z + \frac{1}{2!} D_Z^{\otimes 2} + \frac{1}{3!} D_Z^{\otimes 3} + \cdots \\ &= 1 + \sum_i \langle Z, e_i\rangle D_i + \frac{1}{2!} \sum_{i,j} \langle Z, e_i\rangle\langle Z, e_j\rangle D_i D_j \\ &\qquad\qquad + \frac{1}{3!} \sum_{i,j,k} \langle Z, e_i\rangle\langle Z, e_j\rangle\langle Z, e_k\rangle D_i D_j D_k + \cdots . \end{aligned}$$

We denote $\langle Z, e_i\rangle$ by Z_i, so we may write $\langle Z, e_i\rangle\langle Z, e_j\rangle D_i D_j$ as $Z_i Z_j D_i D_j$ and furthermore $D_Z^{\otimes 2} = \sum_{i,j} Z_i Z_j D_i D_j$ as $\langle Z \otimes Z, \nabla \otimes \nabla\rangle, \cdots, D_Z^{\otimes n} = \langle Z^{\otimes n}, \nabla^{\otimes n}\rangle$, and so on.

Lemma 3.1. *For any $k \in \mathbb{N}$, we have*

$$\begin{aligned}\widetilde{e}^{D_Z}&\Big(H_{n_1}(\int_0^1 \dot{e}_{m_1}dW)\cdots H_{n_k}(\int_0^1 \dot{e}_{m_k}dW)\Big)\\ &= \widetilde{e}^{D_Z}\Big(H_{n_1}(\int_0^1 \dot{e}_{m_1}dW)\Big)\cdots \widetilde{e}^{D_Z}\Big(H_{n_k}(\int_0^1 \dot{e}_{m_k}dW)\Big).\end{aligned} \tag{3.1}$$

Proof. First note that the equation (3.1) is equivalent to

(3.2)

$$\begin{aligned}&\sum_{l=0}^{n_1+\cdots+n_k}\frac{1}{l!}\langle Z^{\otimes l},\nabla^{\otimes l}\rangle\Big(H_{n_1}(\int_0^1 \dot{e}_{m_1}dW)\cdots H_{n_k}(\int_0^1 \dot{e}_{m_k}dW)\Big)\\ &= \sum_{l_1=0}^{n_1}\frac{1}{l_1!}\langle Z^{\otimes l_1},\nabla^{\otimes l_1}\rangle H_{n_1}(\int_0^1 \dot{e}_{m_1}dW)\cdots \sum_{l_k=0}^{n_k}\frac{1}{l_k!}\langle Z^{\otimes l_k},\nabla^{\otimes l_k}\rangle H_{n_k}(\int_0^1 \dot{e}_{m_k}dW).\end{aligned}$$

Fixing $l_1, \cdots l_k$ such that $l_1 \le n_1, \cdots, l_k \le n_k$, it suffices to prove that the coefficients of

$$\nabla^{\otimes l_1}H_{n_1}\nabla^{\otimes l_2}H_{n_2}\cdots\nabla^{\otimes l_k}H_{n_k}$$

of the left-hand after applying Leibniz rule correspond to those of right-hand. The coefficients of the left-hand are the following.

$$\frac{1}{(l_1+l_2+\cdots+l_k)!}\binom{l_1+l_2+\cdots+l_k}{l_1}\binom{l_2+\cdots+l_k}{l_2}\cdots\binom{l_k}{l_k}.$$

This is equal to $\frac{1}{l_1!l_2!\cdots l_k!}$, so we get (3.2). □

Proposition 3.2. *For $f \in \mathbf{P}$, we have*

$$\widetilde{e}^{D_Z}(f(W)) = f(W+Z). \tag{3.3}$$

Proof. Since $\widetilde{e}^{D_Z}$ is linear and by Lemma 3.1, we only prove in the case of $f(W) = H_n(\int_0^1 \dot{e}_i(s)dW_s)$, that is, it suffices to show

$$\widetilde{e}^{D_Z}\left(H_n(\int_0^1 \dot{e}_i(s)dW_s)\right) = H_n\left(\int_0^1 \dot{e}_i(s)dW_s + \langle Z, e_i\rangle\right).$$

By the definition, we have

$$\widetilde{e}^{D_Z}\left(H_n(\int_0^1 \dot{e}_i(s)dW_s)\right) = \sum_{k=0}^{n}\binom{n}{k}\langle Z, e_i\rangle^k H_{n-k}(\int_0^1 \dot{e}_i(s)dW_s).$$

For this, apply $H_n(x+y) = \sum_{k=0}^n \binom{n}{k}H_{n-k}(x)y^k$, then we get (3.3). □

3.2. **The operator L_n^Z.** To prove Maruyama-Girsanov formula, we additionally introduce a sequence $\{L_n^Z\}$ of new operators associated with Z as follows. For any $n \in \mathbb{N}$, L_n^Z is defined by $L_0^Z = \mathrm{id}$ and

$$L_n^Z = -\sum_{k=1}^{n} \binom{n}{k} \hat{H}_{n-k}\left(\int_0^1 \dot{Z}(s)dW(s),\ \|Z\|^2_{\mathscr{H}}\right) D^{\otimes k}_{-Z}, \quad n \in \mathbf{N} \tag{3.4}$$

where the polynomials $\hat{H}_n(x,y)$, $n = 1, 2, \cdots$, are defined by means of the formula

$$e^{\lambda x - \frac{\lambda^2}{2}y^2} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \hat{H}_n(x,y).$$

With this notation, the Hermite polynomials we have used so far are can be written as

$$H_n[x] = \hat{H}_n(x,1).$$

Theorem 3.3. *For any $F \in \mathbf{P}$, we have*

$$E\Big[\sum_{n=0}^{\infty} \frac{1}{n!} L_n^Z F\Big] = E\Big[\mathcal{E}(\int_0^{\cdot} \dot{Z}(s)dW(s))_1 \cdot F\Big]. \tag{3.5}$$

Proof. It suffices to show

$$E\Big[L_n^Z F\Big] = E\Big[\hat{H}_n\Big(\int_0^1 \dot{Z}(s)dW_s, \|Z\|^2_{\mathscr{H}}\Big) \cdot F\Big] \tag{3.6}$$

for each $n \in \mathbf{N}$ and $F \in \mathbf{P}$. If we can prove that

$$E\Big[L_n^Z\Big(\mathcal{E}(\int \dot{f}dW)_1\Big)\Big] = E\Big[\hat{H}_n\Big(\int_0^1 \dot{Z}(s)dW_s, \|Z\|^2_{\mathscr{H}}\Big) \cdot \mathcal{E}(\int \dot{f}dW)_1\Big] \tag{3.7}$$

for arbitrary $f \in \mathscr{H}$, then (3.6) is deduced. In fact, for a finite orthonormal system $\{e_1, \cdots, e_m\}$, take $f := \lambda_1 e_1 + \cdots \lambda_m e_m$ for $\lambda_1, \cdots, \lambda_m \in \mathbf{R}$. Then,

$$\begin{aligned}\mathcal{E}(\int \dot{f}dW)_1 &= \prod_{i=1}^{m} \mathcal{E}(\lambda_i \int \dot{e}_i dW)_1 \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{n_1+\cdots+n_m=N} \frac{N!}{n_1! \cdots n_m!} \prod_{i=1}^{m} \lambda_i^{n_i} H_{n_i}(\int_0^1 \dot{e}_i(s)dW(s)),\end{aligned}$$

and we notice that $\sum_{N=0}^{\infty} a_N$ where

$$a_N = E\left[\sum_{n_1+\cdots+n_m=N} \frac{N!}{n_1! \cdots n_m!} \prod_{i=1}^{m} \lambda_i^{n_i} H_{n_i}(\int_0^1 \dot{e}_i(s)dW(s))\right] = \begin{cases} 1 & \text{if } N = 0 \\ 0 & \text{otherwise} \end{cases}$$

is absolutely convergent. This means that (3.6) is valid for arbitrary monomials and hence for all polynomials.

So, let us prove (3.7). First we note that

$$
\begin{aligned}
&E\Big[L_n^Z\Big(\mathcal{E}(\int \dot f dW)_1\Big)\Big] \\
&= E\Big[\sum_{k=1}^n (-1)^{k+1}\binom{n}{k}\hat H_{n-k}\Big(\int_0^1 \dot Z(s)dW_s, \|Z\|^2_{\mathscr{H}}\Big) D_Z^{\otimes k}\mathcal{E}(\int \dot f dW)_1\Big],
\end{aligned}
$$

where $\hat H_n(s)$ denotes $\hat H_n(\int_0^s \dot Z(u)dW_u, \int_0^s \dot Z(u)^2 du)$ and $\hat H_n := \hat H_n(1)$. Since $D_i\mathcal{E}(\int \dot f dW)_1 = \langle f, e_i\rangle \mathcal{E}(\int \dot f dW)_1$, we have

$$
\begin{aligned}
&E\Big[L_n^Z\Big(\mathcal{E}(\int \dot f dW)_1\Big)\Big] \\
&= E\Big[\mathcal{E}(\int \dot f dW)_1\Big\{\sum_{k=1}^n (-1)^{k+1}\binom{n}{k}\hat H_{n-k}\sum_{i_1,\cdots,i_k} Z_{i_1}\cdots Z_{i_k}\langle f, e_{i_1}\rangle\cdots\langle f, e_{i_k}\rangle\Big\}\Big] \\
&= E\Big[\mathcal{E}(\int \dot f dW)_1\Big\{\sum_{k=1}^n (-1)^{k+1}\binom{n}{k}\hat H_{n-k}\langle Z, f\rangle^k\Big\}\Big].
\end{aligned}
$$

We will use the following formulas to obtain (3.7) which will complete the proof;

$$
\hat H_n(t) = n\int_0^t \hat H_{n-1}(s)\dot Z(s)dW(s),
$$

$$
\mathcal{E}\Big(\int \dot f dW\Big)_t = 1 + \int_0^t \mathcal{E}(\int \dot f dW)_s \dot f(s) dW(s),
$$

and

$$
d\langle \hat H_n, \mathcal{E}(\int \dot f dW)\rangle_s = n\hat H_{n-1}(s)\mathcal{E}(\int \dot f dW)_s \dot f(s)\dot Z(s)ds. \tag{3.8}
$$

As a first step we have

$$
\begin{aligned}
&E\Big[\hat H_n(\int_0^1 \dot Z(s)dW_s, \int_0^1 \dot Z(s)^2 ds)\cdot \mathcal{E}(\int \dot f dW)_1\Big] \\
&= E\Big[n\int_0^1 \hat H_{n-1}(s)\dot Z(s)dW(s)\Big] \\
&\qquad + E\Big[n\int_0^1 \hat H_{n-1}(s)\dot Z(s)dW(s)\int_0^1 \mathcal{E}(\int \dot f dW)_s \dot f(s)dW(s)\Big] \\
&= E\Big[n\int_0^1 \hat H_{n-1}(s)\mathcal{E}(\int \dot f dW)_s \dot f(s)\dot Z(s)ds\Big] =: I.
\end{aligned}
$$

By Ito's formula, we have

$$\begin{aligned}
&\hat{H}_{n-1}(1)\mathcal{E}(\int \dot{f}dW)_1 \int_0^1 \dot{f}(s)\dot{Z}(s)ds \\
&= \int_0^1 \hat{H}_{n-1}(s)\mathcal{E}\Big(\int \dot{f}dW\Big)_s \dot{f}(s)\dot{Z}(s)ds + \int_0^1\int_0^s \dot{f}(u)\dot{Z}(u)du \; d\langle \hat{H}_{n-1}, \mathcal{E}(\int \dot{f}dW)\rangle_s \\
&\qquad + \text{a martingale.}
\end{aligned}$$

Then by using (3.8), we have

$$\begin{aligned}
I &= E\Big[n\hat{H}_{n-1}\mathcal{E}(\int \dot{f}dW)_1 \int_0^1 \dot{f}(s)\dot{Z}(s)ds\Big] \\
&\qquad - E\Big[n(n-1)\int_0^1 \dot{f}(s)\dot{Z}(s)\int_0^s \dot{f}(u)\dot{Z}(u)du \; \hat{H}_{n-2}(s)\; \mathcal{E}(\int \dot{f}dW)_s ds\Big] \\
&=: E\Big[n\hat{H}_{n-1}\mathcal{E}(\int \dot{f}dW)_1 \langle f, Z\rangle\Big] - I\!I.
\end{aligned}$$

Again we apply Ito's formula to get

$$\begin{aligned}
&\hat{H}_{n-2}(1)\mathcal{E}(\int \dot{f}dW)_1 \langle f, Z\rangle^2 \\
&= 2\int_0^1 \hat{H}_{n-2}(s)\mathcal{E}(\int \dot{f}dW)_s \int_0^s \dot{f}(u)\dot{Z}(u)du \; f(s)Z(s)ds \\
&\quad + \int_0^1 \Big\{\int_0^s \dot{f}(u)\dot{Z}(u)du\Big\}^2 d\langle \hat{H}_{n-2}, \mathcal{E}(\int \dot{f}dW)\rangle_s + \text{a martingale}
\end{aligned}$$

and by using (3.8) again, we obtain

$$\begin{aligned}
I\!I &= E\Big[\frac{n(n-1)}{2}\hat{H}_{n-2}\mathcal{E}(\int \dot{f}dW)_1 \langle f, Z\rangle^2\Big] \\
&\quad - E\Big[\frac{n(n-1)(n-2)}{2}\int_0^1 \hat{H}_{n-3}(s)\mathcal{E}(\int \dot{f}dW)_s \dot{f}(s)\dot{Z}(s)\Big\{\int_0^s \dot{f}(u)\dot{Z}(u)du\Big\}^2 ds.\Big]
\end{aligned}$$

Hence we have

$$\begin{aligned}
&E\Big[\hat{H}_n(\int_0^1 \dot{Z}(s)dW_s, \int_0^1 \dot{Z}(s)^2 ds)\cdot \mathcal{E}(\int \dot{f}dW)_1\Big] = I \\
&= E\Big[n\hat{H}_{n-1}\mathcal{E}(\int \dot{f}dW)_1 \langle f, Z\rangle\Big] \\
&\quad - E\Big[\frac{n(n-1)}{2}\hat{H}_{n-2}\mathcal{E}(\int \dot{f}dW)_1 \langle f, Z\rangle^2\Big] \\
&\quad + E\Big[\frac{n(n-1)(n-2)}{2}\int_0^1 \dot{f}(s)\dot{Z}(s)\Big\{\int_0^s \dot{f}(u)\dot{Z}(u)du\Big\}^2 \; \hat{H}_{n-3}(s)\mathcal{E}(\int \dot{f}dW)_s ds\Big].
\end{aligned}$$

By repeating this procedure until $\hat{H}_*(s)$ in the integrand vanishes, we obtain

$$
\begin{aligned}
&E\Big[\hat{H}_n\Big(\int_0^1 Z(s)dW_s, \int_0^1 Z(s)^2 ds\Big)\cdot \mathcal{E}(\int \dot{f}dW)_1\Big] \\
&\quad = E\Big[\mathcal{E}(\int f dW)_1\Big\{\sum_{k=1}^n (-1)^{k+1}\binom{n}{k}\hat{H}_{n-k}\langle Z, f\rangle^k\Big\}\Big].
\end{aligned}
$$

□

3.3. **Passage to the Cameron-Martin-Maruyama-Girsanov formula.** From Proposition 3.2 and Theorem 3.3, we will give a new proof of Maruyama-Girsanov formula in the case of $f \in \mathbf{P}$.

Lemma 3.4. *As an operator acting on* $\mathbf{P}$,

$$
\sum_{n=1}^\infty \frac{1}{n!}L_n^Z = \exp\Big\{\int_0^1 \dot{Z}(t)dW(t) - \frac{1}{2}\int_0^1 \dot{Z}(t)^2 dt\Big\}(1 - \widetilde{e}^{D_Z}).
$$

Proof.

$$
\begin{aligned}
\sum_{n=0}^\infty \frac{1}{n!}L_n^Z &= 1 - \sum_{n=1}^\infty \frac{1}{n!}\sum_{k=1}^n \binom{n}{k}\hat{H}_{n-k}(\int_0^1 \dot{Z}(s)dW(s), \int_0^1 \dot{Z}(s)^2 ds)D_{-Z}^{\otimes k} \\
&= 1 - \sum_{k=1}^\infty\Big(\sum_{n=k}^\infty \frac{1}{k!(n-k)!}\hat{H}_{n-k}(\int_0^1 \dot{Z}(s)dW(s), \int_0^1 \dot{Z}(s)^2 ds)\Big)D_{-Z}^{\otimes k} \\
&= 1 - \sum_{k=1}^\infty \frac{1}{k!}\Big(\sum_{m=0}^\infty \frac{1}{m!}\hat{H}_m(\int_0^1 \dot{Z}(s)dW(s), \int_0^1 \dot{Z}(s)^2 ds)\Big)D_{-Z}^{\otimes k} \\
&= 1 - \mathcal{E}(\int \dot{Z}dW)_1 \sum_{k=1}^\infty \frac{1}{k!}D_{-Z}^{\otimes k} \\
&= 1 - \mathcal{E}(\int \dot{Z}dW)_1 \sum_{k=0}^\infty \frac{1}{k!}D_{-Z}^{\otimes k} + \mathcal{E}(\int \dot{Z}dW)_1.
\end{aligned}
$$

□

Corollary 3.5 (Cameron-Martin-Maruyama-Girsanov formula)**.** *For* $f \in \mathbf{P}$, *the following formula holds*

$$
E\Big[\mathcal{E}(\int \dot{Z}dW)_1 f\Big(W - \int_0^\cdot \dot{Z}(s)ds\Big)\Big] = E\Big[f(W)\Big]. \tag{3.9}
$$

Proof. By Lemma 3.4, we have

$$
\begin{aligned}
(3.10)\quad & E\Big[\sum_{n=0}^{\infty}\frac{1}{n!}L_n\Big(f(W)\Big)\Big] \\
&= E\Big[f(W)-\mathcal{E}(\int \dot{Z}dW)_1\sum_{k=0}^{\infty}\frac{1}{k!}D_{-Z}^{\otimes k}f(W)+\mathcal{E}(\int \dot{Z}dW)_1 f(W)\Big] \\
&= E\Big[f(W)-\mathcal{E}(\int \dot{Z}dW)_1\widetilde{e}^{D_{-Z}}f(W)+\mathcal{E}(\int \dot{Z}dW)_1 f(W)\Big] \\
&= E\Big[f(W)-\mathcal{E}(\int \dot{Z}dW)_1 f\Big(W-\int_0^{\cdot}\dot{Z}(s)ds\Big)+\mathcal{E}(\int \dot{Z}dW)_1 f(W)\Big].
\end{aligned}
$$

Then by Theorem 3.3, we obtain (3.9). □

4. Another Algebraic Proof for CMMG Formula.

As we have mentioned in the introduction, we give an alternative proof which is "purely" algebraic in the sense that we do not use stochastic calculus essentially, though we restrict ourselves in the case of piecewise constant (=finite-dimensional) case.

Let $\mathcal{F}\equiv\{\mathcal{F}_t\}_{0\le t\le 1}$ be the natural filtration of $\mathscr{W}$. Let us consider a simple $\mathcal{F}$-predictable process

$$
z(w,t)=\sum_{k=1}^{2^s}2^{s/2}z_k(w)\,1_{(\frac{k-1}{2^s},\frac{k}{2^s}]}(t) \tag{4.1}
$$

where z_k, $k=1,\cdots,2^s$ are $\mathcal{F}_{\frac{k-1}{2^s}}$- measurable random variables. Define $\sigma_k^s\in\mathscr{H}$, $k=1,\cdots,2^s$ by

$$
\sigma_k^s(t):=2^{s/2}\int_0^t 1_{(\frac{k-1}{2^s},\frac{k}{2^s}]}(u)\,du.
$$

We will suppress the superscript s whenever it is clear from the context. Clearly,

$$
D_{\sigma_k}F=0 \tag{4.2}
$$

for any $\mathcal{F}_{\frac{k-1}{2^s}}$-measurable random variable F. Put

$$
D_{z_k}:=z_kD_{\sigma_k}\ \text{and}\ D_{z_k}^*:=z_kD_{\sigma_k}^*,
$$

for $k=1,\cdots,2^s$.

Lemma 4.1. *For any $n\in\mathbb{N}$ and $f\in\mathbf{P}$, we have*

$$
D_{z_k}^n f=\underbrace{z_kD_{\sigma_k}\cdots z_kD_{\sigma_k}}_{n\ times}f=z_k^nD_{\sigma_k}^n f \tag{4.3}
$$

and

$$(D^*_{z_k})^n f = \underbrace{z_k D^*_{\sigma_k} \cdots z_k D^*_{\sigma_k}}_{n \text{ times}} f = z_k^n (D^*_{\sigma_k})^n f \tag{4.4}$$

Proof. These are direct from the following "commutativity":

$$D_{\sigma_j}(z_i f) = z_i D_{\sigma_j} f, \text{ and } D^*_{\sigma_j}(z_i f) = z_i D^*_{\sigma_j} f, \quad \text{if } i \le j$$

for differentiable f. These follows since $D_{\sigma_j}(z_i) = 0$. □

Define the exponentials as

$$e^{D_{z_k}} := \sum_{n=0}^{\infty} \frac{1}{n!} D^n_{z_k}, \quad k = 1, 2, \cdots, N$$

and

$$e^{D^*_{z_k}} := \sum_{n=0}^{\infty} \frac{1}{n!} (D^*_{z_k})^n, \quad k = 1, 2, \cdots N$$

formally. By Lemma 4.1 we have

$$e^{D_{z_k}} = \sum_{n=0}^{\infty} \frac{z_k^n}{n!} D^n_{\sigma_k}$$

and thus we can include $\mathbf{P}$ in the domain of $e^{D_{z_k}}$.

Let us introduce a subspace $\mathbf{P}_H$ of $\mathbf{P}$, which consists of polynomials with respect to $\{[e_i](w)\}$, where $\{e_i\}$ is the Haar system. Note that $\mathbf{P}_H$ is also characterized as all the polynomials with respect to $\{[\dot{\sigma}^s_k](w) : k = 1, \cdots, 2^s, s \in \mathbb{N}\}$.

The following is a main result in our program.

Theorem 4.2. *(i) For any $F \in \mathbf{P}_H$, we have*

$$e^{D_{z_{2^s}}} \cdots e^{D_{z_1}} F(w) = F(w + \int_0^\cdot z(w, u)\, du). \tag{4.5}$$

(ii) For any $\mathcal{F}_{(k-1)/2^s}$-measurable random variable F,

$$e^{D^*_{z_k}} F = F e^{D^*_{z_k}}(1). \tag{4.6}$$

In particular, the function F is in the domain of $e^{D^*_{z_k}}$. Furthermore, we have

$$e^{D^*_{z_{2^s}}} \cdots e^{D^*_{z_1}}(1) = \exp\Big\{ \int_0^1 z(w, s) dw(s) - \frac{1}{2} \int_0^1 z(w, s)^2 ds \Big\}, \tag{4.7}$$

(iii) Fix $k \in \mathbb{N}$. Let $F \in \mathbf{P}$ and let G be an arbitrary $\mathcal{F}_{(k-1)/2^s}$-measurable integrable function. Then

$$E[e^{D_{z_k}}(F) G] = E[F e^{D^*_{z_k}}(G)]. \tag{4.8}$$

Proof. (i) First, notice that $F \in \mathbf{P}_H$ is always expressed as a linear combination of $\prod_{k=1}^{2^s} F_k$, where each F_k is a polynomial in

$$\left\{ [\sigma_l^t](w) : \Big(\frac{l-1}{2^t}, \frac{l}{2^t}\Big] \subset \Big(\frac{k-1}{2^s}, \frac{k}{2^s}\Big] \right\}, \tag{4.9}$$

so that we can assume that F is of the form

$$F = \sum_{i=1}^{N} \prod_{k=1}^{2^s} F_{k,i},$$

where each $F_{k,i}$ is a polynomial in (4.9). By Proposition 2.2 and the definition of D_{σ_k}, we have

$$e^{D_{z_k}} F_{l,i}(w) = \begin{cases} F_{k,i}(w + z_k \sigma_k) & (l = k) \\ F_{l,i}(w) & (l \neq k). \end{cases}$$

Then by Lemma 2.1,

$$e^{D_{z_k}} \prod_{l=1}^{2^s} F_{l,i}(w) = F_{k,i}(w + z_k \sigma_k) \prod_{l \neq k} F_{l,i}(w).$$

Since z_k is $\mathcal{F}_{t_k}$-measurable, we also have, if $j > k$,

$$\begin{aligned} &e^{D_{z_j}} e^{D_{z_k}} \prod_{l=1}^{2^s} F_{l,i}(w) \\ &= e^{D_{z_j}} F_{k,i}(w + z_k \sigma_k) e^{D_{z_j}} \prod_{l \neq k} F_{l,i}(w) \\ &= F_{k,i}(w + z_k \sigma_k) F_{j,i}(w + z_j \sigma_j) \prod_{l \neq j,k} F_{l,i}(w). \end{aligned}$$

Then, inductively we have

$$e^{D_{z_{2^s}}} \cdots e^{D_{z_1}} \prod_{l=1}^{2^s} F_{l,i}(w) = \prod_{l=1}^{2^s} F_{l,i}(w + z_l \sigma_l),$$

and by linearity we obtain (4.5) since

$$\sum_{l=1}^{2^s} z_l(w) \sigma_l(t) = \int_0^t z(w,u)\, du.$$

(ii) Noting that $D_{\sigma_k} F = 0$ for $\mathcal{F}_{(k-1)/2^s}$ - measurable random variable F, we have

$$\begin{aligned} D_{z_k}^* F &= z_k \{ -D_{\sigma_k} + 2^{s/2} (w_{k/2^s} - w_{(k-1)/2^s}) \} F \\ &= F z_k 2^{s/2} (w_{k/2^s} - w_{(k-1)/2^s}) = F D_{z_k}^*(1) \end{aligned}$$

since z_k is also $\mathcal{F}_{(k-1)/2^s}$-measurable. Inductively, we then have

$$(D^*_{z_k})^n F = F(D^*_{z_k})^n(1),$$

and hence we have (4.6), which in turn implies (4.7). In fact, we have by induction

$$e^{D^*_{z_{2^s}}} \cdots e^{D^*_{z_1}}(1) = \prod_{k=1}^{2^s} \{e^{D^*_{z_k}}(1)\}$$

since $e^{D^*_{z_{k-1}}} \cdots e^{D^*_{z_1}}(1)$ is $\mathcal{F}_{(k-1)/2^s}$-measurable for any k, and for each $i = 1, 2, \cdots, 2^s$, we have

$$\begin{aligned} e^{D^*_{z_i}}(1) &= \sum_{n=0}^{\infty} \frac{z_i^n}{n!}(D^*_{\sigma_i})^n(1) = \sum_{n=0}^{\infty} \frac{z_i^n}{n!} H_n[\int_0^1 \sigma_k(t)\, dw_t] \\ &= \exp\Big\{ z_i(w) 2^{s/2}(w_{k/2^s} - w_{(k-1)/2^s}) - \frac{1}{2} z_i(w)^2 \Big\}. \end{aligned}$$

(iii) Since F is a polynomial,

$$e^{D_{z_k}} F = \sum_{n=0}^{M} \frac{z_k^n}{n!} D^n_{\sigma_k} F$$

for some $M \in \mathbb{N} \cup \{0\}$. Therefore, the left-hand-side of (4.8) is rewritten as

$$\sum_{n=0}^{M} \frac{1}{n!} E[z_k^n D^n_{\sigma_k} F \cdot G].$$

Since z_k and G are $\mathcal{F}_{(k-1)/2^s}$-measurable, we have, for $n \leq M$

$$\begin{aligned} &E[z_k^n D^n_{\sigma_k} F \cdot G] = E[F \cdot (D^*_{\sigma_k})^n z_k^n G] \\ &= E[F \cdot z_k^n (D^*_{\sigma_k})^n G] = E[F \cdot (D^*_{z_k})^n G]. \end{aligned}$$

The relation is valid for $n > M$ since

$$(D^*_{\sigma_k})^n G = G(D^*_{\sigma_k})^n(1) = G H_n(\int_0^1 \sigma_k(t)\, dw_t),$$

and the degree of F as a polynomial of $\int_0^1 \sigma_k(t)\, dw_t$ is less than M, we have

$$E[z_k^n D^n_{\sigma_k} F \cdot G] = E[F \cdot D^{*n}_{z_k} G] = 0.$$

Thus we have

$$E[\sum_{n=0}^{\infty} \frac{1}{n!} D^n_{z_k} F \cdot G] = E[\sum_{n=0}^{\infty} \frac{1}{n!} F \cdot D^{*n}_{z_k} G],$$

which is the desired relation. $\square$

Remark 4.3. (i) We do not assume smoothness for F in (4.6). (ii) In (4.5) and (4.7), the order of application of the operators is important. If it is changed anywhere, neither holds anymore.

By using the above algebraic results, we can prove the following

Corollary 4.4 (Cameron-Martin-Maruyama-Girsanov formula)**.** *For a simple predictable z in (4.1) and $F \in \mathbf{P}_H$, it holds*

$$
\begin{aligned}
(4.10)\qquad & E[F(w - \int_0^{\cdot} z(w,u)\,du) \exp\Big\{ \int_0^1 z(w,t)\,dw_t - \frac{1}{2}\int_0^1 |z(w,t)|^2\,dt \Big\} \\
& = E[F].
\end{aligned}
$$

Proof. As a formal series, we have

$$e^{D_{z_k}} e^{-D_{z_k}} = 1,$$

for $k = 1, \cdots 2^s$. Then, for $F \in \mathbf{P}_H$, we have

$$F = e^{D_{z_1}} e^{-D_{z_1}} F$$

and since $e^{-D_{z_1}} F$ is a polynomial, by Theorem 4.2 (iii), we have

$$
\begin{aligned}
(4.11)\qquad E[F] &= E[e^{D_{z_1}} e^{-D_{z_1}} F] \\
&= E[e^{-D_{z_1}} F \cdot e^{D^*_{z_1}}(1)].
\end{aligned}
$$

Inductively, since

$$e^{-\partial_{z_k}} \cdots e^{-\partial_{z_1}} f(\xi)$$

still is a polynomial in

$$\Big\{ [\sigma_l^t](w) : \Big(\frac{l-1}{2^t}, \frac{l}{2^t}\Big] \subset \Big(\frac{k-1}{2^s}, \frac{k}{2^s}\Big] \Big\},$$

and

$$e^{D^*_{z_{k-1}}} \cdots e^{D^*_{z_1}}(1)$$

is $\mathcal{F}_{(k-1)/2^s}$-measurable, we have

$$
\begin{aligned}
(4.12)\qquad & E[F] \\
&= E[e^{D_{z_k}} e^{-D_{z_k}} e^{-D_{z_{k-1}}} \cdots e^{-D_{z_1}} F \cdot e^{D^*_{z_{k-1}}} \cdots e^{D^*_{z_1}}(1)] \\
&= E[e^{-D_{z_k}} \cdots e^{-D_{z_1}} F \cdot e^{D^*_{z_k}} \cdots e^{D^*_{z_1}}(1)].
\end{aligned}
$$

Combining this with (4.5) and (4.7) in Theorem 4.6, we have the formula (4.10). □

Appendix A. Continuity of the translation

The following lemma extends the translation on the dense subset of polynomials to an operator on L_q to L_p, and hence ensure the MG formula (4.10) for any bounded measurable F.

Lemma A.1. *Let z be a predictable process as (4.1). Suppose that*

$$E\left[\exp\left\{c\int_0^1 z(t)^2\,dt\right\}\right] < \infty \tag{A.1}$$

for some $c > 0$. Then, for $p \in [1,\infty)$, there exists $q \in (p,\infty)$ and a positive constant C_p such that

$$\|e^{-D_{z_{2s}}}\cdots e^{-D_{z_1}}F\|_p \le C_p\|F\|_q$$

for any $F \in \mathbf{P}_H$.

Proof. We will denote $Z := \int_0^\cdot z(t)\,dt$ and

$$\mathcal{E}(z) := \exp\left\{\int_0^1 z(t)\,dw(t) - \frac{1}{2}\int_0^1 z(t)^2\,dt\right\}.$$

Let $n \ge 1$ be an integer and $p < 2n$. By Hölder's inequality,

$$\begin{aligned}
E\left[|F(w - Z(w))|^p\right] &= E\left[|F(w - Z(w))|^p \{\mathcal{E}(z)\}^{\frac{p}{2n}}\{\mathcal{E}(z)\}^{-\frac{p}{2n}}\right] \\
&\le E\left[|F(w - Z(w))|^{p\cdot\frac{2n}{p}}\{\mathcal{E}(z)\}^{\frac{p}{2n}\cdot\frac{2n}{p}}\right]^{\frac{p}{2n}} \cdot E\left[\{\mathcal{E}(z)\}^{-\frac{p}{2n}\cdot\frac{2n}{2n-p}}\right]^{\frac{2n-p}{2n}} \\
&= E\left[|F(w - Z(w))|^{2n}\mathcal{E}(z)\right]^{\frac{p}{2n}} \cdot E\left[\{\mathcal{E}(z)\}^{-\frac{p}{2n-p}}\right]^{\frac{2n-p}{2n}}.
\end{aligned}$$

Since F is a polynomial, so is $|F|^{2n}$. Therefore, we can apply the MG formula for polynomials (4.10) in Corollary 4.4, to obtain

$$E\left[|F(w - Z(w))|^{2n}\mathcal{E}(z)\right]^{\frac{p}{2n}} = E\left[|F|^{2n}\right]^{\frac{p}{2n}} = \|F\|_{2n}^p.$$

Now it suffices to show that

$$E\left[\{\mathcal{E}(z)\}^{-\frac{p}{2n-p}}\right] < \infty. \tag{A.2}$$

Let us denote $L_t := \int_0^t z(u)\,dw(u)$. Then $\langle L\rangle_t = \int_0^t z(u)^2\,du$. Now, since we have

$$\begin{aligned}
\{\mathcal{E}(z)\}^{-\frac{p}{2n-p}} &= \exp\left\{-\frac{p}{2n-p}L - \frac{p^2}{(2n-p)^2}\langle L\rangle\right\} \\
&\quad\cdot \exp\left\{\left(\frac{p}{2(2n-p)} + \frac{p^2}{(2n-p)^2}\right)\langle L\rangle\right\},
\end{aligned}$$

by Schwartz inequality we have

$$\begin{aligned} & E\left[\{\mathcal{E}(z)\}^{-\frac{p}{2n-p}}\right] \\ & \leq E\left[\exp\left\{-\frac{2p}{2n-p}L - \frac{2p^2}{(2n-p)^2}\langle L\rangle\right\}\right]^{1/2} \\ & \qquad \cdot E\left[\exp\left\{\left(\frac{p}{(2n-p)} + \frac{2p^2}{(2n-p)^2}\right)\langle L\rangle\right\}\right]^{1/2}. \end{aligned}$$

Clearly, $\frac{p}{(2n-p)} + \frac{2p^2}{(2n-p)^2} \to 0$ as $n \to \infty$, and hence we can take large enough n to have the estimate (A.2) by using the assumption (A.1). □

Remark A.2. By a similar but easier procedure we can also prove a continuity lemma for e^{D_θ} with $\theta \in \mathscr{H}$, to extend (2.9) in Corollary 2.7 to obtain a full version of CM formula.

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