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NOTE ON THE HOMOTOPY OF THE SPACE OF MAPS BETWEEN REAL PROJECTIVE SPACES

Kohhei Yamaguchi

Abstract. We study the homotopy types of the space consisting of all base-point preseving continuous maps from the m dimensional real projective space into the n dimensional real projective space. When $2 \leq m < n$, it has two path connected components and we investigate whether these two path-components have the same homotopy type or not.

1. Introduction

1.1. **Notation and introduction of the previous works.** Let $2 \leq m < n$ be integers and we choose $\mathbf{e}_k = [1 : 0 : 0 : \cdots : 0] \in \mathbb{R}\mathrm{P}^k$ as the base point of $\mathbb{R}\mathrm{P}^k$ $(k = m, n)$. We denote by $\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ (resp. $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$) the space consisting of all maps $f : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ (resp. of all base-point preserving maps $f : (\mathbb{R}\mathrm{P}^m, \mathbf{e}_m) \to (\mathbb{R}\mathrm{P}^n, \mathbf{e}_n)$). For each $\epsilon \in \mathbb{Z}/2 = \{0, 1\} = \pi_0(\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n))$, let $\mathrm{Map}_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ denote the corresponding path component of $\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$. Similarly, we denote by $\mathrm{Map}^*_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ the corresponding path component of $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$. It is known that there is an isomorphism $\widetilde{KO}(\mathbb{R}\mathrm{P}^m) \cong \mathbb{Z}/2^{a(m)}$, where $a(m)$ denotes *the Hurewicz-Radon number* given by

$$a(m) = \begin{cases} 4k + \epsilon & \text{if } m = 8k + \epsilon \quad (\epsilon = 0, 1), \\ 4k + 2 & \text{if } m = 8k + 2 + \epsilon \quad (\epsilon = 0, 1), \\ 4k + 3 & \text{if } m = 8k + l \quad (4 \leq l \leq 7). \end{cases}$$

Now we recall the following two results.

Theorem 1.1 (M.C. Crabb and W.A. Sutherland, [1]). *Let $2 \leq m < n$ be integers.*

(i) *If $m \leq n - 2$, there is a homotopy equivalence* $\mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \simeq \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ *if and only if* $n + 1 \equiv 0 \pmod{2^{a(m)}}$.

(ii) *If $n \geq 3$ and $n+1 \equiv 0 \pmod{2^{a(n-1)}}$, there is a homotopy equivalence* $\mathrm{Map}_0(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n) \simeq \mathrm{Map}_1(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n)$.

(iii) *If $n \equiv 0 \pmod 2$, two components $\mathrm{Map}^*_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ for $\epsilon \in \{0, 1\}$ have the different rational homotopy types.* □

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Theorem 1.2 ([10]). *Let $2 \leq m < n$ be integers.*

(i) *If $n \equiv 1$ and $m \equiv 0 \pmod 2$, there are rational homotopy equivalences* $\mathrm{Map}_1^*(\mathbb{RP}^m, \mathbb{RP}^n) \simeq_{\mathbb{Q}} \{*\}$ *and* $\mathrm{Map}_1(\mathbb{RP}^m, \mathbb{RP}^n) \simeq_{\mathbb{Q}} S^n$.

(ii) *If $n \equiv 1$ and $m \equiv 1 \pmod 2$,*

$$\pi_k(\mathrm{Map}_1^*(\mathbb{RP}^m, \mathbb{RP}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n-m, \\ 0 & \text{otherwise.} \end{cases}$$

$$\pi_k(\mathrm{Map}_1(\mathbb{RP}^m, \mathbb{RP}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n-m,\ k = n, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If $n \equiv 0$ and $m \equiv 0 \pmod 2$,*

$$\pi_k(\mathrm{Map}_1^*(\mathbb{RP}^m, \mathbb{RP}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n-1,\ k = n-m, \\ 0 & \text{otherwise.} \end{cases}$$

$$\pi_k(\mathrm{Map}_1(\mathbb{RP}^m, \mathbb{RP}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n-m,\ k = 2n-1, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) *If $n \equiv 0$ and $m \equiv 1 \pmod 2$,*

$$\pi_k(\mathrm{Map}_1^*(\mathbb{RP}^m, \mathbb{RP}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n-1,\ k = 2n-m-1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\pi_k(\mathrm{Map}_1(\mathbb{RP}^m, \mathbb{RP}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = 2n-m-1,\ k = 2n-1, \\ 0 & \text{otherwise.} \end{cases}$$

(v) *If $m \leq n-2$,*

$$\pi_{n-m}(\mathrm{Map}_1(\mathbb{RP}^m, \mathbb{RP}^n)) = \begin{cases} \mathbb{Z} & \text{if } n-m \equiv 0 \pmod 2, \\ \mathbb{Z}/2 & \text{if } n-m \equiv 1 \pmod 2, \end{cases}$$

(vi) *If $m = n-1 \geq 2$,*

$$\pi_1(\mathrm{Map}_1(\mathbb{RP}^m, \mathbb{RP}^n)) = \begin{cases} \mathbb{Z}/4 & \text{if } m \equiv 0, 1 \pmod 4, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } m \equiv 2, 3 \pmod 4. \end{cases} \qquad \square$$

1.2. **The main results.** As stated as Theorem 1.1 above, it was already known when two path components $\mathrm{Map}_\epsilon(\mathbb{RP}^m, \mathbb{RP}^n)$ ($\epsilon = 0$ or 1) homotopy equivalent or not. However, it is difficult to construct the explicit homotopy equivalences between them. So we cannot apply it for studying whether two components $\mathrm{Map}_\epsilon^*(\mathbb{RP}^m, \mathbb{RP}^n)$ ($\epsilon = 0, 1$) of based maps are homotopy equivalent to each other. In this paper, we shall study the homotopy types of path-components $\mathrm{Map}_\epsilon^*(\mathbb{RP}^m, \mathbb{RP}^n)$ ($\epsilon = 0$ or 1).

Next, note that the rational homotopy types of path-components of spaces of maps between complex or quaternion projective spaces are well studied ([6], [7], [8]), but the case of real projective spaces is not well studied until now (cf. [4], [10]). So we shall also investigate the rational homotopy types of them explicitly. In fact, the main results of this paper are as follows.

Theorem 1.3. *Let $2 \leq m < n$ be integers and $\epsilon \in \{0,1\}$.*

(i) *The space* $\mathrm{Map}^*_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ *is $(n-m-1)$-connected, and*

$$\pi_{n-m}(\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 1 \pmod 2, \\ \mathbb{Z}/2 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

$$\pi_{n-m}(\mathrm{Map}^*_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) = \begin{cases} \mathbb{Z} & \text{if } n-m \equiv 0 \pmod 2, \\ \mathbb{Z}/2 & \text{if } n-m \equiv 1 \pmod 2. \end{cases}$$

(ii) *If $m \leq n-2$,*

$$\pi_k(\mathrm{Map}_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) = \begin{cases} \mathbb{Z}/2 & \text{if } k = 1, \\ 0 & \text{if } 2 \leq k < n-m, \end{cases}$$

$$\pi_{n-m}(\mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 1 \pmod 2, \\ \mathbb{Z}/2 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

(iii) *If $m = n-1 \geq 2$,*

$$\pi_1(\mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } m \equiv 1 \pmod 2, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

Theorem 1.4. *Let $2 \leq m < n$ be integers.*

(i) *If $n \equiv 1$ and $m \equiv 0 \pmod 2$, there are rational homotopy equivalences*

$$\begin{cases} \mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) & \simeq_{\mathbb{Q}} \{*\}, \\ \mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) & \simeq_{\mathbb{Q}} S^n. \end{cases}$$

(ii) *If $n \equiv 1$ and $m \equiv 1 \pmod 2$,*

$$\pi_k(\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n-m, \\ 0 & \text{otherwise.} \end{cases}$$

$$\pi_k(\mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n-m,\ k = n, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If $n \equiv 0$ and $m \equiv 0 \pmod 2$, there is a rational homotopy equivalence* $\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \simeq_{\mathbb{Q}} \{*\}$ *and*

$$\pi_k(\mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n,\ k = 2n-1, \\ 0 & \text{otherwise.} \end{cases}$$

(vi) *If $n \equiv 0$ and $m \equiv 1 \pmod 2$,*

$$\pi_k(\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n-m,\ k = 2n-m-1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\pi_k(\mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k \in \{n-m, n, 2n-m-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 1.5. *Let $2 \leq m < n$ be integers.*

(i) *If* $n \equiv 1 \pmod 2$, *there are rational homotopy equivalences*

$$\begin{cases} \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \simeq_{\mathbb{Q}} \mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n), \\ \mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \simeq_{\mathbb{Q}} \mathrm{Map}_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n). \end{cases}$$

(ii) *If* $m = n-1 \geq 3$ *and* $n+1 \not\equiv 0 \pmod 4$, $\mathrm{Map}_0(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n)$ *and* $\mathrm{Map}_1(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n)$ *have the different homotopy types.*

(iii) *If* $n \equiv 0 \pmod 2$, *the* $(n-m)$*-dimensional rational homotopy groups of* $\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ *and* $\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ *are different.* □

This paper is organized as follows. In section 2 we compute the homotopy groups of $\mathrm{Map}_\epsilon^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ ($\epsilon = 0, 1$) of low dimensions. In section 3 we compute their rational homotopy groups explicitly by using the standard techniques of rational homotopy theory ([2], [9]). Finally in section 4, we give the proofs of Theorem 1.3 and Theorem 1.4.

2. The space $\mathrm{Map}_\epsilon^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$.

Definition 1. (i) Let $1 \leq m < n$ be integers, and let $V_{n,m}$ denote the real Stiefel manifold of orthogonal m-frames in $\mathbb{R}^n$.

(ii) Define the map $f_{m,n} : O(n) \to \mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ by the matrix multiplication

$$f_{m,n}(A)([x_0 : \cdots : x_m] = [x_0 : \cdots : x_m : 0 : \cdots : 0] \begin{bmatrix} 1 & \mathbf{0}_n \\ {}^t\mathbf{0} & A \end{bmatrix}$$

for $(A, [x_0 : \cdots : x_m]) \in O(n) \times \mathbb{R}\mathrm{P}^m$. Since the subgroup which fixes $\mathbb{R}\mathrm{P}^m$ is $\{E_{m+1}\} \times O(n-m)$, the map $f_{m,n}$ induces the map

$$\alpha_{m,n} : V_{n,m} = O(n)/O(n-m) \to \mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n), \tag{2.1}$$

where E_k denotes the $(k \times k)$-unit matrix.

It is well known that $V_{n,1} = S^{n-1}$ and that there is a homeomorphism $V_{n,m} \cong O(n)/O(n-m)$. If we use this homeomorphism, it is easy to see that there is a fibaration sequence

$$S^{n-m} \to V_{n,m} \to V_{n,m-1}. \tag{2.2}$$

Lemma 2.1. *Let $1 \leq m < n$ be integers.*

(i) $V_{n,m}$ *is* $(n-m-1)$*-connected.*

(ii) $\pi_{n-m}(V_{n,m}) = \begin{cases} \mathbb{Z} & \text{if } n-m \equiv 0 \pmod 2, \\ \mathbb{Z}/2 & \text{if } n-m \equiv 1 \pmod 2). \end{cases}$

Proof. (i) If we use the fibration sequence (i), we can show the assertion (i) very easily by using the induction over m and we omit the detail.

(ii) First, assume that the case $n-m \equiv 1 \pmod 2$, and note that there are isomorphisms (cf. [5])

$$\begin{cases} H^*(V_{n,m}, \mathbb{Z}/2) = \Delta(e_{n-m}, e_{n-m+1}, \cdots, e_{2n-3}) \\ H^*(V_{n,m}, \mathbf{k}) = \begin{cases} E[x_{2(n-m)+1}, x_{2(n-m)+5}, \cdots, x_{2n-3}] & \text{if } m \equiv 0 \pmod 2, \\ E[x_{2(n-m)+1}, x_{2(n-m)+5}, \cdots, x_{n-1}] & \text{if } m \equiv 1 \pmod 2, \end{cases} \end{cases}$$

where $\mathbf{k} = \mathbb{Z}/p$ (p: odd prime) or $\mathbf{k} = \mathbb{Q}$, and $Sq^1(e_{n-m}) = e_{n-m+1}$ ($\deg e_k = k$, $\deg x_j = j$). Then we can easily see that the $(n-m+1)$-skeleton of $V_{n,m}$ is $S^{n-m} \cup_2 e^{n-m+1}$ (up to homotopy equivalence), and we have that $\pi_{n-m}(V_{n,m}) = \mathbb{Z}/2$ if $n-m \equiv 1 \pmod 2$.

Next, assume that $n-m \equiv 0 \pmod 2$. Since $V_{n,m-1}$ is $(n-m)$-connected (by (i)) and $\pi_{n-m+1}(V_{n,m-1}) = \mathbb{Z}/2$, it follows from (2.2) that there is an exact sequence

$$\begin{array}{ccccccc} \pi_{n-m+1}(V_{n,m-1}) & \xrightarrow{\partial} & \pi_{n-m}(S^{n-m}) & \longrightarrow & \pi_{n-m}(V_{n,m}) & \longrightarrow & 0. \\ \| & & \| & & & & \\ \mathbb{Z}/2 & & \mathbb{Z} & & & & \end{array}$$

Hence, $\pi_{n-m}(V_{n,m}) \cong \pi_{n-m}(S^{n-m}) \cong \mathbb{Z}$, and this completes the proof. □

Now recall the following:

Theorem 2.2 ([10]). *If* $1 \leq m < n$, $\alpha_{m,n} : V_{n,m} \to \mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ *is a homotopy equivalence up to dimension* $2(n-m)-1$ *and there is a homotopy commutative diagram*

(2.3)
$$\begin{array}{ccccc} S^{n-m} & \longrightarrow & V_{n,m} & \longrightarrow & V_{n,m-1} \\ {\scriptstyle E^m}\downarrow & & {\scriptstyle \alpha_{m,n}}\downarrow & & {\scriptstyle \alpha_{m-1,n}}\downarrow \\ \Omega^m S^n & \longrightarrow & \mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) & \xrightarrow{r} & \mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n) \end{array}$$

where two horizontal sequences are fibration sequences and E^m *denotes the* m*-fold suspension map.* □

Lemma 2.3. *Let* $m \geq 3$ *be an integer.*

(i) *If* $m \equiv 1 \pmod 2$, *there is a fibration sequence*

(2.4) $$\Omega^{m-1}S^n \times \Omega^m S^n \to \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \xrightarrow{r} \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^{m-2}, \mathbb{R}\mathrm{P}^n).$$

(ii) *If* $m \equiv 0 \pmod 2$, *there is a fibration sequence*

(2.5) $$\mathrm{Map}^*(\Sigma^{m-2}\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^n) \to \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \xrightarrow{r} \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^{m-2}, \mathbb{R}\mathrm{P}^n).$$

Proof. The cofiber sequence $\mathbb{R}\mathrm{P}^{m-2} \to \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{m-2} = \mathbb{R}\mathrm{P}^m_{m-1}$ induces the fibration sequence

$$\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m_{m-1}, \mathbb{R}\mathrm{P}^n) \to \mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \xrightarrow{r} \mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n).$$

Because there is a homotopy equivalence

$$\mathbb{R}\mathrm{P}^m_{m-1} = \mathbb{R}\mathrm{P}^m/\mathbb{R}\mathrm{P}^{m-2} \simeq \begin{cases} S^{m-1} \vee S^m & \text{if } m \equiv 1 \pmod 2 \\ S^{m-1} \cup_2 e^m = \Sigma^{m-2}\mathbb{R}\mathrm{P}^2 & \text{if } m \equiv 0 \pmod 2 \end{cases}$$

there is a homotopy equivalence

$$\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m_{m-1}, \mathbb{R}\mathrm{P}^n) \simeq \begin{cases} \Omega^{m-1}S^n \times \Omega^m S^n & \text{if } m \equiv 1 \pmod 2, \\ \mathrm{Map}^*(\Sigma^{m-2}\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^n) & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

Hence, the assertions follow. □

Theorem 2.4. *Let $1 \leq m < n$ and $\epsilon \in \{0, 1\}$.*

(i) $\mathrm{Map}^*_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ *is $(n-m-1)$-connected.*

(ii) $\pi_{n-m}(\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) = \begin{cases} \mathbb{Z} & \textit{if } m \equiv 1 \pmod 2, \\ \mathbb{Z}/2 & \textit{if } m \equiv 0 \pmod 2. \end{cases}$

(iii) $\pi_{n-m}(\mathrm{Map}^*_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) = \begin{cases} \mathbb{Z} & \textit{if } n-m \equiv 0 \pmod 2, \\ \mathbb{Z}/2 & \textit{if } n-m \equiv 1 \pmod 2. \end{cases}$

Proof. (i) By using Theorem 2.2 there is an epimorphism

$$\alpha_{m,n_*} : \pi_k(V_{n,m}) \to \pi_k(\mathrm{Map}^*_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n))$$

for any $k \leq 2(n-m)-1$. Hence, because $V_{n,m}$ is $(n-m-1)$-connected, $\mathrm{Map}^*_1(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ is also $(n-m-1)$-connected. So the assertion (i) is true for $\epsilon = 1$.

The proof for $\epsilon = 0$ is based on the induction over m. If $m = 1$, since there is a homotopy equivalence $\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^n) \simeq \Omega S^n$, the space $\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^n)$ is $(n-2)$-connected. So the case $m = 1$ is true. Suppose that the space $\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n)$ is $(n-m)$-connected and consider the restriction fibration sequence

$$\Omega^m S^n \to \mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \xrightarrow{r} \mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n). \tag{2.6}$$

Since $\Omega^m S^n$ and it is $(n-m-1)$-connected, by using (2.6) $\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ is $(n-m-1)$-connected. Hence, (i) is proved.

(ii) If $m = 1$, $\pi_{n-m}(\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) = \pi_{n-1}(\Omega\mathbb{R}\mathrm{P}^n) \cong \pi_n(S^n) = \mathbb{Z}$. So the assertion (ii) is true for $m = 1$. Next, because $\mathbb{R}\mathrm{P}^2 = S^1 \cup_2 e^2$, there is a cofibration sequence $S^1 \xrightarrow{2} S^1 \to \mathbb{R}\mathrm{P}^2$. Hence, by identifying $\Omega\mathbb{R}\mathrm{P}^n = \Omega S^n$ this sequence induces a fibration sequence

$$\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^n) \to \Omega S^n \xrightarrow{[2]} \Omega S^n. \tag{2.7}$$

Since $\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^2,\mathbb{R}\mathrm{P}^n)$ is $(n-2)$-connected, the exact sequence induced from (2.7) is reduced to the following exact sequence

$$\pi_{n-1}(\Omega S^n) \xrightarrow{[2]_*} \pi_{n-1}(\Omega S^n) \xrightarrow{\partial} \pi_{n-2}(\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^2,\mathbb{R}\mathrm{P}^n)) \to 0.$$

Because $[2]_*$ is the multiplication by 2, we have $\pi_{n-2}(\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^2,\mathbb{R}\mathrm{P}^n)) = \mathbb{Z}/2$. So the assertion (ii) is also true for $m=2$.

Now suppose that $m \geq 3$. First, consider the case $m \equiv 1 \pmod 2$. Since $\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^{m-2},\mathbb{R}\mathrm{P}^n)$ is $(n-m+1)$-connected, if we consider the homotopy exact sequence induced from the fibration sequence (2.4), there is an isomorphism

$$\pi_{n-m}(\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m,\mathbb{R}\mathrm{P}^n)) \cong \pi_{n-m}(\Omega^{m-1}S^n \times \Omega^m S^n) \cong \mathbb{Z}.$$

Hence, the assertion (ii) is true if $m \equiv 1 \pmod 2$.

Next consider the case $m \equiv 0 \pmod 2$. Because $\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^{m-2},\mathbb{R}\mathrm{P}^n)$ is $(n-m+1)$-connected, if we consider the homotopy exact sequence induced from the fibration sequence (2.5), we also have the isomorphism

$$\pi_{n-m}(\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m,\mathbb{R}\mathrm{P}^n)) \cong \pi_{n-m}(\mathrm{Map}^*(\Sigma^{m-2}\mathbb{R}\mathrm{P}^2,\mathbb{R}\mathrm{P}^n)).$$

On the other hand, the cofiber sequence $S^{m-1} \xrightarrow{2} S^{m-1} \to \Sigma^{m-2}\mathbb{R}\mathrm{P}^2$ induces the fibration sequence

$$\mathrm{Map}^*(\Sigma^{m-2}\mathbb{R}\mathrm{P}^2,\mathbb{R}\mathrm{P}^n) \to \Omega^{m-1}S^n \xrightarrow{[2]} \Omega^{m-1}S^n. \tag{2.8}$$

If we use the exact sequence induced from (2.8), it is easy to see that

$$\pi_{n-m}(\mathrm{Map}^*(\Sigma^{m-2}\mathbb{R}\mathrm{P}^2,\mathbb{R}\mathrm{P}^n)) = \mathbb{Z}/2.$$

Hence, $\pi_{n-m}(\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m,\mathbb{R}\mathrm{P}^n)) = \mathbb{Z}/2$ if $m \equiv 0 \pmod 2$, and (ii) is proved.

(iii) First, consider the case $n-m \geq 2$. Then because $2(n-m)-1 > n-m$, by using Theorem 2.2 there is an isomorphism

$$\alpha_{m,n_*} : \pi_{n-m}(V_{n,m}) \xrightarrow{\cong} \pi_{n-m}(\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m,\mathbb{R}\mathrm{P}^n)).$$

Hence, (iii) follows from Lemma Lemma 2.1.

So it remains to show (iii) for the case $m = n-1$. It suffices to show that $\alpha_{n,n-1_*} : \pi_1(V_{n,n-1}) \xrightarrow{\cong} \pi_1(\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^{n-1},\mathbb{R}\mathrm{P}^n))$ is an isomorphism. Because two spaces $V_{n,n-2}$ and $\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^{n-2},\mathbb{R}\mathrm{P}^n)$ are 1-connected, the diagram (2.3) for $m = n-1$ induces the following commutative exact sequences:

$$\begin{tikzcd}
\pi_2(V_{n,n-2}) \arrow[r, "\partial"] \arrow[d, "\alpha_{n,n-2*}"', "\cong"] & \pi_1(S^1) \arrow[r] \arrow[d, "E^{n-1}"', "\cong"] & \pi_1(V_{n,n-1}) \arrow[r] \arrow[d, "\alpha_{n-1,n*}"'] & 0 \\
\pi_2(\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^{m-2},\mathbb{R}\mathrm{P}^n)) \arrow[r, "\partial'"] & \pi_n(S^n) \arrow[r] & \pi_1(\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^{n-1},\mathbb{R}\mathrm{P}^n)) \arrow[r] & 0
\end{tikzcd}$$

We already proved that $\alpha_{n,n-2_*} : \pi_2(V_{n,n-2}) \xrightarrow{\cong} \pi_2(\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^{m-2}, \mathbb{R}\mathrm{P}^n))$ is an isomorphism. Hence, by the Five Lemma we see that the homomorphism $\alpha_{n,n-1_*} : \pi_1(V_{n,n-1}) \xrightarrow{\cong} \pi_1(\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n))$ is also an isomorphism. $\square$

Corollary 2.5. $\alpha_{n-1,n_*} : \pi_1(V_{n,n-1}) \xrightarrow{\cong} \pi_1(\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^{n-1}, \mathbb{R}\mathrm{P}^n)) \cong \mathbb{Z}/2$ *is an isomorphism for* $n \geq 2$. $\square$

Corollary 2.6 ([1]). *If* $2 \leq m < n$ *and* $n \equiv 0 \pmod 2$, $\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ *and* $\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ *are never homotopy equivalent.*

Proof. By Theorem 2.4, $\pi_{n-m}(\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \not\cong \pi_{n-m}(\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n))$ if $2 \leq m < n$ and $n \equiv 0 \pmod 2$, and the assertion follows. $\square$

Remark. Crabb and Sutherland show that $\pi_{n-1}(\mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = \mathbb{Q}$ and $\pi_{n-1}(\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = 0$ in [1], and they obtain the above result.

3. Rational homotopy types.

Definition 2. Let $\gamma_n : S^n \to \mathbb{R}\mathrm{P}^n$ denote the usual double covering and define the map $\gamma_{n\#} : \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n) \to \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ by $\gamma_{n\#}(f) = \gamma_n \circ f$.

Lemma 3.1. *If* $1 \leq m < n$, $\gamma_{n\#} : \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n) \xrightarrow{\simeq} \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ *is a homotopy equivalence.*

Proof. The proof is based on the induction over m. Because $\mathbb{R}\mathrm{P}^1 = S^1$, the assertion clearly holds for $m = 1$. Assume that the assertion is true for the case $m-1$, and note that $\Omega^m\gamma_n : \Omega^m S^n \xrightarrow{\simeq} \Omega^m\mathbb{R}\mathrm{P}^n$ is a homotopy equivalence. If we consider the commutative diagram of fibration sequences

$$\begin{array}{ccccc}
\Omega^m S^n & \longrightarrow & \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n) & \xrightarrow{r} & \mathrm{Map}^*(\mathbb{R}\mathrm{P}^{m-1}, S^n) \\
{\scriptstyle \Omega^m\gamma_n}\downarrow\simeq & & {\scriptstyle \gamma_{n\#}}\downarrow & & {\scriptstyle \gamma'_{n\#}}\downarrow\simeq \\
\Omega^m\mathbb{R}\mathrm{P}^n & \longrightarrow & \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) & \xrightarrow{r} & \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n)
\end{array}$$

the assertion easily follows. $\square$

We denote by $X_{(0)}$ the $\mathbb{Q}$-localization of a nilpotent space X. Then by using Lemma 3.1, there is a rational homotopy equivalence (cf. [2])

$$\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \simeq_{\mathbb{Q}} \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, S^n_{(0)}). \tag{3.1}$$

It is easy to see that there are homotopy equivalences

$$S^n_{(0)} \simeq \begin{cases} K(\mathbb{Q}, n) & \text{if } n \equiv 1 \pmod 2, \\ K(\mathbb{Q}, n) \times K(\mathbb{Q}, 2n-1) & \text{if } n \equiv 0 \pmod 2, \end{cases} \tag{3.2}$$

$$\mathrm{Map}_0^*(X, K(G,n)) \simeq \prod_{i=1}^{n} K(H^{n-i}(X,G), i) \tag{3.3}$$

for a connected space X ([9]). Then we have:

Lemma 3.2. *If* $2 \le m < n$ *and* $m \equiv 0 \pmod 2$, *there is a rational homotopy equivalence* $\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \ \simeq_{\mathbb{Q}}\ \{*\}$.

Proof. Since $\tilde{H}^*(\mathbb{R}\mathrm{P}^m, \mathbb{Q}) = 0$, the assertion follows from (3.1), (3.2) and (3.3). □

Lemma 3.3. *Let* $2 \le m < n$ *be integers such that* $m \equiv 1 \pmod 2$.

(i) *If* $n \equiv 0 \pmod 2$,

$$\pi_k(\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n-m,\ k = 2n-m-1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If* $n \equiv 1 \pmod 2$, $\pi_k(\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n-m, \\ 0 & \text{otherwise.} \end{cases}$

Proof. Since the proof is completely analogous to that of Lemma 3.2, we omit the detail. □

Corollary 3.4. *Let* $2 \le m < n$ *be integers.*

(i) *If* $n \equiv 1 \pmod 2$ *and* $m \equiv 0 \pmod 2$, *there is a rational homotopy equivalence* $\mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \ \simeq_{\mathbb{Q}}\ S^n$.

(ii) *If* $n \equiv 1 \pmod 2$ *and* $m \equiv 1 \pmod 2$,

$$\pi_k(\mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n-m,\ k = n, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If* $n \equiv 0$ *and* $m \equiv 0 \pmod 2$,

$$\pi_k(\mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k = n,\ k = 2n-1, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) *If* $n \equiv 0$ *and* $m \equiv 1 \pmod 2$,

$$\pi_k(\mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } k \in \{n-m, n, 2n-m-1\}, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

4. Proofs of the main results.

Proof of Theorem 1.3. The assertion (i) follows from Theorem 2.4. Let us consider the evaluation fibration sequence

$$\mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \to \mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \xrightarrow{ev} \mathbb{R}\mathrm{P}^n,$$

where ev is defined by $ev(f) = f(\mathbf{e}_m)$ for $f \in \mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$. If we consider the constant maps $\mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$, we can see that there is a splitting

$s : \mathbb{R}\mathrm{P}^m \to \mathrm{Map}_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ such that $ev \circ s = \mathrm{id}$. Hence, the other assertions easily follow from (i). $\square$

Proof of Theorem 1.4. The assertions follows from Lemma 3.2, Lemma 3.3 and Corollary 3.4. $\square$

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Kohhei Yamaguchi
Department of Mathematics
University of Electro-Communications
Chofu, Tokyo, 182-8585 Japan
e-mail address: kohhei@im.uec.ac.jp

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